# $\delta$-FREENESS OF A CLASS OF LINEAR SYSTEMS

**Nicolas Petit***, **Yann Creff**†, **Pierre Rouchon**‡

*Centre Automatique et Systèmes, Ecole des Mines de Paris, 60, bd. Saint-Michel, 75272 Paris Cedex 06, France. Tel: (33) 01 40 51 93 29. E-mail: petit@cas.ensmp.fr

†ELF ANTAR FRANCE, CRES, BP 22 69 360 Solaize, France. Tel: 04 78 02 64 67. E-mail: yann.creff@s1.elf-antar-france.elf1.fr

‡Centre Automatique et Systèmes, Ecole des Mines de Paris, 60, bd. Saint-Michel, 75272 Paris Cedex 06, France. Tel: 01 40 51 91 15. E-mail: rouchon@cas.ensmp.fr

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## Abstract

Starting from a simple example of linear delayed system (with 2 inputs and 2 outputs) commonly used in process control, we show that, as for flat systems (see [1]), an explicit parametrization of all the trajectories can be found. Once more this leads to an easy motion planning. More generally speaking, we prove that this property, called $\delta$-freeness (see [2, 4]) is general among higher dimensions linear delayed systems.

More theoretically speaking, we use the module framework and consider a linear delayed system as a finitely generated module over the ring $R[\frac{d}{dt}, \delta]$, where $\delta$ is one or a set of delay operators. We show that this system is $\delta$-free. That is we can find a basis of its corresponding module over the localized ring $R[\frac{d}{dt}, \delta, \delta^{-1}]$. An applicable way to exhibit such a basis is explicitly described.

## 1 An introductory example to motion planning using $\delta$-freeness

Let us start by considering a simple system with two inputs and two outputs:

$$y = \begin{pmatrix} \dfrac{K_1^1\ e^{-\delta_1^1\ s}}{1+\tau_1^1\ s} & \dfrac{K_1^2\ e^{-\delta_1^2\ s}}{1+\tau_1^2\ s} \\ \dfrac{K_2^1\ e^{-\delta_2^1\ s}}{1+\tau_2^1\ s} & \dfrac{K_2^2\ e^{-\delta_2^2\ s}}{1+\tau_2^2\ s} \end{pmatrix} u$$

with $s$ the Laplace variable, $i \in \{1,2\}$, $j \in \{1,2\}$, $\tau_i^j \in R^{*+}, K_i^j \in R^*, \delta_i^j \in R^+$.

We want to determine the commands $u$ that will steer the system from the steady state $(\bar{y}, \bar{u})$ to the steady state $(\tilde{y}, \tilde{u})$ within a desired time $T$ that must be well choosen.

Let us introduce $\xi = (\xi^1, \xi^2)$, that we call **$\delta$-flat outputs**:

$$\begin{aligned}
\xi^1(t) \quad = \quad & \frac{\tau_1^1\ K_2^1}{\frac{1}{\tau_1^2} - \frac{1}{\tau_1^1}} \left( \dot{y}_1(t+\delta_1^1) + \frac{y_1(t+\delta_1^1)}{\tau_1^2} \right) \\
& - \frac{\tau_2^1\ K_1^1}{\frac{1}{\tau_2^2} - \frac{1}{\tau_2^1}} \left( \dot{y}_2(t+\delta_2^1) + \frac{y_2(t+\delta_2^1)}{\tau_2^2} \right) \\
& + \left( \frac{K_1^1\ K_2^1}{\frac{1}{\tau_2^2} - \frac{1}{\tau_2^1}} - \frac{K_1^1\ K_2^1}{\frac{1}{\tau_1^2} - \frac{1}{\tau_1^1}} \right) u^1(t) \\
& - \frac{\tau_1^1\ K_2^1\ K_1^2}{1 - \frac{\tau_1^2}{\tau_1^1}}\ u^2(t - \delta_1^2 + \delta_1^1) \\
& + \frac{\tau_2^1\ K_1^1\ K_2^2}{1 - \frac{\tau_2^2}{\tau_2^1}}\ u^2(t - \delta_2^2 + \delta_2^1) \\
\xi^2(t) \quad = \quad & \frac{\tau_1^2\ K_2^2}{\frac{1}{\tau_1^1} - \frac{1}{\tau_1^2}} \left( \dot{y}_1(t+\delta_1^2) + \frac{y_1(t+\delta_1^2)}{\tau_1^1} \right) \\
& - \frac{\tau_2^2\ K_1^2}{\frac{1}{\tau_2^1} - \frac{1}{\tau_2^2}} \left( \dot{y}_2(t+\delta_2^2) + \frac{y_2(t+\delta_2^2)}{\tau_2^1} \right) \\
& + \left( \frac{K_1^2\ K_2^2}{\frac{1}{\tau_2^1} - \frac{1}{\tau_2^2}} - \frac{K_1^2\ K_2^2}{\frac{1}{\tau_1^1} - \frac{1}{\tau_1^2}} \right) u^2(t) \\
& - \frac{\tau_1^2\ K_2^2\ K_1^1}{1 - \frac{\tau_1^1}{\tau_1^2}}\ u^1(t - \delta_1^1 + \delta_1^2) \\
& + \frac{\tau_2^2\ K_1^2\ K_2^1}{1 - \frac{\tau_2^1}{\tau_2^2}}\ u^1(t - \delta_2^1 + \delta_2^2).
\end{aligned}$$

One can determine all the quantities of the system from $\xi, \dot{\xi}, \ddot{\xi}$ by linear combinations, provided that the $\tau_i^j$ are all different. Explicitly:

$$\begin{aligned}
y_1(t) \quad = \quad & \frac{\xi^1(t-\delta_1^1) + \tau_2^1\ \dot{\xi}^1(t-\delta_1^1)}{(\tau_1^1 - \tau_2^1)\ K_2^1} \\
& + \frac{\xi^2(t-\delta_1^2) + \tau_2^2\ \dot{\xi}^2(t-\delta_1^2)}{(\tau_1^2 - \tau_2^2)\ K_2^2}
\end{aligned}$$

$$
\begin{aligned}
y_2(t) &= \frac{\xi^1(t-\delta_2^1)+\tau_1^1\ \dot{\xi}^1(t-\delta_2^1)}{(\tau_2^1-\tau_1^1)\ K_1^1} \\
&\quad + \frac{\xi^2(t-\delta_2^2)+\tau_1^2\ \dot{\xi}^2(t-\delta_2^2)}{(\tau_2^2-\tau_1^2)\ K_1^2} \\
u^1(t) &= \frac{\xi^1(t)+(\tau_1^1+\tau_2^1)\ \dot{\xi}^1(t)+\tau_1^1\ \tau_2^1\ \ddot{\xi}^1(t)}{K_1^1\ K_2^1\ (\tau_1^1-\tau_2^1)} \\
u^2(t) &= \frac{\xi^2(t)+(\tau_1^2+\tau_2^2)\ \dot{\xi}^2(t)+\tau_1^2\ \tau_2^2\ \ddot{\xi}^2(t)}{K_1^2\ K_2^2\ (\tau_1^2-\tau_2^2)}.
\end{aligned}
$$

These last relations show the invertible transformation exchanging the trajectories of $\xi$ and those of $(y,u)$. The boundary conditions can be equivalently written for these $\delta$-flat outputs.

