Problem Set 2

(M2 Dynamics and control of open quantum systems 2023-2024)

This problem set is due on Monday, October 9th, 2023, at 23:59. The solutions should be emailed as a single PDF (handwritten or typeset) to alexandru.petrescu@minesparis.psl.eu by the deadline. If you collaborate with a colleague, please write their names at the top of your solution. Cite your references (books, websites, chatbots etc.). If you submit late without a satisfactory reason, the set will be accepted with a 10% penalty in the score.

I. SIMPLE HARMONIC OSCILLATOR COUPLED TO A BOSONIC BATH

Consider a harmonic oscillator coupled to a bosonic bath in thermal equilibrium at temperature T, as described by the Hamiltonian

$$\begin{aligned} H =& H_S + H_I + H_B \\ H_S =& \hbar\omega_0 a^\dagger a,\ H_B = \sum_l \hbar\omega_l b_l^\dagger b_l,\ H_I = \sum_l g_l (a + a^\dagger) \otimes (b_l + b_l^\dagger). \end{aligned} \tag{1}$$

a) Show that the Lindblad master equation for the reduced density matrix describing the simple harmonic oscillator mode annihilated by operator a can be written as

$$\begin{aligned} \dot{\rho}_S =& -i\left[\omega_0' a^\dagger a, \rho_S\right] + \frac{\gamma}{2}(\bar{n}+1)\left(2a\rho_S a^\dagger - a^\dagger a \rho_S - \rho_S a^\dagger a\right) \\ &+ \frac{\gamma}{2}\bar{n}\left(2a^\dagger \rho_S a - a a^\dagger \rho_S - \rho_S a a^\dagger\right). \end{aligned} \tag{2}$$

Do this without repeating derivations already in the lecture notes, and give expressions for γ, $\bar{n}$, and ω_0'.

b) Write down differential equations for the population of the n^{th} state of the simple harmonic oscillator, $p(n,t) \equiv \langle n|\rho_S(t)|n\rangle$. Suppose the system is in initial Fock state $|1\rangle$ at the beginning of the evolution under Eq. (2). Supposing the bath temperature is $T = 0$, what are $p(n,t)$? Let's keep $T \neq 0$ for the remainder of this problem.

c) Write down and solve the ordinary differential equation for $\langle a\rangle(t) = \mathrm{tr}_S\{\rho_S(t) a\}$.

d) Write down and solve the ordinary differential equation for $\langle n\rangle(t) = \mathrm{tr}_S\{\rho_S(t)n\}$, with $n = a^\dagger a$.

e) Show that

$$\rho_{\rm eq} = \frac{e^{-H_S/k_BT}}{\mathrm{tr}\left(e^{-H_S/k_BT}\right)} = \frac{e^{-\hbar\omega_0 a^\dagger a/k_BT}}{1-e^{-\hbar\omega_0/k_BT}} \tag{3}$$

is a steady state of Eq. (2).

SOLUTION

a) Using the Lecture 2, Eq. (26) for the collapose operator,

$$A_\alpha(\omega) \equiv \sum_{\varepsilon'-\varepsilon=\omega} \Pi(\varepsilon) A_\alpha \Pi\left(\varepsilon'\right), \tag{4}$$

and the eigenenergies and eigenprojectors are (putting $\hbar = 1$ for the rest of this solution)

$$\varepsilon = n\omega_0, \Pi(n\omega_0) = |n\rangle\langle n|, \tag{5}$$

so (there is only one decay channel so we drop the subscript α)

$$\begin{aligned} A(\omega_0) &\equiv \sum_{\varepsilon'-\varepsilon=\omega_0} \Pi(\varepsilon) A_\alpha \Pi\left(\varepsilon'\right) = \sum_{n\geq 0} |n\rangle\langle n| \left(a+a^\dagger\right)|n+1\rangle\langle n+1| \\ &= \sum_{n\geq 0} \sqrt{n+1}\,|n\rangle\langle n+1| = a \\ A(-\omega_0) &= A(\omega_0)^\dagger = a^\dagger. \end{aligned} \tag{6}$$

Furthermore, by identifying the coefficient in front of each dissipator and using Eq. (61) in the notes for Lectures 2-3,

$$\begin{aligned} \frac{1}{2}\gamma(1+\bar{n}) &\equiv 2\pi \sum_l g_l^2 \left[1+n_B\left(\omega_0\right)\right]\delta\left(\omega_0-\omega_l\right), \\ \frac{1}{2}\gamma\bar{n} &\equiv 2\pi \sum_l g_l^2 n_B\left(\omega_0\right)\delta\left(-\omega_0+\omega_l\right), \end{aligned} \tag{7}$$

or simply using Eq. (62) and (63) of the notes for Lectures 2 and 3,

$$\begin{aligned} \frac{1}{2}\gamma &= J(|\omega_0|), \\ J(\omega) &= 2\pi \sum_l g_l^2 \delta\left(\omega-\omega_l\right), \\ \bar{n} &= n_B(\omega_0) = \frac{1}{e^{\beta\hbar\omega_0}-1}. \end{aligned} \tag{8}$$

The Lamb shift is Eq. (45) of the course notes (mind the summation over frequency covers ω_0 *and* $-\omega_0$)

$$\omega_0' - \omega_0 = S(\omega_0) + S(-\omega_0), \tag{9}$$

with $S(\omega)$ as given in Eq. (60). We typically absorb this shift into a redefinition of ω_0.

b) Writing $\rho_S(t) \equiv \sum_{mn\geq 0} \rho_{mn} |m\rangle \langle n|$, and plugging into Eq. (2) we have

$$\begin{aligned}
\sum_{mn\geq 0} \dot{\rho}_{mn} |m\rangle \langle n| =& - i \left[\omega_0' a^\dagger a, \sum_{mn\geq 0} \rho_{mn} |m\rangle \langle n|\right] \\
&+ \frac{\gamma}{2}(\bar{n}+1) \sum_{mn\geq 0} \left(2a\rho_{mn} |m\rangle \langle n| a^\dagger - a^\dagger a \rho_{mn} |m\rangle \langle n| - \rho_{mn} |m\rangle \langle n| a^\dagger a\right) \\
&+ \frac{\gamma}{2}\bar{n} \sum_{mn\geq 0} \left(2a^\dagger \rho_{mn} |m\rangle \langle n| a - a a^\dagger \rho_{mn} |m\rangle \langle n| - \rho_{mn} |m\rangle \langle n| a a^\dagger\right) \\
=& - i\omega_0' \sum_{mn\geq 0} \rho_{mn} m |m\rangle \langle n| + i\omega_0' \sum_{mn\geq 0} \rho_{mn} |m\rangle \langle n| n \\
&+ \frac{\gamma}{2}(\bar{n}+1) \sum_{mn\geq 0} \left(2\sqrt{m}\rho_{mn} |m-1\rangle \langle n-1| \sqrt{n} - m\rho_{mn} |m\rangle \langle n| - \rho_{mn} |m\rangle \langle n| n\right) \\
&+ \frac{\gamma}{2}\bar{n} \sum_{mn\geq 0} \Big(2\sqrt{m+1}\rho_{mn} |m+1\rangle \langle n+1| \sqrt{n+1} \\
&\qquad - (m+1)\rho_{mn} |m\rangle \langle n| - \rho_{mn} |m\rangle \langle n| (n+1)\Big).
\end{aligned} \tag{10}$$

