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Non-linear observer on Lie Groups for left-invariant dynamics with right-left equivariant output ⋆

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Abstract: We consider a left-invariant dynamics on an arbitrary Lie group. We show that it is possible, when the output map is right-left equivariant, to build non-linear observers such that the error equation is autonomous. The theory is illustrated by an inertial navigation example.

Keywords: Lie group, symmetries, invariance, nonlinear asymptotic observers, inertial navigation.

1. INTRODUCTION

Symmetries have been used in the design of controllers and in optimal control theory (Fagnani and Willems [1993], Bloch *et al.* [1996], Koon and Marsden [1997], Grizzle and Marcus [1985], Jakubczyk [1998], Respondek and Tall [2002], Martin *et al.* [2004]). To our knowledge far less for the design of observers. In Mahony *et al.* [2005], Hamel and Mahony [2006], Creamer [1996] some observers are developed for attitude estimation using in particular gyros measurements and quaternions. In Mahony *et al.* [2005], Hamel and Mahony [2006] the authors use the group structure of $\mathrm{SO}(3)$ and its parameterizations by quaternions to derive observers. In section 2 we make an observer for attitude estimation using gyrometers, accelereometers, and magnetic measurements as in Mahony *et al.* [2005]. The construction of the observer relies on symmetries too and the construction method can be found in Aghannan and Rouchon [2002], Bonnabel and Rouchon [2005], Bonnabel *et al.* [2006] . The observer obtained is very close to the one of Mahony *et al.* [2005] but it does not strictly respects the symmetries and it is such that the error equation is autonomous. A preliminary version of this paper in french can be found in the proceedings of the CIFA 2005. Several modifications have been made. The example has evolved since the observer does not rely anymore on an algebraic inversion of the attitude quaternion and is thus much more robust. Moreover the link between the theory section of this paper and the general theory of invariant observers (Bonnabel *et al.* [2006]) was not fully understood at the time of the CIFA (it is now clear that we are considering a particular case of the general theory as explained in section 3.2). Moreover the theory developed in this paper explains why the error equation of the velocity-aided inertial navigation example of Bonnabel *et al.* [2006] is completely autonomous in the general case. We prove now that this result is compatible with the general theory of Bonnabel *et al.* [2006].

The first contribution of the paper (section 2) is to derive an observer (system (2)) for attitude estimation using gyros, acceleros, and magnetic measurements such that the error equation is autonomous (equation (3)). Thus the state-error dynamics is independent of the trajectory and of the time t. The second contribution (section 3) is the generalization to the case of a left-invariant dynamics with right-left equivariant output on a Lie group. It is shown, that "mixing" left and right invariance one can obtain observers (equation (7)) such that the error equation (equation (8)) is autonomous. Their construction relies on methods relative to the construction of invariant observers Aghannan and Rouchon [2002], Bonnabel and Rouchon [2005], Bonnabel *et al.* [2006].

2. EXAMPLE

2.1 Magnetic-aided inertial navigation

It is necessary in order to pilot a flying body to have at least a good knowledge of its orientation. This holds for manual, or semi automatic or automatic piloting. In low-cost or "strap-down" navigation systems the measurements of angular velocity ω and acceleration a by rather cheap gyrometers and accelerometers are completed by a measure of the earth magnetic field B. These various measurements are merged (data fusion) according to the motion equations of the system. The estimation of the orientation is generally performed by an extended Kalman filter. But the use of extended Kalman filter requires much calculus capacity because of the matrix inversions. The orientation can be described by an element of the group of rotations $\mathrm{SO}(3)$, which is the configuration space of a body fixed at a point. We identify the orientation to the rotation which

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maps the earth frame (frame attached to the earth) to the body frame (frame attached to the body).

2.2 Motion equations

In the appendix we recall the link between quaternions and rotation matrices. Indeed elements of $\mathbb{R}^3$ can be looked at as quaternions. We rather use quaternions than rotation matrices since they are more suited to computation and simulations. Indeed quaternions whose norm is 1 provide a global parameterization of SO(3) and it is easier to maintain the norm of a quaternion q equal to 1 than maintain a rotation matrix in SO(3). The motion equations of kinematics for a flying rigid body write thanks to quaternions $\mathbb{H}$

$$\frac{d}{dt}q = \frac{1}{2}q \cdot \omega \tag{1}$$

where

- $q \in \mathbb{H}$ is the quaternion of norm one which represents the rotation which maps the earth frame to the body frame,
- ω(t) is the instantaneous angular velocity vector measured by gyroscopes,
- $\cdot$ is the non commutative quaternion multiplication.

