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Continuous measurement of a statistic quantum ensemble

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Abstract— **We consider an ensemble of quantum systems whose average evolution is descried by a density matrix solution of a Lindbladian differential equation. We will suppose that the decoherence is only due to a highly unstable excited state. We measure then the spontaneously emitted photons. Whenever we consider resonant laser fields, we can remove the fast dynamics associated to the excited state in order to obtain another differential equation for the slow part. We show that this slow differential equation is still of Lindblad type. The decoherence terms are then of second order and the measurement structure depends explicitly on the resonant laser field. The later can be adjusted to give information on a specific linear combination of the density matrix entries. The case of a 3-level system is treated in details and before the general case. On this 3-level system, we show how a simple PI regulator allows us to robustly control the 2-level slow subsystem.**

I. INTRODUCTION

The main difference between the control by feedback in its classical sense and in its quantum sense is due to the fact that measuring a quantum system perturbs it. The exploited idea in former works consists in opening the system toward the environmentally induced quantum fluctuations. The system is therefore entangled to its environment. Open systems are essentially described in density operator language which replaces that of wave functions in elementary quantum mechanics. We have therefore the possibility to analyze the behavior of a statistical ensemble of systems [13], [8], [1] as well as that of a unique system (Monte Carlo trajectories) [7], [14], [15].

In order to avoid the wave packet collapse phenomena induced by the measurement, we consider here the weak measurement of an ensemble of quantum systems independently and identically prepared. This approach has already been considered in other contexts (see for example [13], [1]).

Each member of the ensemble undergoes an identical evolution. The considered dynamics represent then the average effect of all the dynamics corresponding to each element.

The system is coupled to a measurement tool, perturbing weakly the system. We will see that, in certain cases, this perturbation can be neglected whenever the dynamics associated to the measurement are much faster than those of the main system.

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This paper has for its first subject to show on a 3-level system (with a highly unstable level) how we can reduce the dynamics to that of a 2-level system with scalar input and output. We will see then how a simple PI can be used to regulate this reduced model. Next we will treat the general case with density matrix description verifying a Markovian master equation of Lindblad type [6]. This differential equation involves both the coherent and dissipative evolutions of the system. We assume the slow/fast system to verify a Lindblad type equation and then we show that the reduced slow dynamics also undergoes a Linbladian behavior.

In Section II, we present the dynamics for a 3-level system coupled to a coherent and resonant laser field. We also take into account the spontaneous emissions of the system. We assume that one of the states is highly unstable and relaxes rapidly toward the ground state by emitting a photon, photons that we observe by a photo-detector. Whenever the laser is resonant with this unstable state, we show, using the slow/fast systems theory, that the average number of photons per time unit (photo-detectors signal) can be interpreted as a continuous measurement of certain entries of the density matrix for the slow dynamics. The later corresponds to the dynamics of a 2-level system. Furthermore, we show that this continuous measurement perturbs the slow dynamics only by second order dissipative terms. Therefore, in a first approximation, we can neglect these decoherence terms. We have thus a conservative 2-level system, represented by a vector on the Block sphere together with a continuous measurement (see the system (4)).

Such kind of structure is relevant for the physical experiences. Experiments on a $^{40}Ca^{+}$ trapped ion (performed by the physicists at the “Institute for Physical Experiments” at the university of Innsbruck) [12], [11] take advantage of such kind of structure. Figure 1 shows the relevant energy levels and couplings of the $^{40}Ca^{+}$ ion in a cavity. The three level considered in this scheme is spanned by the $S_{1/2}-P_{3/2}$ transition (wavelength of 393 nm), driven by an external laser field, and the levels $P_{3/2}-D_{5/2}$ which are coupled by a cavity mode (wavelength of 854 nm). The lifetime of the $D_{5/2}$ state is $1.16s$, which can be considered as stable compared with the time scale of the population transfer process of $50\mu s$. The lifetime of the $P_{3/2}$ state is $7.4ns$, and the branching ratios from $P_{3/2}$ to $S_{1/2}$, $D_{5/2}$ and $D_{3/2}$ are 1, $1/17.6$, $1/150.8$, respectively [10]. The situation is thus quite similar to that considered in the Section II.

In Section III, we show that the reduced model obtained in Section II can be efficiently controlled using a simple PI regulator. In Section IV, we extend the approach of the Section II to the case of an arbitrary number n of energy levels. The dynamics of the density matrix is of

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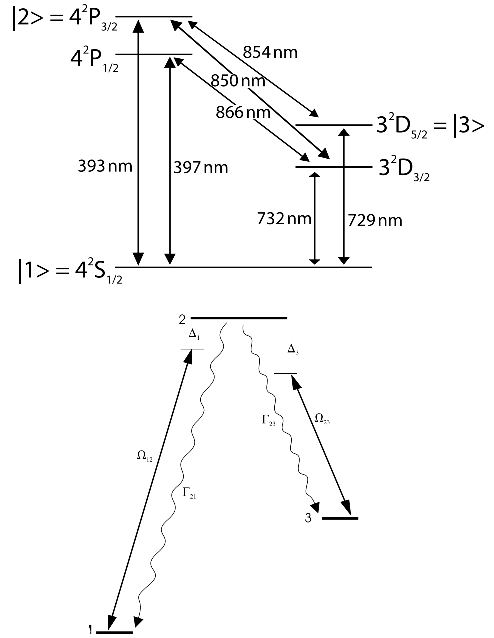

Fig. 1. Relevant energy levels of the ${}^{40}Ca^{+}$ ion (derived from [11]).

