OBSERVABLE SYSTEMS TRANSFORMABLE INTO IMPLICIT AFFINE FORMS

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Abstract: A smooth observable system, $\dot{x} = f(x)$, $y = h(x)$, that can be put, after a change of state coordinates $x \mapsto X$, into $\dot{X} = A(y)X + b(y)$ with an implicit observation equation $C(y)X = d(y)$, is said to be transformable into an implicit affine form. Characterization of such systems is an open problem. For single output systems, a necessary constructive condition on the structure of the differential equation satisfied by the output is given. When $\dim(x) = 2$, this necessary condition, $\ddot{y} = f_0(y) + f_1(y)\dot{y} + f_2(y)\dot{y}^2 + f_3(y)\dot{y}^3$, is shown to be sufficient.

Keywords: observers, nonlinear control, affine systems, input injection

1. INTRODUCTION

Observers for nonlinear systems were much studied in the last decade, and real advances were done during this period. One way of building observers is to find new coordinates in which the system is in a special form. Two important classes of systems were successively studied with such techniques, systems linearizable by output injection (Krener and Respondek, 1985), and systems which are affine regarding to the unmeasured states (Besançon and Bornard, 1997; Hammouri and Gauthier, 1992). In both classes, a part of the states can be considered as measured, e.g, the observation equation admits an explicit structure.

In this paper, a larger class of systems is considered: systems transformable into an implicit affine form. This new class of systems, for which affine asymptotic observers can be designed, are systems $\dot{x} = f(x)$, $y = h(x)$, that can be put, after a change of state coordinates $x \mapsto X$, into $\dot{X} = A(y)X + b(y)$ with an implicit observation equation of the form $C(y)X = d(y)$. For single output systems, a necessary condition for arbitrary state dimension is derived here. This necessary condition (see lemma 4) is shown to be sufficient when $\dim(x) = 2$ (see theorem 1). This indicates, even for single output planar systems, that exist systems transformable into implicit affine forms that are not affine regarding to the unmeasured states in the sense of (Krener and Respondek, 1985; Besançon and Bornard, 1997; Hammouri and Gauthier, 1992): taking implicit observation equations enlarges the class of systems for which asymptotic observers derived from linear and least squares techniques can be designed.

This study of single output systems is local and around "generic" states x: open subsets where x can be expressed as a smooth function of $(h, L_f h, \ldots, L_f^{n-1} h) = (y, \dot{y}, \ldots, y^{(n-1)})$ ($n = \dim(x)$), are only considered here (i.e. open subsets where the tangent manifold is observable).

More precisely, we consider here single output observable systems of the form

$$\dot{x} = f(x), \quad y = h(x) \tag{1}$$

where f and h are smooth and where x lies an open subset of $\mathbb{R}^n$ such that exists a local diffeomorphism between x and $(y, \dot{y}, \ldots, y^{(n-1)})$. Such systems are called observable, in the sequel.

They are said to be transformable into an implicit affine form, if, and only if, exists a local state diffeomorphism $x = \phi(X)$ such that the observation equation $y = h(\phi(X))$ defines locally an affine subspace in the X-space and the restriction of the dynamics in the X-space, $\dot{X} = F(X)$, to the affine subspace $y = h(\phi(X))$ is affine (the restriction of F to $y = h(\phi(X))$ is affine). In other words, this means that in the X coordinates (1) reads

$$\dot{X} = A(y)X + b(y), \quad C(y)X = d(y) \tag{2}$$

where the matrices A, b, C and d are smooth functions of y.

Such systems admit an affine observer of the form:

$$\begin{aligned} \dot{\hat{X}} &= A(y)\hat{X} + b(y) && (3)\\ &\quad -P^{-1}C'(y)(C(y)\hat{X} - d(y)) \\ \dot{P} &= -\theta P - A^t(y)P - PA(y) \\ &\quad +C^t(y)C(y) && (4) \end{aligned}$$

which converges exponentially when y is persistent (in the sense of (Guillaume and Rouchon, 1997), i.e. $P > \mu I_d$ for some $\mu > 0$).

