KRONECKER'S CANONICAL FORMS FOR NONLINEAR IMPLICIT DIFFERENTIAL SYSTEMS

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Abstract: The structure algorithm provides an extension to nonlinear systems of the Kronecker canonical forms relative to linear constant-coefficient implicit differential systems. A connection with the index problem is sketched in the conclusion.

Key words: implicit differential systems, index, Kronecker's canonical form, structure algorithm, inversion.

1 Introduction

The structure of linear constant-coefficient systems

$$A\frac{dx}{dt} + Bx = e(t), \tag{1}$$

where A and B are real square matrices of order n, x is the n-tuple of unknown variables, $e(t)$ is an n-tuple of smooth time functions, is rather well known.

In [14], Sincovec et al. introduce the notion of index for (1) by using the Kronecker canonical form of matrix pencils [8]. If the matrix pencil $\lambda A + B$ is regular[1], there exist P and Q, two regular square matrices of dimension n, and an integer p, between 0 and n, such that

$$PAQ = \begin{pmatrix} 1_p & 0 \\ 0 & E \end{pmatrix} \text{ and } PBQ = \begin{pmatrix} R & 0 \\ 0 & 1_{n-p} \end{pmatrix} \tag{2}$$

where 1_p and 1_{n-p} are the identity matrices of order p and $n-p$, respectively, E is a square nilpotent matrix of order $n-p$ and R a square matrix of order p; the nilpotency index of the matrix E is called the index (see [9, 1, 6, 7]).

This means that, with a linear change of coordinates and linear combinations of the equations, (1) becomes

$$\begin{cases} \frac{dy}{dt} &= Ry + f(t) \\ E\frac{dz}{dt} &= z + g(t) \end{cases} \tag{3}$$

where $Q^{-1}x = (y, z)'$ and $Pe(t) = (f(t), g(t))'$. This system is generally called the Kronecker canonical form of (1).

We show here (see also [12]) that the structure algorithm [15, 11] provides a natural extension of this particular form to implicit nonlinear systems

$$F\left(\frac{dx}{dt}, x, e(t)\right) = 0 \tag{4}$$

where $F = (F_1, \ldots, F_n)'$ is an n-tuple of analytic functions of their arguments on some open connected domain, $e = (e_1, \ldots, e_n)'$ is an n-tuple of known analytic time functions, and $x = (x_1, \ldots, x_n)'$ is an n-tuple of unknown time functions. Such nonlinear canonical forms are generic : we do not address the problems of singularities; as for the structure algorithm, the rank of all the Jacobian matrices are assumed constant.

The paper is organized as follows. In section 2 we recall the structure algorithm and its suitable version due to Li and Feng [11]. In section 3, the nonlinear Kronecker canonical form is established. In conclusion, we sketch some connection with the index [7, 6].

2 Inversion

Consider the square system $\frac{dx}{dt} = f(x, u, t)$, $y = h(x, u, t)$. the state vector x belongs to an open connected domain of $\mathbb{R}^n$; u, the control vector, belongs to an open connected domain of $\mathbb{R}^m$; $y \in \mathbb{R}^m$ is the the output vector; f and h are analytic functions of their arguemts. The inversion of such systems has been studied by many authors in control theory. It consists in finding the control $u(t)$ when the output $y(t)$ is a known smooth time function. In this section, we only refer to the structure algorithm [15] and to a paper of Li and Feng [11] where the control variables appear nonlinearly. For linear systems, Silverman [13] establishes a necessary and sufficent condition for the existence and unicity of $u(t)$. This condition is constructive and based on an elimination principle. Hirschorn [10], Singh [15] and Descusse and Moog [2] use this elimination principle and propose inversion algorithms for nonlinear systems where f and h are nonlinear functions of x and linear functions of u. Li and Feng [11] use the same

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[1] $\lambda A + B$ is said to be regular if, and only if, the polynomial $\det(\lambda A + B)$ in the complex varaible λ is different from zero.

elimination principle for inverting systems where f and h are arbitrary analytic functions.

2.1 Algorithm

For simplicity's sake, we have eliminated all the restrictions relative to singularities. We assume, once for all, that the ranks of all the Jacobian matrices are constant. Consider

$$(S)\left\{\begin{array}{rcl}\frac{dx}{dt} & = & f(x,u,t)\\ \\ 0 & = & h(x,u,t)\end{array}\right.$$

Our purpose is to calculate $x(t)$ and $u(t)$. The inversion algorithm is then as follows.