So the state of the system is the quaternion q. We make an additional assumption. Indeed the accelerometers measure $a = \frac{d}{dt}v + q^{-1} \cdot G \cdot q$ where $\frac{d}{dt}v$ is the acceleration of the center of mass of the body and G is the gravity vector. We suppose the acceleration of the center of mass is small with respect to $\| G \|$ (quasi-stationary flight). Thus we make the approximation $a \simeq q^{-1} \cdot G \cdot q$. The output is the specific acceleration $y_G = a$ and the magnetic field B measured by magnetometers in the body frame $y_B = q^{-1} \cdot B \cdot q$. The measured output y are thus

$$y = (y_G, y_B) = (q^{-1} \cdot G \cdot q, q^{-1} \cdot B \cdot q)$$

2.3 A non-linear observer

A Luenberger observer or an extended Kalman filter is very "linear" by construction. If $\hat{q}$ denotes the estimated state, the usual correction terms are linear functions of the output errors $(q^{-1} \cdot B \cdot q - \hat{q}^{-1} \cdot B \cdot \hat{q}, q^{-1} \cdot G \cdot q - \hat{q}^{-1} \cdot G \cdot \hat{q})$. We consider a class of non-linear observers which take into account the geometric structure of the dynamics (1). They have the following form:

$$\frac{d}{dt}\hat{q} = \frac{1}{2}\hat{q} \cdot \omega + \left(\sum_{i=1}^{3} E_i \, e_i \right) \cdot \hat{q} \tag{2}$$

where the E_i are smooth scalar functions of the output errors $\hat{q} \cdot y_B \hat{q}^{-1} - B$ and $\hat{q} \cdot y_G \cdot \hat{q}^{-1} - G$ vanishing when these errors vanish:

$$E_i \left(\hat{q} \cdot y_B \cdot \hat{q}^{-1} - B \, , \, \hat{q} \cdot y_G \cdot \hat{q}^{-1} - G \right)$$

with $E_i(0,0) = 0$. The e_i are the quaternions associated to the canonical basis of $\mathbb{R}^3$ (see appendix).

2.4 The error system

Instead of considering linear state-errors of the type $\Delta q := \hat{q} - q$ we consider the equivalent state-errors which use the quaternions multiplication $\cdot$ (instead of $-$):

$$r := \hat{q} \cdot q^{-1}$$

Let

$$E(r) = \sum_{i=1}^{3} E_i \left(r \cdot B \cdot r^{-1} - B \, , \, r \cdot G \cdot r^{-1} - G \right) \, e_i$$

It is an invariant quantity by right multiplication $\mathbb{H} \ni q \mapsto q \cdot h \in \mathbb{H}$ for any $h \in \mathbb{H}$. We have

$$\begin{aligned} \dot{r} &= \dot{\hat{q}} \cdot q^{-1} + q \cdot \left(-q^{-1} \cdot \dot{q} \cdot q^{-1} \right) \\ &= \left(\frac{1}{2}q \cdot \omega - E(r) \cdot \hat{q} - \frac{1}{2}q \cdot \omega \right) \cdot q^{-1} \\ &= E(r) \cdot r \end{aligned}$$

Thus

$$\dot{r} = E(r) \cdot r \tag{3}$$

where $E_i(0,0) = 0$. The error dynamics $r = \hat{q} \cdot q^{-1}$ does not depend on the trajectory $t \mapsto q(t)$. This reminds linear stationary theory, since the error obeys an autonomous differential equation.