The paper is organized as follows. In section 2, single output planar observable systems transformable into implicit affine forms are characterized. Section 3 deals with the necessary condition for single output systems with an arbitrary state dimension.

2. SINGLE INPUT PLANAR SYSTEMS

2.1 *Necessary condition*

Assume that the observable system (1) is transformable into an implicit affine form. Set

$$A(y) = \begin{pmatrix} a_{11}(y) & a_{12}(y) \\ a_{21}(y) & a_{22}(y) \end{pmatrix}$$
$$b(y) = \begin{pmatrix} b_1(y) & b_2(y) \end{pmatrix}^t$$
$$C(y) = \begin{pmatrix} c_1(y) & c_2(y) \end{pmatrix}$$

After derivation, ($'$ stands for derivation with respect to y):

$$CX = d$$
$$(C'\dot{y} + CA)X + Cb = d'\dot{y}$$
$$\frac{d(CX)}{dt} = PX + Q = R$$

with $P = (C''\dot{y}^2 + C'\ddot{y} + 2C'\dot{y}A + CA'\dot{y} + CA^2)$, $Q = 2C'b\dot{y} + CAb + Cb'\dot{y}$ and $R = d''\dot{y}^2 + d'\ddot{y}$. Since $\begin{pmatrix} C \\ C'\dot{y} + CA \end{pmatrix}$ is assumed invertible, X can be eliminated from the last equation. This yields a second order differential equation for y. Examine the structure of this equation. Up to renumbering the component of x, assume that $c_1(y) \neq 0$. Thus $X_1 = d(y)/c_1(y) - c_2(y)/c_1(y)X_2$. Without lost of generality, assume also that the first column of A is zero (substitute X_1 into the state equation) and $c_1 \equiv 1$. This leads to the following simplified structure:

$$\begin{aligned} A(y) &= \begin{pmatrix} 0 & a_1(y) \\ 0 & a_2(y) \end{pmatrix} \\ b(y) &= \begin{pmatrix} b_1(y) & b_2(y) \end{pmatrix}^t \\ C(y) &= \begin{pmatrix} 1 & c(y) \end{pmatrix}. \end{aligned} \tag{5}$$

Denote

$$\alpha_1 = a_1 + ca_2, \quad \beta_1 = b_1 + cb_2. \tag{6}$$

Elimination of X leads to

$$g(y)\ddot{y} = g_0(y) + g_1(y)\ \dot{y} + g_2(y)\ \dot{y}^2 + g_3(y)\ \dot{y}^3$$

with

$$\begin{aligned} g &= (c'\beta_1 + d'\alpha_1) \\ g_0 &= (-\beta_1\alpha_1 a_2 + \alpha_1^2 b_2) \\ g_1 &= (-\beta_1(\alpha_1' + c'a_2) + d'\alpha_1 a_2 + 2c'\alpha_1 b_2 + \alpha_1\beta_1') \\ g_2 &= (-d''\alpha_1 - c''\beta_1 + (\alpha_1' + c'a_2)d' + (\beta_1' + c'b_2)c') \\ g_3 &= (d'c'' - c'd''). \end{aligned}$$

Observability of (1), means here that exists a local diffeomorphism between X and $(y, \dot{y})$. This implies that, locally, g cannot vanish. Thus the following lemma is proved:

Lemma 1. (Necessary condition). If the observable planar system (1) is transformable into an implicit affine system (3) then, its output y satisfies a second order differential equation of the form

$$\ddot{y} = f_0(y) + f_1(y)\ \dot{y} + f_2(y)\ \dot{y}^2 + f_3(y)\ \dot{y}^3$$

with f_0, f_1, f_2, f_3 smooth functions.

2.2 *Sufficient condition*

Assume that the output y of planar observable system (1) satisfies a second order differential equation of the form

$$\ddot{y} = f_0(y) + f_1(y)\ \dot{y} + f_2(y)\ \dot{y}^2 + f_3(y)\ \dot{y}^3$$

with the f_0, f_1, f_2 and f_3 smooth function of y. A diffeomorphism $x = \phi(X)$ that put the system into an implicit affine form 3 can explicitely be built.

