Calcul symbolique et génération de trajectoires pour certaines EDP avec contrôle frontière

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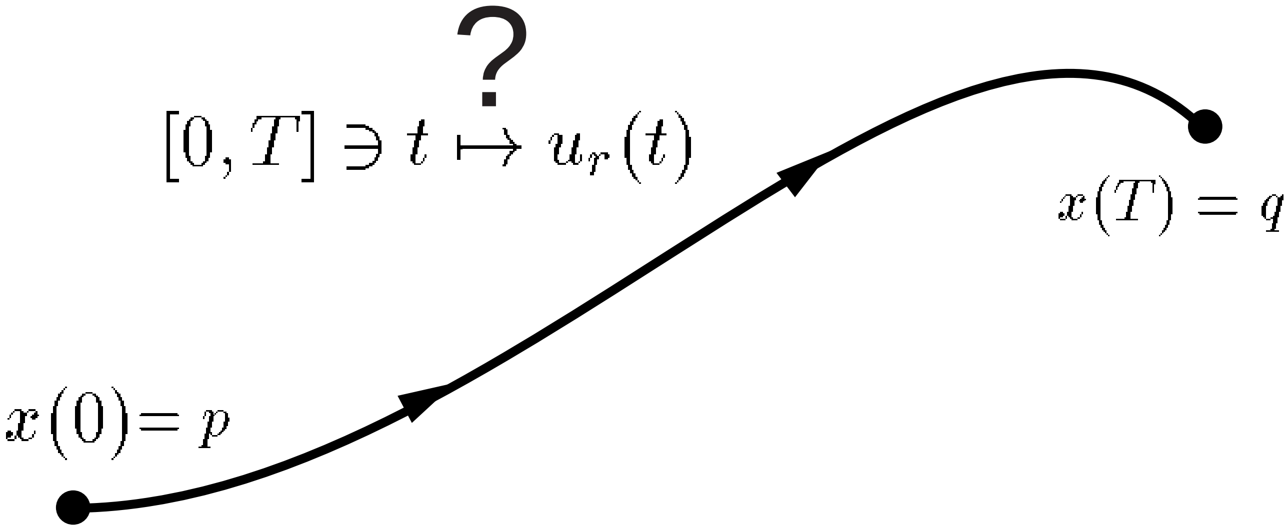
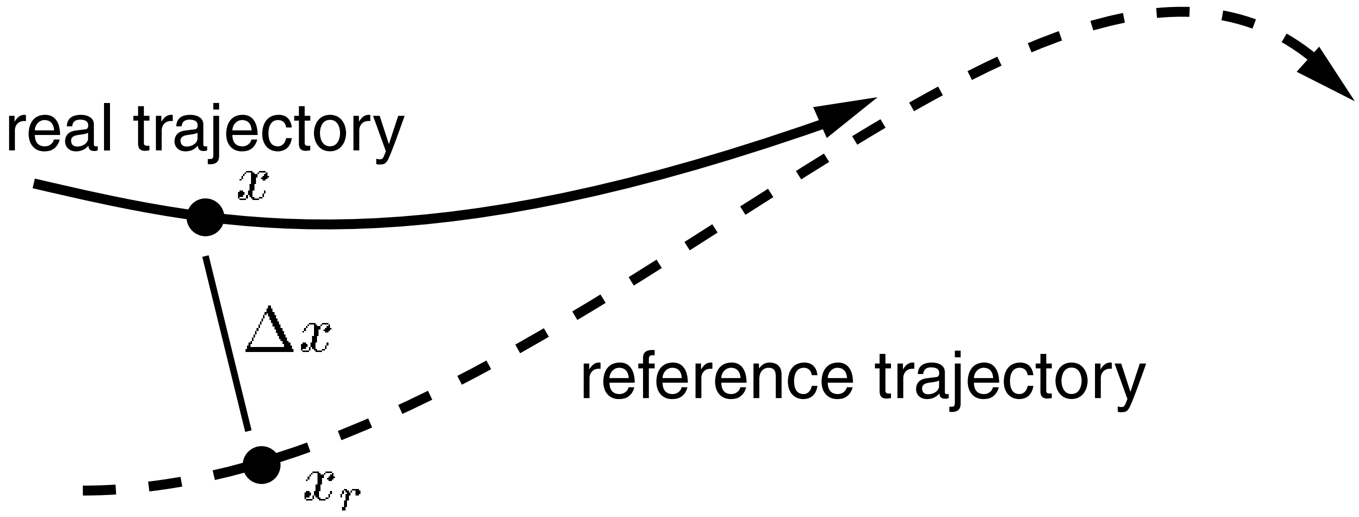
Trajectory planning for $\frac{d}{dt}x = f(x, u)$: controllability



Trajectory tracking: stabilization

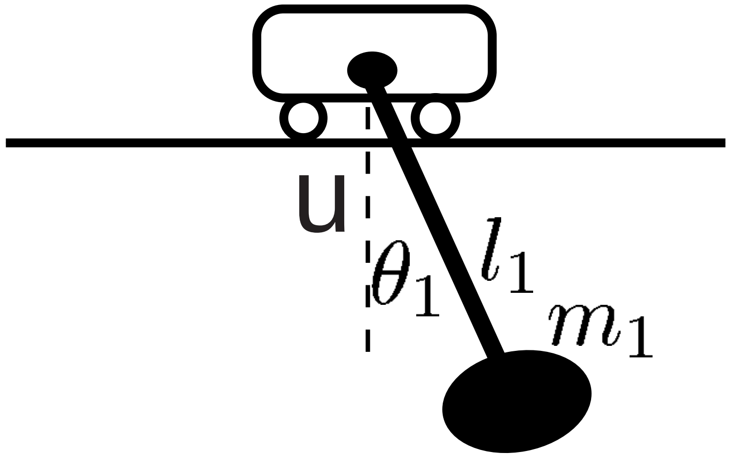


Compute Δu, $u = u_r + \Delta u$, such that $\Delta x = x - x_r$ converges to 0.

Outline

- Pendulum dynamics.
- Water in a moving box
- Heat equation
- Quantum particle in a moving box
- Conclusion: distribution of zeros, analytic functions and operational calculus.

One linearized pendulum



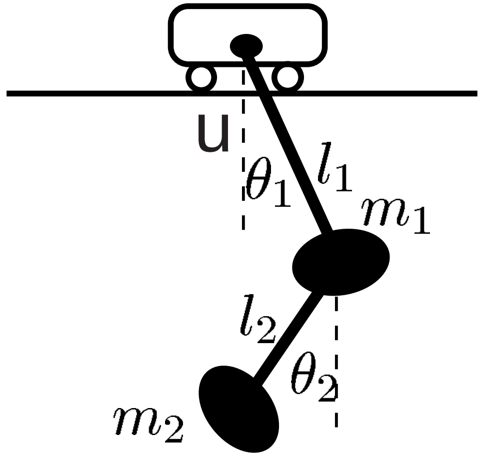
Newton equation with $y = u + l_1\theta_1$:

$$\frac{d^2y}{dt^2} = -g\theta_1 = \frac{g}{l_1}(y-u).$$

Computed torque method:

$$\theta_1 = -\frac{\ddot{y}}{g}, \quad u = y - l_1\theta_1.$$

Two linearized pendulums in series

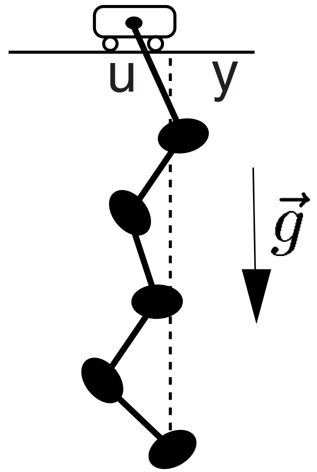


Brunovsky (flat) output $y = u + l_1\theta_1 + l_2\theta_2$:

$$\theta_2 = -\frac{\ddot{y}}{g}, \quad \theta_1 = -\frac{m_1(\overbrace{y - l_2\theta_2}^{\cdot\cdot})}{(m_1+m_2)g} + \frac{m_2}{m_1+m_2}\theta_2$$

and $u = y - l_1\theta_1 - l_2\theta_2$ is a linear combination of $(y, y^{(2)}, y^{(4)})$.

