Any (controllable) driftless system with 3 inputs and 5 states is flat *

Philippe MARTIN† Pierre ROUCHON‡

Systems and Control Letters. Vol: 25, pp:167–173, 1995

Abstract

A very simple classification of Pfaffian systems of dimension 2 in 5 variables is given. It is used to show that any controllable driftless system with 3 inputs and 5 states is 0-flat and can be put into multi-input chained form by dynamic feedback and coordinate change.

Key words: nonlinear control, flatness, dynamic feedback, driftless systems, chained form, Pfaffian systems, absolute equivalence, nonholonomic systems.

1 Introduction

In this paper, driftless systems with 3 inputs and 5 states

$$\dot{x} = u^1 f_1(x) + u^2 f_2(x) + u^3 f_3(x), \quad x \in X \subset \mathbb{R}^5,$$

are investigated. Driftless systems often appear in engineering as kinematics of mechanical systems subjected to nonholonomic constraints, and for that reason have recently received considerable attention in control theory (see e.g. [14, 10] and the bibliography therein). We show here that driftless systems with 3 inputs and 5 states are particularly simple: they are *flat* [6, 11, 7], and even 0-*flat*, as soon as they are controllable, which means that their differential behavior is summarized in a map depending only on the state x. Moreover, around a regular point, they can be converted by dynamic feedback and coordinate change into the "multi-input chained form" [14, 19]

$$\dot{x}^1 = u^1, \quad \dot{x}^2 = x^3 u^1, \quad \dot{x}^3 = u^2, \quad \dot{x}^4 = x^5 u^1, \quad \dot{x}^5 = x^6 u^1, \quad \dot{x}^6 = u^3.$$

*This work was supported in part by INRIA, NSF grant ECS-9203491, G.R. "Automatique" (CNRS) and DRED (Ministère de l'Éducation Nationale).

†Center for Control Engineering and Computation, Engr. I, University of California, Santa Barbara, CA 93106, USA. Email: martin@xanadu.ece.ucsb.edu

‡Centre Automatique et Systèmes, École des Mines de Paris, 60 Bd Saint-Michel, 75272 Paris Cedex 06, FRANCE. Tel: 33 (1) 40 51 91 15. Email: rouchon@cas.ensmp.fr

The proof uses the language of differential forms and Pfaffian systems, which is increasingly popular in control theory (see e.g. [8, 17, 15, 13, 12, 18]). More precisely we show, and then interpret in control terms, that any totally nonholonomic Pfaffian system of dimension 2 in 5 variables is absolutely equivalent [4] to a very simple one. It is interesting to compare this simple result with the much finer (and complicated) classification up to a coordinate change obtained by Cartan in his famous 1910 paper [3].

The paper is organized as follows: section 2 shortly discusses flatness and its links with driftless, Pfaffian and chained systems. The heart of the paper is section 3, where Pfaffian systems of dimension 2 in 5 variables are shown to have a very simple structure. The result is then rephrased for driftless systems in section 4.

2 Flatness and driftless, chained and Pfaffian systems

Let Σ be a (smooth) control system, $\dot{x} = f(x,u)$, defined on an open subset $X \times U$ of $\mathbb{R}^n \times \mathbb{R}^m$. The *prolongation* of a (smooth) map, $y := h(x,u,\dot{u},\ddot{u},\ldots,u_{(k)})$, depending on x and finitely many derivatives of u is the map

$$\dot{y} := \partial_x h(x,u,\dot{u},\ddot{u},\ldots,u_{(k)}).f(x,u) + \sum_{i=0}^{k} \partial_{u_{(i)}} h(x,u,\dot{u},\ddot{u},\ldots,u_{(k)}).u_{(i+1)}.$$

As we will deal only with a finite number of maps and prolongations, we consider them as maps of $X \times U \times U_1 \times \ldots \times U_r$ for some "large" integer r, where the U_i's are open subsets of $\mathbb{R}^m$. We use the short-hand notations $\overline{u} := (u,\dot{u},\ddot{u},\ldots,u_{(r)})$, $\overline{U} := U \times U_1 \times \ldots \times U_r$, etc. We can now define the concept of *flatness* [6, 11, 7]:

Definition 1. *We say* Σ *is* flat at $(x_0,\overline{u}_0)$ *if there exist (smooth) maps* $y := h(x,\overline{u})$ *defined around* $(x_0,\overline{u}_0)$ *and* φ,α *defined around* $\overline{y}_0 := \overline{h}(x_0,\overline{u}_0)$ *such that* $x = \varphi(\overline{y})$ *and* $u = \alpha(\overline{y})$ *hold around* $(x_0,\overline{u}_0)$*. We say* Σ *is* flat *if it is flat at every point of a dense open subset of* $X \times \overline{U}$*.*

In other words every choice of a map $y(t)$ leads to an allowable trajectory $(x(t),u(t))$ only via differentiation. No integration is needed.

The map $y := h(x,\overline{u})$ in the definition is called a *flat* (or *linearizing*) *output*. We say the system is *k-flat* if it has a flat output depending on derivatives of u of order at most $k-1$. Hence a 0-flat system has a flat output depending only on the state x.

Flatness is an important structural property: for instance, the knowledge of a flat output gives a simple answer to the motion planning problem [16] and allows to transform the system into a linear one by (dynamic) feedback [11]. Moreover, many engineering systems (or at least "idealized" systems) happen to be flat.

Checking that a system is flat is a challenging and widely open problem. Checking k-flatness for a given k amounts to studying the integrability of a system of partial differentials equations, which is in theory doable though extremely tedious. Yet, the main conceptual problem is that it is not known whether k may be *a priori* bounded.

Things look a little simpler for *driftless* systems $\dot{x} = \sum_{i=1}^{m} u^i f_i(x)$, which frequently arise in nonholonomic mechanics. The problem is solved for 2-input systems [12], and in that case flatness is equivalent to 0-flatness. No similar result is known for general 2-input systems (and in particular flatness does not imply 0-flatness). A reason for the lesser complexity is that time plays no role in a driftless system: solutions are invariant by a time reparametrization. A driftless system is nothing but a (time-independent) Pfaffian system; the machinery of exterior differential systems directly applies, whereas in the general case time must be carefully handled [8, 17].

Of course, a flat driftless system is never flat at an equilibrium point $(x_0, \overline{0})$, which is often a point of interest, and thus cannot be feedback linearized around that point (the linear tangent approximation is not controllable [5]). Hence, a further question is to find a simple local model in which the system could be converted by feedback and coordinate change, provided the point of interest is "regular" in some sense to be defined. For the 2-input case, a natural candidate is the so-called "chained form" [14]

$$\dot{x}^1 = u^1, \quad \dot{x}^2 = x^3 u_1 \ldots, \quad \dot{x}^{n-1} = x^n u^1, \quad \dot{x}^n = u^2.$$

Combining results of [12, 13], a flat 2-input driftless system can be converted around a "regular" point into chained form by static feedback and coordinate change. Generalizing to driftless systems with more than 2 inputs raises new problems, and conversion into an adequate simple local model may require not only static, but also dynamic feedback [19], which is not completely surprising.

We finally recall some facts about Pfaffian systems (see [2, chapter 2] for a modern introduction). Let X be an open subset of $\mathbb{R}^n$ and $C^\infty(X)$ the ring of smooth functions on X. A *Pfaffian system* on X is a $C^\infty(X)$-module[1] of smooth differential 1-forms on X. Some important properties of a Pfaffian system I can be characterized by its *derived flag*, i.e. the descending chain of Pfaffian systems $I^0 := I \supset I^1 \supset \ldots$ defined by

$$I^{k+1} := \{\alpha \in I^k,\ d\alpha \equiv 0 \bmod I^k\}.$$

We say I is *totally nonholonomic* if $I^k = 0$ for k large enough. We then define a point-wise notion of regularity by considering the real vector spaces[2] $\tilde{I}^0_x := (I^0)_x$ and

$$\tilde{I}^{k+1}_x := \{\alpha(x), \alpha \in I^k,\ d\alpha(x) \equiv 0 \bmod (I^k)_x\}.$$

