Active signal restoration for the telegraph equation

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Abstract

We compensate by a prefilter the distortion of an input signal along an electric line modeled by the telegraph equation. The prefilter is based on the so-called *flatness* property of the telegraph equation. We derive the explicit equation of the filter and illustrate the relevance of our approach by a few simulations.

Key words. Telegraph equation, control, signal, distortion, flatness, modules, operational calculus, convolution, Bessel functions.

Introduction

A standard problem in signal processing is the restoration of an input signal $u(t)$ from a degraded output signal $y(t)$. When the line can be modeled by the telegraph equation, it is possible to design a prefilter to get an undistorted signal at the end of the line. Indeed, the inverse of the transfer function can be seen as a noncausal filter $\mathcal{F}$

$$u = \mathcal{F} \star y.$$

An important feature of $\mathcal{F}$ is that it has compact support, which means it can be actually computed in a finite amount of time: $\mathcal{F}$ combines a delay operator, an advance operator and Bessel functions expressing distributed delays.

The prefilter can be seen as a *motion planner*, a notion often encountered in control theory. It relies on the property of *flatness*, originally developed for ordinary differential systems [7, 8, 14], and later extended to partial differential equations (delay systems [17, 9, 20], the wave equation [18], the heat equation [14, 12], the Euler-Bernoulli equation [10], the Saint-Venant equation [5]).

The paper is organized as follows: we first briefly recall the physical model of an electric line and then derive the equation of the prefilter with operational calculus. Finally, we illustrate the relevance of our approach with numerical simulations.

The appendix deals with the technical aspects, namely a presentation of Heaviside's operational calculus thanks to Mikusiński's algebraic formalism [15, 16] and an interpretation of controllability in the modules theoretic framework [6, 17].

Finally, notice that the point of view of this paper is quite different from the more established approach in the control of partial differential equations [1, 2, 13].

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1 The physical model

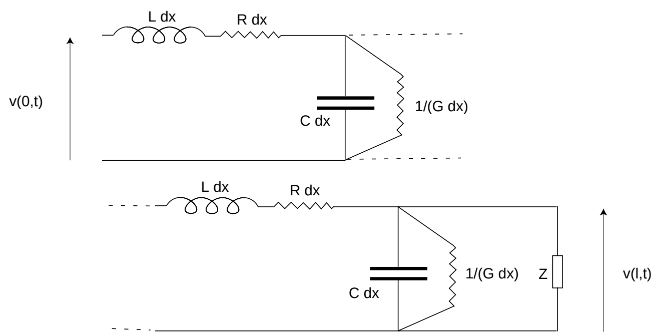

Figure 1: An electric line

We are interested in the propagation of an electric signal through an electric line of length ℓ. Per unit of length, the resistance is R, the inductance is L, the capacity is C and the perditance is G. Kirschoff's laws read (see for instance [21]):

$$L\frac{\partial i}{\partial t} = -Ri - \frac{\partial v}{\partial x}$$
$$C\frac{\partial v}{\partial t} = -\frac{\partial i}{\partial x} - Gv.$$

where $0 \le x \le \ell$, $t \ge 0$. Eliminating the current, we get the *telegraph equation*

$$\frac{\partial^2 v(x,t)}{\partial x^2} = (R + L\frac{\partial}{\partial t})(G + C\frac{\partial}{\partial t})v(x,t). \qquad (1)$$

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The boundary conditions are

$$v(0,t) = u(t)$$
$$v(\ell,t) = Zi(\ell,t).$$

The input and the output of the system are respectively $u(t) = v(0,t)$ and $y(t) = v(\ell,t)$.