We can now get matrix-element equations of motion, by equating coefficients of transition operators $|m\rangle \langle n|$ on the lhs and rhs. Focusing on $|n\rangle \langle n|$ we have (coefficient of $|n\rangle \langle n|$)

$$\begin{aligned}
\dot{\rho}_{nn} =& - i\omega_0' \rho_{nn} n + i\omega_0' \rho_{nn} n \\
&+ \frac{\gamma}{2}(\bar{n}+1) \left[2(n+1)\rho_{n+1,n+1} - 2n\rho_{nn}\right] \\
&+ \frac{\gamma}{2}\bar{n} \left[2\rho_{n-1,n-1} - 2(n+1)\rho_{n,n}\right],
\end{aligned} \tag{11}$$

or, regrouping terms and calling $\rho_{nn} = p_n$, we get

$$\begin{aligned}
\dot{p}_n =& - \gamma \left[(\bar{n}+1)n + \bar{n}(n+1)\right] p_n \\
&+ \gamma(\bar{n}+1)(n+1)p_{n+1} \\
&+ \gamma\bar{n} p_{n-1}.
\end{aligned} \tag{12}$$

At zero temperature $\bar{n} = 0$, so

$$\dot{p}_n = - \gamma n p_n + \gamma(n+1)p_{n+1}. \tag{13}$$

Starting in Fock state 1 means $p_1(t=0)=1$ and $p_{n\neq 1}(t=0)=0$. The cheapest way to solve these equations is to first solve Eq. (13) for $n>1$, which gives the constant solution $p_{n>1}(t)=0$ (at zero temperature there is no way to excite the oscillator into a state higher than the initial state $|1\rangle$), then solve

$$\dot{p}_1(t) = -\gamma p_1(t) \tag{14}$$

with $p_1(t)=e^{-\gamma t}$, then finally plug into the equation for the ground state population

$$\dot{p}_0(t) = \gamma p_1(t) = \gamma e^{-\gamma t}, \tag{15}$$

which is solved by $p_0(t) = 1-e^{-\gamma t}$.

c) The equation for $\mathrm{tr}_S\,\{\rho_S(t)a\}$ derives from Eq. (2)

$$\begin{aligned}
\mathrm{tr}_S\{\dot{\rho}_S a\} =& \,\mathrm{tr}_S \Big\{ -i\left[\omega_0' a^\dagger a, \rho_S\right] a \\
&\qquad + \frac{\gamma}{2}(\bar{n}+1)\left(2a\rho_S a^\dagger - a^\dagger a \rho_S - \rho_S a^\dagger a\right) a \\
&\qquad + \frac{\gamma}{2}\bar{n}\left(2a^\dagger \rho_S a - a a^\dagger \rho_S - \rho_S a a^\dagger\right) a \Big\} \\
=& \,\mathrm{tr}_S \Big[-i\omega_0' a^\dagger a \rho_S a + i\rho_S \omega_0' a^\dagger a a \\
&\qquad + \frac{\gamma}{2}(\bar{n}+1)\left(2a\rho_S a^\dagger a - a^\dagger a \rho_S a - \rho_S a^\dagger a a\right) \\
&\qquad + \frac{\gamma}{2}\bar{n}\left(2a^\dagger \rho_S a a - a a^\dagger \rho_S a - \rho_S a a^\dagger a\right) \Big] \\
=& \,\mathrm{tr}_S \Big[-i\omega_0'(\rho_S a a^\dagger a - \rho_S a^\dagger a a = \rho_S a) \\
&\qquad + \frac{\gamma}{2}(\bar{n}+1)\left(2\rho_S a^\dagger a a - \rho_S a a^\dagger a - \rho_S a^\dagger a a\right) \\
&\qquad + \frac{\gamma}{2}\bar{n}\left(2\rho_S a a a^\dagger - \rho_S a a a^\dagger - \rho_S a a^\dagger a\right) \Big] \\
=& \,\mathrm{tr}_S \Big[-i\omega_0' \rho_S a \\
&\qquad + \frac{\gamma}{2}(\bar{n}+1)\left(\rho_S a^\dagger a a - \rho_S a a^\dagger a\right) \\
&\qquad + \frac{\gamma}{2}\bar{n}\left(\rho_S a a a^\dagger - \rho_S a a^\dagger a\right) \Big] \\
=& -i\omega_0'\,\mathrm{tr}_S\,\rho_S a - \frac{\gamma}{2}(\bar{n}+1)\,\mathrm{tr}_S\,\rho_S a + \frac{\gamma}{2}\bar{n}\,\mathrm{tr}_S\,\rho_S a \\
=& -\left(i\omega_0' + \frac{\gamma}{2}\right)\mathrm{tr}_S\,\rho_S a.
\end{aligned} \tag{16}$$

In the above we have used the cyclicity of the trace multiple times. Employing the notation introduced in the problem,

$$\dot{\langle a \rangle} = -\left(i\omega_0' + \frac{\gamma}{2}\right)\langle a \rangle, \tag{17}$$

at any temperature.

d) Proceeding as above directly from Eq. (2),

$$
\begin{aligned}
\mathrm{tr}_S\{\dot\rho_S a\} =& \,\mathrm{tr}_S \Big\{ -i\left[\omega_0' a^\dagger a, \rho_S\right] a^\dagger a \\
&\qquad + \frac{\gamma}{2}(\bar n + 1)\left(2a\rho_S a^\dagger - a^\dagger a \rho_S - \rho_S a^\dagger a\right) a^\dagger a \\
&\qquad + \frac{\gamma}{2}\bar n \left(2a^\dagger \rho_S a - a a^\dagger \rho_S - \rho_S a a^\dagger\right) a^\dagger a \Big\} \\
=& \,\mathrm{tr}_S \Big[-i\omega_0' a^\dagger a \rho_S a^\dagger a + i\rho_S \omega_0' a^\dagger a a^\dagger a \\
&\qquad + \frac{\gamma}{2}(\bar n + 1)\left(2a\rho_S a^\dagger a^\dagger a - a^\dagger a \rho_S a^\dagger a - \rho_S a^\dagger a a^\dagger a\right) \\
&\qquad + \frac{\gamma}{2}\bar n \left(2a^\dagger \rho_S a a^\dagger a - a a^\dagger \rho_S a^\dagger a - \rho_S a a^\dagger a^\dagger a\right) \Big] \\
=& \,\mathrm{tr}_S \Big[-i\omega_0' (\cancel{\rho_S a^\dagger a a^\dagger a} - \cancel{\rho_S a^\dagger a a^\dagger a}) \\
&\qquad + \frac{\gamma}{2}(\bar n + 1)\left(2\rho_S a^\dagger a^\dagger a a - \rho_S a^\dagger a a^\dagger a - \rho_S a^\dagger a a^\dagger a\right) \\
&\qquad + \frac{\gamma}{2}\bar n \left(2\rho_S a a^\dagger a a^\dagger - \rho_S a^\dagger a a a^\dagger - \rho_S a a^\dagger a^\dagger a\right) \Big] \\
=& \,\mathrm{tr}_S \Big[-\gamma(\bar n + 1)\rho_S a^\dagger a + \frac{\gamma}{2}\bar n \left(2\rho_S (a^\dagger a + 1)(a^\dagger a + 1) - \rho_S a^\dagger a (a^\dagger a + 1) - \rho_S (a^\dagger a + 1) a^\dagger a\right) \Big] \\
=& \,\mathrm{tr}_S \Big[-\gamma(\bar n + 1)\rho_S a^\dagger a + \gamma \bar n \rho_S a^\dagger a + \gamma \bar n \rho_S \Big] \\
=& -\gamma\, \mathrm{tr}_S\, \rho_S a^\dagger a + \gamma \bar n .
\end{aligned}
\tag{18}
$$