Clearly $(I^k)_x \subset \tilde{I}^k_x$. We say $x_0 \in X$ is a *weakly regular point* if $\tilde{I}^k_x$ has constant dimension in a neighborhood of x_0 for $k \geq 0$, in which case $\dim(I^k)_x = \dim \tilde{I}^k_x = \dim I^k$ in that neighborhood. Weakly regular points form a dense open subset of X.

To a driftless control system $D:\ \dot{x} = \sum_{i=1}^m u^i f_i(x)$, with $f_1, \ldots, f_m$ independent vector fields on X, is naturally associated a Pfaffian system $I := \{f_1, \ldots, f_m\}^\perp$ of dimension $n-m$. I represents the kinematic constraints obtained by eliminating the controls in D. Notice D is controllable if and only if I is totally nonholonomic. We say x_0 is a *weakly regular point* of D if it is a weakly regular point of I.

Notice that weak regularity is equivalent for Pfaffian systems of dimension 2 to the stronger property considered in [13], but may in general hide subtler singularities [9].

[1] To discard irrelevant non-local problems, we always assume that a $C^\infty(X)$-module has the same dimension as its restrictions to any open subset of X.

[2] We denote $S_x := \{\eta(x),\ \eta \in S\}$ the real vector space induced at a point $x \in X$ by a $C^\infty(X)$-module S.

3 Pfaffian systems of dimension 2 in 5 variables

We now turn to the study of driftless systems with 3 inputs and 5 states, or equivalently Pfaffian systems of dimension 2 in 5 variables. If I is such a Pfaffian system, the rank structure of its derived flag splits into four cases:

1. $\dim I^1 = 2$
2. $\dim I^1 = 1, \ \dim I^2 = 1$
3. $\dim I^1 = 1, \ \dim I^2 = 0$
4. $\dim I^1 = 0$.

The first three cases are not very difficult, and we refer the reader to the beginning of Cartan's paper [3] (see also [9] for a more careful treatment regarding regularity problems). The first two cases correspond to non controllable, hence non-flat control systems [7]. In suitable local coordinates, we have respectively $I = \{dx^1,\ dx^2\}$ and $I = \{dx^1,\ dx^2 - x^3 dx^4\}$. The third case is more interesting:

Theorem 1. *Let I be a Pfaffian system of dimension 2 and x_0 be a weakly regular point. Assume $\dim I^1 = 1$ and $\dim I^2 = 0$. Then $I = \{dx^1 - x^2 dx^4,\ dx^2 - x^3 dx^4\}$ in suitable coordinates defined around x_0.*

The idea of the proof is to show that the system depends in fact on only 4 variables and to apply Engel's theorem [2, chapter 2].

The fourth case (which is the generic situation) is the heart of Cartan's paper, and its fine study up to a coordinate change is quite intricate. We will show that it can nonetheless be reduced to a unique normal form when considering a larger class of transformations. Before stating the result, we prove a useful lemma:

Lemma 1. *Let I be a totally nonholonomic Pfaffian system of dimension 2 in 5 variables and x_0 be a weakly regular point. Then I contains a form ω such that, around x_0,*

$$\textbf{(i)}\ d\omega \wedge d\omega \wedge \omega = 0 \quad \textit{and} \quad \textbf{(ii)}\ d\omega \wedge \omega \neq 0.$$

A form ω satisfying (i) and (ii) is said to have *rank* 1 at x_0, and a classical result asserts that, in suitable coordinates defined around x_0, ω is colinear to $dx^1 + x^2 dx^3$ [2, chapter 2].