2 Derivation of the prefilter

2.1 Operational solution

With zero initial conditions, i.e. $v(x,0) = \frac{\partial v}{\partial t}(x,0) = 0$, we turn (1) into the following ODE thanks to operational calculus

$$\hat{v}''(x,s) = \varpi(s)\hat{v}(x,s), \qquad (2)$$

where $\varpi(s) = (R + Ls)(G + Cs)$ and s stands for the time derivative. The boundary conditions now read

$$\hat{v}(0,s) = \hat{u}(s), \quad (R + Ls)\hat{v}(\ell,s) = Z\hat{v}'(\ell,s). \qquad (3)$$

$\hat{u}$ et $\hat{v}$ are the operational images[1] of u and v. Clearly the general solution of (2) is

$$\hat{v}(x,s) = A(s)\mathrm{ch}((\ell-x)\sqrt{\varpi(s)}) + B(s)\mathrm{sh}((\ell-x)\sqrt{\varpi(s)}),$$

where $A(s)$ and $B(s)$ are independent of x and are determined by the boundary conditions (3). Now, instead of writing the relation between $\hat{v}$ and $\hat{u}$, we write the relation between $\hat{v}$ and $\hat{y}(s) = \hat{v}(\ell,s)$:

$$\hat{v}(x,s) = \Big(\mathrm{ch}((\ell - x)\sqrt{\varpi(s)}) + \frac{R + Ls}{Z}\frac{\mathrm{sh}((\ell - x)\sqrt{\varpi(s)})}{\sqrt{\varpi(s)}}\Big)\hat{y}(s). \qquad (4)$$

Notice the remarkable fact that the transfer function from $\hat{y}$ to $\hat{v}$ has only zeroes and no poles, i.e., is an analytic function (it is even an entire analytic function). Therefore the motion of the whole system is defined by the motion of $\hat{y}$: in other words $\hat{y}$ is a *flat output* (see [7, 8, 14]).

In particular, for $x = 0$ (4) reads

$$\hat{u}(s) = \left(\mathrm{ch}\left(\ell\sqrt{\varpi(s)}\right) + \frac{R + Ls}{Z}\frac{\mathrm{sh}\left(\ell\sqrt{\varpi(s)}\right)}{\sqrt{\varpi(s)}}\right)\hat{y}(s). \qquad (5)$$

The last formula explicitly solves the motion planning problem: indeed, if we want the output y to follow some desired trajectory then the required input u is given by 5.

[1] In the traditional justification of operational calculus, $\hat{u}$ and $\hat{v}$ are the Laplace transform of u et v, i.e. $\hat{u}(s) = \int_0^{+\infty} e^{-st}u(t)dt$ and $\hat{v}(x,s) = \int_0^{+\infty} e^{-st}v(x,t)dt$. Another approach due to [15, 16] (see also [24]), is exposed in the appendix.

2.2 Time domain solution

We now express formula (5) back into the time domain. For the sake of simplicity but without loss of generality, we assume $G = 0$. Let $\lambda = \ell\sqrt{LC}$, $\alpha = \frac{R}{2L}$. Then $\varpi(s) = RCs + LCs^2$ and (5) gives

$$\begin{aligned} u(t) &= \frac{1}{2}e^{-\alpha\lambda}(1 - \frac{1}{Z}\sqrt{\frac{L}{C}})y(t - \lambda) \\ &+ \frac{1}{2}e^{\alpha\lambda}(1 + \frac{1}{Z}\sqrt{\frac{L}{C}})y(t + \lambda) \\ &+ \int_{-\lambda}^{+\lambda}\Big(\frac{R}{4Z\sqrt{LC}}e^{-\alpha\tau}J_0(i\alpha\sqrt{\tau^2 - \lambda^2}) \\ &+ \frac{e^{-\alpha\tau}i\alpha}{2\sqrt{\tau^2 - \lambda^2}} \\ &(\lambda - \frac{1}{Z}\sqrt{\frac{L}{C}}\tau)J_1(i\alpha\sqrt{\tau^2 - \lambda^2})\Big)y(t - \tau)d\tau \end{aligned} \qquad (6)$$

where J_0 et J_1 are Bessel functions of the first kind. We have used the results of [3], formulas 2.4.180 and 2.4.183 (see also [15], pp. 207–208 or [24], pp. 136–138)[2].

Notice that the last formula is indeed the equation of a noncausal prefilter $\mathcal{F}$ with compact support ($u(t)$ is expressed in terms of the values of y over the finite interval $[t - \lambda, t + \lambda]$).