Therefore

$$
\langle \dot{a^\dagger a} \rangle = -\gamma(\langle a^\dagger a\rangle - \bar n), \tag{19}
$$

that is the number operator expectation value relaxes exponentially to the thermal population, on a time scale that is twice as fast as that of the annihilation operator.

e) To prove the statement, we show that ρ_{eq} makes the rhs of Eq. (2) vanish. The normalization constant factors out, so we need to evaluate

$$
\begin{aligned}
&-i\left[\cancel{\omega_0' a^\dagger a, e^{-\beta\hbar\omega_0 a^\dagger a}}\right] + \frac{\gamma}{2}(\bar n + 1)\left(2a e^{-\beta\hbar\omega_0 a^\dagger a} a^\dagger - a^\dagger a e^{-\beta\hbar\omega_0 a^\dagger a} - e^{-\beta\hbar\omega_0 a^\dagger a} a^\dagger a\right) \\
&+ \frac{\gamma}{2}\bar n \left(2a^\dagger e^{-\beta\hbar\omega_0 a^\dagger a} a - a a^\dagger e^{-\beta\hbar\omega_0 a^\dagger a} - e^{-\beta\hbar\omega_0 a^\dagger a} a a^\dagger\right)
\end{aligned}
\tag{20}
$$

The tricky one to do is $e^{-\beta\hbar\omega_0 a^\dagger a} a - a e^{-\beta\hbar\omega_0 a^\dagger a}$. We can easily get $[a, n] = [a, a^\dagger a] = a$, so $an = (n+1)a$. Then $an^2 = (an)n = (n+1)an = (n+1)^2 a$, and $an^p = (n+1)an^{p-1} = (n+1)^2 an^{p-2} = \ldots = (n+1)^p a$. So then for any function of the number operator that admits a Taylor series we can write $af(n) = f(n+1)a$, and by taking the Hermitian conjugate we get $f(n)a^\dagger = a^\dagger f(n+1)$. Going back to Eq. (20) and using these facts

$$\begin{aligned}
&+\frac{\gamma}{2}(\bar{n}+1)\left(2aa^\dagger e^{-\beta\hbar\omega_0(a^\dagger a+1)} - a^\dagger a e^{-\beta\hbar\omega_0 a^\dagger a} - e^{-\beta\hbar\omega_0 a^\dagger a}a^\dagger a\right)\\
&+\frac{\gamma}{2}\bar{n}\left(2a^\dagger a e^{-\beta\hbar\omega_0(a^\dagger a-1)} - aa^\dagger e^{-\beta\hbar\omega_0 a^\dagger a} - e^{-\beta\hbar\omega_0 a^\dagger a}aa^\dagger\right)\\
=&+\frac{\gamma}{2}(\bar{n}+1)\left[2(a^\dagger a+1)e^{-\beta\hbar\omega_0(a^\dagger a+1)} - a^\dagger a e^{-\beta\hbar\omega_0 a^\dagger a} - e^{-\beta\hbar\omega_0 a^\dagger a}a^\dagger a\right]\\
&+\frac{\gamma}{2}\bar{n}\left[2a^\dagger a e^{-\beta\hbar\omega_0(a^\dagger a-1)} - (a^\dagger a+1)e^{-\beta\hbar\omega_0 a^\dagger a} - e^{-\beta\hbar\omega_0 a^\dagger a}(a^\dagger a+1)\right]\\
=&+\frac{\gamma}{2}(\bar{n}+1)\left[2(a^\dagger a+1)e^{-\beta\hbar\omega_0} - a^\dagger a - a^\dagger a\right]e^{-\beta\hbar\omega_0 a^\dagger a}\\
&+\frac{\gamma}{2}\bar{n}\left[2a^\dagger a e^{\beta\hbar\omega_0} - (a^\dagger a+1) - (a^\dagger a+1)\right]e^{-\beta\hbar\omega_0 a^\dagger a}\\
\propto&+\frac{\gamma}{2}(\bar{n}+1)\left[2(a^\dagger a+1)e^{-\beta\hbar\omega_0} - a^\dagger a - a^\dagger a\right]\\
&+\frac{\gamma}{2}\bar{n}\left[2a^\dagger a e^{\beta\hbar\omega_0} - (a^\dagger a+1) - (a^\dagger a+1)\right]\\
=&+\gamma(\bar{n}+1)e^{-\beta\hbar\omega_0} - \gamma\bar{n} + \left[\gamma(\bar{n}+1)(e^{-\beta\hbar\omega_0}-1) + \gamma\bar{n}(e^{\beta\hbar\omega_0}-1)\right]a^\dagger a
\end{aligned} \tag{21}$$

Note that $\bar{n} = \frac{1}{e^{\beta\hbar\omega_0}-1}$ so that $\bar{n}+1 = \frac{e^{\beta\hbar\omega_0}}{e^{\beta\hbar\omega_0}-1}$. This makes the c-number part of the expression above vanish. For the coefficient of $a^\dagger a$,

$$\left[\gamma\frac{e^{\beta\hbar\omega_0}}{e^{\beta\hbar\omega_0}-1}(e^{-\beta\hbar\omega_0}-1) + \gamma\frac{1}{e^{\beta\hbar\omega_0}-1}(e^{\beta\hbar\omega_0}-1)\right] = 0, \tag{22}$$

and thus ρ_{eq} makes the rhs of Eq. (2) vanish. The thermal state is therefore a steady-state of the master equation.

II. SPIN-1/2 COUPLED TO BOSONIC BATHS

Consider a spin-1/2 coupled to two bosonic baths, both at temperature T,

$$\begin{aligned}
H_S =& \frac{1}{2}\hbar\omega_{01}\sigma_z,\\
H_I =& \sigma_x \otimes \sum_l g_{x,l}(b_{x,l} + b_{x,l}^\dagger) + \sigma_z \otimes \sum_l g_{z,l}(b_{z,l} + b_{z,l}^\dagger),\\
H_B =& \sum_{\alpha=x,z}\sum_l \hbar\omega_{\alpha,l} b_{\alpha,l}^\dagger b_{\alpha,l},
\end{aligned} \tag{23}$$

where the canonical commutators hold $[b_{\alpha,l}, b_{\beta,m}^\dagger] = \delta_{\alpha\beta}\delta_{lm}$.

a) Show that the Lindblad master equation for the dynamics of the reduced density matrix of the system is (express all quantities below in terms of two-point correlation functions of the baths at finite temperature):

$$\frac{d}{dt}\rho_S(t) = -i\left[\frac{1}{2}\omega'_{01}\sigma_z, \rho_S(t)\right] + \gamma_\downarrow \mathcal{D}\left[\sigma_-\right]\rho_S(t) + \gamma_\uparrow \mathcal{D}\left[\sigma_+\right]\rho_S(t) + \frac{1}{2}\gamma_\varphi \mathcal{D}\left[\sigma_z\right]\rho_S(t). \tag{24}$$

b) Find equations of motion for $\langle\sigma_{\pm,z}\rangle(t) = \mathrm{tr}_S\{\rho_S(t)\sigma_{\pm,z}\}$. Hint: one way to do this is to write down equations of motion first for the four entries of the reduced density matrix in the qubit Hilbert space, $\rho_S(t)$.

c) Show that the expectation value $\langle\sigma_z\rangle$ has an exponential decay with characteristic time T_1, i.e. $\langle\sigma_z\rangle(t) \propto e^{-t/T_1}$, and express T_1 in terms of the constants in Eq. (24). Show that $\langle\sigma_-\rangle(t)$ oscillates in time with an exponentially decaying envelope, with a characterstic timescale T_2, again to be expressed in terms of the constants in Eq. (24). Express $1/T_2$ in terms of $1/T_1$ and $1/T_\varphi \equiv \gamma_\varphi$.

d) Show that the density matrix ρ_S is pure if and only if $|\langle\vec{\sigma}\rangle| = 1$. Can you write down an equation of motion for the purity of the density matrix, $\mathrm{Tr}\{\rho^2\}$?

e) Assume that $T \to \infty$. What is the steady-state density matrix? Same question for $T \to 0$. What is $\langle\vec{\sigma}\rangle$ for each of these steady states?

f) How does the answer in part a) change if now you have $H_S = \hbar\omega_{01}\sigma_z + \hbar\omega_x\sigma_x$? What about the particular case when $\omega_{01} = 0$ and $\omega_x > 0$?