n pendulums in series

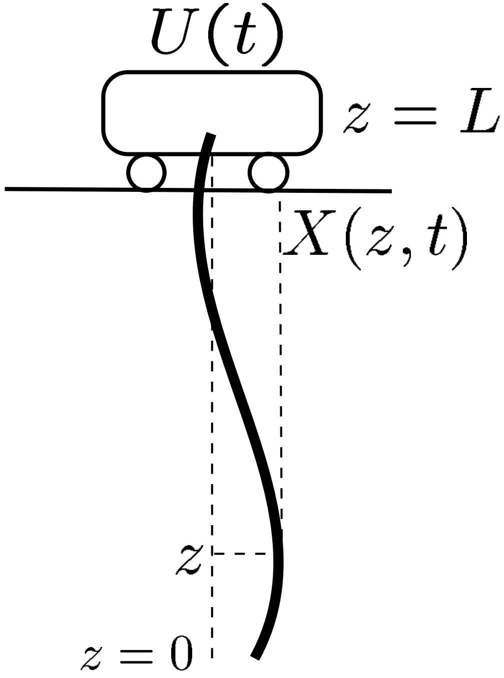


Brunovsky (flat) output $y = u + l_1\theta_1 + \ldots + l_n\theta_n$:

$$u = y + a_1 y^{(2)} + a_2 y^{(4)} + \ldots + a_n y^{(2n)}.$$

When n tends to ∞ the system tends to a partial differential equation.

The heavy chain (Petit-R 2001).



$$\frac{\partial^2 X}{\partial t^2} = \frac{\partial}{\partial z}\left(gz\frac{\partial X}{\partial z}\right)$$
$$X(L,t) = U(t)$$

Flat output $y(t) = X(0,t)$ with

$$U(t) = \frac{1}{2\pi} \quad \int_0^{2\pi} y\left(t - 2\sqrt{L/g}\ \sin\zeta\right)\ d\zeta$$

With the same flat output, for a discrete approximation (n pendulums in series, n large) we have

$$u(t) = y(t) + a_1 \ddot{y}(t) + a_2 y^{(4)}(t) + \ldots + a_n y^{(2n)}(t),$$

for a continuous approximation (the heavy chain) we have

$$U(t) = \frac{1}{2\pi} \quad \int_0^{2\pi} y\left(t + 2\sqrt{L/g}\ \sin\zeta\right)\ d\zeta.$$

Why? Because formally

$$y(t + 2\sqrt{L/g}\ \sin\zeta) = y(t) + \ldots + \frac{\left(2\sqrt{L/g}\ \sin\zeta\right)^n}{n!}\ y^{(n)}(t) + \ldots$$

But integral formula is preferable (divergence of the series. . .).

The general solution of the PDE

$$\frac{\partial^2 X}{\partial t^2} = \frac{\partial}{\partial z}\left(gz\frac{\partial X}{\partial z}\right)$$

is

$$X(z,t) = \frac{1}{2\pi} \quad \int_0^{2\pi} y\left(t - 2\sqrt{z/g}\ \sin\zeta\right)\ d\zeta$$

where $t \mapsto y(t)$ is any time function.

Proof: replace $\frac{d}{dt}$ by s, the Laplace variable, to obtain a singular second order ODE in z with bounded solutions. Symbolic computations and operational calculus on

$$s^2 X = \frac{\partial}{\partial z}\left(gz\frac{\partial X}{\partial z}\right).$$

Symbolic computations in the Laplace domain

Thanks to $x = 2\sqrt{\frac{z}{g}}$, we get

$$x\frac{\partial^2 X}{\partial x^2}(x,t) + \frac{\partial X}{\partial x}(x,t) - x\frac{\partial^2 X}{\partial t^2}(x,t) = 0.$$

Use Laplace transform of X with respect to the variable t

$$x\frac{\partial^2 \hat{X}}{\partial x^2}(x,s) + \frac{\partial \hat{X}}{\partial x}(x,s) - xs^2\hat{X}(x,s) = 0.$$

This is a the Bessel equation defining J_0 and Y_0:

$$\hat{X}(z,s) = a(s)\ J_0(2\imath s\sqrt{z/g}) + b(s)\ Y_0(2\imath s\sqrt{z/g}).$$

Since we are looking for a bounded solution at $z = 0$ we have $b(s) = 0$ and (remember that $J_0(0) = 1$):

$$\hat{X}(z,s) = J_0(2\imath s\sqrt{z/g})\hat{X}(0,s).$$

$$\widehat{X}(z,s) = J_0(2\imath s\sqrt{z/g})\widehat{X}(0,s).$$

Using Poisson's integral representation of J_0

$$J_0(\zeta) = \frac{1}{2\pi}\int_0^{2\pi} \exp(\imath\zeta \sin\theta)\ d\theta, \quad \zeta \in \mathbb{C}$$

we have

$$J_0(2\imath s\sqrt{x/g}) = \frac{1}{2\pi}\int_0^{2\pi} \exp(2s\sqrt{x/g}\sin\theta)\ d\theta.$$

In terms of Laplace transforms, this last expression is a combination of delay operators:

$$X(z,t) = \frac{1}{2\pi}\int_0^{2\pi} y(t + 2\sqrt{z/g}\sin\theta)\ d\theta$$

with $y(t) = X(0,t)$.

Explicit parameterization of the heavy chain

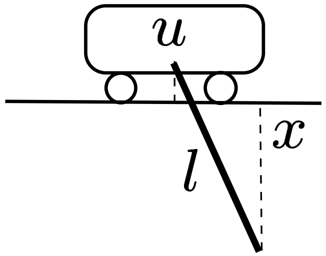
The general solution of

$$\frac{\partial^2 X}{\partial t^2} = \frac{\partial}{\partial z}\left(gz\frac{\partial X}{\partial z}\right), \qquad U(t) = X(L,t)$$

reads

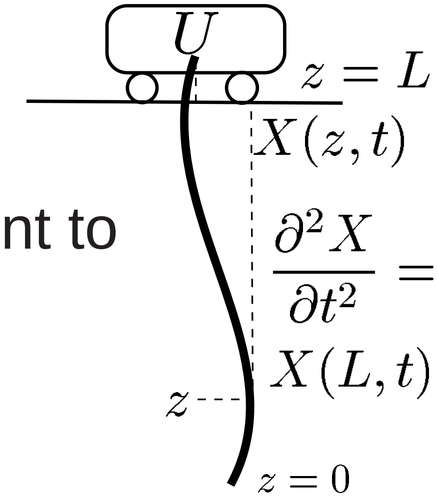
$$X(z,t) = \frac{1}{2\pi}\int_0^{2\pi} y(t + 2\sqrt{z/g}\sin\theta)\ d\theta$$