Lindblad type [6], [8] and one of the states is supposed to be highly unstable. We show that the slow dynamics is still of Lindblad type, and that the decoherence due to the highly unstable state is of second order and thus can be neglected.

II. 3-LEVEL SYSTEM

Let us take a 3-level atom coupled to resonant laser fields. We note the three states of the atom by $|1\rangle$, $|2\rangle$ and $|3\rangle$ corresponding respectively to the energies E_1, E_2 and E_3 with $E_1 < E_2, E_3$. We consider here the Bolch-optic equations which take into account the effect of the spontaneous emission (see for example, [4], [2]). The system without control is given by the following equation with $\Omega_{ij} = \frac{E_i - E_j}{\hbar}$:

$$\begin{aligned}
\frac{d}{dt}\sigma_1 &= \Gamma_2\sigma_2 + \Gamma_3\sigma_3 \\
\frac{d}{dt}\sigma_2 &= -\Gamma_2\sigma_2 \\
\frac{d}{dt}\sigma_3 &= -\Gamma_3\sigma_3 \\
\frac{d}{dt}\sigma_{12} &= -\imath\Omega_{12}\sigma_{12} - \frac{\Gamma_2}{2}\sigma_{12} \\
\frac{d}{dt}\sigma_{23} &= -\imath\Omega_{23}\sigma_{23} - \frac{(\Gamma_2+\Gamma_3)}{2}\sigma_{23} \\
\frac{d}{dt}\sigma_{31} &= -\imath\Omega_{31}\sigma_{31} - \frac{\Gamma_3}{2}\sigma_{31}.
\end{aligned}$$

The terms $\{\sigma_i\}$ correspond to the populations of different states, and the terms σ_{ij} are the coherence terms. The quantity $\mathrm{Tr}[\sigma] = \sigma_1 + \sigma_2 + \sigma_3$ is conserved during the evolution. We have supposed that the relaxation is just due to the spontaneous emission with Γ_2^{-1} and Γ_3^{-1} being the atomic lifetimes of the states $|2\rangle$ and $|3\rangle$, respectively.

Let us now consider the effect of the laser. We suppose that the laser acts on the system through the transitions between the states $|1\rangle$ and $|2\rangle$, the states $|2\rangle$ and $|3\rangle$ and the states $|3\rangle$ and $|1\rangle$. The interaction Hamiltonian is thus given by :

$$\begin{aligned}
&\hbar[a_{12}\ u\ (|1\rangle\langle 2| + |2\rangle\langle 1|) \\
&\quad + a_{23}\ u\ (|2\rangle\langle 3| + |3\rangle\langle 2|) + a_{31}\ u\ (|3\rangle\langle 1| + |1\rangle\langle 3|)]
\end{aligned}$$

where a_{12}, a_{23} and a_{31} are real coefficients. A simple computation shows that in the density matrix description and whenever the effect of the control field is taken into account, the equations can be written in the following form:

$$\begin{aligned}
\frac{d}{dt}\sigma_1 &= \Gamma_2\sigma_2 + \Gamma_3\sigma_3 - 2\ a_{12}\ u\ \Im(\sigma_{12}) + 2\ a_{31}\ u\ \Im(\sigma_{31}) \\
\frac{d}{dt}\sigma_2 &= -\Gamma_2\sigma_2 - 2\ a_{23}\ u\ \Im(\sigma_{23}) + 2\ a_{12}\ u\ \Im(\sigma_{12}) \\
\frac{d}{dt}\sigma_3 &= -\Gamma_3\sigma_3 - 2\ a_{31}\ u\ \Im(\sigma_{31}) + 2\ a_{23}\ u\ \Im(\sigma_{23}) \\
\frac{d}{dt}\sigma_{12} &= -\imath\Omega_{12}\sigma_{12} - \frac{\Gamma_2}{2}\sigma_{12} \\
&\quad + \imath\ a_{12}\ u\ \sigma_1 - \imath\ a_{12}\ u\ \sigma_2 - \imath\ a_{31}\ u\ \sigma_{23}^* + \imath a_{23} u \sigma_{31}^* \\
\frac{d}{dt}\sigma_{23} &= -\imath\Omega_{23}\sigma_{23} - \frac{(\Gamma_2+\Gamma_3)}{2}\sigma_{23} \\
&\quad + \imath\ a_{23}\ u\ \sigma_2 - \imath\ a_{23}\ u\ \sigma_3 - \imath\ a_{12}\ u\ \sigma_{31}^* + \imath a_{31} u \sigma_{23}^* \\
\frac{d}{dt}\sigma_{31} &= -\imath\Omega_{31}\sigma_{13} - \frac{\Gamma_3}{2}\sigma_{31} \\
&\quad + \imath\ a_{31}\ u\ \sigma_3 - \imath\ a_{31}\ u\ \sigma_1 - \imath\ a_{23}\ u\ \sigma_{12}^* + \imath a_{12} u \sigma_{31}^*
\end{aligned}$$