2.5 First order approximation

A small error corresponds to r close to 1, where 1 is the unit quaternion. The choice of e_3 being arbitrary one can assume $e_3 = G$. Moreover one can suppose the earth magnetic field B to be horizontal by considering rather $B - (B.G)G$ than B, where . represents here the scalar product. It is easier to prove global convergence with Lyapunov arguments when we consider B to be horizontal (see section 2.6). Thus one can choose $e_1 = B$. We write the small error $r = 1 + \xi$ with ξ small. We have up to second order terms in ξ

$$\frac{d}{dt}\xi = \sum_{1 \le i \le 3 \; 1 \le j \le 6} \frac{\partial E_i}{\partial x_j} (\xi \times e_3, \xi \times e_1)^j \; e_i$$

where "$\times$" represents the usual vectorial product of $\mathbb{R}^3$. ξ is looked at as a vector and $(\xi, \eta)^j$ represents the j-th coordinate of (ξ, η) on the canonical basis of $\mathbb{R}^3 \times \mathbb{R}^3$. Thus we have

$$\frac{d}{dt} \begin{pmatrix} \xi^1 \\ \xi^2 \\ \xi^3 \end{pmatrix} = \left(\frac{\partial E_i}{\partial x_j} \right)_{1 \le i \le 3 \; 1 \le j \le 6} \begin{pmatrix} \xi^2 & 0 \\ -\xi^1 & \xi^3 \\ 0 & -\xi^2 \end{pmatrix}$$

Thus it is easy to choose the E_i's such that

$$\frac{d}{dt} \begin{pmatrix} \xi^1 \\ \xi^2 \\ \xi^3 \end{pmatrix} = -K_1 \begin{pmatrix} \xi^1 \\ \xi^2 \\ 0 \end{pmatrix} - K_2 \begin{pmatrix} 0 \\ \xi^2 \\ \xi^3 \end{pmatrix} \tag{4}$$

We have exponential convergence of the linearized system around any trajectory and we have the choice of the time constants $1/K_1$ and $1/K_2$.

2.6 A class of non-linear asymptotic observers

Inspiring from Bonnabel and Rouchon [2005], consider the following non-linear observer

$$\begin{cases} \frac{d}{dt}\hat{q} = \frac{1}{2}\hat{q} \cdot \omega \\ \qquad - \; [K_1 G \times (\hat{q} \cdot q^{-1} \cdot G \cdot q \cdot \hat{q}^{-1} - G) \\ \qquad + K_2 B \times (\hat{q} \cdot q^{-1} \cdot B \cdot q \cdot \hat{q}^{-1} - B)] \cdot \hat{q} \\ \frac{d}{dt}\hat{q} = \frac{1}{2}\hat{q} \cdot \omega - \; [K_1 G \times (\hat{q} \cdot y_G \cdot \hat{q}^{-1} - G) \\ \qquad + K_2 B \times (\hat{q} \cdot y_B \cdot \hat{q}^{-1} - B)] \cdot \hat{q} \end{cases}$$

It corresponds indeed to (2). Simple computations show that its linear approximation for r close to 1 is exactly (4). Moreover for any $K_1, K_2 > 0$:

$$\lim_{t \mapsto +\infty} \hat{q}(t) - q(t) = 0 \text{ or } \lim_{t \mapsto +\infty} \hat{q}(t) + q(t) = 0$$

for any initial condition. This results was partly shown in Bonnabel and Rouchon [2005] and is shown in Mahony *et al.*

[2005]. Notice q and $-q$ correspond to the same rotation of SO(3).

The proof consists in taking the following Lyapunov function $V(r) = \|r \cdot B \cdot r^{-1} - B\|^2 + \|r \cdot G \cdot r^{-1} - G\|^2$ for (3).

3. THEORY

The fact that the error dynamics obeys an autonomous differential equation can be extended in the general case where the state space (subset of $\mathbb{H}$ of quaternions whose norm is 1) is replaced by a Lie group G, the dynamics (1) by a left-invariant vector field which can depend on t, and the output space by a smooth manifold. In this paper we use the Lie group structure to design observers. See Jurdevic and Sussman [1972] as one of the pioneering papers on control systems on Lie group.

In all this section we use the same notations as in Arnold [1976]. We suppose the real Lie group G to be of dimension n and we consider a left-invariant dynamics on G. One will show now that under some assumptions on the output map (equivariance versus right translations), one can build a non-linear observer such that the error equation satisfies an autonomous differential equation.