SOLUTION

a) We start again from Eq. (26) in the course notes for the collapse operator

$$A_\alpha(\omega) \equiv \sum_{\varepsilon'-\varepsilon=\omega} \Pi(\varepsilon)A_\alpha\Pi(\varepsilon'), \tag{25}$$

and we apply this to $A_x = \sigma_x$ and $A_z = \sigma_z$ corresponding to the two baths and to the system operators coupling to them. Note that the qubit spectrum, in terms of eigenvalues and eigenprojectors, reads

$$\epsilon_\pm = \pm\frac{1}{2}\omega_{01},\ \Pi_\pm = \left|\uparrow / \downarrow\right\rangle\left\langle\uparrow / \downarrow\right|. \tag{26}$$

For $A_x = \sigma_x$, we have

$$\begin{aligned} A_x(+\omega_{01}) =& \Pi_- \sigma_x \Pi_+ = \left|\downarrow\right\rangle\left\langle\downarrow\right| \left(\left|\downarrow\right\rangle\left\langle\uparrow\right| + \left|\uparrow\right\rangle\left\langle\downarrow\right|\right) \left|\uparrow\right\rangle\left\langle\uparrow\right| = \left|\downarrow\right\rangle\left\langle\uparrow\right| = \sigma_-, \\ A_x(-\omega_{01}) =& A_x(\omega_{01})^\dagger = \sigma_+. \end{aligned} \tag{27}$$

For $A_z = \sigma_z$, we have

$$\begin{aligned} A_z(0) = \Pi_- \sigma_z \Pi_- + \Pi_+ \sigma_z \Pi_+ = & \left|\downarrow\right\rangle\left\langle\downarrow\right| \left(- \left|\downarrow\right\rangle\left\langle\downarrow\right| + \left|\uparrow\right\rangle\left\langle\uparrow\right|\right) \left|\downarrow\right\rangle\left\langle\downarrow\right| \\ & + \left|\uparrow\right\rangle\left\langle\uparrow\right| \left(- \left|\downarrow\right\rangle\left\langle\downarrow\right| + \left|\uparrow\right\rangle\left\langle\uparrow\right|\right) \left|\uparrow\right\rangle\left\langle\uparrow\right| = \sigma_z. \end{aligned} \tag{28}$$

We can now identify the terms in Eq. (24). For the Lamb shift, we write the Lamb shift Hamiltonian according to Eq. (45) of the course notes, reproduced below

$$H_{LS} = \sum_\omega \sum_{\alpha,\beta} S_{\alpha\beta}(\omega) A_\alpha^\dagger(\omega) A_\beta(\omega) \tag{29}$$

Note that there are no cross-correlations between the two baths, so $S_{xz} = S_{zx} = 0$. We provide the explicit expressions of the bath correlation functions below. The Lamb-shift Hamiltonian then evaluates to

$$\begin{aligned} H_{LS} &= S_{xx}(+\omega_{01}) A_x^\dagger(+\omega_{01}) A_x(+\omega_{01}) + S_{xx}(-\omega_{01}) A_x^\dagger(-\omega_{01}) A_x(-\omega_{01}) + S_{zz}(0) A_z^\dagger(0) A_z(0) \\ &= S_{xx}(+\omega_{01}) \sigma_- \sigma_+ + S_{xx}(-\omega_{01}) \sigma_+ \sigma_- + S_{zz}(0) \sigma_z \sigma_z \\ &= S_{xx}(+\omega_{01}) \sigma_- \sigma_+ + S_{xx}(-\omega_{01}) \sigma_+ \sigma_- + S_{zz}(0) \sigma_z \sigma_z. \end{aligned} \tag{30}$$

Now recall that $\sigma^-\sigma^+ = (\sigma_x - i\sigma_y)(\sigma_x + i\sigma_y)/4 = \frac{1}{2} + i[\sigma_x, \sigma_y] = \frac{1}{2} + \frac{i}{4} 2i\sigma_z = \frac{1}{2} - \frac{1}{2}\sigma_z = \frac{1-\sigma_z}{2}$, whereas $\sigma^+\sigma^- = (\sigma_x + i\sigma_y)(\sigma_x - i\sigma_y)/4 = \frac{1}{2} - i[\sigma_x, \sigma_y] = \frac{1}{2} - \frac{i}{4} 2i\sigma_z = \frac{1}{2} + \frac{1}{2}\sigma_z = \frac{1+\sigma_z}{2}$. So

$$\begin{aligned} H_{LS} &= S_{xx}(+\omega_{01}) \frac{I_2 - \sigma_z}{2} + S_{xx}(-\omega_{01}) \frac{I_2 + \sigma_z}{2} + S_{zz}(0) I_2 \\ &= \frac{S_{xx}(-\omega_{01}) - S_{xx}(\omega_{01})}{2} \sigma_z + \text{c-numbers}, \end{aligned} \tag{31}$$

where we have neglected constant offsets to all energies, since they are not observable. Therefore

$$\omega_0' - \omega_0 = S_{xx}(-\omega_{01}) - S_{xx}(\omega_{01}). \tag{32}$$

Moreover, the decay rates in front of the dissipators in Eq. (24) are

$$\begin{aligned} \gamma_\downarrow &= \frac{1}{2} \gamma_{xx}(\omega_{01}), \\ \gamma_\uparrow &= \frac{1}{2} \gamma_{xx}(-\omega_{01}), \\ \frac{1}{2}\gamma_\varphi &= \frac{1}{2} \gamma_{zz}(0). \end{aligned} \tag{33}$$

For each $\alpha = x, z$, the rates are given by half the (real) bilateral power spectral density, as in Eq. (43)

$$\gamma_{\alpha\alpha}(\omega) = \int_{-\infty}^{+\infty} ds e^{i\omega s} \left\langle B_\alpha^\dagger(s) B_\alpha(0) \right\rangle , \tag{34}$$

where we have used the interaction picture bath operators as

$$B_\alpha(t) = \sum_l g_{\alpha,l} \left(b_{\alpha,l} e^{-i\omega_{\alpha,l}t} + b_{\alpha,l}^\dagger e^{i\omega_{\alpha,l}t} \right) , \tag{35}$$

these can be recast in terms of the spectral functions $J_\alpha(\omega)$ as done in the previous problem and detailed in the course notes. The Lamb shift is given by

$$S_{\alpha\alpha}(\omega) = \frac{1}{2i} \left(\Gamma_{\alpha\alpha}(\omega) - \Gamma_{\alpha\alpha}^*(\omega) \right) , \tag{36}$$

with

$$\Gamma_{\alpha\alpha}(\omega) = \Gamma_{\alpha\alpha}(\omega) \equiv \int_0^\infty ds e^{i\omega s} \left\langle B_\alpha^\dagger(s) B_\alpha(0) \right\rangle . \tag{37}$$