From previous computation,it appears that the problem to solve is equivalent to find smooth y-function c, d, α_1, β_1, b_2 and a_2 such that

$$\begin{aligned} f_0(c'\beta_1 + d'\alpha_1) &= (-\beta_1\alpha_1 a_2 + \alpha_1^2 b_2) \\ f_1(c'\beta_1 + d'\alpha_1) &= (-\beta_1(\alpha_1' + c'a_2) + d'\alpha_1 a_2 \\ &\quad +2c'\alpha_1 b_2 + \alpha_1\beta_1') \\ f_2(c'\beta_1 + d'\alpha_1) &= (-d''\alpha_1 - c''\beta_1 \\ &\quad +(\alpha_1' + c'a_2)d' + (\beta_1' + c'b_2)c') \\ f_3(c'\beta_1 + d'\alpha_1) &= (d'c'' - c'd''). \end{aligned} \quad (7)$$

Basic algebraic manipulations of this system yield to the following three equations:

$$\begin{aligned} &\alpha_1\beta_1\alpha_1' = -(c'\beta_1 + d'\alpha_1)(2c'f_0 + \alpha_1 f_1) \\ &-c'\alpha_1\beta_1 a_2 + d'\alpha_1^2 a_2 + \alpha_1^2\beta_1' - 2c'\beta_1\alpha_1 a_2 \\ &\alpha_1^2\beta_1(c'\beta_1 + d'\alpha_1)c'' = c'F + \alpha_1^3\beta_1 f_3(c'\beta_1 + d'\alpha_1) \\ &\alpha_1^2\beta_1(c'\beta_1 + d'\alpha_1)d'' = d'F - \alpha_1^2\beta_1^2 f_3(c'\beta_1 + d'\alpha_1) \end{aligned} \quad (8)$$

where

$$\begin{aligned} F = \beta_1(&\alpha_1^2(c'a_2d' + \beta_1'c') - c'^2\beta_1\alpha_1 a_2) \\ &+\alpha_1 d'(\alpha_1(-\beta_1 c'a_2 + d'\alpha_1 a_2 + \alpha_1\beta_1') \\ &-(c'\beta_1 + d'\alpha_1)(\alpha_1 d'(2c'f_0 + \alpha_1 f_1) \\ &+\beta_1(c'^2 f_0 + \alpha_1^2 f_2)) - 2c'\beta_1\alpha_1 a_2). \end{aligned}$$

Denote by $\bar{x}$ the state around which the transformation $x = \phi(X)$ is looked for and denote by $(\bar{y}, \dot{\bar{y}})$ the corresponding values of $(y, \dot{y})$.

Set $\beta_1(y) \equiv 1$ and $a_2(y) \equiv 0$. By the Cauchy-Lipschitz theorem, there exists locally around $\bar{y}$, a unique solution $y \mapsto (\alpha_1, c, d)$ of the differential system (8) with the following initial values: $\alpha_1(\bar{y}) = (\dot{\bar{y}})^2 + 1$, $c(\bar{y}) = d(\bar{y}) = d'(\bar{y}) = 0$, $c'(\bar{y}) = 1$. Such a solution yields to $b_2(y)$ via $\alpha_1^2 b_2 = f_0(c'\beta_1 + d'\alpha_1) + \beta_1\alpha_1 a_2$.

For A, b and C derived from the above solution $y \mapsto (\alpha_1, \beta_1, a_2, b_2, c, d)$ of (7) via the formulae (5) and (6), the relations

$$\begin{pmatrix} C \\ C'\dot{y} + CA \end{pmatrix} X = \begin{pmatrix} d \\ d'\dot{y} - Cb \end{pmatrix} \quad (9)$$

define a local diffeomorphism between $(y, \dot{y})$ and X, that is between x and X. This results from the following arguments:

- since $c'(\bar{y})\dot{\bar{y}} + \alpha_1(\bar{y}) = \bar{y}^2 + \bar{y} + 1 > 0$, the matrix
$$\begin{pmatrix} C \\ C'\dot{y} + CA \end{pmatrix}$$
is locally invertible; thus X is a smooth function of $(y, \dot{y})$.
- denote by $\bar{X}$ the solution of (9) for $(\bar{y}, \dot{\bar{y}})$; since
$$(\alpha_1(\bar{y}) + c'(\bar{y})\dot{\bar{y}})\bar{X}_2 = d'(\bar{y})\dot{\bar{y}} - \beta_1(\bar{y}),$$
$\bar{X}_2 \neq 0$ and $c'(\bar{y})\bar{X}_2 - d'(\bar{y}) \neq 0$; the implicit function theorem for $X_1 + c(y)X_2 = d(y)$ ensures that y can be expressed as a smooth function of X;
- similarly, $\dot{y}$ is also a smooth function of X.

Thus, the following lemma is proved:

Lemma 2. (Sufficient condition). Assume that the scalar output of the planar observable systems (1) satisfies a smooth second order differential equation of the form

$$\ddot{y} = f_0(y) + f_1(y)\ \dot{y} + f_2(y)\ \dot{y}^2 + f_3(y)\ \dot{y}^3.$$

Then, locally, there exists a change of state coordinates putting the system into an implicit affine form (3).

Lemmas 1 et 2 lead to the following

Theorem 1. A single output observable planar system (1) is transformable into an implicit affine form (3), if, and only if, the second-order derivative of the output is a polynomial of third degree in the first-order derivative of the output with coefficients smooth function of the output.

Remark that:

- planar observable systems transformable into implicit affine forms admit the following normal forms
$$\begin{aligned} \dot{X}_1 &= a(y)X_2 + b_1(y) \\ \dot{X}_2 &= b_2(y) \\ X_1 + c(y)X_2 &= d(y). \end{aligned}$$
This results from the proof of the sufficient condition.
- The class of planar observable systems, affine regarding to the unmeasured states in the sense of (Krener and Respondek, 1985; Besançon and Bornard, 1997; Hammouri and Gauthier, 1992), admit the following output differential equation:
$$\ddot{y} = k_0(y) + k_1(y)\ \dot{y} + k_2(y)\ \dot{y}^2.$$
This former class is thus smaller than the class considered here.

3. NECESSARY CONDITION FOR GENERAL SINGLE OUTPUT SYSTEMS

Denote by n the state dimension and take the implicit affine form. After n derivations of the observation equation, the following system:

$$
\begin{aligned}
CX &= d \\
\frac{d(CX)}{dt} &= d'\dot{y} \\
\frac{d^2(CX)}{dt^2} &= d"\dot{y}^2 + d'\ddot{y} \\
\ldots &= \ldots \\
\frac{d^n(CX)}{dt^n} &= d'y^{(n)} + \ldots + d^{(n)}\dot{y}^n
\end{aligned} \tag{10}
$$

is a linear overdetermined system with respect to X. Eliminating of X yields to a polynomial implicit scalar equation with respect to $(\dot{y}, \ldots, y^{(n)})$. A careful inspection of the different partial degrees of this polynomial equation yields to a necessary condition. Introduce the following definitions borrowed from (Krener and Respondek, 1985).

Definition 1. Note $\mathcal{P}$ the ring of polynomials with indeterminates the successive derivative $y^{(k)}$ of y and with coefficients $\mathcal{C}^\infty$ functions of y. By definition the degree of $y^{(k)}$ is k and the degree of $y^{(j_1)} \ldots y^{(j_r)}$ is $j_1 + \ldots + j_r$. Note $\mathcal{P}^k$ the subring of polynomials, the degree of which is less than or equal to k and $\mathcal{P}_i^k$ the subring of polynomials, the degree of which is less than or equal to k spanned by $\mathcal{P}^i$, with $i \leq k$.

