Dynamics and Control of Open Quantum Systems
Lectures 6 and 7 (November 2023):

Quantum measurement and discrete-time open systems

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1 Example of the photon-box

This section is devoted to the case study of a photon box consisting of a cavity quantum electrodynamics setup developed within Laboratoire Kastler-Brossel (LKB) at École Normale Supérieure.

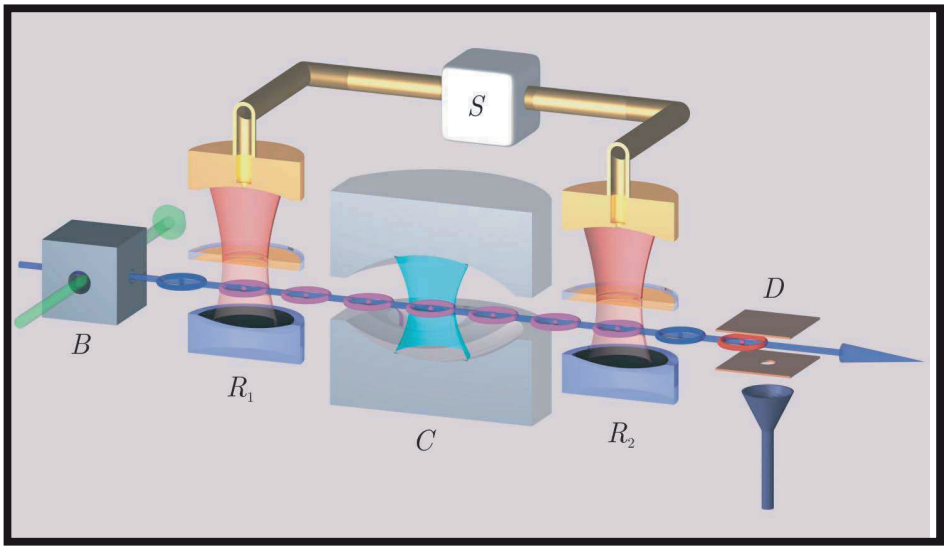

Figure 1: The LKB photon box; atoms get out box B one by one, undergo then a first Rabi pulse in Ramsey zone R_1, become entangled with electromagnetic field trapped in C, undergo a second Rabi pulse in Ramsey zone R_2 and finally are measured in the detector D.

1.1 Markov chain model

Here S corresponds to a quantized trapped mode inside the cavity. It is described by a wave function $|\psi\rangle$ in the Hilbert space $\mathcal{H}_S$ (see section **??**)

$$\mathcal{H}_S = \left\{ \sum_{n=0}^{\infty} \psi_n |n\rangle \ \mid (\psi_n)_{n=0}^{\infty} \in l^2(\mathbb{C}) \right\},$$

where $|n\rangle$ represents the Fock state associated to exactly n photons inside the cavity and $l^2(\mathbb{C})$ is the space of square summable sequences in $\mathbb{C}$ ($\sum_{n=0}^{\infty} |\psi_n|^2 = 1$). The meter M is

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associated to atoms : $\mathcal{H}_M = \mathbb{C}^2$, each atom admits two energy levels and is described by a wave function $c_g |g\rangle + c_e |e\rangle$ with $|c_g|^2 + |c_e|^2 = 1$.

Let us follow an atom leaving B where it is prepared in state $|g\rangle$. It is symbolized by a small horizontal and blue torus in Figure 1. When atom comes out B, the state of the composite system atom/field is separable and is denoted by $|\Psi\rangle_B \in \mathcal{H}_M \otimes \mathcal{H}_S$

$$|\Psi\rangle_B = |g\rangle \otimes |\psi\rangle . \tag{1}$$

When atom comes out the first Ramsey zone R_1 (pink torus between R_1 and C), the state remains separable but has changed to

$$|\Psi\rangle_{R_1} = (\boldsymbol{U}_{R_1} \otimes \boldsymbol{I}) |\Psi\rangle_B = (\boldsymbol{U}_{R_1} |g\rangle) \otimes |\psi\rangle \tag{2}$$

where the unitary transformation performed in R_1 only affects the atom:

$$\boldsymbol{U}_{R_1} = e^{-i\frac{\theta_1}{2}(x_1\boldsymbol{\sigma_x}+y_1\boldsymbol{\sigma_y}+z_1\boldsymbol{\sigma_z})} = \cos(\tfrac{\theta_1}{2}) - i\sin(\tfrac{\theta_1}{2})(x_1\boldsymbol{\sigma_x} + y_1\boldsymbol{\sigma_y} + z_1\boldsymbol{\sigma_z}) \tag{3}$$

corresponds, in the Bloch sphere representation, to a rotation of angle θ_1 around the oriented axis defined by the unit-length vector $x_1\vec{\imath} + y_1\vec{\jmath} + z_1\vec{k}$ ($x_1^2 + y_1^2 + z_1^2 = 1$), see section **??**.