where $\Im$ and $*$ significate the imaginary part and the complex conjugate of a complex number.

We suppose the laser field to be resonant with the transition frequencies:

$$u = \frac{v_{12}}{a_{12}} e^{\imath\Omega_{12}t} + \frac{v_{23}}{a_{23}} e^{\imath\Omega_{23}t} + \frac{v_{31}}{a_{31}} e^{\imath\Omega_{31}t} + C.C.$$

where we have denoted by C.C. the complex conjugate and where v_{12}, v_{23} and v_{31} are complex amplitudes.

A. Averaging

We suppose here that the frequencies $|\Omega_{12}$, $|\Omega_{23}$ and $|\Omega_{31}|$ are all different and that the atomic lifetime of the states $|2\rangle$ and $|3\rangle$ are sufficiently large to have

$$\Gamma_2, \Gamma_3 \ll |\Omega_{12}|, |\Omega_{23}|, |\Omega_{31}|.$$

We assume moreover that the amplitudes for the control field u are not so large (weak field limit):

$$|v_{12}|, |v_{23}|, |v_{31}| \ll |\Omega_{12}|, |\Omega_{23}|, |\Omega_{31}|.$$

All these assumptions together permit us to apply the *rotating wave approximation*, also named as the secular approximation. Such kind of approximation can easily be justified using averaging type arguments.

Considering the time-dependent change of variables

$$\tilde{\sigma}_{12} = e^{\imath\Omega_{12}t}\sigma_{12}, \quad \tilde{\sigma}_{23} = e^{\imath\Omega_{23}t}\sigma_{23} \qquad \text{et} \quad \tilde{\sigma}_{31} = e^{\imath\Omega_{31}t}\sigma_{31}$$

and removing the high frequency terms, we obtain the following average system :

$$\begin{aligned}
\frac{d}{dt}\sigma_1 &= \Gamma_2\sigma_2 + \Gamma_3\sigma_3 \\
&\quad + \imath\ (v_{12}\widetilde{\sigma}_{12} - v_{12}^*\widetilde{\sigma}_{12}^*) - \imath\ (v_{31}\widetilde{\sigma}_{31} - v_{31}^*\widetilde{\sigma}_{31}^*) \\
\frac{d}{dt}\sigma_2 &= -\Gamma_2\sigma_2 \\
&\quad + \imath\ (v_{23}\widetilde{\sigma}_{23} - v_{23}^*\widetilde{\sigma}_{23}^*) - \imath\ (v_{12}\widetilde{\sigma}_{12} - v_{12}^*\widetilde{\sigma}_{12}^*) \\
\frac{d}{dt}\sigma_3 &= -\Gamma_3\sigma_3 \\
&\quad + \imath\ (v_{31}\widetilde{\sigma}_{31} - v_{31}^*\widetilde{\sigma}_{31}^*) - \imath\ (v_{23}\widetilde{\sigma}_{23} - v_{23}^*\widetilde{\sigma}_{23}^*) \\
\frac{d}{dt}\widetilde{\sigma}_{12} &= -\frac{\Gamma_2}{2}\widetilde{\sigma}_{12} \\
&\quad + \imath\ v_{12}^*(\sigma_1 - \sigma_2) - \imath v_{31}\widetilde{\sigma}_{23}^* + \imath v_{23}\widetilde{\sigma}_{31}^* \\
\frac{d}{dt}\widetilde{\sigma}_{23} &= -\frac{(\Gamma_2+\Gamma_3)}{2}\widetilde{\sigma}_{23} \\
&\quad + \imath\ v_{23}^*(\sigma_2 - \sigma_3) - \imath v_{12}\widetilde{\sigma}_{31}^* + \imath v_{31}\widetilde{\sigma}_{12}^* \\
\frac{d}{dt}\widetilde{\sigma}_{31} &= -\frac{\Gamma_3}{2}\widetilde{\sigma}_{31} \\
&\quad + \imath\ v_{31}^*(\sigma_3 - \sigma_1) - \imath v_{23}\widetilde{\sigma}_{12}^* + \imath v_{12}\widetilde{\sigma}_{23}^*.
\end{aligned}$$