Again, we will not write the detailed form in terms of the $g_{\alpha,l}$, which was given in the problem above. These forms are interchangeable. Note also that expressions of rates and the dressed frequency in terms of the microscopic details of the bath (coupling constants $g_{\alpha,l}$, or the spectral functions $J_\alpha(\omega)$) are typically not used, since one usually takes the relevant frequency and decay rates from some experimental data. We seldom worry about the microscopic couplings $g_{\alpha,l}$, unless we specifically try to model the bath as a many-body system.

b-c) We represent the density matrix as

$$\rho_S(t) = \begin{pmatrix} \rho_{\uparrow\uparrow} & \rho_{\uparrow\downarrow} \\ \rho_{\downarrow\uparrow} & \rho_{\downarrow\downarrow} \end{pmatrix} \tag{38}$$

and, while we could use Pauli algebra we opt for using the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{39}$$

We can go ahead and evaluate terms in Eq. (24) as follows

$$
\begin{aligned}
\frac{d}{dt}\rho_S(t) &= \begin{pmatrix} \dot{\rho_{\uparrow\uparrow}} & \dot{\rho_{\uparrow\downarrow}} \\ \dot{\rho_{\downarrow\uparrow}} & \dot{\rho_{\downarrow\downarrow}} \end{pmatrix}, \\
-i\left[\frac{1}{2}\omega'_{01}\sigma_z, \rho_S(t)\right] &= -i\frac{1}{2}\omega'_{01}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \rho_{\uparrow\uparrow} & \rho_{\uparrow\downarrow} \\ \rho_{\downarrow\uparrow} & \rho_{\downarrow\downarrow} \end{pmatrix} + i\frac{1}{2}\omega'_{01}\begin{pmatrix} \rho_{\uparrow\uparrow} & \rho_{\uparrow\downarrow} \\ \rho_{\downarrow\uparrow} & \rho_{\downarrow\downarrow} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= -i\frac{1}{2}\omega'_{01}\begin{pmatrix} \rho_{\uparrow\uparrow} & \rho_{\uparrow\downarrow} \\ -\rho_{\downarrow\uparrow} & -\rho_{\downarrow\downarrow} \end{pmatrix} + i\frac{1}{2}\omega'_{01}\begin{pmatrix} \rho_{\uparrow\uparrow} & -\rho_{\uparrow\downarrow} \\ \rho_{\downarrow\uparrow} & -\rho_{\downarrow\downarrow} \end{pmatrix} \\
&= -i\frac{1}{2}\omega'_{01}\begin{pmatrix} \rho_{\uparrow\uparrow} & \rho_{\uparrow\downarrow} \\ -\rho_{\downarrow\uparrow} & -\rho_{\downarrow\downarrow} \end{pmatrix} - i\frac{1}{2}\omega'_{01}\begin{pmatrix} -\rho_{\uparrow\uparrow} & +\rho_{\uparrow\downarrow} \\ -\rho_{\downarrow\uparrow} & +\rho_{\downarrow\downarrow} \end{pmatrix} \\
&= -i\frac{1}{2}\omega'_{01}\begin{pmatrix} 0 & 2\rho_{\uparrow\downarrow} \\ -2\rho_{\downarrow\uparrow} & 0 \end{pmatrix} = -i\omega'_{01}\begin{pmatrix} 0 & \rho_{\uparrow\downarrow} \\ -\rho_{\downarrow\uparrow} & 0 \end{pmatrix} \\
\gamma_\downarrow\mathcal{D}\left[\sigma_-\right]\rho_S(t) &= \gamma_\downarrow\left[\sigma_-\rho_S(t)\sigma_+ - \frac{1}{2}\left\{\sigma_+\sigma_-, \rho_S(t)\right\}\right] \\
&= \gamma_\downarrow\left[\sigma_-\rho_S(t)\sigma_+ - \frac{1}{2}\left\{\frac{1+\sigma_z}{2}, \rho_S(t)\right\}\right] \\
&= \gamma_\downarrow\Big[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \rho_{\uparrow\uparrow} & \rho_{\uparrow\downarrow} \\ \rho_{\downarrow\uparrow} & \rho_{\downarrow\downarrow} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
&\quad -\frac{1}{2}\left\{\frac{1+\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}{2}\begin{pmatrix} \rho_{\uparrow\uparrow} & \rho_{\uparrow\downarrow} \\ \rho_{\downarrow\uparrow} & \rho_{\downarrow\downarrow} \end{pmatrix} + \begin{pmatrix} \rho_{\uparrow\uparrow} & \rho_{\uparrow\downarrow} \\ \rho_{\downarrow\uparrow} & \rho_{\downarrow\downarrow} \end{pmatrix}\frac{1+\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}{2}\right\}\Big] \\
&= \gamma_\downarrow\Big[\begin{pmatrix} 0 & 0 \\ \rho_{\uparrow\uparrow} & \rho_{\uparrow\downarrow} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \rho_{\uparrow\uparrow} & \rho_{\uparrow\downarrow} \\ \rho_{\downarrow\uparrow} & \rho_{\downarrow\downarrow} \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \rho_{\uparrow\uparrow} & \rho_{\uparrow\downarrow} \\ \rho_{\downarrow\uparrow} & \rho_{\downarrow\downarrow} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\Big] \\
&= \gamma_\downarrow\Big[\begin{pmatrix} 0 & 0 \\ 0 & \rho_{\uparrow\uparrow} \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \rho_{\uparrow\uparrow} & \rho_{\uparrow\downarrow} \\ 0 & 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \rho_{\uparrow\uparrow} & 0 \\ \rho_{\downarrow\uparrow} & 0 \end{pmatrix}\Big] \\
&= \gamma_\downarrow\begin{pmatrix} -\rho_{\uparrow\uparrow} & -\rho_{\uparrow\downarrow}/2 \\ -\rho_{\downarrow\uparrow}/2 & \rho_{\uparrow\uparrow} \end{pmatrix}
\end{aligned}
\tag{40}
$$

$$
\begin{aligned}
\gamma_\uparrow \mathcal{D}\left[\sigma_+\right]\rho_S(t) &= \gamma_\uparrow \left[\sigma_+\rho_S(t)\sigma_- - \frac{1}{2}\left\{\sigma_-\sigma_+, \rho_S(t)\right\}\right] \\
&= \gamma_\uparrow \left[\sigma_+\rho_S(t)\sigma_- - \frac{1}{2}\left\{\frac{1-\sigma_z}{2}, \rho_S(t)\right\}\right] \\
&= \gamma_\uparrow \Big[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \rho_{\uparrow\uparrow} & \rho_{\uparrow\downarrow} \\ \rho_{\downarrow\uparrow} & \rho_{\downarrow\downarrow} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\
&- \frac{1}{2}\left\{ \frac{1-\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}{2} \begin{pmatrix} \rho_{\uparrow\uparrow} & \rho_{\uparrow\downarrow} \\ \rho_{\downarrow\uparrow} & \rho_{\downarrow\downarrow} \end{pmatrix} + \begin{pmatrix} \rho_{\uparrow\uparrow} & \rho_{\uparrow\downarrow} \\ \rho_{\downarrow\uparrow} & \rho_{\downarrow\downarrow} \end{pmatrix} \frac{1-\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}{2} \right\} \Big] \\
&= \gamma_\uparrow \Big[\begin{pmatrix} \rho_{\downarrow\uparrow} & \rho_{\downarrow\downarrow} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \rho_{\uparrow\uparrow} & \rho_{\uparrow\downarrow} \\ \rho_{\downarrow\uparrow} & \rho_{\downarrow\downarrow} \end{pmatrix} - \frac{1}{2} \begin{pmatrix} \rho_{\uparrow\uparrow} & \rho_{\uparrow\downarrow} \\ \rho_{\downarrow\uparrow} & \rho_{\downarrow\downarrow} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \Big] \\
&= \gamma_\uparrow \Big[\begin{pmatrix} \rho_{\downarrow\downarrow} & 0 \\ 0 & 0 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 0 & 0 \\ \rho_{\downarrow\uparrow} & \rho_{\downarrow\downarrow} \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 0 & \rho_{\uparrow\downarrow} \\ 0 & \rho_{\downarrow\downarrow} \end{pmatrix} \Big] \\
&= \gamma_\uparrow \begin{pmatrix} \rho_{\downarrow\downarrow} & -\rho_{\uparrow\downarrow}/2 \\ -\rho_{\downarrow\uparrow}/2 & -\rho_{\downarrow\downarrow} \end{pmatrix}
\end{aligned}
\tag{41}
$$