$$
\begin{aligned}
\bar{\xi}^1 &= \frac{\tau_1^1\ K_2^1}{1-\frac{\tau_1^2}{\tau_1^1}}\ (\bar{y}_1-K_1^2\ \bar{u}^2) - \frac{\tau_2^1\ K_1^1}{1-\frac{\tau_2^2}{\tau_2^1}}\ (\bar{y}_2-K_2^2\ \bar{u}^2) \\
&\quad +\left(\frac{K_1^1\ K_2^1}{\frac{1}{\tau_2^2}-\frac{1}{\tau_2^1}} - \frac{K_1^1\ K_2^1}{\frac{1}{\tau_1^2}-\frac{1}{\tau_1^1}}\right)\ \bar{u}^1 \\
\bar{\xi}^2 &= \frac{\tau_1^2\ K_2^2}{1-\frac{\tau_1^1}{\tau_1^2}}\ (\bar{y}_1-K_1^1\ \bar{u}^1) - \frac{\tau_2^2\ K_1^2}{1-\frac{\tau_2^1}{\tau_2^2}}\ (\bar{y}_2-K_2^1\ \bar{u}^1) \\
&\quad +\left(\frac{K_1^2\ K_2^2}{\frac{1}{\tau_2^1}-\frac{1}{\tau_2^2}} - \frac{K_1^2\ K_2^2}{\frac{1}{\tau_1^1}-\frac{1}{\tau_1^2}}\right)\ \bar{u}^2 \\
\tilde{\xi}^1 &= \frac{\tau_1^1\ K_2^1}{1-\frac{\tau_1^2}{\tau_1^1}}\ (\tilde{y}_1-K_1^2\ \tilde{u}^2) - \frac{\tau_2^1\ K_1^1}{1-\frac{\tau_2^2}{\tau_2^1}}\ (\tilde{y}_2-K_2^2\ \tilde{u}^2) \\
&\quad +\left(\frac{K_1^1\ K_2^1}{\frac{1}{\tau_2^2}-\frac{1}{\tau_2^1}} - \frac{K_1^1\ K_2^1}{\frac{1}{\tau_1^2}-\frac{1}{\tau_1^1}}\right)\ \tilde{u}^1 \\
\tilde{\xi}^2 &= \frac{\tau_1^2\ K_2^2}{1-\frac{\tau_1^1}{\tau_1^2}}\ (\tilde{y}_1-K_1^1\ \tilde{u}^1) - \frac{\tau_2^2\ K_1^2}{1-\frac{\tau_2^1}{\tau_2^2}}\ (\tilde{y}_2-K_2^1\ \tilde{u}^1) \\
&\quad +\left(\frac{K_1^2\ K_2^2}{\frac{1}{\tau_2^1}-\frac{1}{\tau_2^2}} - \frac{K_1^2\ K_2^2}{\frac{1}{\tau_1^1}-\frac{1}{\tau_1^2}}\right)\ \tilde{u}^2.
\end{aligned}
$$

Smoothness implies $\dot{\xi}(t\le 0)=0$, $\ddot{\xi}(t\le 0)=0$, $\dot{\xi}(t\ge\Delta)=0$, $\ddot{\xi}(t\ge\Delta)=0$ with $\Delta = T - max_{i,j}(\delta_i^j)$, provided that $T > max_{i,j}(\delta_i^j)$. This permits the continuity of the commands.

Any smooth function $[0,\Delta]\ni t\longmapsto \xi(t)$ satisfying the conditions above will provide us a set of commands for the desired motion planning.

For example, one could choose

$$\xi(t) = (1-\pi(\frac{t}{\Delta}))\bar{\xi} + \pi(\frac{t}{\Delta})\tilde{\xi}$$

where

$$\pi(\sigma) = \begin{cases} 0 & \sigma < 0 \\ 6\ \sigma^5 - 15\ \sigma^4 + 10\ \sigma^3 & \\ 1 & \sigma > 1 \end{cases}$$

but many other choices are possible.

Similarly it is possible to steer the system from a past trajectory to a future one. One just have to replace $(\bar{y},\bar{u})$ and $(\tilde{y},\tilde{u})$ by $(\bar{y},\bar{u})(t)$ and $(\tilde{y},\tilde{u})(t)$, then calculate $\bar{\xi}(t)$ and $\tilde{\xi}(t)$ and use

$$\xi(t) = (1-\pi(\frac{t}{\Delta}))\bar{\xi}(t) + \pi(\frac{t}{\Delta})\tilde{\xi}(t).$$

This proves that such systems are controllable in the sense of [5] and [7].

We have singled out the fundamental topic, namely the existence of a parametrization of the trajectories. In the following we will exhibit the same property in the general case.