When atom leaves the cavity C, the state is not anymore separable: atom and field become entangled and the state is described by

$$|\Psi\rangle_C = \boldsymbol{U}_C |\Psi\rangle_{R_1} \tag{4}$$

where the unitary transformation $\boldsymbol{U}_C$ on $\mathcal{H}_M \otimes \mathcal{H}_S$ is associated to a Jaynes-Cummings Hamiltonian for describing the atom/field interaction:

$$\boldsymbol{H}_C = \tfrac{\Delta}{2}\boldsymbol{\sigma_z} + i\tfrac{\Omega}{2}(\boldsymbol{\sigma_-}\boldsymbol{a}^\dagger - \boldsymbol{\sigma_+}\boldsymbol{a}) \tag{5}$$

is the Jaynes-Cumming Hamiltonian after the rotating wave approximation ($\Delta = \omega_{\text{eg}} - \omega_c$ de-tuning between atom and cavity field, Ω the vacuum Rabi pulsation, see section **??** and (**??**) with $v = 0$, $\omega_r = \omega_c$ and $\Delta_c = 0$ and $\Delta_{eg} = \Delta$). The precise form of $\boldsymbol{U}_C$ is given in next subsection for resonant and dispersive cases.

When the atom leaves the second Ramsey zone R_2, the state becomes

$$|\Psi\rangle_{R_2} = (\boldsymbol{U}_{R_2} \otimes \boldsymbol{I}) |\Psi\rangle_C$$

where $\boldsymbol{U}_{R_2}$ is similar to $\boldsymbol{U}_{R_1}$ but with different parameters θ_2, x_2, y_2, z_2,

$$U_{R_2} = e^{-i\frac{\theta_2}{2}(x_2\boldsymbol{\sigma_x}+y_2\boldsymbol{\sigma_y}+z_2\boldsymbol{\sigma_z})} = \cos(\tfrac{\theta_2}{2}) - i\sin(\tfrac{\theta_2}{2})(x_2\boldsymbol{\sigma_x} + y_2\boldsymbol{\sigma_y} + z_2\boldsymbol{\sigma_z}). \tag{6}$$

This means that, just before the measurement in D, the state is given by

$$|\Psi\rangle_{R_2} = \boldsymbol{U} |g\rangle \otimes |\psi\rangle = |g\rangle \otimes \boldsymbol{M}_g |\psi\rangle + |e\rangle \otimes \boldsymbol{M}_e |\psi\rangle \tag{7}$$

where $\boldsymbol{U} = \boldsymbol{U}_{R_2}\boldsymbol{U}_C\boldsymbol{U}_{R_1}$ is the total unitary transformation defining the linear measurement operators $\boldsymbol{M}_g$ and $\boldsymbol{M}_e$ on $\mathcal{H}_S$.

Denote by $s \in \{g, e\}$ the measurement outcome in detector D: with probability $p_s = \langle\psi|\boldsymbol{M}_s^\dagger\boldsymbol{M}_s|\psi\rangle$ we get s. Just after the measurement outcome s, the state becomes separable. It has partially collapsed to

$$|\Psi\rangle_D = \tfrac{1}{\sqrt{p_s}} \ |s\rangle \otimes (\boldsymbol{M}_s |\psi\rangle) = \frac{|s\rangle \otimes (\boldsymbol{M}_s |\psi\rangle)}{\sqrt{\langle\psi|\boldsymbol{M}_s^\dagger\boldsymbol{M}_s|\psi\rangle}}.$$

We have a Markov process: after the complete passage of an atom, the cavity state initially equal to $|\psi\rangle$ undergoes an irreversible and stochastic jump to $|\psi\rangle_+$ driven by $\boldsymbol{M}_g$ and $\boldsymbol{M}_e$ defined via unitary operator $\boldsymbol{U} = \boldsymbol{U}_{R_2}\boldsymbol{U}_C\boldsymbol{U}_{R_1}$ and (7):

$$|\psi\rangle_+ = \begin{cases} \dfrac{\boldsymbol{M}_g|\psi\rangle}{\sqrt{\langle\psi|\boldsymbol{M}_g^\dagger\boldsymbol{M}_g|\psi\rangle}}, & \text{with probability } p_g = \langle\psi|\boldsymbol{M}_g^\dagger\boldsymbol{M}_g|\psi\rangle; \\ \dfrac{\boldsymbol{M}_e|\psi\rangle}{\sqrt{\langle\psi|\boldsymbol{M}_e^\dagger\boldsymbol{M}_e|\psi\rangle}}, & \text{with probability } p_e = \langle\psi|\boldsymbol{M}_e^\dagger\boldsymbol{M}_e|\psi\rangle. \end{cases} \tag{8}$$