Proof. If $\dim I^1 = 1$ and $\dim I^2 = 0$, the assertion stems from theorem 1. We thus assume $\dim I^1 = 0$; now (ii) is satisfied for any form in I, otherwise $\dim \tilde{I}^1_{x_0} > 0$. Let ω^1, ω^2 span I around x_0; we want to find a function λ defined around x_0 such that $\omega := \omega^1 + \lambda\omega^2$ satisfies (i). Notice that $(d\omega)^2 \wedge \omega$ is colinear to $dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \wedge dx^5$ (it is a form a degree 5 in a space of dimension 5) and that

$$\begin{aligned}(d\omega)^2 \wedge \omega \ = \ & (d\omega^1)^2 \wedge \omega^1 + \lambda(2d\omega^1 \wedge d\omega^2 \wedge \omega^1 + (d\omega^1)^2 \wedge \omega^2) \\ & + \lambda^2(2d\omega^1 \wedge d\omega^2 \wedge \omega^2 + (d\omega^2)^2 \wedge \omega^1) + \lambda^3 (d\omega^2)^2 \wedge \omega^2 \\ & + 2d\lambda \wedge (d\omega^1 \wedge \omega^2 \wedge \omega^1 + \lambda d\omega^2 \wedge \omega^2 \wedge \omega^1).\end{aligned}$$

Hence $(d\omega)^2 \wedge \omega = 0$ reduces to the first-order quasilinear partial differential equation

$$d\lambda(x).(f(x) + \lambda(x)g(x)) = a(x, \lambda(x)), \tag{1}$$

where the two vectors fields f, g and the function a are known expressions. Moreover since $\dim \tilde{I}^1_x = 0$ around x_0, $d\omega^1 \wedge \omega^2 \wedge \omega^1$ and $d\omega^2 \wedge \omega^2 \wedge \omega^1$, thus f and g, are independent at x_0. Hence $f(x_0) + \lambda(x_0)g(x_0) \neq 0$ whatever $\lambda(x_0)$, and it is well-known [1, chapter 2] that equation (1) then admits solutions λ defined around x_0. $\square$

Theorem 2. *Let I be a totally nonholonomic Pfaffian system of dimension 2 in 5 variables and x_0 be a weakly regular point. Then, in suitable coordinates defined around x_0, I takes one of the two following normal forms:*

- $I = \{dx^1 - x^2 dx^4,\ dx^2 - x^3 dx^4\}$ *if* $\dim I^1 = 1$ *and* $\dim I^2 = 0$
- $I = \{dx^1 - a(x)dx^3 - x^5 dx^4,\ dx^2 - x^3 dx^4\}$ *if* $\dim I^1 = 0$,

where a is some function. Moreover I can be prolonged around x_0 into

$$J := \{dz^1 - z^5 dz^4,\ dz^2 - z^3 dz^4,\ dz^3 - z^6 dz^4\}.$$

Using Cartan's terminology [4], this result asserts that any totally nonholonomic systems of dimension 2 in 5 variables is *absolutely equivalent* to a "contact" system whose general solution depends on two arbitrary functions of one argument.

Proof. We first prove the second claim. If I is in the first normal form, prolong it into $J := I + \{dx^5 - x^6 dx^4\}$ and perform the coordinate change $z := (x^5, x^1, x^2, x^4, x^6, x^3)$. If I is in the second normal form, prolong it into $J := I + \{dx^3 - x^6 dx^4\}$ and perform the coordinate change $z := (x^1, x^2, x^3, x^4, x^5 + x^6 a(x), x^6)$, which is well-defined as soon as x^6 is small enough. In the new coordinates, J is obviously in the requested form.

We now turn to the first claim. By lemma 1, I contains a form of rank 1 around x_0 and, up to a coordinate change, is spanned by the forms

$$\omega^1 := \alpha dx^1 + \beta dx^3 + \gamma dx^4 + \delta dx^5 \quad \text{and} \quad \omega^2 := dx^2 - x^3 dx^4,$$

for some functions $\alpha, \beta, \gamma, \delta$. Since $\dim \tilde{I}^0_{x_0} = 2$, one of these four functions must be different from 0 at x_0.