# 2 Main result

From now on, the system under consideration has $p$ outputs and $m$ independent inputs and is called **the original system**. It is frequently used in process control [6] :

| Inputs / Outputs | $u^1$ | $\ldots$ | $u^m$ |
|---|---|---|---|
| $y_1 = z_1^1 + \ldots + z_1^m$ | $z_1^1$ | $\ldots$ | $z_1^m$ |
| $\vdots$ | $\vdots$ | | $\vdots$ |
| $y_p = z_p^1 + \ldots + z_p^m$ | $z_p^1$ | $\ldots$ | $z_p^m$ |

where the $z_i^j$ stand for

$$z_i^j = \frac{K_i^j\ e^{-\delta_i^j\ s}}{1+\tau_i^j\ s}\ u^j,$$

with $s$ the Laplace variable, $i\in\{1,\ldots,p\}$, $j\in\{1,\ldots,m\}$ $\tau_i^j\in R^{*+}$, $K_i^j\in R$, $\delta_i^j\in R^+$.

Note that $K_i^j = 0$ means that $u^j$ does not affect $y_i$. Moreover we assume that every input does affect the system, which means that for each column $j$ there exists $i_j$ such as $K_{i_j}^j \neq 0$.

**Definition 1** *In each column $j$, let us denote $\{z_{nz_1}^j,\ldots,z_{nz_{t_j}}^j\}$ the set of partial states $z_i^j$ whose $K_i^j\neq 0$ (one could call them the non-zero (nz) states). These and only these act upon the outputs. Among these, let $\{z_{i_1}^j,\ldots,z_{i_{n_j}}^j\}$ be a maximal set of partial states such as $\tau_{i_k}^j \neq \tau_{i_l}^j$ for all $k,l\le n_j$. We call them* **the essential partial states** *of the column. One can easily check that there is at least one essential partial state per column.*

**Main result** *Let $\delta = \{\delta_i^j, i=1,\ldots,p, j=1,\ldots,m\}$. For each column $j$, one can exhibit $\xi^j$, a $R[\delta^{-1}]$ combination of elements of $\{z_{i_1}^j,\ldots,z_{i_{n_j}}^j\}$ that is a basis of the $R[\frac{d}{dt},\delta,\delta^{-1}]$ module corresponding to $\{u^j, z_{i_1}^j,\ldots,z_{i_{n_j}}^j\}$. This does not require any rational relation between the $\delta_i^j$.*

*As a result one gets $\{\xi^1,\ldots,\xi^m\}$ which is a basis of the $R[\frac{d}{dt},\delta,\delta^{-1}]$ module corresponding to the original system,* *that is the module spanned by the essential partial states and the inputs, which is thus $\delta$-free.*

## 2.1 Building up $\{\xi^1, \dots, \xi^m\}$

Let us consider any of the $m$ columns, say the $j^{th}$ column. Denote $\{z^j_{i_1}, \dots, z^j_{i_{n_j}}\}$ the set of its essential partial states. Obviously $n_j$, which is the number of partial states of the $j^{th}$ column, depends on $j$. To streamline notation we now denote $\{z^j_{i_1}, \dots, z^j_{i_{n_j}}\}$ as $\{z_1, \dots, z_q\}$. That means that subsequently we won't keep in mind the number of the column we work in, and that we will use a dedicated reordering of the partial states of the column. Now we are looking for a basis of the $R[\frac{d}{dt}, \delta, \delta^{-1}]$ module corresponding to $\{u^j, z^j_{i_1}, \dots, z^j_{i_{n_j}}\}$. In other words we are looking for a basis of the $R[\frac{d}{dt}, \delta, \delta^{-1}]$ module corresponding to $\{u, z_1, \dots, z_q\}$. We can try this kind of $R[\delta^{-1}]$ combination:

$$\xi = a_1\ z_1(t+\delta_1) + \dots + a_q\ z_q(t+\delta_q)$$

where the appropriate $a_1, \dots, a_q$ are to be found. Let us calculate the derivatives of $\xi$. First:

$$\begin{aligned}\dot{\xi} &= -\left[\frac{a_1}{\tau_1}\ z_1(t+\delta_1) + \dots + \frac{a_q}{\tau_q}\ z_q(t+\delta_q)\right] \\ &\quad +\left(\frac{a_1\ K_1}{\tau_1} + \dots + \frac{a_q\ K_q}{\tau_q}\right)\ u(t)\end{aligned}$$

since

$$\dot{z_k}(t+\delta_k) = \frac{K_k\ u(t) - z_k(t+\delta_k)}{\tau_k}.$$

Now we choose to get rid off $u(t)$. In order to do so we make:

$$\frac{a_1\ K_1}{\tau_1} + \dots + \frac{a_q\ K_q}{\tau_q} = 0.$$

Assuming this, the next derivative is:

$$\begin{aligned}\ddot{\xi} &= \left[\frac{a_1}{(\tau_1)^2}\ z_1(t+\delta_1) + \dots + \frac{a_q}{(\tau_q)^2}\ z_q(t+\delta_q)\right] \\ &\quad +\left(\frac{a_1\ K_1}{(\tau_1)^2} + \dots + \frac{a_q\ K_q}{(\tau_q)^2}\right)\ u(t).\end{aligned}$$

Once more we want to get rid off $u(t)$, which means:

$$\frac{a_1\ K_q}{(\tau_1)^2} + \dots + \frac{a_q\ K_q}{(\tau_q)^2} = 0.$$

We go on successively until the $(q-1)^{th}$ derivative

$$\begin{aligned}(\xi)^{(q-1)} &= \left[\frac{a_1}{(\tau_1)^{(q-1)}}\ z_1(t+\delta_1) + \dots \right.\\ &\quad \left. + \frac{a_q}{(\tau_q)^{(q-1)}}\ z_q(t+\delta_q)\right]\ (-1)^{(q-1)} \\ &\quad +\left(\frac{a_1\ K_1}{(\tau_1)^{(q-1)}} + \dots + \frac{a_q\ K_q}{(\tau_q)^{(q-1)}}\right)\ u(t).\end{aligned}$$

The final condition is:

$$\frac{a_1\ K_1}{(\tau_1)^{(q-1)}} + \dots + \frac{a_q\ K_q}{(\tau_q)^{(q-1)}} = 0.$$

In the end, assuming the $q-1$ equations of $C$ over the $q$ variables $a_i$ we guarantee $D$:

$$C: \left\{\begin{array}{rcl} \frac{a_1\ K_1}{\tau_1} + \dots + \frac{a_q\ K_q}{\tau_q} &=& 0 \\ \frac{a_1\ K_1}{(\tau_1)^2} + \dots + \frac{a_q\ K_q}{(\tau_q)^2} &=& 0 \\ &\vdots& \\ \frac{a_1\ K_1}{(\tau_1)^{(q-1)}} + \dots + \frac{a_q\ K_q}{(\tau_q)^{(q-1)}} &=& 0 \end{array}\right.$$

$$D: \left\{\begin{array}{rcl} \xi &=& a_1\ z_1(t+\delta_1) + \dots + a_q\ z_q(t+\delta_q) \\ \dot{\xi} &=& -(\frac{a_1}{\tau_1}\ z_1(t+\delta_1) + \dots + \frac{a_q}{\tau_q}\ z_q(t+\delta_q)) \\ &\vdots& \\ (\xi)^{(q-1)} &=& (-1)^{(q-1)}\ (\frac{a_1}{(\tau_1)^{(q-1)}}\ z_1(t+\delta_1) + \dots \\ && +\frac{a_q}{(\tau_{i_q})^{(q-1)}}\ z_q(t+\delta_q). \end{array}\right.$$

## Some fundamental issues

**Proposition 1** *The system of equation $C$ is underdetermined. Subjected to an extra condition of normality, say $a_1 = 1$, all the $(a_i)_{i=1\dots q}$ are different from $0$.*

