Inverse Scattering and Optic Fiber design

Nabil Belouardy [*] Pierre Rouchon [†]

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Abstract

Suitable radial profiles of refractive index are designed in order to reduce signal dispersion and increase fiber capacity. It is shown that for any radial index profile independent of the frequency, the phase velocity cannot be constant, even locally around any frequency. On the contrary, it is proved, that the phase velocity can be maintained constant when the refractive index is a function of space and frequency. Some simple and explicit computations show how to design such an index profile (as a function of space and frequency) in order to ensure a constant phase velocity.

1 The direct problem

The modal analysis (propagation modes, see [1]) of an optic fiber with cylindric symmetry (cylindric coordinates (r, θ, z)) leads to the following problem: for each frequency ω, finding a propagation constant $\beta > 0$ and non zero electric and magnetic fields of the form

$$\begin{cases} \mathcal{E} = \exp(i(\omega t - \beta z))\mathcal{E}_0(r, \theta) \\ \mathcal{H} = \exp(i(\omega t - \beta z))\mathcal{H}_0(r, \theta) \end{cases}$$

satisfying the Maxwell equations

$$\begin{cases} \nabla \times \mathcal{E} = -\frac{\partial \mathcal{B}}{\partial t} \\ \nabla \times \mathcal{H} = \frac{\partial \mathcal{D}}{\partial t} \\ \nabla \cdot \mathcal{D} = 0 \\ \nabla \cdot \mathcal{B} = 0 \end{cases}$$

where $\mathcal{D}$ and $\mathcal{B}$ obey

$$\begin{cases} \mathcal{D} = \epsilon_0 \epsilon_r \mathcal{E} \\ \mathcal{B} = \mu_0 \mathcal{H}. \end{cases}$$

[*]Ecole Polyetchnique, Promotion X98, 91128, Palaiseau cedex, France.

[†]Ecole des Mines de Paris, Centre Automatique et Systèmes, 60 Bd Saint-Michel, 75272 Paris cedex 06 FRANCE. rouchon@cas.ensmp.fr.

The relative permittivity ϵ_r is equal to square of the refractive index n. It is real (no absorption, conservative case) and depends on r and ω (radial and material dispersion). $\mathcal{E}$ and $\mathcal{B}$ must tend to zero when r tends to $+\infty$ (boundary conditions). Since $\nabla \times \mathcal{E} = -i\omega\mathcal{H}$, we can eliminate $\mathcal{H}$ and concentrates on finding β such that (c is the light velocity in vaccum)

$$\Delta\mathcal{E} + \nabla[(2/n)\nabla n \cdot \mathcal{E}] = -(\omega^2 n^2)/c^2\mathcal{E}, \quad \nabla \cdot (n^2\mathcal{E}) = 0.$$

admits non zero solution of form $\mathcal{E} = \mathcal{E}_0(r,\theta).e^{i(\omega t - \beta z)}$ vanishing at infinity. Throughout the paper, we say that a spatial field tends to 0 at infinity when the module of any derivatives of order less or equal to 2 decreases enough rapidly.

$\mathcal{E}_0$ admits 3 components (E_r, E_θ, E_z) in the cylindric frame (e_r, e_θ, e_z). Let us denote by $A = E_r e_r + E_\theta e_\theta$, $C = E_z/(i\beta)$ and $\nu = c\beta/\omega$. Then we have the following system

$$\begin{aligned}
\frac{\omega^2}{c^2}(\nu^2 - n^2)A &= \Delta A + \nabla(\nabla \log(n^2) \cdot A) \\
\frac{\omega^2}{c^2}(\nu^2 - n^2)C &= \Delta C + \nabla \log(n^2) \cdot A \\
\frac{\omega^2\nu^2}{c^2}C &= \nabla \cdot A + \nabla \log(n^2) \cdot A
\end{aligned}$$

where the two components of ∇ correspond to derivation with respect to $x = r\cos\theta$ and $y = r\sin\theta$. Simple computations show that if A is solution of the first equation, and C is given by the last equation, then C automatically satisfies the second equation. Thus we can just consider the first equation:

$$\frac{\omega^2}{c^2}(\nu^2 - n^2)A = \Delta A + \nabla(\nabla \log(n^2) \cdot A) \tag{1}$$