Step k = 0 Denote by $h_0(x,u,t)$ the function $h(x,u,t)$. If $x(t)$ and $u(t)$ are solutions of the inversion problem, then, for all time t, $h_0(x(t),u(t),t)=0$. ∎

Step k ≥ 0 Assume we know the analytic function $h_k(x,u,t)$ ($\dim h_k = m$) such that, if $x(t)$ and $u(t)$ are solution of the inversion problem, then $h_k(x(t),u(t),t)\equiv 0$.

Denote by μ_k the rank of h_k with respect to u, i.e. the rank of the Jacobian matrix $\frac{\partial h_k}{\partial u}$. If we permute the rows of h_k, we may assume that its first μ_k rows, $\overline{h}_k=(h_k^1,\ldots,h_k^{\mu_k})'$, are such that the rank of $\frac{\partial \overline{h}_k}{\partial u}$ is maximum and equal to μ_k. Consequently, the last $m-\mu_k$ rows, $\tilde{h}_k=(h_k^{\mu_k+1},\ldots,h_k^m)'$, of h_k depend on u only through $\overline{h}_k$: there exists an analytic function $\Phi_k(x,t,\blacksquare)$ such that $\tilde{h}_k(x,u,t)=\Phi_k(x,t,\overline{h}_k(x,u,t))$. h_{k+1} is then defined by

$$h_{k+1}(x,u,t)=\left(\begin{array}{c}\overline{h}_k(x,u,t)\\ \left(\frac{\partial\Phi_k}{\partial x}\right)_{(x,t,0)} f(x,u)+\left(\frac{\partial\Phi_k}{\partial t}\right)_{(x,t,0)}\end{array}\right).$$

Notice that

$$\frac{d}{dt}[\Phi_k(x,t,0)]=\left(\frac{\partial\Phi_k}{\partial x}\right)_{(x,t,0)} f(x,u)+\left(\frac{\partial\Phi_k}{\partial t}\right)_{(x,t,0)}$$

is equal to $\frac{d}{dt}\left[\tilde{h}_k(x,u,t)\right]$, when $\overline{h}_k(x(t),u(t),t)=0$ for all t. This implies that, if $x(t)$ and $u(t)$ are solution of the inversion problem, then $h_{k+1}(x(t),u(t),t)=0$. We impose additionally that the first μ_k rows of $\overline{h}_{k+1}$ coincide with the ones of $\overline{h}_k$. ∎

2.2 Algorithmic analysis

The μ_k's constitute a nondecreasing series of integers less or equal to m. One can prove [3] that this series does not depend on the arbitrary choices that we impose at each step of the algorithm. The μ_k correspond to structural invariants attached to the system. Clearly, the μ_k are constant for k large enough. The following definition[2] is thus natural.

[2] [11], definition 2.

Definition 1. *If there exists $k\geq 0$ such that $\mu_k=m$, then the relative order α of (S) is the smallest integer k such that $\mu_k=m$. If, for all $k\geq 0$, $\mu_k<m$, then the relative order α of (S) is equal to $+\infty$.*

One has also the following result[3] :

Lemma 1. *If the relative order α of (S) is finite, then $\alpha\leq n$ and the rank of the Jacobian matrix*

$$\frac{\partial}{\partial x}\left(\begin{array}{c}\Phi_0(x,t,0)\\ \vdots\\ \Phi_{\alpha-1}(x,t,0)\end{array}\right)$$

is equal to the number of its rows, $\sum_{k=0}^{\alpha}(m-\mu_k)$.

The stationary value of the μ_k's is the *differential output rank* of the system (S) [4, 3]. If this output rank is equal to m, then the system is invertible[4] : the relative order α of (S) is then finite and the square algebraic system $h_\alpha(x,u,t)=0$ provides u as a function of x and t.

If the output rank is less than m, the system is not invertible : the relative order α of (S) is infinite and, generically, (S) has no solution.

3 Canonical form

In the following theorem, we have replaced $F\left(\frac{dx}{dt},x,e(t)\right)$ by $F\left(\frac{dx}{dt},x,t\right)$ for clarity's sake.