**Proof:** By adding the extra condition $a_1 = 1$ we get a square linear system:

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ \frac{K_1}{\tau_1} & \frac{K_2}{\tau_2} & \dots & \frac{K_q}{\tau_q} \\ \frac{K_1}{(\tau_1)^2} & \frac{K_2}{(\tau_2)^2} & \dots & \frac{K_q}{(\tau_q)^2} \\ \frac{K_1}{(\tau_1)^{(q-1)}} & \frac{K_2}{(\tau_2)^{(q-1)}} & \dots & \frac{K_q}{(\tau_q)^{(q-1)}} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \\ a_q \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \\ 0 \end{pmatrix}.$$

• First we aim at showing that this system is invertible. It is easy to check it by looking at its determinant:

$$det = 1 \begin{vmatrix} \frac{K_2}{\tau_2} & \dots & \frac{K_q}{\tau_q} \\ \frac{K_2}{(\tau_2)^2} & \dots & \frac{K_q}{(\tau_q)^2} \\ \frac{K_2}{(\tau_2)^{(q-1)}} & \dots & \frac{K_q}{(\tau_q)^{(q-1)}} \end{vmatrix}$$

$$= \frac{K_2}{\tau_2} \cdots \frac{K_q}{\tau_q} \begin{vmatrix} 1 & \cdots & 1 \\ \frac{1}{(\tau_2)} & \cdots & \frac{1}{(\tau_q)} \\ \\ \frac{1}{(\tau_2)^{(q-2)}} & \cdots & \frac{1}{(\tau_q)^{(q-2)}} \end{vmatrix}.$$

Here one can recognize a Vandermonde determinant, then:

$$det = \prod_{2\le m\le q} \frac{K_m}{\tau_m} \prod_{2\le k<l\le q} (\frac{1}{\tau_l} - \frac{1}{\tau_k}).$$

Since $\{z_1, \ldots, z_q\}$ are the essential partial states, we know that:

- for all $m$ such as $2 \le m \le n_j$: $K_m \neq 0$
- for all $k, l$ such as $2 \le k < l \le n_j$: $\tau_l \neq \tau_k$.

Therefore the determinant of the system is different from 0 which means that the system is invertible.

• Second, let us show that all the $(a_i)_{i=1\ldots q}$ are different from 0. Using Cramer formulae we can write for each $k \in \{2, \ldots, q\}$:

$$a_k = \begin{vmatrix} 1 & 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 \\ \frac{K_1}{\tau_1} & \frac{K_2}{\tau_2} & \cdots & \frac{K_{k-1}}{\tau_{k-1}} & \cdots & 0 & \frac{K_{k+1}}{\tau_{k+1}} & \cdots & \frac{K_q}{\tau_q} \\ \frac{K_1}{(\tau_1)^2} & \frac{K_2}{(\tau_2)^2} & \cdots & \frac{K_{k-1}}{(\tau_{k-1})^2} & \cdots & 0 & \frac{K_{k+1}}{(\tau_{k+1})^2} & \cdots & \frac{K_q}{(\tau_q)^2} \\ \\ \frac{K_1}{(\tau_1)^{(q-1)}} & \frac{K_2}{(\tau_2)^{(q-1)}} & \cdots & \frac{K_{k-1}}{(\tau_{k-1})^{(q-1)}} & \cdots & 0 & \frac{K_{k+1}}{(\tau_{k+1})^{(q-1)}} & \cdots & \frac{K_q}{(\tau_q)^{(q-1)}} \end{vmatrix} \times \begin{vmatrix} 1 & 0 & \cdots & 0 \\ \frac{K_1}{\tau_1} & \frac{K_2}{\tau_2} & \cdots & \frac{K_q}{\tau_q} \\ \frac{K_1}{(\tau_1)^2} & \frac{K_2}{(\tau_2)^2} & \cdots & \frac{K_q}{(\tau_q)^2} \\ \\ \frac{K_1}{(\tau_1)^{(q-1)}} & \frac{K_2}{(\tau_2)^{(q-1)}} & \cdots & \frac{K_q}{(\tau_q)^{(q-1)}} \end{vmatrix}^{-1}.$$