One can also directly prove that $\mathcal{F}$ has compact support by the Paley-Wiener theorem (see [22]). On the other hand $\mathcal{F}^{-1}$ has not a compact support.

3 Simulations

For the following simulations we take $R = 2.16e - 3$, $L = 18.42e - 7$, $C = 1.8e - 11$, $Z = 100$, $l = 1e - 6$ (S.I. units). We use a discrete model to simulate the partial differential system with $N = 80$ cells (see figure 1). The input test signals (voltages) are square signals with $50Hz$ and $300Hz$ frequencies.

The natural $(1 + \frac{Rl}{Z})$ attenuation of the line is compensated by an adequate static gain.

3.1 50 hz frequency

Without precompensation the input signal is quite distorted (figure 2). With precompensation the output signal is much less distorted, though more attenuated (figure 3). Notice the same voltage range is used in both cases.

3.2 300 hz frequency

Without precompensation the input signal is hardly recognizable (figure 4). The precompensation still gives

[2] One may compare the demonstrations based on Mikusiński operational calculus and Laplace transform (see, for instance, [4]).

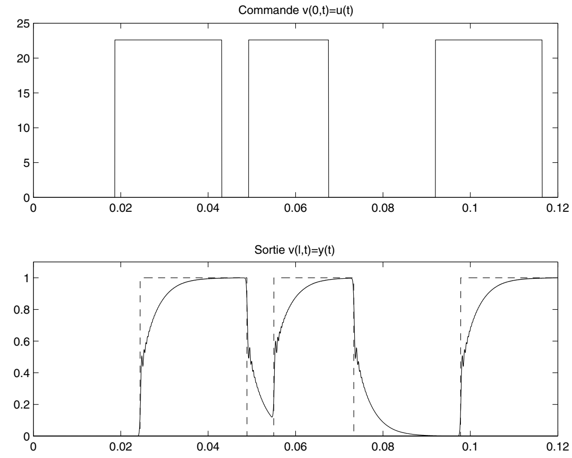

Figure 2: No compensation. $50Hz$ frequency

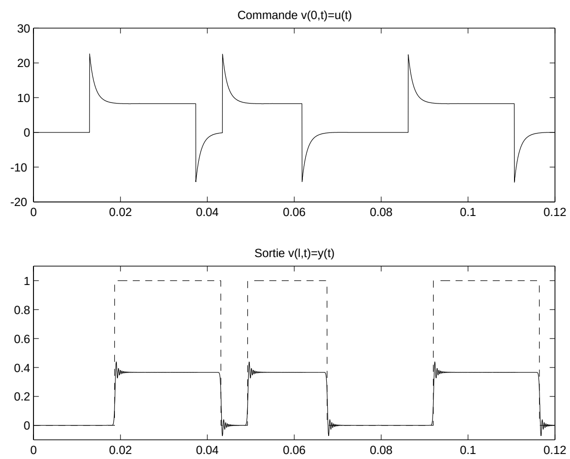

Figure 3: Precompensation. 50 Hz frequency

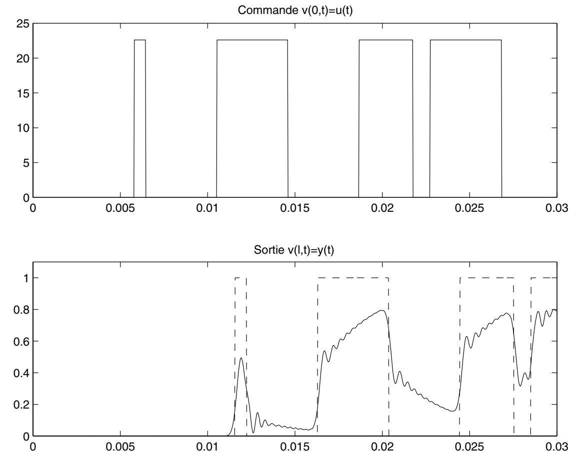

Figure 4: No compensation. 300 Hz frequency

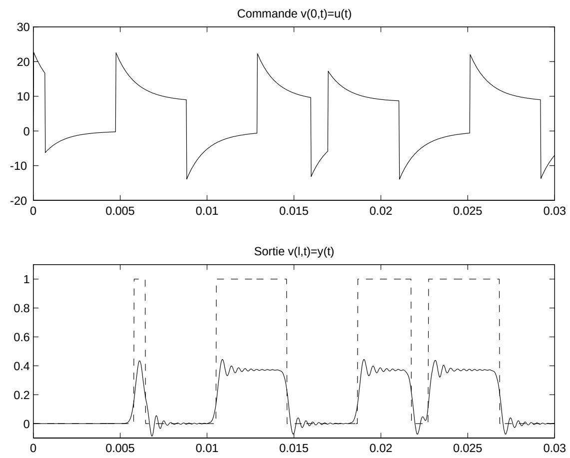