For the density matrix formulation we have thus

$$\boldsymbol{\rho}_+ = \begin{cases} \mathbb{M}_g(\boldsymbol{\rho}) = \dfrac{\boldsymbol{M}_g\boldsymbol{\rho}\boldsymbol{M}_g}{\mathrm{Tr}\left(\boldsymbol{M}_g\boldsymbol{\rho}\boldsymbol{M}_g^\dagger\right)}, & \text{with probability } p_g = \mathrm{Tr}\left(\boldsymbol{M}_g\boldsymbol{\rho}\boldsymbol{M}_g^\dagger\right); \\ \mathbb{M}_e(\boldsymbol{\rho}) = \dfrac{\boldsymbol{M}_e\boldsymbol{\rho}\boldsymbol{M}_e}{\mathrm{Tr}\left(\boldsymbol{M}_e\boldsymbol{\rho}\boldsymbol{M}_e^\dagger\right)}, & \text{with probability } p_e = \mathrm{Tr}\left(\boldsymbol{M}_e\boldsymbol{\rho}\boldsymbol{M}_e^\dagger\right). \end{cases} \tag{9}$$

Exercice 1. *Consider $\boldsymbol{M}_g$ and $\boldsymbol{M}_e$ defined by (7). Show that, for any density matrix $\boldsymbol{\rho}$ the operator (defining a Kraus map, see appendix A)*

$$\boldsymbol{M}_g\boldsymbol{\rho}\boldsymbol{M}_g^\dagger + \boldsymbol{M}_e\boldsymbol{\rho}\boldsymbol{M}_e^\dagger$$

does not depend on $(\theta_2, x_2, y_2, z_2)$, the parameters of the second Ramsey pulse $\boldsymbol{U}_{R_2}$.

1.2 Jaynes-Cummings propagator

In the resonant case, $\Delta = 0$. The atom/cavity propagator $\boldsymbol{U}_C$ based on Jaynes-Cummings Hamiltonian (5) admits the following form (see [6] for the detailed derivations including Gaussian radial dependence of the quantized mode and atom velocity):

$$\begin{aligned} \boldsymbol{U}_C = |g\rangle\langle g|\cos\left(\tfrac{\Theta}{2}\sqrt{\boldsymbol{N}}\right) + |e\rangle\langle e|\cos\left(\tfrac{\Theta}{2}\sqrt{\boldsymbol{N}+\boldsymbol{I}}\right) \\ + |g\rangle\langle e|\left(\frac{\sin\left(\frac{\Theta}{2}\sqrt{\boldsymbol{N}}\right)}{\sqrt{\boldsymbol{N}}}\right)\boldsymbol{a}^\dagger - |e\rangle\langle g|\,\boldsymbol{a}\left(\frac{\sin\left(\frac{\Theta}{2}\sqrt{\boldsymbol{N}}\right)}{\sqrt{\boldsymbol{N}}}\right) \end{aligned} \tag{10}$$

where $\boldsymbol{N} = \boldsymbol{a}^\dagger\boldsymbol{a}$ is the photon number operator, the adjustable parameter Θ being the Rabi angle with zero photon.

In the dispersive case, $|\Delta| \gg |\Omega|$, U_C based on Jaynes-Cummings Hamiltonian (5) admits the following form (see [6] for the detailed derivations based on adiabatic invariance):

$$\boldsymbol{U}_C = |g\rangle\langle g|\,e^{-i\phi(\boldsymbol{N})} + |e\rangle\langle e|\,e^{i\phi(\boldsymbol{N}+\boldsymbol{I})} \tag{11}$$

where the dephasing $\phi(\boldsymbol{N})$ depends on the photon number and can be approximated by a linear real function: $\phi(\boldsymbol{N}) = \vartheta_0 + \vartheta\boldsymbol{N}$, the phases ϑ_0 and ϑ being adjustable parameters.

The exercise below can be seen as a simplified derivation of the above formulae for $\boldsymbol{U}_C$.

Exercice 2. *Let us assume that the Jaynes-Cummings propagator $\boldsymbol{U}_C$ admits the following form*

$$\boldsymbol{U}_C = e^{-i\tau\left(\frac{\Delta\left(|e\rangle\langle e| - |g\rangle\langle g|\right)}{2} + i\frac{\Omega\left(|g\rangle\langle e|\boldsymbol{a}^\dagger - |e\rangle\langle g|\boldsymbol{a}\right)}{2}\right)}$$

where τ is an interaction time.