There is a one to one correspondence between the (smooth) solutions of the PDE and the (smooth) functions $t \mapsto y(t)$.



$$\frac{d^2x}{dt^2} = \frac{g}{l}(u - x)$$

lent to



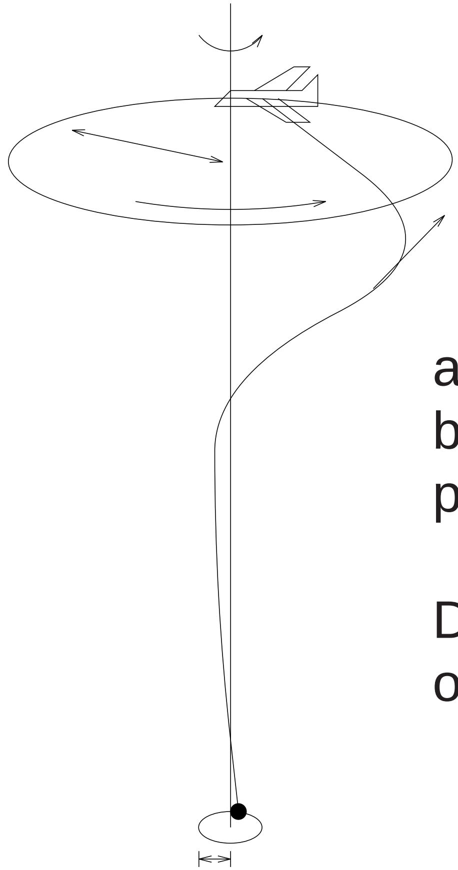
$$\frac{\partial^2 X}{\partial t^2} = \frac{\partial}{\partial z}\left(gz\frac{\partial X}{\partial z}\right)$$

$$X(L,t) = U$$

The following maps exchange the trajectories:

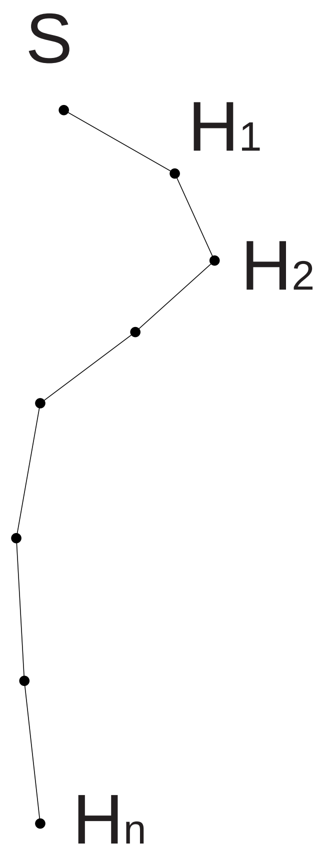
$$\begin{cases} x(t) = X(0,t) \\ u(t) = \frac{\partial^2 X}{\partial t^2}(0,t) \end{cases} \quad \begin{cases} X(z,t) = \frac{1}{2\pi}\int_0^{2\pi} x\left(t - 2\sqrt{z/g}\sin\zeta\right)d\zeta \\ U(t) = \frac{1}{2\pi}\int_0^{2\pi} x\left(t - 2\sqrt{L/g}\sin\zeta\right)d\zeta \end{cases}$$

The Towed Cable Flight Control System, Murray (1996)

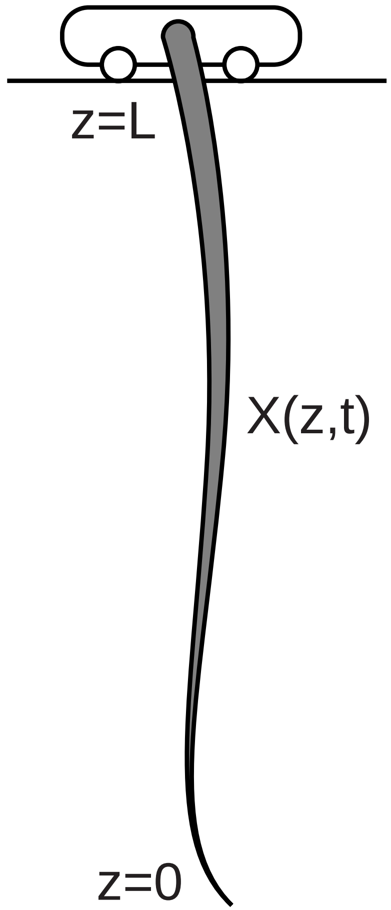


approximated
by a series of small
pendulums

Derivatives
of order 2n



Heavy chain with a variable section

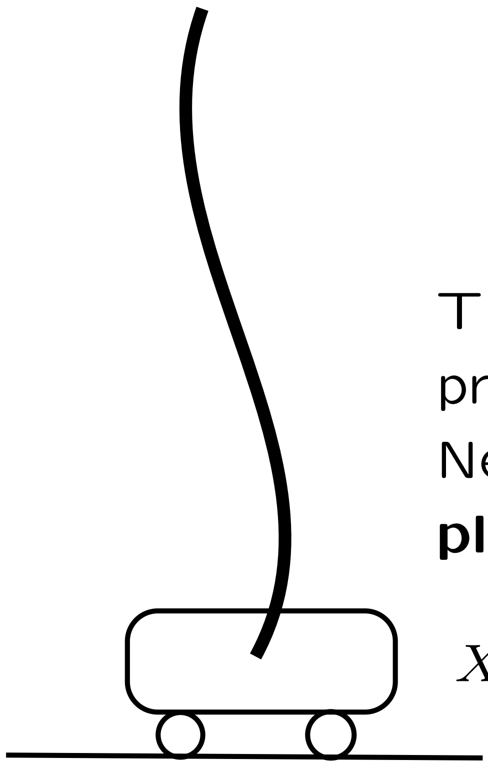


The general solution of

$$\begin{cases} \dfrac{\tau'(z)}{g} \quad \dfrac{\partial^2 X}{\partial t^2} = \dfrac{\partial}{\partial z}\left(\tau(z)\dfrac{\partial X}{\partial z}\right) \\ \\ X(L,t) = u(t) \end{cases}$$

where $\tau(z) \geq 0$ is the tension in the rope, can be parameterized by an arbitrary time function $y(t)$, the position of the free end of the system $y = X(0,t)$, via delay and advance operators with **compact** support.

The Indian rope.