B. Slow/Fast approximation

The probability conservation is still satisfied : $\sigma_1+\sigma_2+\sigma_3 = 1$ remains constant during the evolution of the system. Thus we can remove the equation corresponding to $\frac{d}{dt}\sigma_1$, by replacing σ_1 with $1-\sigma_2-\sigma_3$. We have

$$\begin{aligned}
\frac{d}{dt}\sigma_2 &= -\Gamma_2\sigma_2 \\
&\quad + \imath\ (v_{23}\widetilde{\sigma}_{23} - v_{23}^*\widetilde{\sigma}_{23}^*) - \imath\ (v_{12}\widetilde{\sigma}_{12} - v_{12}^*\widetilde{\sigma}_{12}^*) \\
\frac{d}{dt}\sigma_3 &= -\Gamma_3\sigma_3 \\
&\quad + \imath\ (v_{31}\widetilde{\sigma}_{31} - v_{31}^*\widetilde{\sigma}_{31}^*) - \imath\ (v_{23}\widetilde{\sigma}_{23} - v_{23}^*\widetilde{\sigma}_{23}^*) \\
\frac{d}{dt}\widetilde{\sigma}_{12} &= -\frac{\Gamma_2}{2}\widetilde{\sigma}_{12} \\
&\quad + \imath\ v_{12}^*(1 - 2\sigma_2 - \sigma_3) - \imath v_{31}\widetilde{\sigma}_{23}^* + \imath v_{23}\widetilde{\sigma}_{31}^* \\
\frac{d}{dt}\widetilde{\sigma}_{23} &= -\frac{(\Gamma_2+\Gamma_3)}{2}\widetilde{\sigma}_{23} \\
&\quad + \imath\ v_{23}^*(\sigma_2 - \sigma_3) - \imath v_{12}\widetilde{\sigma}_{31}^* + \imath v_{31}\widetilde{\sigma}_{12}^* \\
\frac{d}{dt}\widetilde{\sigma}_{31} &= -\frac{\Gamma_3}{2}\widetilde{\sigma}_{31} \\
&\quad + \imath\ v_{31}^*(-1 + \sigma_2 + 2\sigma_3) - \imath v_{23}\widetilde{\sigma}_{12}^* + \imath v_{12}\widetilde{\sigma}_{23}^*.
\end{aligned}$$

Suppose that the atomic lifetime of the third state (Γ_3^{-1}) is much shorter than that of the second state (Γ_2^{-1}). So the population dynamics for the third state will be much faster than that of the second state. The system remains in the third state for a very short time. As soon as the system reaches the third state, it emits spontaneously a photon and relaxes toward $|1\rangle$, the ground state. If we couple to this third transition between $|1\rangle$ and $|3\rangle$ a resonant light and detect the fluorescent photons emitted from $|3\rangle$, we can deduce a partial measure on the system. This third state can be seen as a part of the measurement tool.

Let us return to the equations. The inequalities

$$|v_{12}|, |v_{23}|, |v_{31}|, \Gamma_2 \ll \Gamma_3$$

allow us to apply the singular perturbation theory (also called adiabatic approximation) [9]. Take $\Gamma_3 = \overline{\Gamma}_3/\varepsilon$ where $0<\varepsilon\ll 1$ and $\overline{\Gamma}_3$ is of the same magnitude order as Γ_2. The dynamics read

$$\begin{aligned}
\frac{d}{dt}\sigma_2 &= -\Gamma_2\sigma_2 + \imath\ (v_{23}\widetilde{\sigma}_{23} - v_{23}^*\widetilde{\sigma}_{23}^*) - \imath\ (v_{12}\widetilde{\sigma}_{12} - v_{12}^*\widetilde{\sigma}_{12}^*) \\
\frac{d}{dt}\sigma_3 &= -\frac{\overline{\Gamma}_3}{\varepsilon}\sigma_3 + \imath\ (v_{31}\widetilde{\sigma}_{31} - v_{31}^*\widetilde{\sigma}_{31}^*) - \imath\ (v_{23}\widetilde{\sigma}_{23} - v_{23}^*\widetilde{\sigma}_{23}^*) \\
\frac{d}{dt}\widetilde{\sigma}_{12} &= -\frac{\Gamma_2}{2}\widetilde{\sigma}_{12} + \imath\ v_{12}^*(1 - 2\sigma_2 - \sigma_3) - \imath v_{31}\widetilde{\sigma}_{23}^* + \imath v_{23}\widetilde{\sigma}_{31}^* \\
\frac{d}{dt}\widetilde{\sigma}_{23} &= -\frac{\overline{\Gamma}_3}{2\varepsilon}\widetilde{\sigma}_{23} - \frac{\Gamma_2}{2}\widetilde{\sigma}_{23} \\
&\quad + \imath\ v_{23}^*(\sigma_2 - \sigma_3) - \imath v_{12}\widetilde{\sigma}_{31}^* + \imath v_{31}\widetilde{\sigma}_{12}^* \\
\frac{d}{dt}\widetilde{\sigma}_{31} &= -\frac{\overline{\Gamma}_3}{2\varepsilon}\widetilde{\sigma}_{31} \\
&\quad + \imath\ v_{31}^*(-1 + \sigma_2 + 2\sigma_3) - \imath v_{23}\widetilde{\sigma}_{12}^* + \imath v_{12}\widetilde{\sigma}_{23}^*.
\end{aligned}$$