3.1 Left invariant dynamics and right-left equivariant output

Consider the following dynamics :

$$\frac{d}{dt}g(t) = F(g,t) \tag{5}$$

where g is an element of G, and F is a smooth vector field on G. Let us suppose the dynamics is *left*-invariant, i.e:

$$\forall g,h \in G \;\; F(L_h(g),t) = L_{h*}F(g,t)$$

where $L_h : g \mapsto h \cdot g$ is the left multiplication on G, and L_{h*} the induced map on the tangent space. L_{h*} maps the tangent space $TG|_g$ to $TG|_{hg}$. G is a group of symmetries for itself : for all $h \in G$, the change of variables $g_2(t) = h \cdot g_1(t)$ leaves the dynamics equations unchanged :

$$\frac{d}{dt}g_2(t) = F(g_2(t),t)$$

As in Arnold [1976] let

$$\omega_s = L_{g^{-1}*}\dot{g} \in \mathfrak{g}$$

ω_s is an element of the Lie algebra $\mathfrak{g}$ of G. Indeed one can look at any left invariant dynamics on G as a motion of a "generalized rigid body" with configuration space G. Thus one can look at $\omega_s(t) = F(e,t)$ as the "angular velocity in the body", where e is the group identity element. We will systematically write the left-invariant dynamics (5)

$$\frac{d}{dt}g(t) = L_{g*}\omega_s(t) \tag{6}$$

Let us suppose that $H : G \to Y$ is a *right*-left equivariant smooth output map. Y is a smooth manifold, it can be in particular a Lie group or an euclidian space as in the example. Inspiring from Aghannan and Rouchon [2002] one can define the right-left equivariance the following way : for all $h \in G$, there exists a smooth map $\rho_h : Y \to Y$, such that for all $g \in G$, $H(g \cdot h) = \rho_h(H(g))$ i,e

$$H(R_h(g)) = \rho_h(H(g))$$

where R_h denotes the right multiplication on G (and R_{h*} the induced map on tangent spaces). This means the group action on itself by right multiplication corresponds to another group action on the output space. We consider left-invariant systems with right-left equivariant output.

In the example the dynamics is left invariant indeed $\frac{d}{dt}q(t) = q \cdot \omega = L_{q*}\omega(t)$ and the output $H(q) = q^{-1} \cdot B \cdot q$ is right invariant

$$\begin{aligned} H(R_p(q)) &= ((q \cdot p)^{-1} \cdot G \cdot (q \cdot p), (q \cdot p)^{-1} \cdot B \cdot (q \cdot p)) \\ &= (p^{-1} \cdot y_G \cdot p, p^{-1} \cdot y_B \cdot p) = \rho_p(H(q)) \end{aligned}$$

3.2 The dynamics is right-invariant for a different definition of the group action

Consider the dynamics (5). It can be viewed as a right-invariant dynamics on G! Indeed let us look at $\omega_s(t)$ as an input : $u(t) = \omega_s(t) \in \mathscr{U}$, where $\mathscr{U} \subset \mathbb{R}^n$ is the input space. Let us define the action of G on $\mathscr{U}$ via the diffeormorphisms ψ_g for all g:

$$\psi_g = DL_{g^{-1}}DR_g$$

It means ψ_g is the differential of the interior automorphism of G. It is a group action indeed since for all $g,h \in G$ we have $\psi_g \circ \psi_h = \psi_{gh}$. And the dynamics (5) writes

$$\frac{d}{dt}x = F(x,u) = DL_x u$$

and can be seen as a right-invariant dynamics . For all x,g we have indeed:

$$\begin{aligned} \frac{d}{dt}R_g(x) = DR_g DL_x \omega_s(t) &= DL_x DL_g DL_{g^{-1}} DR_g \omega_s(t) \\ &= DL_{R_g(x)} \psi_g(\omega_s(t)) \\ &= F(R_g(x), \psi_g(u)) \end{aligned}$$