1. *Show by recurrence on integer k that*

$$\Big(\Delta\big(\,|e\rangle\langle e| - |g\rangle\langle g|\,\big) + i\Omega\big(\,|g\rangle\langle e|\,\boldsymbol{a}^\dagger - |e\rangle\langle g|\,\boldsymbol{a}\big)\Big)^{2k} = \\ |e\rangle\langle e|\left(\Delta^2 + (\boldsymbol{N}+1)\Omega^2\right)^k + |g\rangle\langle g|\left(\Delta^2 + \boldsymbol{N}\Omega^2\right)^k$$

and that

$$\Big(\Delta\big(\,|e\rangle\langle e| - |g\rangle\langle g|\,\big) + i\Omega\big(\,|g\rangle\langle e|\,\boldsymbol{a}^\dagger - |e\rangle\langle g|\,\boldsymbol{a}\big)\Big)^{2k+1} = \\ |e\rangle\langle e|\,\Delta\left(\Delta^2 + (\boldsymbol{N}+1)\Omega^2\right)^k - |g\rangle\langle g|\,\Delta\left(\Delta^2 + \boldsymbol{N}\Omega^2\right)^k \\ + i\Omega\Big(\,|g\rangle\langle e|\left(\Delta^2 + \boldsymbol{N}\Omega^2\right)^k\boldsymbol{a}^\dagger - |e\rangle\langle g|\,\boldsymbol{a}\left(\Delta^2 + \boldsymbol{N}\Omega^2\right)^k\Big).$$

2. *Deduce that*

$$\boldsymbol{U}_C = |g\rangle\langle g|\left(\cos\left(\tfrac{\tau\sqrt{\Delta^2+\boldsymbol{N}\Omega^2}}{2}\right) + i\frac{\Delta\sin\left(\frac{\tau\sqrt{\Delta^2+\boldsymbol{N}\Omega^2}}{2}\right)}{\sqrt{\Delta^2+\boldsymbol{N}\Omega^2}}\right) \\ + |e\rangle\langle e|\left(\cos\left(\tfrac{\tau\sqrt{\Delta^2+(\boldsymbol{N}+1)\Omega^2}}{2}\right) - i\frac{\Delta\sin\left(\frac{\tau\sqrt{\Delta^2+(\boldsymbol{N}+1)\Omega^2}}{2}\right)}{\sqrt{\Delta^2+(\boldsymbol{N}+1)\Omega^2}}\right) \\ + |g\rangle\langle e|\left(\frac{\Omega\sin\left(\frac{\tau\sqrt{\Delta^2+\boldsymbol{N}\Omega^2}}{2}\right)}{\sqrt{\Delta^2+\boldsymbol{N}\Omega^2}}\right)\boldsymbol{a}^\dagger - |e\rangle\langle g|\,\boldsymbol{a}\left(\frac{\Omega\sin\left(\frac{\tau\sqrt{\Delta^2+\boldsymbol{N}\Omega^2}}{2}\right)}{\sqrt{\Delta^2+\boldsymbol{N}\Omega^2}}\right). \quad (12)$$

3. *In the resonant case, $\Delta = 0$, express the vacuum Rabi angle Θ appearing in (10) with respect to Ω and τ.*

4. *In the dispersive case, $|\Delta| \gg |\Omega|$, and when the interaction time τ is large, $\Delta\tau \sim \left(\frac{\Delta}{\Omega}\right)^2$, show that, up to first order terms in Ω/Δ, we get*

$$e^{-i\tau\left(\frac{\Delta\left(|e\rangle\langle e| - |g\rangle\langle g|\right)}{2} + i\frac{\Omega\left(|g\rangle\langle e|\boldsymbol{a}^\dagger - |e\rangle\langle g|\boldsymbol{a}\right)}{2}\right)} = |g\rangle\langle g|\,e^{i\left(\frac{\Delta\tau}{2} + \frac{\Omega^2\tau}{4\Delta}\boldsymbol{N}\right)} \\ + |e\rangle\langle e|\,e^{-i\left(\frac{\Delta\tau}{2} + \frac{\Omega^2\tau}{4\Delta}(\boldsymbol{N}+1)\right)}.$$

Express the phases ϑ_0 and ϑ appearing in (11) with respect to τ, Δ and Ω.