If $\alpha(x_0) = \delta(x_0) = 0$, we may assume $\beta(x_0) \neq 0$ and without loss of generality $\beta(x) = -1$ around x_0 (if $\beta(x_0) = 0$ but $\gamma(x_0) \neq 0$, we exchange the roles of β and γ with the coordinate change $z := (x^1, x^2 - x^3 x^4, -x^4, x^3, x^5)$). Hence

$$d\omega^2 = dx^4 \wedge dx^3 = dx^4 \wedge (\alpha dx^1 + \gamma dx^4 + \delta dx^5 - \omega^1) \equiv \alpha dx^4 \wedge dx^1 + \delta dx^4 \wedge dx^5 \mod I,$$

which implies $\dim \tilde{I}^1_{x_0} \leq 1$. But $\dim \tilde{I}^1_x$ must have non-zero constant dimension around x_0, thus α and δ equal 0 in a neighborhood of x_0. We conclude that, around x_0, $\dim \tilde{I}^1_x$ has constant rank 1 and $\dim \tilde{I}^2_x$ constant rank 0, and we may use theorem 1.

Otherwise $\alpha(x_0) \neq 0$ (up to a renumbering of x), and the system is spanned by

$$\omega^1 := dx^1 - \beta dx^3 - \gamma dx^4 - \delta dx^5 \quad \text{and} \quad \omega^2 := dx^2 - x^3 dx^4.$$

Perform now the coordinate change $z^1 := g(x)$, $z^i := x^i$, $i = 2, \dots, 5$, where g is a function such that $g_1(x_0) \neq 0$ and is yet to be determined[3]. It follows

$$\begin{aligned} dz^1 &= g_1(\omega^1 + \beta dx^3 + \gamma dx^4 + \delta dx^5) + g_2(\omega^2 + x^3 dx^4) + g_3 dx^3 + g_4 dx^4 + g_5 dx^5 \\ &= (\beta g_1 + g_3) dx^3 + (\gamma g_1 + x^3 g_2 + g_4) dx^4 + (\delta g_1 + g_5) dx^5 + g_1 \omega^1 + g_2 \omega^2. \end{aligned}$$

If g is a solution of the first-order linear partial differential equation $\delta g_1 + g_5 = 0$ such that $g_1(x_0) \neq 0$ (such a solution always exists [1, chapter 2]), I is clearly spanned by

$$\eta^1 := dz^1 - a(z) dz^3 - b(z) dz^4 \quad \text{and} \quad \eta^2 := dz^2 - z^3 dz^4,$$

where a, b follow from the expression of dz^1. Computing the exterior derivative, we get

$$d\eta^1 \equiv c dz^3 \wedge dz^4 + a_5 dz^3 \wedge dz^5 + b_5 dz^4 \wedge dz^5 \mod I, \quad d\eta^2 = -dz^3 \wedge dz^4,$$

where c is some function involving a, b and their partial derivatives.

If $a_5(z_0) = b_5(z_0) = 0$, then $\dim \tilde{I}^1_{x_0} = 1$. But $\dim \tilde{I}^1_x$ must have constant dimension 1 around z_0, thus a_5 and b_5 equal 0 in a neighborhood of z_0, i.e. the system does not depend on z_5. Once again we conclude by theorem 1.