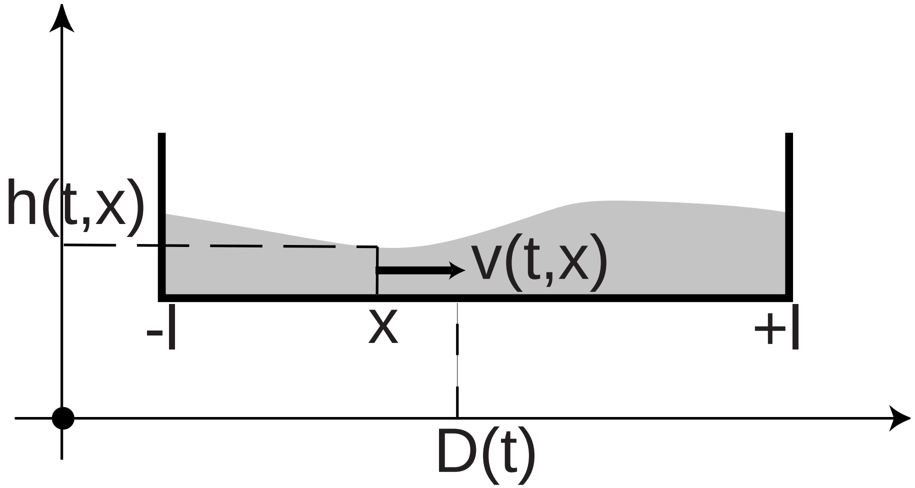
$$\frac{\partial}{\partial z}\left(gz\frac{\partial X}{\partial z}\right)+\frac{\partial^2 X}{\partial t^2} = 0$$

$$X(L,t) = U(t)$$

The equation becomes elliptic and the Cauchy problem is not well posed in the sense of Hadamard. Nevertheless **formulas are still valid with a complex time and y holomorphic**

$$X(z,t)=\frac{1}{2\pi}\int_0^{2\pi} y\left(t-(2\sqrt{z/g}\ \sin\zeta)\ \sqrt{-1}\right)\ d\zeta.$$

1D Tank: shallow water approximation

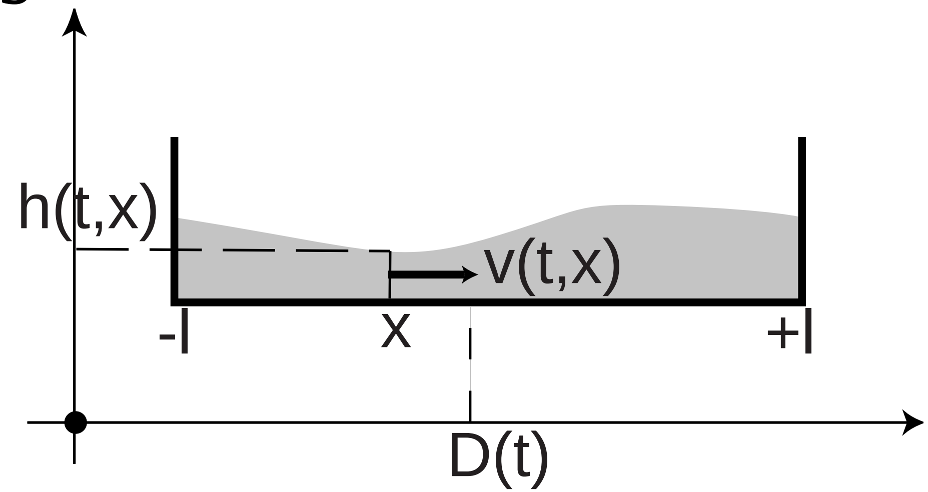


$$\frac{\partial h}{\partial t} + \frac{\partial (hv)}{\partial x} = 0, \qquad \frac{\partial v}{\partial t} + \ddot{D} + v\frac{\partial v}{\partial x} = -g\frac{\partial h}{\partial x}$$

with $v(t,-l) = v(t,l) = 0$.

The nonlinear dynamics is controllable (Coron 2002) but the tangent linearization is not controllable (Petit-R, 2002).

1D tank: tangent linearization.



Assumptions: $h = \bar{h} + H$, $|H| \ll \bar{h}$; $|\ddot{D}| \ll g$, $|v| \ll c = \sqrt{g\bar{h}}$.

$$\frac{\partial^2 H}{\partial t^2} = g\bar{h}\frac{\partial^2 H}{\partial x^2}, \qquad \frac{\partial H}{\partial x}(t,-l) = \frac{\partial H}{\partial x}(t,l) = -\frac{1}{g}\ \ddot{D}(t)$$

Non controllable system

Since $H = \phi(t + x/c) + \psi(t - x/c)$, with ϕ and ψ arbitrary, one gets

$$\begin{cases} \phi'(t+\Delta) - \psi'(t-\Delta) = -c\ddot{D}(t)/g \\ \phi'(t-\Delta) - \psi'(t+\Delta) = -c\ddot{D}(t)/g \end{cases}$$

with $2\Delta = l/c$. Elimination of D yields

$$\phi'(t+\Delta) + \psi'(t+\Delta) = \phi'(t-\Delta) + \psi'(t-\Delta).$$

So the quantity $\pi(t) = \phi(t) + \psi(t)$ satisfies an autonomous equation (torsion element of the underlying module, Fliess, Mounier, ...)

$$\pi(t + 2\Delta) = \pi(t).$$

The system is not controllable.

Trajectories passing through a steady-state

Since $\pi(t) = \phi(t) + \psi(t) \equiv 0$ we have

$$\phi'(t + \Delta) + \phi'(t - \Delta) = -c\ddot{D}(t)/g$$

thus

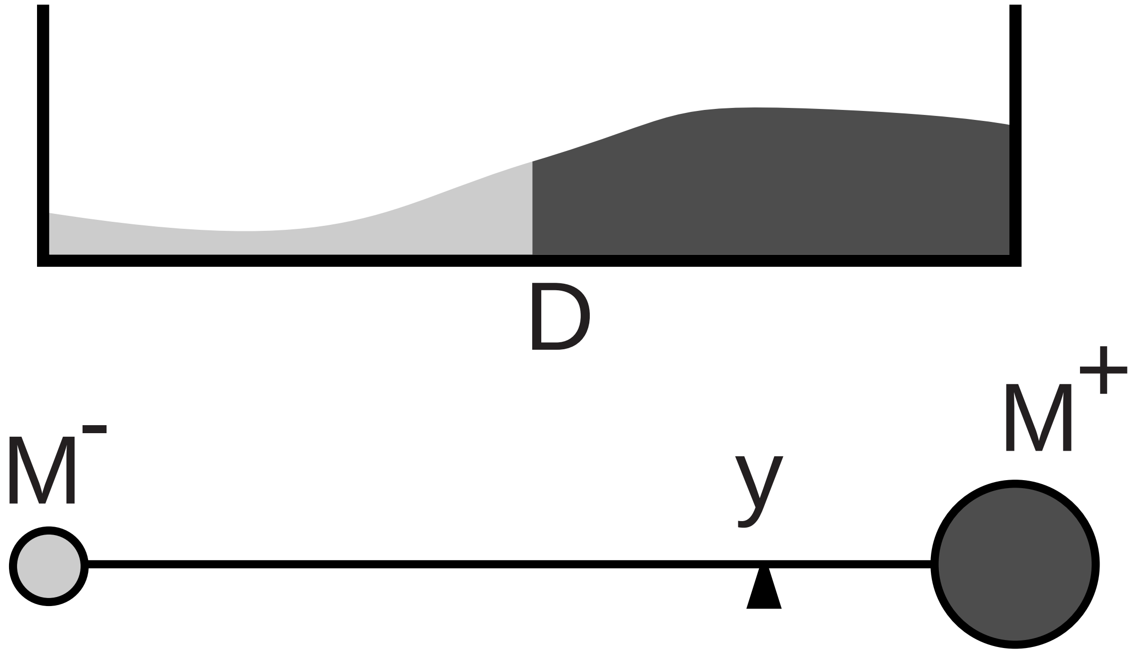
$$\phi(t) = -\left(\frac{c}{2g}\right) y'(t), \qquad D(t) = (y(t + \Delta) + y(t - \Delta))/2$$

and

$$\begin{cases} H(t,x) = \dfrac{1}{2}\sqrt{\dfrac{\bar{h}}{g}}\left[y'(t + x/c) - y'(t - x/c)\right] \\ D(t) = \dfrac{1}{2}\left[y(t + \Delta) + y(t - \Delta)\right] \end{cases}$$