$$
\begin{aligned}
\frac{1}{2}\gamma_\varphi \mathcal{D}\left[\sigma_z\right]\rho_S(t) &= \frac{1}{2}\gamma_\varphi \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \rho_{\uparrow\uparrow} & \rho_{\uparrow\downarrow} \\ \rho_{\downarrow\uparrow} & \rho_{\downarrow\downarrow} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} \rho_{\uparrow\uparrow} & \rho_{\uparrow\downarrow} \\ \rho_{\downarrow\uparrow} & \rho_{\downarrow\downarrow} \end{pmatrix} \right] \\
&= \frac{1}{2}\gamma_\varphi \left[\begin{pmatrix} \rho_{\uparrow\uparrow} & \rho_{\uparrow\downarrow} \\ -\rho_{\downarrow\uparrow} & -\rho_{\downarrow\downarrow} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} \rho_{\uparrow\uparrow} & \rho_{\uparrow\downarrow} \\ \rho_{\downarrow\uparrow} & \rho_{\downarrow\downarrow} \end{pmatrix} \right] \\
&= \frac{1}{2}\gamma_\varphi \left[\begin{pmatrix} \rho_{\uparrow\uparrow} & -\rho_{\uparrow\downarrow} \\ -\rho_{\downarrow\uparrow} & \rho_{\downarrow\downarrow} \end{pmatrix} - \begin{pmatrix} \rho_{\uparrow\uparrow} & \rho_{\uparrow\downarrow} \\ \rho_{\downarrow\uparrow} & \rho_{\downarrow\downarrow} \end{pmatrix} \right] \\
&= \gamma_\varphi \begin{pmatrix} 0 & -\rho_{\uparrow\downarrow} \\ -\rho_{\downarrow\uparrow} & 0 \end{pmatrix}.
\end{aligned}
\tag{42}
$$

Reassembling Eq. (24) in matrix form we get

$$\begin{pmatrix} \dot{\rho_{\uparrow\uparrow}} & \dot{\rho_{\uparrow\downarrow}} \\ \dot{\rho_{\downarrow\uparrow}} & \dot{\rho_{\downarrow\downarrow}} \end{pmatrix} = -i\omega'_{01}\begin{pmatrix} 0 & \rho_{\uparrow\downarrow} \\ -\rho_{\downarrow\uparrow} & 0 \end{pmatrix} + \gamma_\downarrow \begin{pmatrix} -\rho_{\uparrow\uparrow} & -\rho_{\downarrow\uparrow}/2 \\ -\rho_{\uparrow\downarrow}/2 & \rho_{\uparrow\uparrow} \end{pmatrix}$$
$$+\gamma_\uparrow \begin{pmatrix} \rho_{\downarrow\downarrow} & -\rho_{\uparrow\downarrow}/2 \\ -\rho_{\downarrow\uparrow}/2 & -\rho_{\downarrow\downarrow} \end{pmatrix} + \gamma_\varphi \begin{pmatrix} 0 & -\rho_{\uparrow\downarrow} \\ -\rho_{\downarrow\uparrow} & 0 \end{pmatrix} \tag{43}$$

Finally, we express the expectation values of Pauli matrices in terms of the entries of the density matrix

$$\langle\sigma_z\rangle = \mathrm{tr}\left\{\begin{pmatrix} \rho_{\uparrow\uparrow} & \rho_{\uparrow\downarrow} \\ \rho_{\downarrow\uparrow} & \rho_{\downarrow\downarrow} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\} = \mathrm{tr}\left\{\begin{pmatrix} \rho_{\uparrow\uparrow} & -\rho_{\uparrow\downarrow} \\ \rho_{\downarrow\uparrow} & -\rho_{\downarrow\downarrow} \end{pmatrix}\right\} = \rho_{\uparrow\uparrow} - \rho_{\downarrow\downarrow},$$
$$\langle\sigma_+\rangle = \mathrm{tr}\left\{\begin{pmatrix} \rho_{\uparrow\uparrow} & \rho_{\uparrow\downarrow} \\ \rho_{\downarrow\uparrow} & \rho_{\downarrow\downarrow} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right\} = \mathrm{tr}\left\{\begin{pmatrix} 0 & \rho_{\uparrow\uparrow} \\ 0 & \rho_{\downarrow\uparrow} \end{pmatrix}\right\} = \rho_{\downarrow\uparrow}, \tag{44}$$
$$\langle\sigma_-\rangle = \mathrm{tr}\left\{\begin{pmatrix} \rho_{\uparrow\uparrow} & \rho_{\uparrow\downarrow} \\ \rho_{\downarrow\uparrow} & \rho_{\downarrow\downarrow} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right\} = \mathrm{tr}\left\{\begin{pmatrix} \rho_{\uparrow\downarrow} & 0 \\ \rho_{\downarrow\downarrow} & 0 \end{pmatrix}\right\} = \rho_{\uparrow\downarrow}.$$

Using Eq. (43) we can then obtain the equations of motion for the expectation values of the Pauli matrices. We begin with σ_z. Note that since $\langle\sigma_z\rangle = \rho_{\uparrow\uparrow} - \rho_{\downarrow\downarrow} = 2\rho_{\uparrow\uparrow} - 1 = 1 - 2\rho_{\downarrow\downarrow}$, we have $\rho_{\uparrow\uparrow} = \frac{\langle\sigma_z\rangle+1}{2}$ and $\rho_{\downarrow\downarrow} = \frac{-\langle\sigma_z\rangle+1}{2}$, so that

$$\begin{aligned}\dot{\langle\sigma_z\rangle} &= \dot{\rho_{\uparrow\uparrow}} - \dot{\rho_{\downarrow\downarrow}} = -\gamma_\downarrow\rho_{\uparrow\uparrow} + \gamma_\uparrow\rho_{\downarrow\downarrow} - \gamma_\downarrow\rho_{\uparrow\uparrow} + \gamma_\uparrow\rho_{\downarrow\downarrow} = -2\gamma_\downarrow\rho_{\uparrow\uparrow} + 2\gamma_\uparrow\rho_{\downarrow\downarrow} \\ &= -2\gamma_\downarrow\frac{\langle\sigma_z\rangle+1}{2} + 2\gamma_\uparrow\frac{-\langle\sigma_z\rangle+1}{2} = -(\gamma_\downarrow+\gamma_\uparrow)\langle\sigma_z\rangle - \gamma_\downarrow + \gamma_\uparrow.\end{aligned} \tag{45}$$

Defining

$$\gamma_1 = \gamma_\downarrow + \gamma_\uparrow \equiv \frac{1}{T_1}, \quad \langle\sigma_z\rangle_{ss} = \frac{-\gamma_\downarrow+\gamma_\uparrow}{\gamma_\downarrow+\gamma_\uparrow}, \tag{46}$$

we have the solution

$$\langle\sigma_z\rangle(t) = e^{-t/T_1}(\langle\sigma_z\rangle(0) - \langle\sigma_z\rangle_{ss}) + \langle\sigma_z\rangle_{ss}, \tag{47}$$

i.e. the expectation value of σ_z starts in some initial value $\langle\sigma_z\rangle(0)$ specified by the initial condition for the reduced density matrix $\rho_S(t=0)$, exponentially decays on a timescale T_1 to the steady-state value $\langle\sigma_z\rangle_{ss}$.