Theorem 1. *Consider the square nonlinear implicit system depending on the time t, (Σ) : $F\left(\frac{dx}{dt},x,t\right)=0$, where F is an analytic function of its arguments and x belongs to an open connected domain of $\mathbb{R}^n$. Assume that the relative order α (definition 1) of*

$$(\Sigma_e)\left\{\begin{array}{rcl}\frac{dx}{dt} & = & u\\ 0 & = & F(u,x,t)\end{array}\right.$$

is finite. Then, there exist, locally, a change of variables on x, $\xi=\Xi(x,t)$,[5] depending on t, and a local diffeomorphism $\Pi_{(\frac{d\xi}{dt},\xi,t)}(F)$ depending on $\left(\frac{d\xi}{dt},\xi,t\right)$ such that : ξ is made of $\alpha+1$ groups of components $\xi=(\xi_1,\ldots,\xi_\alpha,\zeta)'$ with $\dim(\xi_1)\geq\dim(\xi_2),\ldots,\geq\dim(\xi_\alpha)$; $\Pi_{(\frac{d\xi}{dt},\xi,t)}(0)=0$ for all $\frac{d\xi}{dt}$, ξ and t ; $\Pi_{(\frac{d\xi}{dt},\xi,t)}\left(F\left(\frac{\partial\Xi^{-1}}{\partial\xi}\frac{d\xi}{dt}+\frac{\partial\Xi^{-1}}{\partial t},\Xi^{-1}(\xi,t),t\right)\right)$ is equal to

$$\left(\begin{array}{c}\xi_1\\ \xi_2-\phi_1\left(\xi,t,\frac{d\xi_1}{dt}\right)\\ \xi_3-\phi_2\left(\xi,t,\frac{d\xi_1}{dt},\frac{d\xi_2}{dt}\right)\\ \vdots\\ \xi_\alpha-\phi_{\alpha-1}\left(\xi,t,\frac{d\xi_1}{dt},\ldots,\frac{d\xi_{\alpha-1}}{dt}\right)\\ \frac{d\zeta}{dt}-\Omega\left(\zeta,t,\frac{d\xi_1}{dt},\ldots,\frac{d\xi_\alpha}{dt}\right)\end{array}\right);$$

[3] [11], theorem 1 and lemma 4.

[4] For square systems there is no difference between left and right invertibility.

[5] In the theorem proof, we show how the function Ξ is explicitly given by the structure algorithm.

the functions ϕ_k and Ω are analytic ; each function ϕ_k vanishes when $(\frac{d\xi_1}{dt}, \ldots, \frac{d\xi_k}{dt})$ becomes zero ; the rank of ϕ_k with respect to $\frac{d\xi_k}{dt}$ is maximum.

In the coordinates ξ, (Σ) yields :

$$(\Sigma_c) \left\{ \begin{array}{rcl} \xi_1 & = & 0 \\ \xi_2 & = & \phi_1\left(\xi, t, \frac{d\xi_1}{dt}\right) \\ \xi_3 & = & \phi_2\left(\xi, t, \frac{d\xi_1}{dt}, \frac{d\xi_2}{dt}\right) \\ & \vdots & \\ \xi_\alpha & = & \phi_{\alpha-1}\left(\xi, t, \frac{d\xi_1}{dt}, \ldots, \frac{d\xi_{\alpha-1}}{dt}\right) \\ \frac{d\zeta}{dt} & = & \Omega\left(\zeta, t, \frac{d\xi_1}{dt}, \ldots, \frac{d\xi_\alpha}{dt}\right). \end{array} \right.$$

When F is linear with respect to $\frac{dx}{dt}$ and x and independant of t, $F(\frac{dx}{dt}, x, t) = A\frac{dx}{dt} + Bx - e(t)$, (Σ_c) corresponds to the Kronecker's canonical form (3) : $\Xi = Q$, $\Pi = P$ and the nilpotent operator E corresponds to

$$\begin{pmatrix} \frac{d\xi_1}{dt} \\ \vdots \\ \frac{d\xi_\alpha}{dt} \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ \phi_1\left(\xi, \frac{d\xi_1}{dt}\right) \\ \vdots \\ \phi_{\alpha-1}\left(\xi, \frac{d\xi_1}{dt}, \ldots, \frac{d\xi_{\alpha-1}}{dt}\right) \end{pmatrix}.$$

Thus, the coordinates ξ can be called canonical coordinates, and the system (Σ_c) the canonical form of (Σ) associated to the canonical coordinates ξ. Notice that, as in the linear case, such canonical coordinates are not unique.

Proof of theorem 1 We only describe in details the passage to (Σ_c). The obtention of the equation diffeomorphism $\Pi_{(\frac{d\xi}{dt}, \xi, t)}(\blacksquare)$ is then straightforward : it is just the translation, into a more mathematical statement, of sentences like "the system becomes equivalent to" that are used here below.