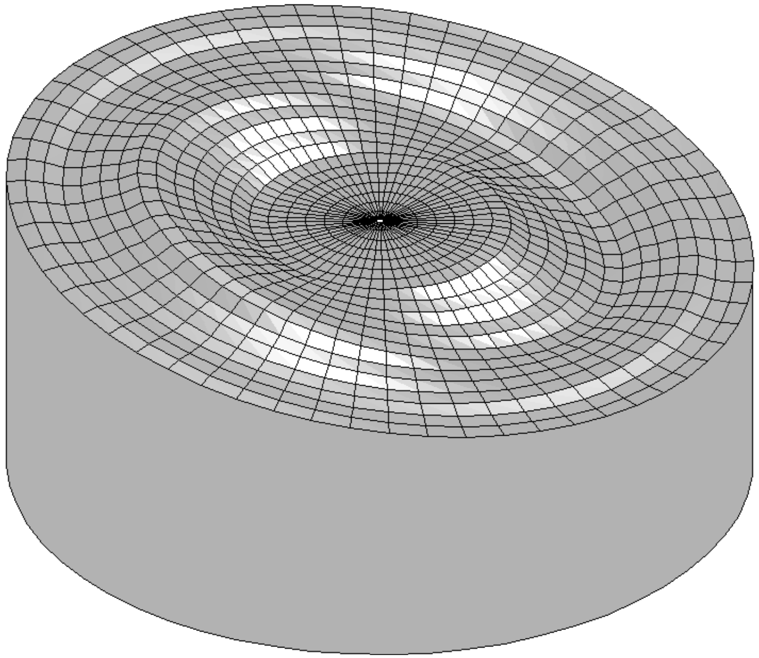
with $t \mapsto y(t)$ an arbitrary time function.

Physical interpretation of y



$$M^{-} = \int_{-l}^{0} h(t,x)\ dx \qquad M^{+} = \int_{0}^{l} h(t,x)\ dx$$

The tumbler in movement: 2D cylindrical tank



Modelling the 2D tank

The liquid occupies a cylinder with vertical edges with the 2D domain Ω as horizontal section. The tangent linear equations are:

$$\frac{\partial^2 H}{\partial t^2} = g\bar{h}\Delta H \quad \text{in } \Omega$$
$$\nabla H \cdot \vec{n} = -\frac{\ddot{D}(t)}{g} \cdot \vec{n} \quad \text{on } \partial\Omega$$

with $D = (D_1, D_2)$, $\vec{n}$ the normal to $\partial\Omega$.

2D Tank, circular shape.

Steady-state motion planning results from a symbolic computations in polar coordinates:

$$
\begin{aligned}
H(t,x_1,x_2) &= \frac{1}{\pi}\sqrt{\bar{h}/g}\int_0^{2\pi}\left[\cos\zeta\ y_1'\left(t-\frac{x_1\cos\zeta+x_2\sin\zeta}{c}\right)\right.\\
&\qquad\qquad\left.+\sin\zeta\ y_2'\left(t-\frac{x_1\cos\zeta+x_2\sin\zeta}{c}\right)\right]\ d\zeta\\
D_1(t) &= \frac{1}{\pi}\int_0^{2\pi}\left[\cos^2\zeta\ y_1\left(t-\frac{l\cos\zeta}{c}\right)\right]\ d\zeta\\
D_2(t) &= \frac{1}{\pi}\int_0^{2\pi}\left[\sin^2\zeta\ y_2\left(t-\frac{l\sin\zeta}{c}\right)\right]\ d\zeta
\end{aligned}
$$

with $t\mapsto y_1(t)$ and $t\mapsto y_2(t)$ as you want.

Open question

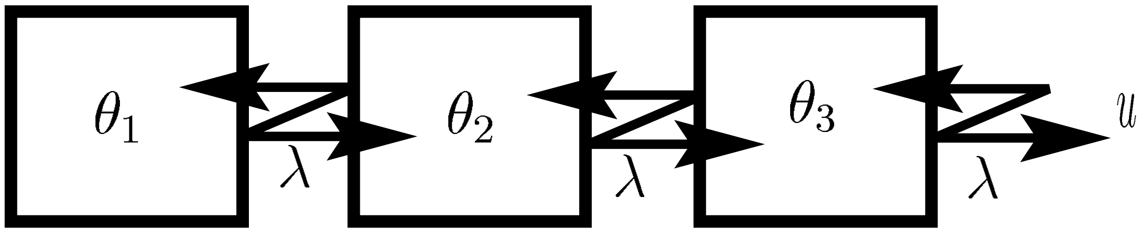
Under which conditions on Ω is the 2D tank described by

$$\frac{\partial^2 H}{\partial t^2} = g\bar{h}\Delta H \quad \text{in } \Omega$$
$$\nabla H \cdot \vec{n} = -\frac{u}{g} \cdot \vec{n} \quad \text{on } \partial\Omega$$
$$\ddot{D}(t) = u$$

steady-state controllable ?

It is true for Ω a disk or a rectangle.

Compartmental approximation of the heat equation



Energy balance equations

$$\begin{cases} \frac{d}{dt}\theta_1 = (\theta_2 - \theta_1) \\ \frac{d}{dt}\theta_2 = (\theta_1 - \theta_2) + (\theta_3 - \theta_2) \\ \frac{d}{dt}\theta_3 = (\theta_2 - \theta_3) + (u - \theta_3). \end{cases}$$

Linear system controllable with $y = \theta_1$ as Brunovsky or flat output: it can be transformed via linear change of coordinates and linear static feedback into $y^{(3)} = v$.

Compartmental approximation of the heat equation (end)

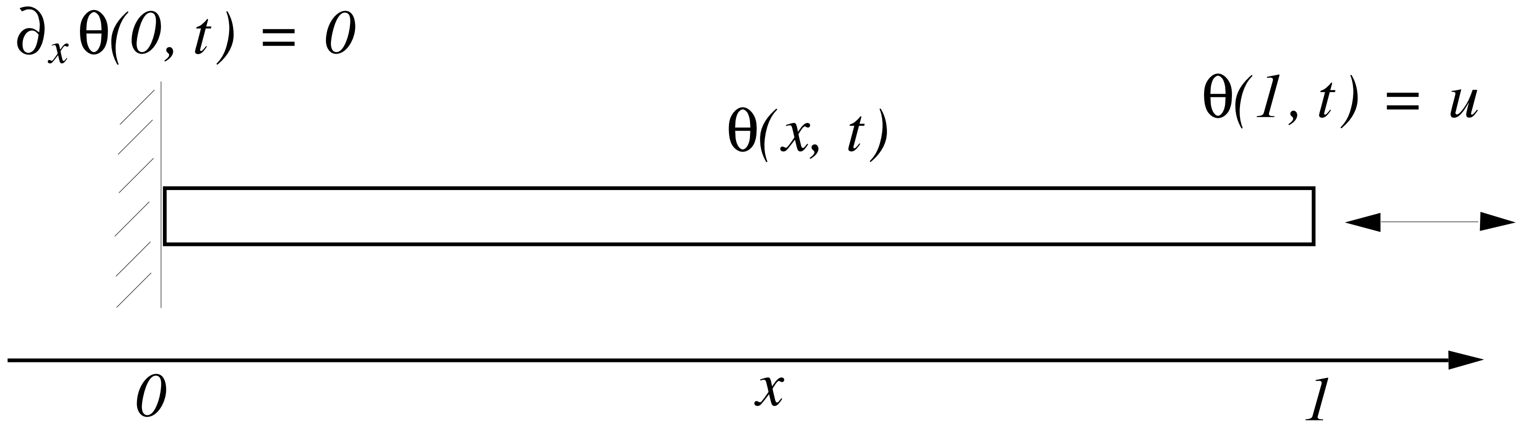
An arbitrary number n of compartments yields