Classically, the term $\nabla \log(n^2) \cdot A$ is neglected when $\frac{\partial n}{\partial r}$ is much smaller than the inverse characteristic wave length c/ω. So we will consider the following approximate direct problem: for a given profile $(r,\omega) \mapsto n(r,\omega)$, for any frequency ω such that $|\omega|/c \ll 1/|\frac{\partial n}{\partial r}|$, find ν such that

$$\frac{\omega^2}{c^2}(\nu^2 - n^2)A = \Delta A \tag{2}$$

admits a non zeros solution $A = (A_x, A_y)$ that tends to O at ∞. For a given ω, we can have several ν. This corresponds to different propagation modes associated to the same pulsation ω.

Let us prove (at least formally) that for a given branch, says $\nu = \phi(\omega)$, we have always $\phi'(\omega) > 0$ when n is independent of ω. This means that for any profile $n(r)$ independent of ω, the phase velocity cannot remain constant. This fact comes from the following analysis that can be made rigorous if necessary. The first variation of (2), yields

$$\frac{\omega^2}{c^2}(\nu^2 - n^2)\delta A + \frac{2\omega}{c^2}(\nu^2 - n^2)A\delta\omega + + \frac{2\omega^2\nu}{c^2}A\delta\nu = \Delta\delta A$$

where δA and A are fields tending to 0 at infinity. We then take the dot-product with A, integrate over the plane (x, y), use the fact that $\frac{\omega^2}{c^2}(\nu^2 - n^2)A = \Delta A$ and integrate by part, to obtain

$$\left(\int\int \frac{2\omega}{c^2}(\nu^2 - n^2)A^2\right)\delta\omega + \left(\int\int \frac{2\omega^2\nu}{c^2}A^2\right)\delta\nu = 0.$$

Similarly, $\int\int \frac{\omega^2}{c^2}(\nu^2 - n^2)A^2 = -\int\int(\nabla A_x)^2 + (\nabla A_y)^2$. Thus we have

$$\frac{d\nu}{d\omega} = \frac{\delta\nu}{\delta\omega} = \frac{c^2}{\omega^3\nu}\frac{\int\int(\nabla A_x)^2 + (\nabla A_y)^2}{\int\int A^2}.$$

We do not know if such obstruction to constant phase velocity ($\nu = cte$ independent of ω), is still valid if we do not neglect the term $\nabla \log(n^2) \cdot A$ in the original equation (1).

2 The inverse problem

Take (2) and set $\omega \mapsto \nu(\omega)$. Find $r \mapsto n(r, \omega)$ such that for any ω, exists a non zero solution of (2) that tends to 0 at infinity. Notice that n depends on ω. We will see that this problem always admits a solution. In fact they are many solutions. We will propose here below a simple and explicit one by taking $A = W(r)e_r$ independent of ω. Then (2) reduces to the following classical equation ($'$ stands for d/dr):

$$W'' + W'/r - W/r^2 + \frac{\omega^2}{c^2}(n^2 - \nu^2)W = 0.$$

This second order differential equation is regular singular at $r = 0$ (see, e.g.,[2, page 197]) and $\alpha = 1$ is a root of the indicial equation. Thus we set $W(r) = r\exp(h(r))$ where $h(r)$ will be chosen later. Then h satisfies

$$rh'' + r(h')^2 + 3h' + \frac{\omega^2}{c^2}(n^2 - \nu^2) = 0.$$

Set, e.g., $h(r) = -r/\sigma$ where σ is a positive length. We have the explicit formula