Figure 5: Precompensation. 300 Hz frequency

good results (figure 5).

3.3 Experimental robustness

In practice R is easy to measure. We numerically investigated the robustness to variations of L and C (figures 6 and 7).

3.4 Experimental conclusion

We have proposed a way to improve the bandwidth of the line. Though the output signal is more attenuated due to a larger input excursion its shape is much easier to recognize.

A Theoretic background

A.1 Mikusiński's operational calculus

$\mathcal{C}$, the set of complex continuous functions defined over the real interval $[0, +\infty[$ endowed with addition $+$,

$$(f+g)(t) = f(t) + g(t)$$

and the convolution $f \star g$,

$$(f \star g)(t) = \int_0^t f(t-\tau) g(\tau) d\tau,$$

is a commutative ring. From Titchmarsh's theorem (see [15, 16, 24]), $\mathcal{C}$ is entire, i.e. without any zero divisor:

$$f \star g = 0 \Leftrightarrow f = 0 \quad \text{ou} \quad g = 0$$

The field of fractions $\mathcal{M}$ of $\mathcal{C}$ is called the *Mikusiński field*. The elements of $\mathcal{M}$ are *operators*.

<u>Notations</u>. 1) every function $f(t)$ when considered as an operator of $\mathcal{M}$ is noted as $\{f(t)\}$. Thus, $\{1\} \in \mathcal{C}$ is the Heaviside function

$$H(t) = \begin{cases} 1 & \text{si} \quad t \geq 0 \\ 0 & \text{si} \quad t < 0 \end{cases}$$

2) The (convolution) product of two operators $a, b \in \mathcal{M}$ is noted ab.