Next, for the coherences, we have

$$\begin{aligned}\dot{\langle\sigma_+\rangle} &= i\omega'_{01}\langle\sigma_+\rangle - \frac{1}{2}(\gamma_\downarrow+\gamma_\uparrow)\langle\sigma_+\rangle - \gamma_\varphi\langle\sigma_+\rangle, \\ \dot{\langle\sigma_-\rangle} &= -i\omega'_{01}\langle\sigma_-\rangle - \frac{1}{2}(\gamma_\downarrow+\gamma_\uparrow)\langle\sigma_-\rangle - \gamma_\varphi\langle\sigma_-\rangle.\end{aligned} \tag{48}$$

Let us now define

$$
\begin{aligned}
T_\varphi &= \gamma_\varphi^{-1}, \\
T_1 &= \gamma_1^{-1} = \frac{1}{\gamma_\uparrow + \gamma_\downarrow}, \\
\gamma_2 &\equiv \frac{\gamma_1}{2} + \gamma_\varphi, \\
\gamma_2 \equiv \frac{1}{T_2} &= \frac{1}{2T_1} + \frac{1}{T_\varphi}.
\end{aligned}
\tag{49}
$$

T_1 is the energy relaxation time, T_2 is the decoherence time, and T_φ is the dephasing time. In terms of these, Eq. (45) and Eq. (48) become

$$
\begin{aligned}
\dot{\langle\sigma_z\rangle} &= -\frac{1}{T_1}\left(\langle\sigma_z\rangle - \langle\sigma_z\rangle_{ss}\right), \\
\dot{\langle\sigma_+\rangle} &= i\omega_{01}'\langle\sigma_+\rangle - \frac{1}{T_2}\langle\sigma_+\rangle, \\
\dot{\langle\sigma_-\rangle} &= -i\omega_{01}'\langle\sigma_-\rangle - \frac{1}{T_2}\langle\sigma_-\rangle,
\end{aligned}
\tag{50}
$$

which is the more standard expression of the Bloch equations.

d) The expectation value of $\vec{\sigma}$ is

$$
\langle\vec{\sigma}\rangle = (\rho_{\downarrow\uparrow} + \rho_{\uparrow\downarrow}, -i(\rho_{\downarrow\uparrow} - \rho_{\uparrow\downarrow}), \rho_{\uparrow\uparrow} - \rho_{\downarrow\downarrow}). \tag{51}
$$

Note that all entries above are real, by the hermiticity of ρ_S. Then

$$
\frac{1 + |\langle\vec{\sigma}\rangle|^2}{2} = \frac{1 + (\rho_{\downarrow\uparrow} + \rho_{\uparrow\downarrow})^2 + (-i\rho_{\downarrow\uparrow} + i\rho_{\uparrow\downarrow})^2 + (\rho_{\uparrow\uparrow} - \rho_{\downarrow\downarrow})^2}{2}. \tag{52}
$$

We may expand the trace-1 density matrix over the basis formed by the identity and the Pauli matrices $\rho_S = \frac{1}{2}I + \vec{\rho}\cdot\vec{\sigma}$. Then (using $\mathrm{tr}\left[\sigma_\alpha\sigma_\beta\right] = 2\delta_{\alpha\beta}$, and thus reading off the components of ρ_S over the Pauli matrices)

$$
\begin{aligned}
\mathrm{tr}\rho_S^2 &= \frac{1}{2} + \mathrm{tr}\{\vec{\rho}\cdot\vec{\sigma}\vec{\rho}\cdot\vec{\sigma}\} = \frac{1}{2} + \sum_{\alpha,\beta=1}^{3}\mathrm{tr}\{\rho_\alpha\rho_\beta\sigma_\alpha\sigma_\beta\} = \frac{1}{2} + \sum_{\alpha,\beta=1}^{3}\rho_\alpha\rho_\beta 2\delta_{\alpha\beta} \\
&= \frac{1}{2} + 2\vec{\rho}^2 = \frac{1}{2} + 2\left[\left(\frac{\rho_{\uparrow\uparrow} - \rho_{\downarrow\downarrow}}{2}\right)^2 + \left(\frac{\rho_{\uparrow\downarrow} + \rho_{\downarrow\uparrow}}{2}\right)^2 + \left(\frac{\rho_{\uparrow\downarrow} - \rho_{\downarrow\uparrow}}{2i}\right)^2\right] \\
&= \frac{1}{2} + \frac{1}{2}\left[(\rho_{\uparrow\uparrow} - \rho_{\downarrow\downarrow})^2 + (\rho_{\uparrow\downarrow} + \rho_{\downarrow\uparrow})^2 + (-i\rho_{\uparrow\downarrow} + i\rho_{\downarrow\uparrow})^2\right] = \frac{1 + |\langle\vec{\sigma}\rangle|^2}{2}
\end{aligned}
\tag{53}
$$

Finally, ρ_S is pure iff $\mathrm{tr}(\rho_S^2) = 1$. It is readily seen that this is true iff $|\langle\vec{\sigma}\rangle| = 1$. That is, the spin-$\frac{1}{2}$ is in a pure state iff the expectation value of the spin resides on the Bloch sphere (the unit sphere).

e) For $T \to \infty$, $\gamma_\uparrow = \gamma_\downarrow = \gamma_1/2$ and in the steady state the Bloch equations give $\langle\sigma_z\rangle_{\rm ss} = \langle\sigma_\pm\rangle_{\rm ss} = 0$, and the steady state is the fully mixed state $\rho_{ss} = I/2$.

For $T \to 0$, $\gamma_\uparrow = 0$ and $\gamma_\downarrow = \gamma_1$, and $\langle\sigma_z\rangle_{\rm ss} = -1$, whereas $\langle\sigma_\pm\rangle_{\rm ss} = 0$, while $\rho_{ss} = {\rm diag}(0,1)$. The system is in a pure state.

f) According to standard conventions, we are missing a half in the expression of the Hamiltonian. Putting this back in, and renaming $\omega_{01} = \omega_z$, we have

$$H_S/\hbar = \frac{1}{2}(\omega_z\sigma_z + \omega_x\sigma_x). \tag{54}$$

The easiest way to solve this is to recall that the eigensystem of this is

$$E_\pm = \pm\frac{1}{2}\sqrt{\omega_x^2+\omega_z^2},\ \Pi_\pm = \frac{1}{2}(1 \pm \hat{\omega}\cdot\vec{\sigma}), \tag{55}$$

where $\hat{\omega} = \frac{(\omega_x,0,\omega_z)}{\sqrt{\omega_x^2+\omega_z^2}}$.