1.3 Resonant case

Let us detail the operators $\boldsymbol{M}_g$ and $\boldsymbol{M}_e$ defined in (7) when $\boldsymbol{U}_C$ is given by (10), $\boldsymbol{U}_{R_1} = e^{-i\frac{\theta_1}{2}\boldsymbol{\sigma_y}}$ and $\boldsymbol{U}_{R_2} = \boldsymbol{I}$. Since $\boldsymbol{U}_{R_1} = \cos\left(\frac{\theta_1}{2}\right) + \sin\left(\frac{\theta_1}{2}\right)\left(\,|g\rangle\langle e| - |e\rangle\langle g|\,\right)$, $|\Psi\rangle_{R_1}$ given by (2) reads:

$$|\Psi\rangle_{R_1} = \left(\cos\left(\tfrac{\theta_1}{2}\right)|g\rangle - \sin\left(\tfrac{\theta_1}{2}\right)|e\rangle\right)\otimes|\psi\rangle\,.$$

Then $|\Psi\rangle_C$ given by (4) becomes

$$
\begin{aligned}
|\Psi\rangle_C = \cos\left(\tfrac{\theta_1}{2}\right)\Bigg(|g\rangle \otimes \cos\left(\tfrac{\Theta}{2}\sqrt{\boldsymbol{N}}\right)|\psi\rangle - |e\rangle \otimes \boldsymbol{a}\left(\frac{\sin\left(\frac{\Theta}{2}\sqrt{\boldsymbol{N}}\right)}{\sqrt{\boldsymbol{N}}}\right)|\psi\rangle\Bigg) \\
- \sin\left(\tfrac{\theta_1}{2}\right)\Bigg(|e\rangle \otimes \cos\left(\tfrac{\Theta}{2}\sqrt{\boldsymbol{N}+1}\right)|\psi\rangle + |g\rangle \otimes \left(\frac{\sin\left(\frac{\Theta}{2}\sqrt{\boldsymbol{N}}\right)}{\sqrt{\boldsymbol{N}}}\right)\boldsymbol{a}^\dagger|\psi\rangle\Bigg) \\
= |g\rangle \otimes \left(\cos\left(\tfrac{\theta_1}{2}\right)\cos\left(\tfrac{\Theta}{2}\sqrt{\boldsymbol{N}}\right) - \sin\left(\tfrac{\theta_1}{2}\right)\left(\frac{\sin\left(\frac{\Theta}{2}\sqrt{\boldsymbol{N}}\right)}{\sqrt{\boldsymbol{N}}}\right)\boldsymbol{a}^\dagger\right)|\psi\rangle \\
- |e\rangle \otimes \left(\sin\left(\tfrac{\theta_1}{2}\right)\cos\left(\tfrac{\Theta}{2}\sqrt{\boldsymbol{N}+1}\right) + \cos\left(\tfrac{\theta_1}{2}\right)\boldsymbol{a}\left(\frac{\sin\left(\frac{\Theta}{2}\sqrt{\boldsymbol{N}}\right)}{\sqrt{\boldsymbol{N}}}\right)\right)|\psi\rangle\,.
\end{aligned}
$$

Since $\boldsymbol{U}_{R_2} = \boldsymbol{I}$, $|\Psi\rangle_C = |\Psi\rangle_{R_2}$. The measurement operators are thus given by

$$
\begin{aligned}
\boldsymbol{M}_g &= \cos\left(\tfrac{\theta_1}{2}\right)\cos\left(\tfrac{\Theta}{2}\sqrt{\boldsymbol{N}}\right) - \sin\left(\tfrac{\theta_1}{2}\right)\left(\frac{\sin\left(\frac{\Theta}{2}\sqrt{\boldsymbol{N}}\right)}{\sqrt{\boldsymbol{N}}}\right)\boldsymbol{a}^\dagger \\
\boldsymbol{M}_e &= -\sin\left(\tfrac{\theta_1}{2}\right)\cos\left(\tfrac{\Theta}{2}\sqrt{\boldsymbol{N}+1}\right) - \cos\left(\tfrac{\theta_1}{2}\right)\boldsymbol{a}\left(\frac{\sin\left(\frac{\Theta}{2}\sqrt{\boldsymbol{N}}\right)}{\sqrt{\boldsymbol{N}}}\right)
\end{aligned}
\tag{13}
$$

Exercice 3. *Verify that the operators (measurement operators) given by (13) satisfy* $\boldsymbol{M}_g^\dagger\boldsymbol{M}_g + \boldsymbol{M}_e^\dagger\boldsymbol{M}_e = \boldsymbol{I}$ *(hint: use,* $\boldsymbol{N} = \boldsymbol{a}^\dagger\boldsymbol{a}$, $\boldsymbol{a}\, f(\boldsymbol{N}) = f(\boldsymbol{N}+1)\,\boldsymbol{a}$ *and* $\boldsymbol{a}^\dagger f(\boldsymbol{N}) = f(\boldsymbol{N}-1)\,\boldsymbol{a}^\dagger$*).*