Then by expanding the numerator we get:

$$a_k = (-1)^k \left(\prod_{l\neq k} \frac{K_l}{\tau_l}\right) \begin{vmatrix} 1 & \cdots & 1 & \cdots & \cdots & 1 & \cdots & 1 \\ \frac{1}{(\tau_1)} & \cdots & \frac{1}{(\tau_{k-1})} & \cdots & \cdots & \frac{1}{(\tau_{k+1})} & \cdots & \frac{1}{(\tau_q)} \\ \\ \frac{1}{(\tau_1)^{(q-2)}} & \cdots & \frac{1}{(\tau_{k-1})^{(q-2)}} & \cdots & \cdots & \frac{1}{(\tau_{k+1})^{(q-2)}} & \cdots & \frac{1}{(\tau_q)^{(q-2)}} \end{vmatrix} \times \begin{vmatrix} 1 & 0 & \cdots & 0 \\ \frac{K_1}{\tau_1} & \frac{K_2}{\tau_2} & \cdots & \frac{K_q}{\tau_q} \\ \frac{K_1}{(\tau_1)^2} & \frac{K_2}{(\tau_2)^2} & \cdots & \frac{K_q}{(\tau_q)^2} \\ \\ \frac{K_1}{(\tau_1)^{(q-1)}} & \frac{K_2}{(\tau_2)^{(q-1)}} & \cdots & \frac{K_q}{(\tau_q)^{(q-1)}} \end{vmatrix}^{-1}$$

$$= (-1)^k \frac{\prod_{l\neq k} \frac{K_l}{\tau_l}}{\prod_{l=2}^{q} \frac{K_l}{\tau_l}} \frac{\prod_{l=2,l\neq k}^{l=q} (\frac{1}{\tau_l} - \frac{1}{\tau_1})}{\prod_{l=2,l\neq k}^{l=q} (\frac{1}{\tau_l} - \frac{1}{\tau_k})}$$

$$= (-1)^k \frac{\tau_k}{\tau_1} \frac{K_1}{K_k} \frac{\prod_{l=2,l\neq k}^{l=q} (\frac{1}{\tau_l} - \frac{1}{\tau_1})}{\prod_{l=2,l\neq k}^{l=q} (\frac{1}{\tau_l} - \frac{1}{\tau_k})}.$$

Since $\{z_1, \ldots, z_q\}$ are the essential partial states, we can conclude that for all $k \in \{2, \ldots, q\}$, $a_k \neq 0$.

**Proposition 2** *The linear system $D$ is invertible.*

**Proof:** One can write the linear system $D$ that way:

$$\begin{pmatrix} \xi \\ \dot{\xi} \\ \vdots \\ \xi^{(q-1)} \end{pmatrix} = \begin{pmatrix} a_1 & \cdots & \cdots & a_q \\ -\frac{a_1}{\tau_1} & \cdots & \cdots & -\frac{a_q}{\tau_q} \\ \vdots & & & \vdots \\ (-1)^{(q-1)} \frac{a_1}{(\tau_1)^{(q-1)}} & \cdots & \cdots & (-1)^{(q-1)} \frac{a_q}{(\tau_q)^{(q-1)}} \end{pmatrix} \begin{pmatrix} z_1(t+\delta_1) \\ \vdots \\ z_q(t+\delta_q) \end{pmatrix}.$$

Its determinant is:

$$det = (-1)^{\frac{(q-1)q}{2}} \prod_{i=1}^{q} a_i \begin{vmatrix} 1 & \cdots & 1 \\ \frac{1}{\tau_1} & \cdots & \frac{1}{\tau_q} \\ \vdots & & \vdots \\ \frac{1}{(\tau_1)^{(q-1)}} & \cdots & \frac{1}{(\tau_q)^{(q-1)}} \end{vmatrix}.$$

On the one hand, we know that $\prod_{i=1}^{q} a_i \neq 0$ thanks to proposition 1. On the other hand, we have to deal with another Vandermonde determinant. Since $\{z_1, \ldots, z_q\}$ is the set of essential partial states, it is different from 0. Thus the linear system $D$ is invertible.