Such a system admits exactly the same structure as that of the slow/fast systems discussed in the Appendix where x corresponds to $(\sigma_2, \widetilde{\sigma}_{12})$ and y corresponds to $(\sigma_3, \widetilde{\sigma}_{23}, \widetilde{\sigma}_{31})$. So, for the terms of order less or equal to 1 with respect to ε, we have the following approximation (we have used the scaling $\sigma_3 \mapsto \frac{\overline{\Gamma}_3}{\varepsilon}\sigma_3$ in order to be able to apply directly the approximation of the Appendix):

$$\begin{aligned}
\widetilde{\sigma}_{23} &= \frac{2\imath\varepsilon}{\overline{\Gamma}_3}[v_{23}^*\sigma_2 + v_{31}\widetilde{\sigma}_{12}^*] + O(\varepsilon^2) \\
\widetilde{\sigma}_{31} &= \frac{2\imath\varepsilon}{\overline{\Gamma}_3}[v_{31}^*(\sigma_2 - 1) - v_{23}\widetilde{\sigma}_{12}^*] + O(\varepsilon^2) \\
\frac{\overline{\Gamma}_3}{\varepsilon}\sigma_3 &= \frac{4\varepsilon}{\overline{\Gamma}_3}\left[|v_{31}|^2(1-\sigma_2) + |v_{23}|^2\sigma_2 + 2\Re(v_{23}v_{31}\widetilde{\sigma}_{12}^*)\right] \\
&\quad + O(\varepsilon^2).
\end{aligned}$$

Taking the terms of order less or equal to 1, for the reduced slow dynamics, we have (remember that $\Gamma_3 = \overline{\Gamma}_3/\varepsilon$)

$$\begin{aligned}
\frac{d}{dt}\sigma_2 &= -\Gamma_2\sigma_2 + \imath(v_{12}^*\widetilde{\sigma}_{12}^* - v_{12}\widetilde{\sigma}_{12}) \\
&\quad - \frac{4}{\Gamma_3}\left[|v_{23}|^2\sigma_2 + \Re(v_{23}v_{31}\widetilde{\sigma}_{12}^*)\right] \qquad (1) \\
\frac{d}{dt}\widetilde{\sigma}_{12} &= \imath\ v_{12}^*(1-2\sigma_2) - \frac{\Gamma_2}{2}\widetilde{\sigma}_{12} \\
&\quad - \frac{2}{\Gamma_3}\left[(|v_{23}|^2 + |v_{31}|^2)\widetilde{\sigma}_{12} + v_{23}v_{31}\right] \qquad (2)
\end{aligned}$$

with the output

$$Y = \Gamma_3\sigma_3 = \frac{4}{\Gamma_3}\left[|v_{31}|^2(1-\sigma_2) + |v_{23}|^2\sigma_2 + 2\Re(v_{23}v_{31}\widetilde{\sigma}_{12}^*)\right] \quad (3)$$

and where the controls are the three complex amplitudes v_{12}, v_{23} and v_{31}. Remember that the above slow model is valid whenever

$$|v_{12}|, |v_{23}|, |v_{31}|, \Gamma_2 \ll \Gamma_3$$

and

$$\left|\frac{d}{dt}v_{12}\right| \ll |\Omega_{12}||v_{12}|, \quad \left|\frac{d}{dt}v_{23}\right| \ll |\Omega_{23}||v_{23}|,$$
$$\left|\frac{d}{dt}v_{31}\right| \ll |\Omega_{31}||v_{31}|.$$

In order to conclude, note that these computations justify the following model (Γ_3 large):

$$\left\{\begin{array}{rcl}
\frac{d}{dt}x_1 &=& u_1x_3 - \frac{\Gamma_2}{2}x_1 \\
\frac{d}{dt}x_2 &=& u_2x_3 - \frac{\Gamma_2}{2}x_1 \\
\frac{d}{dt}x_3 &=& -u_1x_1 - u_2x_2 - \Gamma_2(x_3+1) \\
y &=& v_1x_1 + v_2x_2 + v_3x_3
\end{array}\right. \qquad (4)$$

with

$$x_1 + \imath\, x_2 = \widetilde{\sigma}_{12}/2, \quad x_3 = 2\sigma_2 - 1$$
$$u_1 + \imath\, u_2 = -\imath\, v_{12}^*, \quad v_1 + \imath\, v_2 = v_{23}v_{31}$$
$$v_3 = (|v_{23}|^2 - |v_{31}|^2)/2, \quad y = \Gamma_3 Y/4 - (|v_{23}|^2 + |v_{31}|^2)/2.$$

This model shows how we can couple the quantum dynamics of the state $x = (x_1, x_2, x_3)$ belonging to the Bloch sphere (unit sphere of $\mathbb{R}^3$) to the classical inputs $(u_1, u_2) \in \mathbb{R}^2$ and $(v_1, v_2, v_3) \in \mathbb{R}^3$ (the amplitudes of the field resonant modes) and to a classical output y (the average number of photons emitted in the time unit). Notice the direct influence of the inputs v_i on the output y.