We use the same Eq. (26) of the lecture notes to find collapse operators. We need to evaluate 4 operators for each of $A_{x,z}$. Using Einstein summation, letting i be either x or z,

$$\begin{aligned}
4\Pi_\pm\sigma_i\Pi_\pm &= (1\pm\hat{\omega}\cdot\vec{\sigma})\sigma_i(1\pm\hat{\omega}\cdot\vec{\sigma}) = \sigma_i \pm \hat{\omega}_\alpha\{\sigma_\alpha,\sigma_i\} + \hat{\omega}_\alpha\hat{\omega}_\beta\sigma_\alpha\sigma_i\sigma_\beta \\
&= \sigma_i \pm 2\hat{\omega}_i I + \hat{\omega}_\alpha\hat{\omega}_\beta\sigma_\alpha(\delta_{i\beta} + i\epsilon_{i\beta\gamma}\sigma_\gamma) \\
&= \sigma_i \pm 2\hat{\omega}_i I + \hat{\omega}_i\hat{\omega}_\alpha\sigma_\alpha + \hat{\omega}_\alpha\hat{\omega}_\beta i\epsilon_{i\beta\gamma}(\delta_{\alpha\gamma} + i\epsilon_{\alpha\gamma\delta}\sigma_\delta) \\
&= \sigma_i \pm 2\hat{\omega}_i I + \hat{\omega}_i\hat{\omega}_\alpha\sigma_\alpha + \hat{\omega}_\alpha\hat{\omega}_\beta i\epsilon_{i\beta\gamma}(\delta_{\alpha\gamma} + i\epsilon_{\alpha\gamma\delta}\sigma_\delta) \\
&= \sigma_i \pm 2\hat{\omega}_i I + \hat{\omega}_i\hat{\omega}_\alpha\sigma_\alpha + \hat{\omega}_\alpha\hat{\omega}_\beta\epsilon_{i\beta\gamma}\epsilon_{\alpha\delta\gamma}\sigma_\delta \\
&= \sigma_i \pm 2\hat{\omega}_i I + \hat{\omega}_i\hat{\omega}_\alpha\sigma_\alpha + \hat{\omega}_\alpha\hat{\omega}_\beta(\delta_{i\alpha}\delta_{\beta\delta} - \delta_{i\delta}\delta_{\beta\alpha})\sigma_\delta \\
&= \sigma_i \pm 2\hat{\omega}_i I + \hat{\omega}_i\hat{\omega}_\alpha\sigma_\alpha + \hat{\omega}_i\hat{\omega}_\delta\sigma_\delta - \hat{\omega}_\alpha\hat{\omega}_\beta\delta_{i\delta}\delta_{\beta\alpha}\sigma_\delta \\
&= 2\hat{\omega}_i(\hat{\omega}\cdot\vec{\sigma} \pm I) \\
4\Pi_-\sigma_i\Pi_+ &= (1-\hat{\omega}\cdot\vec{\sigma})\sigma_i(1+\hat{\omega}\cdot\vec{\sigma}) = \sigma_i - \hat{\omega}_\alpha[\sigma_\alpha,\sigma_i] - \hat{\omega}_\alpha\hat{\omega}_\beta\sigma_\alpha\sigma_i\sigma_\beta \\
&= \sigma_i - 2i\epsilon_{\alpha il}\hat{\omega}_\alpha\sigma_l - \hat{\omega}_i\hat{\omega}_\alpha\sigma_\alpha - \hat{\omega}_i\hat{\omega}_\delta\sigma_\delta + \hat{\omega}_\alpha\hat{\omega}_\beta\delta_{i\delta}\delta_{\beta\alpha}\sigma_\delta \\
&= 2\sigma_i - 2i\epsilon_{\alpha il}\hat{\omega}_\alpha\sigma_l + \hat{\omega}_i\hat{\omega}_\alpha\sigma_\alpha + \hat{\omega}_i\hat{\omega}_\delta\sigma_\delta \\
&= 2\sigma_i - 2\hat{\omega}_i\hat{\omega}\cdot\vec{\sigma} - 2i\epsilon_{\alpha il}\hat{\omega}_\alpha\sigma_l \\
4\Pi_+\sigma_i\Pi_- &= 2\sigma_i - 2\hat{\omega}_i\hat{\omega}\cdot\vec{\sigma} + 2i\epsilon_{\alpha il}\hat{\omega}_\alpha\sigma_l
\end{aligned} \tag{56}$$

Then the collapse operators should be

$$A_i(0) = \hat{\omega}_i\hat{\omega}\cdot\vec{\sigma}. \tag{57}$$

If $\omega_z \neq 0$ and $\omega_x = 0$, then

$$A_x(0) = \hat{\omega}_x \hat{\omega} \cdot \vec{\sigma} = 0, A_z(0) = \hat{\omega}_z \hat{\omega} \cdot \vec{\sigma} = \sigma_z. \tag{58}$$

If $\omega_x \neq 0$ and $\omega_z = 0$, then

$$A_x(0) = \hat{\omega}_x \hat{\omega} \cdot \vec{\sigma} = \sigma_x, A_z(0) = 0. \tag{59}$$

Thus, if the Zeeman field $\vec{\omega}$ is pointing in the x direction, then dephasing is induced by the bath coupling to σ_x (see discussion of Bloch equation in the previous sections. If the Zeeman field is pointing in the z direction, then dephasing comes from the bath coupling to σ_z.

Moreover, transitions in the qubit are induced by the collapse operators

$$A_i \left(\mp \sqrt{\omega_x^2 + \omega_z^2} \right) = \Pi_\pm \sigma_i \Pi_\mp, \tag{60}$$

which give explicitly

$$\begin{aligned} 4A_x \left(\mp \sqrt{\omega_x^2 + \omega_z^2} \right) &= 2\sigma_x - 2\hat{\omega}_x \hat{\omega} \cdot \vec{\sigma} \pm 2i\epsilon_{\alpha x l} \hat{\omega}_\alpha \sigma_l, \\ 4A_z \left(\mp \sqrt{\omega_x^2 + \omega_z^2} \right) &= 2\sigma_z - 2\hat{\omega}_z \hat{\omega} \cdot \vec{\sigma} \pm 2i\epsilon_{\alpha z l} \hat{\omega}_\alpha \sigma_l. \end{aligned} \tag{61}$$

If $\omega_z \neq 0$ and $\omega_x = 0$, then

$$\begin{aligned} A_x \left(\mp \omega_z \right) &= \frac{1}{2}\sigma_x \pm \frac{1}{2} i \sigma_y = \sigma_\pm, \\ A_z \left(\mp \omega_z \right) &= \frac{1}{2}\sigma_z - \frac{1}{2}\sigma_z \pm \frac{1}{2} i \epsilon_{zzl} \sigma_l = 0. \end{aligned} \tag{62}$$

If $\omega_x \neq 0$ and $\omega_z = 0$, then

$$\begin{aligned} A_x \left(\mp \omega_x \right) &= \frac{1}{2}\sigma_x - \frac{1}{2}\sigma_x \pm \frac{1}{2} i \epsilon_{xxl} \hat{\omega}_x \sigma_l = 0, \\ A_z \left(\mp \omega_x \right) &= \frac{1}{2}\sigma_z \mp \frac{1}{2} i \sigma_y. \end{aligned} \tag{63}$$

That is, if the Zeeman field is pointing in the z direction, then transitions are induced by the bath coupled to σ_x. Else, if the Zeeman field is pointing in the x direction, then the bath coupling to σ_x cannot induce transitions in the qubit, but the bath coupling to σ_z does. This is because σ_z is the bit flip operator if the bit is defined on the eigenstates of σ_x.