1.4 Dispersive case

Let us now describe the measurement operators $\boldsymbol{M}_g$ and $\boldsymbol{M}_e$ defined in (7) when $\boldsymbol{U}_C$ is given by (11), $\boldsymbol{U}_{R_1} = e^{-i\frac{\pi}{4}\boldsymbol{\sigma_y}}$ and $\boldsymbol{U}_{R_2} = e^{-i\frac{\pi}{4}(-\sin\eta\boldsymbol{\sigma_x} + \cos\eta\boldsymbol{\sigma_y})}$ (with angle η chosen below). Since $\boldsymbol{U}_{R_1} = \frac{|g\rangle\langle e| - |e\rangle\langle g|}{\sqrt{2}}$, $|\Psi\rangle_{R_1}$ given by (2) reads:

$$
|\Psi\rangle_{R_1} = \frac{|g\rangle - |e\rangle}{\sqrt{2}} \otimes |\psi\rangle\,.
$$

Then $|\Psi\rangle_C$ given by (4) becomes

$$
|\Psi\rangle_C = \tfrac{1}{\sqrt{2}}\,|g\rangle \otimes e^{-i\phi(\boldsymbol{N})}\,|\psi\rangle - \tfrac{1}{\sqrt{2}}\,|e\rangle \otimes e^{i\phi(\boldsymbol{N}+1)}\,|\psi\rangle\,.
$$

Since $\boldsymbol{U}_{R_2} = \frac{1}{\sqrt{2}}\left(\boldsymbol{I} + e^{i\eta}\,|g\rangle\langle e| - e^{-i\eta}\,|e\rangle\langle g|\right)$, we have

$$
\begin{aligned}
2\,|\Psi\rangle_{R_2} &= \left(\,|g\rangle - e^{-i\eta}\,|e\rangle\,\right) \otimes e^{-i\phi(\boldsymbol{N})}\,|\psi\rangle - \left(e^{i\eta}\,|g\rangle + |e\rangle\,\right) \otimes e^{i\phi(\boldsymbol{N}+1)}\,|\psi\rangle \\
&= |g\rangle \otimes \left(e^{-i\phi(\boldsymbol{N})} - e^{i(\eta+\phi(\boldsymbol{N}+1))}\right)|\psi\rangle - |e\rangle \otimes \left(e^{-i(\eta+\phi(\boldsymbol{N}))} + e^{i\phi(\boldsymbol{N}+1)}\right)|\psi\rangle
\end{aligned}
$$

where $\phi(\boldsymbol{N}) = \vartheta_0 + \boldsymbol{N}\vartheta$. Take φ_0 an arbitrary phase and set $\eta = 2(\varphi_0 - \vartheta_0) - \vartheta - \pi$. Then the measurement operators are given by the simple formulae

$$
\boldsymbol{M}_g = \cos(\varphi_0 + \boldsymbol{N}\vartheta), \quad \boldsymbol{M}_e = \sin(\varphi_0 + \boldsymbol{N}\vartheta) \tag{14}
$$

where we have removed the irrelevant global phase factors $e^{i(\varphi_0 - \vartheta_0)}$ for $\boldsymbol{M}_g$ and $e^{i(\vartheta_0 - \varphi_0 + \pi/2)}$ for $\boldsymbol{M}_e$. In the Fock basis $\{|n\rangle\}_0^\infty$), the operator $\boldsymbol{M}_g$ (resp. $\boldsymbol{M}_e$) is diagonal with diagonal elements $\cos(n\vartheta + \varphi_0)$ (resp. $\sin(n\vartheta + \varphi_0)$. We note in particular that $\boldsymbol{M}_g^\dagger\boldsymbol{M}_g + \boldsymbol{M}_e^\dagger\boldsymbol{M}_e = \boldsymbol{I}$.

Exercice 4. *Take $\boldsymbol{M}_g$ and $\boldsymbol{M}_e$ defined by (7) with $\boldsymbol{U}_C$ given by (11) with ϕ an arbitrary real value function.*

1. *Show that any Fock state $|n\rangle$ is an eigenvector of $\boldsymbol{M}_g$ and $\boldsymbol{M}_e$, whatever $\boldsymbol{U}_{R_1}$ and $\boldsymbol{U}_{R_2}$ are.*

2. *Deduce from preceding question that, for any density operator $\boldsymbol{\rho}$, any integer n and any Ramsey pulses $\boldsymbol{U}_{R_1}$ and $\boldsymbol{U}_{R_2}$, we have*

$$\left\langle n|\boldsymbol{M}_g\boldsymbol{\rho}\boldsymbol{M}_g^\dagger|n\right\rangle + \left\langle n|\boldsymbol{M}_e\boldsymbol{\rho}\boldsymbol{M}_e^\dagger|n\right\rangle = \langle n|\boldsymbol{\rho}|n\rangle\,.$$

3. *What does-it mean for the Markov chain associated to such $\boldsymbol{M}_g$ and $\boldsymbol{M}_e$ and defined by (9).*