3.3 Observability

If the dimension of the output space is strictly smaller than the dimension of the state space ($\dim y < \dim g$) the system is necessarily not observable. This comes from the fact that, in this case, there exists two distinct elements g_1 and g_2 of G such that $H(g_1) = H(g_2)$. If $g(t)$ is a trajectory of the system, we have

$$\frac{d}{dt}g(t) = L_{g*}\omega_s(t)$$

and because of the left-invariance, $g_1 g(t)$ and $g_2 g(t)$ are also trajectories of the system:

$$\frac{d}{dt}(g_1 \cdot g(t)) = L_{g_1 g*}\omega_s(t), \quad \frac{d}{dt}(g_2 \cdot g(t)) = L_{g_2 g*}\omega_s(t).$$

But since H is right-left equivariant:

$$H(g_1 \cdot g(t)) = \rho_{g(t)}H(g_1) = \rho_{g(t)}H(g_2) = H(g_2 g(t)).$$

The trajectories $g_1 \cdot g(t)$ and $g_2 \cdot g(t)$ are distinct and for all t they correspond to the same output. The system is unobservable.

3.4 Construction of the observer

The dynamics is right-invariant, and the output is right-left equivariant, so we follow the method of bonnabel-et-al:Arxiv06. Let $(W_1,...,W_n)$ be a right-invariant frame (see Olver [1995]). An invariant frame is a set of n point-wise linearly independent invariant vector fields forming a basis of the tangent space at any $g \in G$. The vector fields verify

$$W_i(g) = DR_g W_i(e)$$

(by definition). Moreover we know how to build invariant output errors ,i.e, smooth scalar functions E_i of $\hat{g}$ and $H(g)$, which verify :

$$\forall h,g \;\; E_i(R_h(\hat{g}),H(R_h(g))) = E_i(\hat{g},H(g))$$
$$E_i(g,H(g)) = 0$$

It suffices to take

$$E_i(\hat{g},y) = \mathscr{L}_i(\rho_{\hat{g}^{-1}}(y)) = \mathscr{L}_i(H(g\hat{g}^{-1}))$$

with $\mathscr{L}_i(H(e)) = 0$. Consider the class of observers of the form

$$\frac{d}{dt}\hat{g} = DL_{\hat{g}}\omega_s(t) + \sum_{i=1}^{n} E_i(\hat{g},y)W_i(\hat{x})$$
$$= DL_{\hat{g}}\omega_s(t) + DR_{\hat{g}}(\sum_{i=1}^{n} E_i(\hat{g},y)W_i(e)) \quad (7)$$

where the E_i are output errors (for right multiplication), and $(W_1,...,W_n)$ is an invariant frame (for right multiplication) and $y = h(x)$ is the output. These are invariant observers when ω_s is viewed as an input on which G acts via ψ_g.

3.5 The error system

Let us define the error (invariant by right multiplication) $G \ni r = (\hat{g}g^{-1}) = L_{\hat{g}}(g^{-1})$. The error dynamics verifies

$$\dot{r} = R_{r*}(\sum_{i=1}^{n} E_i(e,H(r^{-1}))W_i(e)) \quad (8)$$

Indeed we have

$$\dot{r} = L_{\hat{g}*}(\dot{\overline{g^{-1}}}) + D_g L_{\hat{g}}(g^{-1})\dot{\hat{g}}$$

We have

$$D_g L_{\hat{g}}(g^{-1})\dot{\hat{g}} = R_{g^{-1}*}(\dot{\hat{g}})$$
$$= R_{g^{-1}*}(F(\hat{g},t) + \sum_{i=1}^{n} E_i(\hat{g},y)W_i(\hat{g}))$$
$$= R_{g^{-1}*}L_{\hat{g}*}\; F(e,t) + R_{g^{-1}*}R_{\hat{g}*}\sum_{i=1}^{n} E_i(\hat{g},y)W_i(e)$$
$$= R_{g^{-1}*}L_{\hat{g}*}\; \omega_s + R_{r*}\sum_{i=1}^{n} E_i(\hat{g},y)W_i(e)$$

But invariance implies

$$E_i(\hat{g},y) = E_i(\hat{g},H(g)) = E_i(e,H(r^{-1}))$$

And we have also

$$L_{\hat{g}*}(\dot{g}^{-1}) = -L_{\hat{g}*}R_{g^{-1}*}L_{g^{-1}*}\dot{g} = -L_{\hat{g}*}R_{g^{-1}*}\omega_s$$
$$= -R_{g^{-1}*}L_{\hat{g}*}\omega_s$$