$$n^2 = \nu^2 - \frac{c^2}{\omega^2}(rh'' + r(h')^2 + 3h') = \nu^2 + \frac{c^2}{\sigma\omega^2}(3 - r/\sigma)$$

that can be used to design physical profile n. In particular we recover the fact that for $r = 0$, $n > \nu$ and for r large $n < \nu$. Moreover we know explicitly the propagation mode:

$$\mathcal{E}_0 = \exp h(re_r - i\frac{c}{\omega\nu}(rh' + 2)e_z)$$

For a typical wavelength of $\lambda = \frac{c}{\omega} = 1\mu$m, figure 1 illustrates the obtained profile for $\sigma = 10\lambda$ with a constant phase velocity associated to $\nu = 1.5$. We have checked that the approximation $|dn/dr| \ll 1/\lambda$ is valid.

Notice that similar computations can be made for the complete system (1) where $\frac{\partial n}{\partial r}$ is not neglected.

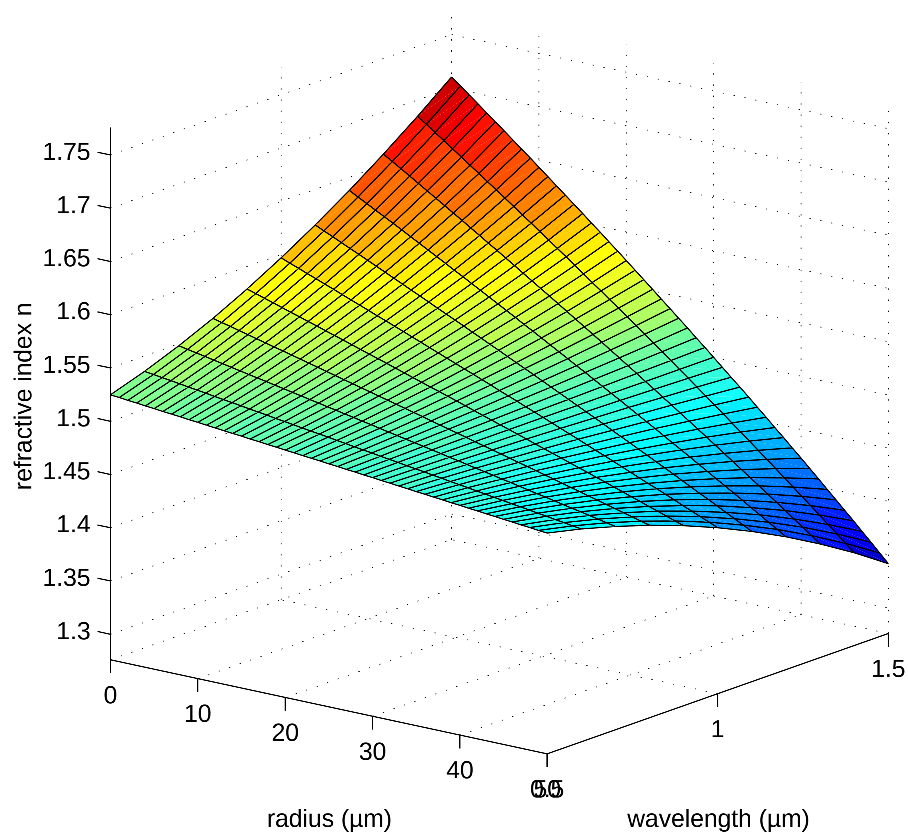

Figure 1: the refractive index $n(r, \omega)$ that ensures a constant phase velocity $\nu = 1.5$: with such profile, the fiber is dispersion free and the corresponding propagation mode is almost independent of ω.

References

[1] Agrawal G.P. *Fiber-Optic Communication Systems.* John Wiley and Sons, 1997.

[2] E.T. Whittaker and G.N. Watson. *A course of modern analysis (4th edition).* Cambridge University Press, Cambridge, 1927.