Since $\alpha < +\infty$, lemma 1 holds. Consequently, we can complete the functions $\Phi_0(x, t, 0)$, $\ldots$, $\Phi_{\alpha-1}(x, t, 0)$ with a function $\Psi(x)$ such that

$$x \longrightarrow \begin{pmatrix} \xi_1 = \Phi_0(x, t, 0) \\ \vdots \\ \xi_\alpha = \Phi_{\alpha-1}(x, t, 0) \\ \zeta = \Psi(x) \end{pmatrix}$$

is a local diffeomorphism. Denote by $\xi = (\xi_1, \ldots, \xi_\alpha, \zeta)' = \Xi(x, t)$ $(\dim(\xi_k) = n - \mu_{k-1}$ and $\dim \zeta = n - \sum_{k=1}^{\alpha}(n - \mu_{k-1}))$. $h_k(x, \dot{x}, t)$ is denoted by $h_k(\xi, \dot{\xi}, t)$, $\Phi_k(x, t, \blacksquare)$ is denoted by $\Phi_k(\xi, t, \blacksquare)$ with $\xi = \Xi(x, t)$ and $\dot{\xi} = \frac{\partial \Xi}{\partial x}\,\dot{x} + \frac{\partial \Xi}{\partial t}$.

By construction, the first μ_0 rows of $\overline{h}_1$ correspond to $\overline{h}_0$. Consequently

$$h_1(\xi, \dot{\xi}, t) = \begin{pmatrix} \overline{h}_0(\xi, \dot{\xi}, t) \\ \dot{\xi}_1 \end{pmatrix} = \begin{pmatrix} \overline{h}_0(\xi, \dot{\xi}, t) \\ \dot{\overline{\xi}}_1 \\ \dot{\tilde{\xi}}_1 \end{pmatrix}$$

with $\overline{h}_1(\xi, \dot{\xi}, t) = \begin{pmatrix} \overline{h}_0(\xi, \dot{\xi}, t) \\ \dot{\overline{\xi}}_1 \end{pmatrix}$, $\dot{\tilde{\xi}}_1 = \Phi_1(\xi, t, (\overline{h}_0(\xi, t, \dot{\xi}), \dot{\overline{\xi}}_1)')$. ξ_1 is made of two groups of components, $\xi_1 = (\overline{\xi}_1, \tilde{\xi}_1)'$ of dimensions, respectively, $\mu_1 - \mu_0$ and $n - \mu_1$.

Similarly, each ξ_k is made of two groups of components, $\xi_k = (\overline{\xi}_k, \tilde{\xi}_k)'$ of dimensions $\mu_k - \mu_{k-1}$ and $n - \mu_k$. By construction,

$$h_k(\xi, \dot{\xi}, t) = \begin{pmatrix} \overline{h}_{k-1}(\xi, \dot{\xi}, t) \\ \dot{\xi}_k \end{pmatrix} = \begin{pmatrix} \overline{h}_k(\xi, \dot{\xi}, t) \\ \dot{\overline{\xi}}_k \\ \dot{\tilde{\xi}}_k \end{pmatrix}$$

with $\overline{h}_k(\xi, \dot{\xi}, t) = \begin{pmatrix} \overline{h}_{k-1}(\xi, \dot{\xi}, t) \\ \dot{\overline{\xi}}_k \end{pmatrix}$, $\dot{\tilde{\xi}}_k = \Phi_k(\xi, t, (\overline{h}_k(\xi, \dot{\xi}, t), \dot{\overline{\xi}}_k)')$.

Since $\mu_\alpha = n$, we have

$$h_\alpha(\xi, \dot{\xi}, t) = \overline{h}_\alpha(\xi, \dot{\xi}, t) = \begin{pmatrix} \overline{h}_0(\xi, \dot{\xi}, t) \\ \dot{\overline{\xi}}_1 \\ \vdots \\ \dot{\overline{\xi}}_\alpha \end{pmatrix}.$$

The rank of h_α with respect to $\dot{\xi} = (\dot{\overline{\xi}}_1, \dot{\tilde{\xi}}_1, \dot{\overline{\xi}}_2, \dot{\tilde{\xi}}_2, \ldots, \dot{\overline{\xi}}_{\alpha-1}, \dot{\tilde{\xi}}_{\alpha-1}, \dot{\overline{\xi}}_\alpha, \dot{\zeta})'$ is equal to n and $\dim(h_\alpha) = n$. Necessarily, the rank of the Jacobian matrix $\frac{\partial \overline{h}_0}{\partial\ \dot{\tilde{\xi}}_1, \ldots, \dot{\tilde{\xi}}_{\alpha-1}, \dot{\zeta}}$ is equal to $n - \sum_{k=1}^{\alpha}(\mu_k - \mu_{k-1}) = \mu_0$. But the dimension of the vector $(\dot{\tilde{\xi}}_1, \ldots, \dot{\tilde{\xi}}_{\alpha-1}, \dot{\zeta})$ is equal to

$$\sum_{k=1}^{\alpha-1}(n - \mu_k) + n - \sum_{k=1}^{\alpha}(n - \mu_{k-1}) = \mu_0.$$