So we have an autonomous differential equation independent from the trajectory $t \mapsto g(t)$:

$$\dot{r} = R_{r*}(\sum_{i=1}^{n} E_i(e,H(r^{-1}))W_i(e)).$$

3.6 First order approximation

We suppose that r is close to e . Let ξ be the small element of the Lie algebra $\mathfrak{g}$ such that $r = \exp\; \xi$. Let $p = \dim y$ be the dimension of the output, we have

$$\frac{d}{dt}\xi = -D_y E\;|_{e,H(e)}\;\; DH\;|_e\;(\xi)$$

where we call $E = (E_1,...,E_n)$.

Let us define a scalar product on the tangent space $\mathfrak{g}$ at e , and let us consider the adjoint operator of $DH\;|_e$ in the sense of the metrics associated to the scalar product. The adjoint operator is denoted by $(DH\;|_e)^T$ and we take for all $\eta \in \mathfrak{g}$ close to zero

$$E(e,\exp\eta) = K(DH\;|_e)^T\,\eta.$$

Thanks to right invariance of E it is possible to define E without ambiguity for all $(\hat{g},g)$ with $\hat{g}g^{-1}$ close to 0.

The first order approximation writes

$$\dot{\xi} = -K\; DH^T\; DH\; \xi \quad (9)$$

and for $K > 0$, admits as Lyapunov function $\|\xi\|^2$ which the length of ξ in the sense of the scalar product.

3.7 A class of non-linear first-order convergent observers

Consider the following observers :

$$\frac{d}{dt}\hat{g} = L_{\hat{g}*}\omega_s(t) + R_{\hat{g}*}[\sum_{i=1}^{n}[E_i(\rho_{\hat{g}}^{-1}(H(g)) - E_i(H(e))]W_i(e)]$$

where the E_i's are smooth scalar functions. Using the first order approximation design, take $E = (E_1,...,E_n)$ such that the symmetric part (in the sense of the scalar product chosen on TG_e) of the linear map

$$\xi \mapsto \frac{\partial E}{\partial y}\;|_{H(e)}\;\frac{\partial H}{\partial g}\;|_e\;\xi$$

is negative. When it is negative definite, we get locally exponentially convergent non-linear observers around any system trajectory.

4. CONCLUSION

We show in this article that when a group of symmetries acts on itself and when the dynamics is left invariant and the output map is right-left equivariant, we can build non-linear observers for which the error equation follows an autonomous differential equation. The construction is based on the notion of invariant output error (Aghannan and Rouchon [2002], Martin *et al.* [2004]).

If the output is left-invariant one can show the error equation only depends on the error r and is still independent on the trajectory but in this case it can depend on the time t if $\omega(t)$ depends on time t.(see Bonnabel *et al.* [2006])

5. APPENDIX: QUATERNIONS

A quaternion p can be looked at a set of a scalar $p_0 \in \mathbb{R}$ and a vector $p \in \mathbb{R}^3$,

$$p = \begin{pmatrix} p_0 \\ p \end{pmatrix}.$$

The quaternion multiplication $\cdot$ writes

$$p \cdot q := \begin{pmatrix} p_0 q_0 - p \cdot q \\ p_0 q + q_0 p + p \times q \end{pmatrix}.$$

The unit element is

$$e := \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and $(p \cdot q)^{-1} = q^{-1} \cdot p^{-1}$.

Any vector $p \in \mathbb{R}^3$ can be looked at as a quaternion

$$p := \begin{pmatrix} 0 \\ p \end{pmatrix},$$

For instance the quaternions associated to the canonical basis of $\mathbb{R}^3$ are $\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$, $\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$, $\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$ and we have the following formulas

$$p \times q := p \times q = \frac{1}{2}(p \cdot q - q \cdot p)$$
$$(p \cdot q)r = -\frac{1}{2}(p \cdot q + q \cdot p) \cdot r.$$

To any quaternion q whose norm is 1 one can associate a rotations matrix $R_q \in \mathrm{SO}(3)$ the following way

$$q^{-1} \cdot p \cdot q = R_q \cdot p \quad \text{for all } p.$$

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