Otherwise we may assume $b_5(z_0) \neq 0$ and without loss of generality $b(z) = z^5$ around z_0 (if $b_5(z_0) = 0$ but $a_5(z_0) \neq 0$, we can exchange the roles of a and b by the coordinate change $x := (z^1, z^2 - z^3 z^4, -z^4, z^3, z^5)$). This shows I is spanned by $dz^1 - a(z) dz^3 - z^5 dz^4$ and $dz^2 - z^3 dz^4$. □

4 Application to driftless systems

We now interpret theorem 2 in terms of driftless systems:

Corollary 1. *Let $D: \ \dot{x} = u^1 f_1(x) + u^2 f_2(x) + u^3 f_3(x)$ be a driftless system with 3 inputs and 5 states. The following three statements are equivalent:*

(i) *D is 0-flat,* **(ii)** *D is flat,* **(iii)** *D is controllable.*

If D satisfies these conditions then, around any weakly regular point, it can be put by dynamic feedback and coordinate change into the multi-input chained form

$$\dot{z}^1 = v^1, \quad \dot{z}^2 = z^3 v^1, \quad \dot{z}^3 = v^2, \quad \dot{z}^4 = z^5 v^1, \quad \dot{z}^5 = z^6 v^1, \quad \dot{z}^6 = v^3.$$

Proof. Clearly (i)⇒(ii)⇒(iii), and we only have to prove (iii)⇒(i). Suppose D is controllable and x_0 is a weakly regular point. By the previous theorem, the Pfaffian system $I := \{f_1, f_2, f_2\}^\perp$ can be assumed in normal form around x_0.

If $\dim I^1 = 0$ then, up to invertible static feedback and coordinate change, D reads

$$\dot{x}^1 = a(x) u^1 + x^5 u^2, \quad \dot{x}^2 = x^3 u^2, \quad \dot{x}^3 = u^1, \quad \dot{x}^4 = u^2, \quad \dot{x}^5 = u^3.$$

[3] If h is some function of x, we denote by h_i the partial derivative $\dfrac{\partial h}{\partial x^i}$, $i = 1, \dots, 5$.

We claim $y := (x^1, x^2, x^4)$ is a flat output. Indeed, simple computations give

$$x^1 = y^1, \quad x^2 = y^2, \quad x^3 = \frac{\dot{y}^2}{\dot{y}^3}, \quad x^4 = y^3, \quad u^1 = \frac{\ddot{y}^2\dot{y}^3 - \ddot{y}^3\dot{y}^2}{(\dot{y}^3)^2}, \quad u^2 = \dot{y}^3$$

and x^5 is obtained by solving the equation

$$\dot{y}^1 = a(y^1, y^2, \frac{\dot{y}^2}{\dot{y}^3}, y^2, x^5)\frac{\ddot{y}^2\dot{y}^3 - \ddot{y}^3\dot{y}^2}{(\dot{y}^3)^2} + x^5\dot{y}^3;$$

the expression for u^3 follows by differentiation. The system is thus flat at every point $(x_0, \overline{u}_0)$ such that x_0 is weakly regular, $u_0^2 \neq 0$ and $a_5(x_0)u_0^1 + u_0^2 \neq 0$ (where as usual $a_i := \frac{\partial a}{\partial x^i}$). These points clearly form a dense open subset of $X \times \overline{U}$.

Apply now the dynamic feedback

$$\dot{x}^6 = v^3, \quad u^1 = x^6 v^1, \quad u^2 = v^1,$$
$$u^3 = \frac{v^2 - [(x^5 + x^6 a(x))a_1(x) + x^3 a_2(x) + x^6 a_3(x) + a_4(x)]x^6 v^1 - a(x)v^3}{1 + x^6 a_5(x)}$$

and the coordinate change $z := (x^4, x^1, x^5 + x^6 a(x), x^2, x^3, x^6)$, which are well-defined as soon as x^6 is small enough, to put the system into multi-input chained form. The dynamic feedback is the counterpart of the prolongation in theorem 2.

If $\dim I^1 = 1$ and $\dim I^2 = 0$, the system reads

$$\dot{x}^1 = x^2 u^2, \quad \dot{x}^2 = x^3 u^2, \quad \dot{x}^3 = u^1, \quad \dot{x}^4 = u^2, \quad \dot{x}^5 = u^3,$$

and $y := (x^1, x^4, x^5)$ is a flat output. Details are left to the reader. □

The dynamic feedback in the proof respects the driftless structure of the system. It is also *endogenous* [11], i.e. built only from x and derivatives of u.