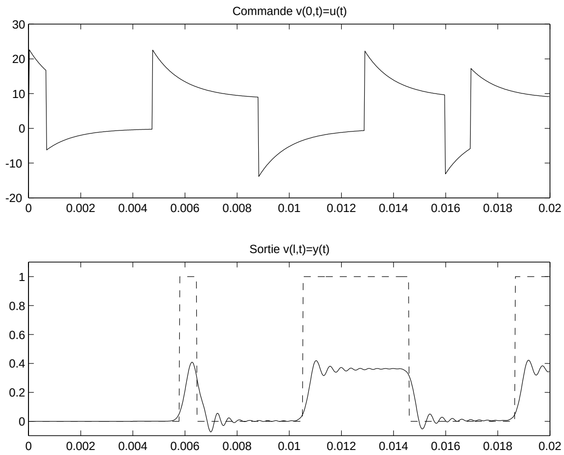

Figure 6: 300 Hz. % 5 under-estimation of C

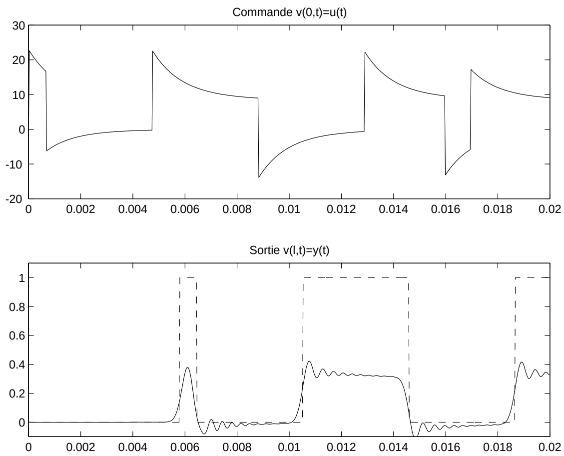

Figure 7: 300 Hz. % 5 over-estimation of L

Examples. 1) The unit element 1 of $\mathcal{M}$ is similar to the Dirac distribution in the theory of distributions of L. Schwartz.

2) The inverse in $\mathcal{M}$ of the Heaviside function $\{1\}$ is the derivation operator s which obeys the following rules: if $f \in \mathcal{C}$ is C^1, then $sf = \{\dot{f}\} - \{f(0)\}$. The operators of the subfield $\mathbf{C}(s)$ of $\mathcal{M}$ have the usual signification. The fractional derivative $\sqrt{s}$ is the inverse of $\{\frac{1}{\sqrt{2\pi t}}\}$.

3) $e^{-\lambda s}$, $\lambda \in \mathbf{R}$, $\lambda > 0$, is the λ delay operator. Its inverse is $e^{\lambda s}$ the advance operator.

An *operational function* is a $I \to \mathcal{M}$ map, where I is a interval of $\mathbf{R}$. One may define its continuity, differentiability and integrability.

Operational calculus turns certain types of linear partial differential equations into linear ordinary differential equations that we call *operational equations*. The telegraph equation (1) is turned into the operational equation (2), where s is considered as a constant parameter. In equations (3), (4) and (5), $\hat{u}$, which corresponds to u, is a transcendental quantity with respect to the field $\mathcal{M}$.

A.2 Modules and flat outputs

Let us write (4) as

$$Q\hat{v} = P(x)\hat{u} \tag{7}$$

where

$$\begin{aligned} P(x) &= \mathrm{ch}(\ell - x)\sqrt{\varpi(s)} \\ &+ \frac{R + Ls}{Z}\frac{\mathrm{sh}(\ell - x)\sqrt{\varpi(s)}}{\sqrt{\varpi(s)}} \\ Q &= \mathrm{ch}\ell\sqrt{\varpi(s)} + \frac{R + Ls}{Z}\frac{\mathrm{sh}\ell\sqrt{\varpi(s)}}{\sqrt{\varpi(s)}} \end{aligned}$$

respectively are an operational function and and operator. We will check the properties of the $\mathbf{C}[P(x), Q]$-module M generated by $\hat{u}$ and $\hat{v}$, satisfying (7).

It can be shown [11] that $\mathbf{C}[P(x), Q]$ is isomorphic to a ring of polynomials in two variables with complex coefficients.The matrix $(Q, -P(x))$, which has generic rank 1, is the presentation matrix of M, i.e.

$$(Q, -P(x))\begin{pmatrix}\hat{v}\\ \hat{u}\end{pmatrix} = 0$$

Hence, by Youla's theorem [25] M is without torsion since the minors of this matrix are coprime. Moreover thanks to the resolution of Serre's conjecture by Quillen [19] and Suslin [23] M is not free; indeed the rank of the presentation matrix degenerates when the indeterminate represented by $P(x)$ and Q are equal to zero. The localized module

$$M_{\mathrm{loc}} = \mathbf{C}[P(x), Q, (P(x)Q)^{-1}] \otimes_{\mathbf{C[P,Q]}} M,$$

where one can multiply by the inverse of $P(x)Q$, is free and has $1 \otimes \hat{u}$ or $1 \otimes \hat{v}$ as a basis. This property is the π-freeness of [9, 17], where $\pi = PQ$.

Theorem. *The $\mathbf{C}[P, Q]$-module M is torsion-free but not free. The localized module M_{loc} is free.*

Since $P(\ell) = 1$, an interesting basis of M_{loc} is $\hat{y} = \hat{v}(0)$,which is called a *flat output*.

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