$$\begin{cases} \dot{\theta}_1 = (\theta_2 - \theta_1) \\ \dot{\theta}_2 = (\theta_1 - \theta_2) + (\theta_3 - \theta_2) \\ \quad \vdots \\ \dot{\theta}_i = (\theta_{i-1} - \theta_i) + (\theta_{i+1} - \theta_i) \\ \quad \vdots \\ \dot{\theta}_{n-1} = (\theta_{n-2} - \theta_{n-1}) + (\theta_n - \theta_{n-1}) \\ \dot{\theta}_n = (\theta_{n-1} - \theta_n) + (u - \theta_n). \end{cases}$$

$y = \theta_1$ remains the Brunovsky output: via linear change of coordinates and linear static feedback we have $y^{(n)} = v$.

When n tends to infinity we recover $\partial_t \theta = \partial_x^2 \theta$

Heat equation



$$\partial_t \theta(x,t) = \partial_x^2 \theta(x,t), \quad x \in [0,1]$$
$$\partial_x \theta(0,t) = 0 \qquad \theta(1,t) = u(t).$$

Series solutions

Set, formally

$$\theta = \sum_{i=0}^{\infty} a_i(t)\frac{x^i}{i!}, \quad \partial_t\theta = \sum_{i=0}^{\infty} \frac{da_i}{dt}\left(\frac{x^i}{i!}\right), \quad \partial_x^2\theta = \sum_{i=0}^{\infty} a_{i+2}\left(\frac{x^i}{i!}\right)$$

and $\partial_t\theta = \partial_x^2\theta$ reads $da_i/dt = a_{i+2}$. Since $a_1 = \partial_x\theta(0,t) = 0$ and $a_0 = \theta(0,t)$ we have

$$a_{2i+1} = 0, \quad a_{2i} = a_0^{(i)}$$

Set $y := a_0 = \theta(0,t)$ we have

$$\theta(x,t) = \sum_{i=0}^{\infty} y^{(i)}(t)\left(\frac{x^{2i}}{(2i)!}\right)$$

Symbolic computations: $s := d/dt$, $s \in \mathbb{C}$

The general solution of $\theta'' = s\theta$ reads ($' := d/dx$)

$$\theta = \cosh(x\sqrt{s})\ a(s) + \frac{\sinh(x\sqrt{s})}{\sqrt{s}}\ b(s)$$

The boundary condition $\theta(1) = u$ and $\theta'(0) = 0$ reads

$$u = \cosh(\sqrt{s})\ a(s) + \frac{\sinh(\sqrt{s})}{\sqrt{s}}\ b(s), \quad b = 0$$

Since $y = \theta(0) = a$ we have

$$\theta(x,s) = \cosh(x\sqrt{s})\ y(s) = \left(\sum_{i \geq 0} \frac{x^{2i}}{(2i)!} s^i \right)\ y(s).$$

The general solution parameterized via $t \mapsto y(t) \in \mathbb{R}$, C^∞ ($y(t) := \theta(0,t)$)

$$\theta(x,t) = \sum_{i=1}^{+\infty} \frac{y^{(i)}(t)}{(2i)!}\ x^{2i}$$

$$u(t) = \sum_{i=1}^{+\infty} \frac{y^{(i)}(t)}{(2i)!}.$$

Convergence issue.

Gevrey function of order α

A C^∞ time function $[0,T] \ni t \mapsto y(t)$ is of Gevrey order α when,

$$\exists\ C, D > 0, \quad \forall t \in [0,T], \forall i \geq 0, \quad |y^{(i)}(t)| \leq C D^i \Gamma(1 + (\alpha+1)i)$$

where Γ is the classical gamma function with $n! = \Gamma(n+1)$, $\forall n \in \mathbb{N}$.

Analytic functions correspond to Gevrey functions of order ≤ 0. When $\alpha > 0$, the class of α-order functions contains non-zero functions with compact supports. Prototype of such functions:

$$t \mapsto y(t) = \begin{cases} e^{-\left(\frac{1}{t(1-t)}\right)^{\frac{1}{\alpha}}} & \text{if } t \in]0,1[\\ 0 & \text{otherwise.} \end{cases}$$

Operators $P(s)$ as entire functions of s, order at infinity

$\mathbb{C} \ni s \mapsto P(s) = \sum_{i\geq 0} a_i s^i$ is an entire function when the radius of convergence is infinite. If its order at infinity is $\sigma > 0$, i.e., $\exists M, K > 0$ such that $\forall s \in \mathbb{C}$, $|P(s)| \leq M \exp(K|s|^\sigma)$, then

$$\exists A, B > 0 \mid \forall i \geq 0, \quad |a_i| \leq A \frac{B^i}{\Gamma(i/\sigma + 1)}.$$

$\cosh(\sqrt{s})$ and $\sinh(\sqrt{s})/\sqrt{s}$ are entire functions of order $\sigma = 1/2$.

Take $P(s)$ of order σ with $s = d/dt$. Then $P(s)y(s)$ corresponds to series with a strictly positive convergence radius

$$P(s)y(s) \equiv \sum_{i=0}^{\infty} a_i \; y^{(i)}(t)$$

when $t \mapsto y(t)$ is a Gevrey function of order $\alpha < 1/\sigma - 1$.

Motion planning of the heat equation

Take $\sum_{i\geq 0} a_i \frac{\xi^i}{i!}$ and $\sum_{i\geq 0} b_i \frac{\xi^i}{i!}$ entire functions of ξ. With $\sigma > 1$

$$y(t) = \left(\sum_{i\geq 0} a_i \frac{t^i}{i!}\right)\left(\frac{e^{\frac{-T^\sigma}{(T-t)^\sigma}}}{e^{\frac{-T^\sigma}{t^\sigma}} + e^{\frac{-T^\sigma}{(T-t)^\sigma}}}\right) + \left(\sum_{i\geq 0} b_i \frac{t^i}{i!}\right)\left(\frac{e^{\frac{-T^\sigma}{t^\sigma}}}{e^{\frac{-T^\sigma}{t^\sigma}} + e^{\frac{-T^\sigma}{(T-t)^\sigma}}}\right)$$