Consequently, $\frac{\partial \overline{h}_0}{\partial\ \dot{\tilde{\xi}}_1, \ldots, \dot{\tilde{\xi}}_{\alpha-1}, \dot{\zeta}}$ is square and invertible. Thus, locally, $\overline{h}_0(\xi, \dot{\xi}, t) = 0$ can be written explicitly with respect to $(\dot{\tilde{\xi}}_1, \ldots, \dot{\tilde{\xi}}_{\alpha-1}, \dot{\zeta})$:

$$\left\{ \begin{array}{rcl} \dot{\tilde{\xi}}_1 & = & \theta_2(\xi, t, \dot{\overline{\xi}}_1, \ldots, \dot{\overline{\xi}}_\alpha) \\ & \vdots & \\ \dot{\tilde{\xi}}_{\alpha-1} & = & \theta_\alpha(\xi, t, \dot{\overline{\xi}}_1, \ldots, \dot{\overline{\xi}}_\alpha) \\ \dot{\zeta} & = & \Theta(\xi, t, \dot{\overline{\xi}}_1, \ldots, \dot{\overline{\xi}}_\alpha). \end{array} \right. \quad (5)$$

One has : $\dot{\tilde{\xi}}_k = \Phi_k(\xi, t, (\overline{h}_0(\xi, \dot{\xi}, t)), \dot{\overline{\xi}}_1, \ldots, \dot{\overline{\xi}}_k)')$. Since $\overline{h}_0(\xi, \dot{\xi}, t) = 0$, we have for $k = 2, \ldots, \alpha$

$$\theta_k(\xi, t, \dot{\overline{\xi}}_1, \ldots, \dot{\overline{\xi}}_\alpha) = \Phi_{k-1}(\xi, t, (0, \dot{\overline{\xi}}_1, \ldots, \dot{\overline{\xi}}_k)').$$

Since

$$h(\xi, \dot{\xi}, t) = h_0(\xi, \dot{\xi}, t) = \begin{pmatrix} \overline{h}_0(\xi, \dot{\xi}, t) \\ \Phi_0(\xi, t, \overline{h}_0(\xi, \dot{\xi}, t)) \end{pmatrix},$$

$h(\xi, \dot{\xi}, t) = 0$ is equivalent to

$$\left\{ \begin{array}{rcl} \overline{h}_0(\xi, \dot{\xi}, t) & = & 0 \\ \Phi_0(\xi, t, 0) & = & 0. \end{array} \right.$$

With (5), the change of variables $x \to \xi$ tranforms the system (Σ) into

$$\left\{ \begin{array}{rcl} \xi_1 &=& 0 \\ \dot{\bar{\xi}}_1 &=& \Phi_1(\xi, t, (0, \dot{\bar{\xi}}_1)') \\ & \vdots & \\ \dot{\bar{\xi}}_{\alpha-1} &=& \Phi_{\alpha-1}(\xi, t, (0, \dot{\bar{\xi}}_1, \ldots, \dot{\bar{\xi}}_{\alpha-1})') \\ \dot{\zeta} &=& \Theta(\xi, t, \dot{\bar{\xi}}_1, \ldots, \dot{\bar{\xi}}_\alpha), \end{array} \right.$$

with Θ an analytic function. It suffices to take

$$\phi_k(\xi, t, \dot{\xi}_1, \ldots, \dot{\xi}_k) = \dot{\bar{\xi}}_k + \xi_{k+1} - \Phi_k(\xi, t, (0, \dot{\bar{\xi}}_1, \ldots, \dot{\bar{\xi}}_k)')$$

and to remark that, locally, $(\xi_1, \ldots, \xi_\alpha)$ is a function of $(\dot{\xi}_1, \ldots, \dot{\xi}_{\alpha-1})$, in order to obtain the canonical form (Σ_c). ∎

4 Concluding remarks

In [6], we give a general algebraic definition of the index for nonlinear systems of form (4) through their linear tangent time-varying systems and non commutative extension of Laplace techniques. One can easily prove that the index is bounded above by the relative order α of the extended system $\frac{dx}{dt} = u$, $0 = F(u, x, e(t))$ and is equal to α when $\frac{\partial F}{\partial e}$ is invertible. In [7] state-variable representation of linear time-varying implicit system are given. Similarly, such nonlinear Kronecker canonical forms provide generalized state-space form representation [5] of the implicit system (4). In [12], it is shown how such canonical forms can be used to analyze the convergence of numerical resolution algorithms.

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