1.5 QND measurements: open-loop asymptotic behavior

Through this subsection, we consider the measurement associated to the dispersive coupling regime between the system (cavity) and the meter (atoms). As discussed through the previous subsection, the measurement operators $\boldsymbol{M}_g$ and $\boldsymbol{M}_e$ are given by (14). These operators being diagonal in the basis $\{|n\rangle\}_{n=0}^{\infty}$ of photon number states, they commute with the physical observable $\boldsymbol{N} = \boldsymbol{a}^\dagger\boldsymbol{a}$ (photon number operator). Indeed, following the definition of Subsection 3.3, they define a quantum non-demolition (QND) measurement of the photon number observable $\boldsymbol{N}$. Here, we study the asymptotic behavior of the Markov chain associated to a repetitive application of such QND measurements. The cavity state after the k'th measurement is represented by $\boldsymbol{\rho}_k$ and follows the Markov chain dynamics

$$\boldsymbol{\rho}_{k+1} = \mathbb{M}_{s_k}(\boldsymbol{\rho}_k),$$

where s_k takes the value g (resp. e) with probability $p_{g,k} = \mathrm{Tr}\left(\boldsymbol{M}_g\boldsymbol{\rho}_k\boldsymbol{M}_g^\dagger\right)$ (resp. with probability $p_{e,k} = \mathrm{Tr}\left(\boldsymbol{M}_e\boldsymbol{\rho}_k\boldsymbol{M}_e^\dagger\right)$). We have the following theorem:

Theorem 1. *Consider the Markov process defined above with an initial density matrix $\boldsymbol{\rho}_0$ defined on the subspace span$\{|n\rangle \mid n = 0, 1, \cdots, n^{\max}\}$. Also, assume the non-degeneracy assumption*

$$\cos^2(\varphi_m) \neq \cos^2(\varphi_n) \quad \forall n \neq m \in \{0, 1, \cdots, n^{\max}\},$$

where $\varphi_n = \varphi_0 + n\vartheta$. Then

- *for any $n \in \{0, \ldots, n^{\max}\}$, $\mathrm{Tr}\left(\boldsymbol{\rho}_k\,|n\rangle\,\langle n|\right) = \langle n|\,\boldsymbol{\rho}_k\,|n\rangle$ is a martingale*
- *$\boldsymbol{\rho}_k$ converges with probability 1 to one of the $n^{\max}+1$ Fock state $|n\rangle\langle n|$ with $n \in \{0, \ldots, n^{\max}\}$.*
- *the probability to converge towards the Fock state $|n\rangle\langle n|$ is given by $\mathrm{Tr}\left(\boldsymbol{\rho}_0\,|n\rangle\,\langle n|\right) = \langle n|\,\boldsymbol{\rho}_0\,|n\rangle$.*

Proof. First, we note that, the measurement operators $\boldsymbol{M}_g$ and $\boldsymbol{M}_e$ being diagonal in the basis of photon number states, and $\boldsymbol{\rho}_0$ being defined on the subspace span$\{|n\rangle \mid n = 0, 1, \cdots, n^{\max}\}$, the state $\boldsymbol{\rho}_k$ remains in this subspace for all $k \geq 0$. We can therefore restrict the proof to this finite dimensional Hilbert space.

Let us prove that $\mathrm{Tr}\left(\boldsymbol{\rho}_k \left|n\right\rangle \left\langle n\right|\right)$ is a martingale. Set $\boldsymbol{\xi} = \left|n\right\rangle \left\langle n\right|$. We have

$$\begin{aligned}\mathbb{E}\left(\mathrm{Tr}\left(\boldsymbol{\xi}\boldsymbol{\rho}_{k+1}\right) \mid \boldsymbol{\rho}_k\right) &= p_{g,k}\,\mathrm{Tr}\left(\boldsymbol{\xi}\frac{\boldsymbol{M}_g\boldsymbol{\rho}_k\boldsymbol{M}_g^\dagger}{p_{g,k}}\right) + p_{e,k}\,\mathrm{Tr}\left(\boldsymbol{\xi}\frac{\boldsymbol{M}_e\boldsymbol{\rho}_k\boldsymbol{M}_e^\dagger}{p_{e,k}}\right)\\ &= \mathrm{Tr}\left(\boldsymbol{\xi}\boldsymbol{M}_g\boldsymbol{\rho}_k\boldsymbol{M}_g^\dagger\right) + \mathrm{Tr}\left(\boldsymbol{\xi}\boldsymbol{M}_e\boldsymbol{\rho}_k\boldsymbol{M}_e^\dagger\right) = \mathrm{Tr}\left(\boldsymbol{\rho}_k(\boldsymbol{M}_g^\dagger\boldsymbol{\xi}\boldsymbol{M}_g + \boldsymbol{M}_e^\dagger\boldsymbol{\xi}\boldsymbol{M}_e)\right).\end{aligned}$$