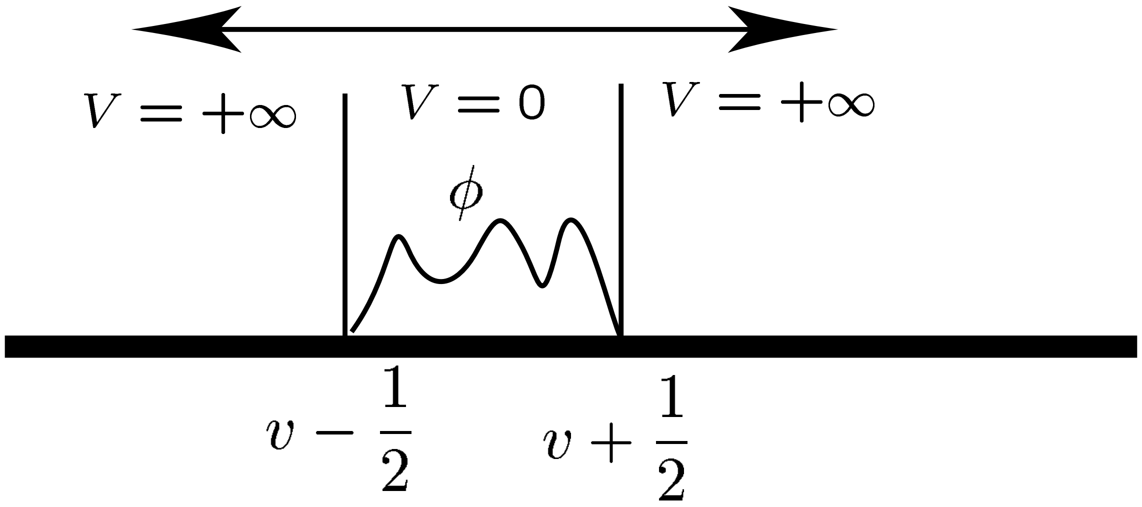
the series

$$\theta(x,t) = \sum_{i=1}^{+\infty} \frac{y^{(i)}(t)}{(2i)!}\ x^{2i}, \quad u(t) = \sum_{i=1}^{+\infty} \frac{y^{(i)}(t)}{(2i)!}.$$

are convergent and provide a trajectory from

$$\theta(x,0) = \sum_{i\geq 0} a_i \frac{x^{2i}}{(2i)!} \quad \text{to} \quad \theta(x,T) = \sum_{i\geq 0} b_i \frac{x^{2i}}{(2i)!}$$

A quantum analogue of the water-tank problem: the quantum box problem (R. 2002)



In a Galilean frame

$$\imath \frac{\partial \phi}{\partial t} = -\frac{1}{2}\frac{\partial^2 \phi}{\partial z^2}, \qquad z \in [v - \frac{1}{2}, v + \frac{1}{2}],$$
$$\phi(v - \frac{1}{2}, t) = \phi(v + \frac{1}{2}, t) = 0$$

Particle in a moving box of position v

In a Galilean frame

$$\imath \frac{\partial \phi}{\partial t} = -\frac{1}{2}\frac{\partial^2 \phi}{\partial z^2}, \quad z \in [v - \frac{1}{2}, v + \frac{1}{2}],$$
$$\phi(v - \frac{1}{2}, t) = \phi(v + \frac{1}{2}, t) = 0$$

where v is the position of the box and z is an absolute position .

In the box frame:

$$\imath \frac{\partial \psi}{\partial t} = -\frac{1}{2}\frac{\partial^2 \psi}{\partial q^2} + \ddot{v} q \psi, \quad q \in [-\frac{1}{2}, \frac{1}{2}],$$
$$\psi(-\frac{1}{2}, t) = \psi(\frac{1}{2}, t) = 0$$

Tangent linearization around eigen-state $\bar{\psi}$ of energy $\bar{\omega}$

$$\psi(t,q) = \exp(-\imath\bar{\omega}t)(\bar{\psi}(q) + \Psi(q,t))$$

and Ψ satisfies

$$\imath\frac{\partial\Psi}{\partial t} + \bar{\omega}\Psi = -\frac{1}{2}\frac{\partial^2\Psi}{\partial q^2} + \ddot{v}q(\bar{\psi} + \Psi)$$
$$0 = \Psi(-\frac{1}{2},t) = \Psi(\frac{1}{2},t).$$

Assume Ψ and $\ddot{v}$ small and neglecte the second order term $\ddot{v}q\Psi$:

$$\imath\frac{\partial\Psi}{\partial t} + \bar{\omega}\Psi = -\frac{1}{2}\frac{\partial^2\Psi}{\partial q^2} + \ddot{v}q\bar{\psi}, \quad \Psi(-\frac{1}{2},t) = \Psi(\frac{1}{2},t) = 0.$$

Operational computations $s = d/dt$

The general solution of

$$(\imath s + \bar{\omega})\Psi = -\frac{1}{2}\Psi'' + s^2 v q \bar{\psi}$$

is

$$\Psi = A(s,q)a(s) + B(s,q)b(s) + C(s,q)v(s)$$

where

$$A(s,q) = \cos\left(q\sqrt{2\imath s + 2\bar{\omega}}\right)$$
$$B(s,q) = \frac{\sin\left(q\sqrt{2\imath s + 2\bar{\omega}}\right)}{\sqrt{2\imath s + 2\bar{\omega}}}$$
$$C(s,q) = (-\imath s q \bar{\psi}(q) + \bar{\psi}'(q)).$$

Case $q \mapsto \bar{\phi}(q)$ even

The boundary conditions imply

$$A(s, 1/2)a(s) = 0, \quad B(s, 1/2)b(s) = -\psi'(1/2)v(s).$$

$a(s)$ is a torsion element: the system is not controllable.

Nevertheless, for steady-state controllability, we have

$$\begin{aligned} b(s) &= -\bar{\psi}'(1/2)\frac{\sin\left(\frac{1}{2}\sqrt{-2\imath s + 2\bar{\omega}}\right)}{\sqrt{-2\imath s + 2\bar{\omega}}}\ y(s) \\ v(s) &= \frac{\sin\left(\frac{1}{2}\sqrt{2\imath s + 2\bar{\omega}}\right)}{\sqrt{2\imath s + 2\bar{\omega}}}\frac{\sin\left(\frac{1}{2}\sqrt{-2\imath s + 2\bar{\omega}}\right)}{\sqrt{-2\imath s + 2\bar{\omega}}} y(s) \\ \Psi(s, q) &= B(s, q)b(s) + C(s, q)v(s) \end{aligned}$$

Series and convergence

$$v(s) = \frac{\sin\left(\frac{1}{2}\sqrt{2\imath s + 2\bar{\omega}}\right)}{\sqrt{2\imath s + 2\bar{\omega}}} \frac{\sin\left(\frac{1}{2}\sqrt{-2\imath s + 2\bar{\omega}}\right)}{\sqrt{-2\imath s + 2\bar{\omega}}} y(s) = F(s)y(s)$$

where the entire function $s \mapsto F(s)$ is of order 1/2,

$$\exists K, M > 0, \forall s \in \mathbb{C}, \quad |F(s)| \leq K \exp(M|s|^{1/2}).$$

Set $F(s) = \sum_{n \geq 0} a_n s^n$ where $|a_n| \leq K^n / \Gamma(1 + 2n)$ with $K > 0$ independent of n. Then $F(s)y(s)$ corresponds in the time domain to

$$\sum_{n \geq 0} a_n y^{(n)}(t)$$

that is convergent when $t \mapsto y(t)$ is a C^∞ time function of Gevrey order $\alpha < 1$: i.e. $\exists M > 0$ such that $|y^{(n)}(t)| \leq M^n \Gamma(1 + (\alpha + 1)n)$

Steady state controllability

Steering from $\Psi = 0$, $v = 0$ at time $t = 0$, to $\Psi = 0$, $v = D$ at $t = T$ is possible with the following Gevrey function of order σ:

$$[0, T] \ni t \mapsto y(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ \bar{D} \dfrac{\exp\left(-\left(\frac{T}{t}\right)^{\frac{1}{\sigma}}\right)}{\exp\left(-\left(\frac{T}{t}\right)^{\frac{1}{\sigma}}\right) + \exp\left(-\left(\frac{T}{T-t}\right)^{\frac{1}{\sigma}}\right)} & \text{for } 0 < t < T \\ \bar{D} & \text{for } t \geq T \end{cases}$$

with $\bar{D} = \frac{2\bar{\omega}D}{\sin^2(\sqrt{\bar{\omega}/2})}$. The fact that this function is of Gevrey order σ results from its exponential decay of order σ around 0 and 1.