**Proposition 3** *$\xi$ is a basis of the $R[\frac{d}{dt}, \delta, \delta^{-1}]$ module corresponding to $\{u, z_1, \ldots, z_q\}$ (in other words $\xi^j$ is a basis of the $R[\frac{d}{dt}, \delta, \delta^{-1}]$ module corresponding to $\{u^j, z^j_{i_1}, \ldots, z^j_{i_{n_j}}\}$).*

**Proof:** Since $D$ is solvable, one can calculate $z_1(t+\delta_1), \ldots z_q(t+\delta_q)$ thanks to $\xi(t), \ldots, \xi^{(q-1)}(t)$.

At last, we can use any equation from the dynamics of the essential partial states to calculate the input $u$. Thus:

$$u(t) = \frac{\tau_1 \; \dot{z_1}(t+\delta_1) + z_1(t+\delta_1)}{K_1}.$$

**Proposition 4** *The set $\{\xi^1, \ldots, \xi^m\}$ constructed as shown is a basis of the $R[\frac{d}{dt}, \delta]$ module spanned by the essential partial states and the inputs of the original system.*

**Proof:** For $j = 1, \ldots, m$, $\xi^j$ is a basis of the module corresponding to the essential partial states of the column and its input $\{u^j, z^j_{i_1}, \ldots, z^j_{i_{n_j}}\}$. Let us consider the set $\{\xi^1, \ldots, \xi^m\}$. This set generates all the essential partial states of the original system. Furthermore this set is free because it generates the $m$ inputs that are independent. We can conclude that it is a basis of the module spanned by the essential partial states and all the inputs of the original system.

**Proposition 5** *The original system has a $\delta$-free representation.*

**Proof:** We have found a basis $\{\xi^1, \ldots, \xi^m\}$ for the $R[\frac{d}{dt}, \delta, \delta^{-1}]$ module corresponding to a representation of the original system. So this representation is $\delta$-free.

**Remark:** If we want to, we can calculate those among the $\{z^j_{nz_1}, \ldots, z^j_{nz_{t_j}}\}$ that are not in the set of the essential partial states $\{z^j_{i_1}, \ldots z^j_{i_{n_j}}\}$. Let us denote these *non-essential partial states* $z^j_{ne_1}, \ldots, z^j_{ne_{tne_j}}$. Obviously $t_j = tne_j + n_j$, which means that the number of non-zero partial states equals the number of non-essential partial states added to the number of essential partial states. For any $z^j_{i_{ne_h}}$ one can find an essential partial state $z^j_{i_{n_p}}$ with $\tau^j_{i_{n_p}} = \tau^j_{i_{ne_h}} = \tau$. Thus we can build a torsion element of the module corresponding to $\{z^j_{nz_1}, \ldots, z^j_{nz_{t_j}}\}$: let

$$w(t) = \frac{z^j_{i_{ne_h}}(t+\delta^j_{i_{ne_h}})}{K^j_{i_{ne_h}}} - \frac{z^j_{i_{n_p}}(t+\delta^j_{i_{n_p}})}{K^j_{i_{n_p}}}.$$

This element is a torsion element since :

$$\tau \; \dot{w}(t) = -w(t).$$

Thus, up to an initial condition, to know $z^j_{i_{n_p}}$ is to know $z^j_{i_{ne_h}}$. So to know $z^j_{i_1}, \ldots z^j_{i_{n_j}}$ is to know the whole set $\{z^j_{nz_1}, \ldots, z^j_{nz_{t_j}}\}$. In fact the non-essential partial states can be viewed as the non-commandable part of a non-minimal realization.

# 3 Concluding remarks

We have shown that a large class of linear delayed systems, which are commonly used as process control models, are $\delta$-free. This means that, as for flat systems [1], we have an explicit parametrization of the trajectories via a finite set of arbitrary time functions and their derivatives. In forthcoming publications, we will use this property, as in [3], for trajectory generation.

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