Performing a dynamic feedback (or prolonging the Pfaffian system) when $\dim I^1 = 1$ and $\dim I^2 = 0$ has little practical interest, since the system is already in a very simple form.

References

[1] V.I. Arnold. *Geometrical Methods in the Theory of Ordinary Differential Equations.* Springer-Verlag, 1983.

[2] R.L. Bryant, S.S. Chern, R.B. Gardner, H.L. Goldschmidt, and P.A. Griffiths. *Exterior Differential Systems.* Springer-Verlag, 1991.

[3] É. Cartan. Les systèmes de Pfaff à cinq variables et les équations aux dérivées partielles du second ordre. *Ann. Ec. Norm.*, 27:109–192, 1910.

[4] É. Cartan. Sur l'équivalence absolue de certains systèmes d'équations différentielles et sur certaines familles de courbes. *Bull. Soc. Math. France*, 42:12–48, 1914.

[5] B. Charlet, J. Lévine, and R. Marino. On dynamic feedback linearization. *Systems & Control Letters*, 13:743–151, 1989.

[6] M. Fliess, J. Lévine, Ph. Martin, and P. Rouchon. Flatness and defect of nonlinear systems: introductory theory and applications. *Int. J. of Control*. To appear.

[7] M. Fliess, J. Lévine, Ph. Martin, and P. Rouchon. Towards a new differential geometric setting in nonlinear control. In *Proc. Int. Geom. Coll., Moscow, 1993*. World Scientific. To appear.

[8] R.B. Gardner and W.F Shadwick. The GS algorithm for exact linearization to Brunovsky normal form. *IEEE Trans. Automat. Control*, 37:224–230, 1992.

[9] A. Kumpera and C. Ruiz. Sur l'équivalence locale des systèmes de Pfaff en drapeau. In *Monge-Ampère equations and related topics. Proc. Semin, Firenze, 1980*, p. 201–248. Inst. Naz. di Alta Math., 1982.

[10] Z. Li and J.F. Canny. *Nonholonomic Motion Planning*. Kluwer, 1993.

[11] Ph. Martin. *Contribution à l'Étude des Systèmes Différentiellement Plats*. PhD thesis, École des Mines de Paris, France, 1992.

[12] Ph. Martin and P. Rouchon. Feedback linearization and driftless systems. Technical Report 446, Centre Automatique et Systèmes, École des Mines de Paris, 1993.

[13] R.M. Murray. Nilpotent bases for a class of non-integrable distributions with applications to trajectory generation for nonholonomic systems. Technical Report CIT/CDS 92-002, California Institute of Technology, 1992.

[14] R.M. Murray and S.S. Sastry. Nonholonomic motion planning: steering with sinusoids. *IEEE Trans. Automat. Control*, 38(5):700–716, 1993.

[15] J.B. Pomet, C. Moog, and E. Aranda. A non-exact Brunovsky form and dynamic feedback linearization. In *Proc. of the* 31*nd IEEE CDC*, p. 2012–2017, Tucson, 1992.

[16] P. Rouchon, M. Fliess, J. Lévine, and Ph. Martin. Flatness, motion planning and trailer systems. In *Proc. of the* 32*nd IEEE CDC*, p. 2700–2705, San Antonio, 1993.

[17] W.M. Sluis. *Absolute Equivalence and its Applications to Control Theory*. PhD thesis, University of Waterloo, Ontario, Canada, 1992.

[18] D. Tilbury, R. Murray, and S. Sastry. Trajectory generation for the n-trailer problem using Goursat normal form. Technical Report UCB/ERL M93/12, Electronics Research Laboratory, University of California at Berkeley, 1993.

[19] D. Tilbury, O. Sørdalen, L. Bushnell, and S. Sastry. A multi-steering trailer system: conversion into chained form using dynamic feedback. Technical Report UCB/ERL M93/55, Electronics Research Laboratory, University of California at Berkeley, 1993.