Lemma 3. The system (10) can be written:

$$
\begin{aligned}
P_0X + Q_0(y) &= R_0 \\
\ldots &= \ldots \\
P_{n-1}X + Q_{n-1} &= R_{n-1} \\
P_nX + Q_n &= R_n
\end{aligned} \tag{11}
$$

with P_i and R_i elements of $\mathcal{P}^i$ for $i = 1 \ldots n$ and Q_i element de $\mathcal{P}^{i-1}$ for $i = 2 \ldots n$.

Proof: Immediate for $i = 1, 2, 3 \ldots$ The derivative of a monomial of the form $f(y)y^{(k)}$ is a monomial of degree $k+1$ because $\frac{d(f(y)y^{(k)})}{dt} = f'(y)\dot{y}y^{(k)} + f(y)y^{(k+1)}$.

The structure of the (vector) polynomials can be more precisely described:

$$
P = (p_{ij}) = \begin{pmatrix} P_0 \\ P_1 \\ \vdots \\ P_{n-1} \end{pmatrix}
$$

with

$$
\begin{aligned}
P_0 &= C \\
P_1 &= C'\dot{y} + CA \\
&\vdots \\
P_{n-1} &= C^{(n-1)}\dot{y}^{n-1} \\
&\quad +(n-1)C^{(n-1)}\ddot{y}\dot{y}^{n-3} + \ldots + CA^{n-2}
\end{aligned}
$$

$Q = \begin{pmatrix} Q_0 \\ Q_1 \\ \vdots \\ Q_n \end{pmatrix}$, $R = \begin{pmatrix} R_0 \\ R_1 \\ \vdots \\ R_n \end{pmatrix}$, $\Delta = \det P$, Δ_{ij} the cofactors of P and $M = (\Delta_{ij})$, the comatrix.

The elimination of X from the previous equations leads to:

$$
P_n P^{-1}(R - Q) + Q_n = R_n
$$

that is:

$$
P_n M^t(R - Q) + \Delta Q_n = \Delta R_n \tag{12}
$$

The coefficient $\mathcal{D}$ of $y^{(n)}$ in (12) is: $\mathcal{D} = \Delta(C'X - d') = C'M^t(R-Q) - d'\Delta$ with R and Q elements of $\mathcal{P}^{n-1}$. M is an element of $\mathcal{P}_{n-1}^{n(n-1)/2-1}$ and Δ an element of $\mathcal{P}_{n-1}^{n(n-1)/2}$. We have the following degrees for the lines $M(i)$, $Q(i)$ and $R(i)$ of the corresponding matrices

$$
\begin{aligned}
d^\circ M(1) &= n(n-1)/2 \\
d^\circ M(i) &= n(n-1)/2 - (i-1), i = 2 \ldots n \\
d^\circ Q(1) &= 0 \\
d^\circ Q(i) &= i - 2, i = 2 \ldots n \\
d^\circ R(i) &= i - 1, i = 1 \ldots n
\end{aligned}
$$

Then, $\mathcal{D}$ is an element of $\mathcal{P}_{n-1}^{n(n-1)/2}$. Isolating the terms with d' in X and using:

$$
\mathcal{D} = \sum_{k=1}^{n} c'(k)(-1)^{n+k}\Delta_{nk} d' y^{(n-1)} - d'\Delta + \ldots
$$

with $\Delta = \sum_{k=1}^{n}(-1)^{n+k}p_{nk}\Delta_{nk}$ and $p_{nk} = c'(k)y^{(n-1)} + \ldots$, the term in $y^{(n-1)}$ disappears. $\mathcal{D}$ is an element of a subring spanned by $\mathcal{P}^{n-2}$.

From the equation $(C'X - d')\dot{y} = -(CAX + Cb)$ and after multiplication by Δ, it appears that the degree of the right term is less than equal to $n(n-1)/2$ (degree de ΔX and of Δ) and the one of the left term less than or equal $n(n-1)/2 + 1$. This is possible only if the degree of $(C'X - d')\Delta$ is less than or equal to $n(n-1)/2 - 1$. There is no term of degree $n(n-1)/2$ in $\mathcal{D}$. Finally, $\mathcal{D}$ is an element of $\mathcal{P}_{n-2}^{(n(n-1)/2-1)}$.