Practical computations via Cauchy formula

$$y^{(n)}(t) = \frac{\Gamma(n+1)}{2\imath\pi} \oint_\gamma \frac{y(t+\xi)}{\xi^{n+1}}\, d\xi$$

where γ is a closed path around zero, $\sum_{n\geq 0} a_n y^{(n)}(t)$ becomes

$$\sum_{n\geq 0} a_n \frac{\Gamma(n+1)}{2\imath\pi} \oint_\gamma \frac{y(t+\xi)}{\xi^{n+1}}\, d\xi = \frac{1}{2\imath\pi} \oint_\gamma \left(\sum_{n\geq 0} a_n \frac{\Gamma(n+1)}{\xi^{n+1}} \right) y(t+\xi)\, d\xi.$$

But

$$\sum_{n\geq 0} a_n \frac{\Gamma(n+1)}{\xi^{n+1}} = \int_{D_\delta} F(s) \exp(-s\xi) ds = B_1(F)(\xi)$$

is the Borel transform of F.

Practical computations via Cauchy formula (end)

In the time domain $F(s)y(s)$ corresponds to

$$\frac{1}{2\imath\pi}\oint_{\gamma} B_1(F)(\xi)y(t+\xi)\ d\xi$$

where γ is a closed path around zero. Such integral representation is very useful when y is defined by convolution with a real signal Y,

$$y(\zeta) = \frac{1}{\varepsilon\sqrt{2\pi}}\int_{-\infty}^{+\infty} \exp(-(\zeta-t)^2/2\varepsilon^2)Y(t)\ dt$$

where $\mathbb{R} \ni t \mapsto Y(t) \in \mathbb{R}$ is any measurable and bounded function:

$$v(t) = \int_{-\infty}^{+\infty}\left[\frac{1}{\imath\varepsilon(2\pi)^{\frac{3}{2}}}\oint_{\gamma} B_1(F)(\xi)\exp(-(\xi-\tau)^2/2\varepsilon^2)\ d\xi\right]Y(t-\tau)\ d\tau.$$

Conclusion

- 1-D wave equation: eigenvalue asymptotics $|\lambda_n| \sim n$:

$$\text{Prototype:} \qquad \prod_{n=1}^{+\infty}\left(1+\frac{s^2}{n^2}\right) = \frac{\sinh(\pi s)}{\pi s}$$

  entire function of exponential type (OK).

- 1-D Heat equation: eigenvalue asymptotics $|\lambda_n| \sim n^2$:

$$\text{Prototype:} \qquad \prod_{n=1}^{+\infty}\left(1-\frac{s}{n^2}\right) = \frac{\sinh(\pi\sqrt{s})}{\pi\sqrt{s}}$$

  entire function of order 1/2 (OK).

Conclusion (continued)

Systems described by 2-D partial differential equation on Ω with 0-D control $u(t)$ on the boundary. An Example

$$\begin{aligned} \frac{\partial H}{\partial t} &= \Delta H \text{ on } \Omega \\ H &= u(t) \text{ on } \Gamma_1 \\ \frac{\partial H}{\partial n} &= 0 \text{ on } \Gamma_2 \end{aligned}$$

where the control is not distributed on Γ_1 ($\partial\Omega = \Gamma_1 \bigcup \Gamma_2$).

Steady-state controllability: steering in finite time from one steady-state to another steady-state.

Conclusion (continued)

- 2D-heat equation: eigenvalue asymptotics $w_n \sim -n$

$$\text{Prototype:} \qquad \prod_{n=1}^{+\infty} \left(1 + \frac{s}{n}\right) \exp(-s/n) = \frac{\exp(-\gamma s)}{s\Gamma(s)}$$

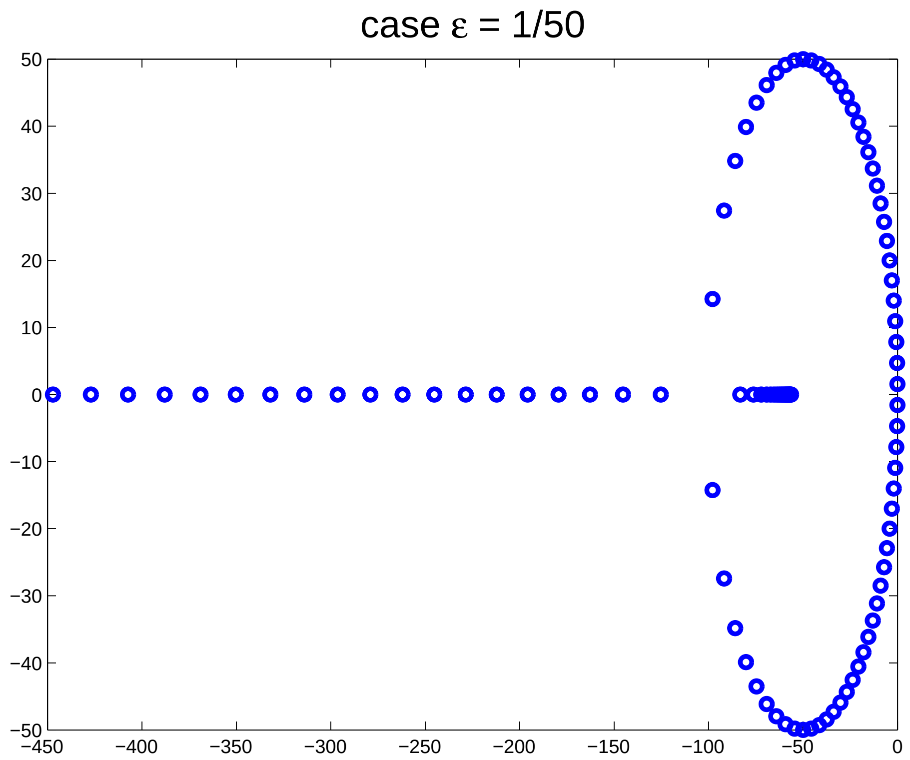
  entire function of order 1 but of infinite type (?).

- 2D-wave equation: eigenvalue asymptotics $|w_n| \sim \sqrt{n}$

$$\text{Prototype:} \qquad \prod_{n=1}^{+\infty} \left(1 - \frac{s^2}{n}\right) \exp(s^2/n) = \frac{-\exp(\gamma s^2)}{s^2\Gamma(-s^2)}$$

  entire function of order 2 but of infinite type (?).

Conclusion (end)



1-D wave equation with internal damping:

$$\frac{\partial^2 H}{\partial t^2} = \frac{\partial^2 H}{\partial x^2} + \epsilon \frac{\partial^3 H}{\partial x^2 \partial t}$$

$$H(0,t) = 0, \quad H(1,t) = u(t)$$

where the eigenvalues are the zeros of analytic function

$$P(s) = \cosh\left(\frac{s}{\sqrt{\epsilon s + 1}}\right).$$