III. A SIMPLE PI REGULATOR

Take (4) as the control model, with $u_2 = v_2 = v_3 = 0$, $v_1 = 1$. Assume moreover that $\Gamma_2 = 0$: the states 1 and 2 are without decoherence (kind of dark states). Suppose that $x_2 = 0$ (this is always the case up to a rotation around the axis x_3). We have thus

$$\frac{d}{dt}x_1 = u_1 x_3, \quad \frac{d}{dt}x_3 = -u_1 x_1, \quad y = x_1.$$

The state (x_1, x_3) evolves on the circle $\mathbb{S}^1$. Suppose that the control goal is to stabilize the system around $x_1 = 0$ and $x_3 = 1$ using the control term $u_1 \in [-u^{\max}, u^{\max}] (u^{\max} > 0)$. The PI regulator with the gains $K_p > 0$ and $K_i > 0$

$$u_1 = \begin{cases} -u^{\max}, & \text{if} -K_p y + I < -u^{\max} \\ -K_p y + I, & \text{if } -u^{\max} \le -K_p y + I \le u^{\max} \\ u^{\max}, & \text{if } u^{\max} < -K_p y + I \end{cases}$$

with the anti-windup (in order to manage the constraints on u_1)

$$\frac{d}{dt}I = \frac{K_i}{K_p}(u_1 - I)$$

ensures the quasi-global stabilization around $(x_1 = 0, x_3 = 1)$. In fact, the closed-loop system (a system on the cylinder $\mathbb{S}^1 \times \mathbb{R}$) admits only two stationary states : $(x_1, x_3) = (0, 1)$ with $I = 0$ is asymptotically stable, $(x_1, x_3) = (0, -1)$ with $I = 0$ is unstable. All the closed-loop trajectories other than the unstable equilibrium state converge asymptotically and exponentially toward the stable equilibrium state.

Except the fact that the original system evolves on $\mathbb{S}^1$ and not on $\mathbb{R}$, this regulator is similar to the classical PI regulator currently applied for the first order system $\frac{d}{dt}y = u$. The practical interest and the robustness of such a regulator is thus well-established.

IV. EXTENSION

In this section, we adapt the computations of the Section II to the case where instead of the Bloch equations, a system of Lindblad type is considered. We show that the reduced slow system verifies likewise an other equation of the Lindblad type where the operators can be directly deduced from the original ones.

Consider the following master equation, for the density matrix ρ (positive $n \times n$ Hermitian matrix) associated to an n-dimensional system :

$$\frac{d}{dt}\rho = -\frac{\imath}{\hbar}[H_0 + u(t)H_1, \rho] + \Gamma\mathscr{D}[Q](\rho),$$
$$Y = \Gamma \mathrm{Tr}\left[Q^\dagger Q \rho\right].$$

The Hermitian operators H_0 and H_1 are, respectively, the free Hamiltonian and the interaction Hamiltonian with a coherent source of photons $u(t) \in \mathbb{R}$. For the arbitrary operators A and B, the super-operator $\mathscr{D}$ is defined by

$$\mathscr{D}[A](B) = \frac{1}{2}(2ABA^\dagger - A^\dagger AB - BA^\dagger A).$$

The operator Q models the decoherence associated to the measurement Y. $\Gamma > 0$ is a normalization constant depending on the inverse of the decoherence time associated to the measurement Y. Q does not have any dimension and we suppose everywhere in this section that Q is a transition operator of the form $|g\rangle\langle e|$, where $|g\rangle$ and $|e\rangle$ are two different eigenstates of the free Hamiltonian H_0. One easily has the following relations:

$$Q^2 = 0, \qquad Q^\dagger Q = P = |e\rangle\langle e|,$$

where P is the projection operator on the excited state $|e\rangle$.

The transition frequencies, $\Omega_{ij} = \lambda_i - \lambda_j$ (where the λ_i's are the eigenvalues of $H_0/\hbar$), are supposed to be much larger than the decoherence Γ. We have thus the possibility to use the secular approximation. We consider the transformation $\rho \mapsto U\rho U^\dagger$, where $U = e^{iH_0 t/\hbar}$ is the propagator of the free evolution. On the other hand, we assume the control field $u(t)$ to be in the resonant regime with respect to the natural frequencies of the system and we modulate the amplitudes $u_{ij}(t)$:

$$u(t) = \sum_{i,j} u_{ij}(t)\sin(\Omega_{ij}t).$$

The amplitudes $u_{ij}(t)$ are varying slowly.