Consider the degree of the remaining term $\mathcal{N}$. $\mathcal{N}$ is got from the equation (12), in which the term with $y^{(n)}$ is suppressed:

$$
\begin{aligned}
\mathcal{N} &= (P_n(y, \ldots, y^{(n-1)})X \\
&+Q_n(y, \ldots, y^{(n-2)}) - R_n(y, \ldots, y^{(n-1)}))\Delta
\end{aligned}
$$

P_n is an element of $\mathcal{P}_{n-1}^n$ (polynomial of degree n without derivatives of order n) and ΔX element of $\mathcal{P}_{n-1}^{(n(n-1)/2)}$ so $\mathcal{N}$ is an element of $\mathcal{P}_{n-1}^{n(n+1)/2}$.

Sum up these results in the following proposition:

Lemma 4. If the single output observable system (1) is transformable into an affine system (3) then the output y satisfies the following differential equation:

$$y^{(n)} = \mathcal{N}/\mathcal{D} \tag{13}$$

with $\mathcal{N}$ element of $\mathcal{P}_{n-1}^{n(n+1)/2}$ and $\mathcal{D}$ element of $\mathcal{P}_{n-2}^{(n(n-1)/2-1)}$ (see definition 1).

For $n = 3$, the following output equation exists necessarily:

$$y^{(3)} = \frac{\mathcal{N}(y, \dot{y}, \ddot{y})}{d_0(y) + d_1(y)\dot{y} + d_2(y)\dot{y}^2}$$

with

$$\begin{aligned}
\mathcal{N} = {} & n_1(y)\ (\ddot{y})^3 + n_2(y)\ (\ddot{y})^2\dot{y}^2 + n_3(y)\ \ddot{y}\dot{y}^4 + n_4(y)\dot{y}^6 \\
& + n_5(y)\ (\ddot{y})^2\dot{y} + n_6(y)\ \ddot{y}\dot{y}^3 + n_7(y)\ \dot{y}^5 \\
& + n_8(y)\ (\ddot{y})^2 + n_9(y)\ \ddot{y}\dot{y}^2 + n_{10}(y)\ \dot{y}^4 \\
& + n_{11}(y)\ \ddot{y}\dot{y} + n_{12}(y)\ \dot{y}^3 \\
& + n_{13}(y)\ \ddot{y} + n_{14}(y)\ \dot{y}^2 \\
& + n_{15}(y)\dot{y} \\
& + n_{16}(y)
\end{aligned}$$

The fact that the denominator depends only on $\dot{y}$ is not obvious.

Remark: the gap between the dimension 2 and 3 appears here. For the 3 dimensional case, there are 19 equations to satisfy for 12 functions (overdetermined), when for the 2 dimensional case, there are 5 equations (4 if g is set to 1) for 6 functions (underdetermined). So we see that there are degrees of freedoms for the 2 dimensional and relations to satisfy in the 3 dimensional case.

4. CONCLUSION

Here is establishesd a necessary condition so that an nonlinear system is equivalent to an affine system (3) with an observation equation, affine regarding to the state and implicit regarding to the the measured output. A such class of systems contains linearizable systems by output injection, and also affine systems described in (Besançon and Bornard, 1997). An observer for the system described here can be found in (Guillaume and Rouchon, 1997). Our present work is on the intrinsic characterization of such systems and also on the way to compute the coordinates change when it exists.

5. REFERENCES

Besançon, G. and G. Bornard (1997). On characterizing classes of observers forms for nonlinear systems. *Proceedings of the 4st European Control Conference ECC97.*

Guillaume, D. and P. Rouchon (1997). Estimation d'état et de paramètres cinétiques pour une classe de bioréacteurs. *Proc AGIS'97* **1**, 173–177.

Hammouri, H. and JP. Gauthier (1992). Global time-varying linearization up to output injection. *SIAM J. Control Optim.* **30**, 1295–1310.

Krener, A.J. and W. Respondek (1985). Nonlinear observers with linearizable error dynamics. *SIAM J. Control Optim.* **23**, 1197–216.