After averaging, we obtain the following master equation :

$$\frac{d}{dt}\rho = -\frac{\imath}{\hbar}[H, \rho] + \frac{\Gamma}{2}(2Q\rho Q^\dagger - Q^\dagger Q\rho - \rho Q^\dagger Q),$$
$$Y = \Gamma \mathrm{Tr}\left[Q^\dagger Q\rho\right] \tag{5}$$

where H corresponds to the secular terms of $uUH_1U^\dagger$. Such an approximation is valid whenever the amplitudes u_{ij} are sufficiently small.

Moreover, we suppose that the relaxation of the state $|e\rangle$ toward the state $|g\rangle$, which ends up by detecting photons, is much faster than the other dynamics of the equation (5) (short atomic lifetime for the state $|e\rangle$). We can therefore take $\Gamma = \overline{\Gamma}/\varepsilon$ where ε is a small positive parameter. Thus, in the interaction frame and for sufficiently low amplitudes u_{ij}, we have the following master equation :

$$\frac{d}{dt}\rho = -\frac{\imath}{\hbar}[H, \rho] + \frac{\overline{\Gamma}}{2\varepsilon}(2Q\rho Q^\dagger - Q^\dagger Q\rho - \rho Q^\dagger Q),$$

where $0 < \varepsilon \ll 1$.

Define

$$\rho_f = P\rho + \rho P - P\rho P$$
$$\rho_s = (1 - P)\rho(1 - P) + Q\rho Q^\dagger.$$

As $\rho = \rho_s + \rho_f - QP\rho_f PQ^\dagger$, there exists a bijective correspondence between ρ and (ρ_f, ρ_s): we have a sort of "change of variables" for ρ which decouples the slow part from the fast part of the dynamics. This change of variables propose us a "Tikhonov standard form" :

$$\frac{d}{dt}\rho_f = -\frac{\overline{\Gamma}}{2\varepsilon}(\rho_f + P\rho_f P) - \frac{\imath}{\hbar}(P[H, \rho] + [H, \rho]P - P[H, \rho]P),$$
$$\imath\hbar\frac{d}{dt}\rho_s = (1 - P)[H, \rho](1 - P) + Q[H, \rho]Q^\dagger.$$

Therefore ρ_f is associated to the fast part of the dynamics and ρ_s represents the slow part. The fast part is asymptotically stable because $-\frac{\overline{\Gamma}}{2\varepsilon}(\rho_f + P\rho_f P)$ defines a negative definite super-operator on the space of Hermitian operators:

$$\mathrm{Tr}\left[-(\rho_f + P\rho_f P)\rho_f\right] = -(\|\rho_f\|^2 + \|P\rho_f P\|^2).$$

Here we can apply the slow manifold approximation described in the Appendix. Computing the first order terms, we find the following approximation for ρ_f with respect to ρ_s:

$$\rho_f = \frac{-2\imath\varepsilon}{\hbar\overline{\Gamma}} \, (PH\rho_s - \rho_s HP) + O(\varepsilon^2).$$

The slow dynamics are given by :

$$\frac{d}{dt}\rho_s = -\frac{\imath}{\hbar}[H_s, \rho_s] + 4\varepsilon\overline{\Gamma}\left(\overline{Q}\rho_s\overline{Q}^\dagger - \frac{1}{2}\overline{Q}^\dagger\overline{Q}\rho_s - \frac{1}{2}\rho_s\overline{Q}^\dagger\overline{Q}\right) + O(\varepsilon^2),$$

where

$$H_s = (1-P)H(1-P)$$

and

$$\overline{Q} = \frac{1}{\hbar\overline{\Gamma}}(1-P)QH(1-P), \quad \overline{Q}^\dagger = \frac{1}{\hbar\overline{\Gamma}}(1-P)HQ^\dagger(1-P).$$

Note that, similarly to Q, the operator $\overline{Q}$ is without dimension. The situation is different for the output Y. We have:

$$Y(t) = \frac{\overline{\Gamma}}{\varepsilon}\mathrm{Tr}\left[Q^\dagger Q\rho\right] = \frac{\overline{\Gamma}}{\varepsilon}\mathrm{Tr}\left[Q^\dagger Q\rho_f\right] = \frac{-2\imath}{\hbar} \, \mathrm{Tr}\left[P(PH\rho_s - \rho_s HP)\right] + O(\varepsilon).$$

but $\mathrm{Tr}\left[P(PH\rho_s - \rho_s HP)\right] = 0$. We should therefore consider the second order terms. Using the Appendix, simple but tedious computations end up to the following natural approximation :

$$Y(t) = 4\varepsilon\overline{\Gamma}\mathrm{Tr}\left[\overline{Q}^\dagger\overline{Q}\rho_s\right] + O(\varepsilon^2).$$

But $\overline{\Gamma}/\varepsilon = \Gamma$. Therefore, we have shown that whenever Γ is large (with respect to $\frac{H}{\hbar}$), the slow master equation associated to (5) reads

$$\frac{d}{dt}\rho_s = -\frac{\imath}{\hbar}[H_s, \rho_s] + \frac{2}{\Gamma}\left(2Q_s\rho_s Q_s^\dagger - Q_s^\dagger Q_s\rho_s - \rho_s Q_s^\dagger Q_s\right),$$

$$Y = \frac{2}{\Gamma}\mathrm{Tr}\left[Q_s Q_s \rho_s\right]$$

where

$$\rho_s = (1-P)\rho(1-P),$$

$$H_s = (1-P)H(1-P), \qquad Q_s = (1-P)Q\left(\frac{H}{\hbar}\right)(1-P)$$

with $P = Q^\dagger Q$. Recall that H is the Hamiltonian in the interaction frame with resonant laser modes and after the secular approximation.

In order to conclude, we remark that for an ensemble of independent and identical quantum systems, and whenever the decoherence dynamics due to the measurement is much faster than the other dynamics, the adiabatic approximation helps us to find the slow dynamics as well as the measurement result with respect to the slow dynamics. Note that in this new system, the decoherence term can be removed in a first order approximation (large Γ). We obtain therefore a system of the form

$$\frac{d}{dt}\rho_s = -\frac{\imath}{\hbar}[H_s, \rho_s],$$

where the control appears linearly in the reduced Hamiltonian H_s. This system corresponds to a bilinear system with the wave-functions as state variables. Furthermore, we have access to a measurement Y given by the reduced slow evolution. We can henceforth consider a control problem with continuous measurement associated to this system.

In order to finish this section let us remark that one can consider more general and complicated situations. In particular, note that as it can be seen in Figure 1 for the $^{40}Ca^+$ ion, the highly unstable state $|2\rangle$ dissipates not only toward the ground state $|1\rangle$ but also toward the metastable state $|3\rangle$. Nevertheless, the above computations can be easily extended to the case where the highly unstable excited state $|e\rangle$ decoheres toward many stable (or metastable) ground states $|g_1\rangle, |g_2\rangle, ..., |g_k\rangle$ with branching ratios $\Gamma_1, \Gamma_2, ..., \Gamma_k$. One only needs to consider a sum of Lindblad terms with quantum jump operators $Q_i = |g_i\rangle\langle e|$, $i = 1, 2, ..., k$. The relevant change of variables will therefore be given by:

$$\rho_f = P\rho + \rho P - P\rho P$$

$$\rho_s = (1-P)\rho(1-P) + \sum_{i=1}^{k}\frac{\Gamma_i}{\sum_{i=1}^{k}\Gamma_i}Q_i\rho Q_i^\dagger,$$

where $P = |e\rangle\langle e|$ is still the projection operator on the excited state $|e\rangle$.

V. Conclusion

In this paper, we have used the Lindblad type equations in order to represent the dynamics of an open quantum system including the measurement process. Applying the singular perturbation techniques, we have reduced the model whenever some assumptions on the parameters of the system are satisfied. This model reduction allows us to understand whether when we can model a quantum system with continuous measurement by a conservative equation of Schrödinger type where the density matrix evolves coherently. This reduction also permits us to better understand in which circumstances it is eligible to put down the feedback problem for a quantum system in classical and usual terms.

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APPENDIX

This appendix has for goal to remind an approximation technique that can be perfectly justified using the geometrical tools of singular perturbation and the center manifold [9], [3], [5].

Consider the slow/fast system (x and y are of arbitrary dimensions, f and g are regular functions)

$$\frac{d}{dt}x = f(x,y), \qquad \frac{d}{dt}y = -\frac{1}{\varepsilon}Ay + g(x,y)$$

where x and y are respectively the slow and fast states (Tikhonov coordinates), all the eigenvalues of the matrix A have strictly positive real parts, and ε is small strictly positive parameter. Therefore the invariant attractive manifold admits for the equation

$$y = \varepsilon A^{-1} g(x,0) + O(\varepsilon^2)$$

and the restriction of the dynamics on this slow invariant manifold reads

$$\begin{aligned}\frac{d}{dt}x &= f(x, \varepsilon A^{-1} g(x,0)) + O(\varepsilon^2)\\ &= f(x,0) + \varepsilon \frac{\partial f}{\partial y}|_{(x,0)} A^{-1} g(x,0) + O(\varepsilon^2).\end{aligned}$$

The Taylor expansion of g can be used to find the higher order terms. For example, the second order term in the expansion of y is given by:

$$\begin{aligned}&y = \varepsilon A^{-1} g(x,0)\\ &+ \varepsilon^2 A^{-1}\left(\frac{\partial g}{\partial y}|_{(x,0)} A^{-1} g(x,0) - A^{-1}\frac{\partial g}{\partial x}|_{(x,0)} f(x,0)\right) + O(\varepsilon^3),\end{aligned}$$

and so on.