Since $\boldsymbol{\xi}$ commutes with $\boldsymbol{M}_g$ and $\boldsymbol{M}_e$ and $\boldsymbol{M}_g^\dagger\boldsymbol{M}_g + \boldsymbol{M}_e^\dagger\boldsymbol{M}_e = \boldsymbol{I}$, we have $\mathbb{E}\left(\mathrm{Tr}\left(\boldsymbol{\xi}\boldsymbol{\rho}_{k+1}\right) \mid \boldsymbol{\rho}_k\right) = \mathrm{Tr}\left(\boldsymbol{\xi}\boldsymbol{\rho}_k\right)$. This implies that $\mathrm{Tr}\left(\boldsymbol{\rho}_k \left|n\right\rangle \left\langle n\right|\right)$ is a martingale.

Now, we consider the function

$$V(\boldsymbol{\rho}) = \sum_{n\neq m} \sqrt{\langle n \mid \boldsymbol{\rho} \mid n\rangle \langle m \mid \boldsymbol{\rho} \mid m\rangle}. \tag{15}$$

Simple calculations show that $V(\boldsymbol{\rho}_k)$ is a strict super-martingale:

$$\mathbb{E}\left(V(\boldsymbol{\rho}_{k+1}) \mid \boldsymbol{\rho}_k\right) = \sum_{n\neq m} (|\cos\varphi_n \cos\varphi_m| + |\sin\varphi_n \sin\varphi_m|)\sqrt{\langle n \mid \boldsymbol{\rho} \mid n\rangle \langle m \mid \boldsymbol{\rho} \mid m\rangle} \leq rV(\boldsymbol{\rho}_k)$$

with $r = \max_{n\neq m}(|\cos\varphi_n \cos\varphi_m| + |\sin\varphi_n \sin\varphi_m|) < 1$. Therefore

$$\mathbb{E}\left(V(\boldsymbol{\rho}_k)\right) \leq r^k V(\boldsymbol{\rho}_0),$$

which implies the convergence in mean of $V(\boldsymbol{\rho}_k)$ to zero. We now argue that this also implies the almost sure convergence of $V(\boldsymbol{\rho}_k)$ to zero. This can be done by combining the Markov's inequality and the Borel-Cantelli lemma, both reminded in Appendix B. Indeed, for any $\epsilon > 0$, using the Markov's inequality, we have

$$\mathbb{P}\left(V(\boldsymbol{\rho}_k) \geq \epsilon\right) \leq \frac{\mathbb{E}\left(V(\boldsymbol{\rho}_k)\right))}{\epsilon} \leq r^k \frac{V(\boldsymbol{\rho}_0)}{\epsilon}.$$

Therefore, we have

$$\sum_{k\geq 0} \mathbb{P}\left(V(\boldsymbol{\rho}_k) \geq \epsilon\right) \leq \frac{1}{1-r}\frac{V(\boldsymbol{\rho}_0)}{\epsilon} < \infty.$$

The Borel-Cantelli lemma implies then that

$$\mathbb{P}\left(\limsup_k V(\boldsymbol{\rho}_k) \geq \epsilon\right) = 0,$$

which leads to

$$\mathbb{P}\left(\lim_k V(\boldsymbol{\rho}_k) \to 0\right) = 1.$$

Now, for any such trajectory (meaning any $\omega \in \Omega$, the sample space, such that $V(\boldsymbol{\rho}_k(\omega)) \to 0$), we note that $\boldsymbol{\rho}_k(\omega)$ lives in a compact set and therefore, from any subsequence we can extract another converging subsequence $\boldsymbol{\rho}_{k_n}(\omega) \to \bar{\boldsymbol{\rho}}(\omega)$. By continuity of V, we now that $V(\boldsymbol{\rho}(\omega)) = 0$. Furthermore, we note that $V(\bar{\boldsymbol{\rho}}) = 0$ implies that $\bar{\boldsymbol{\rho}} = \left|n\right\rangle \left\langle n\right|$ for some $n \in \{0, \cdots, n_{\max}\}$ (we leave the details of this reasoning to be proven by the interested reader). This closes the proof of the second assertion, meaning the almost sure convergence of $\boldsymbol{\rho}_k$ to the set of Fock states $\{\left|n\right\rangle\}_{n=0}^{n_{\max}}$.

We have shown that the probability measure associated to the random variable $\boldsymbol{\rho}_k$ converges to $\sum_{n=0}^{n^{\max}} p_n \delta_{\left|n\right\rangle\left\langle n\right|}$, where $\delta_{\left|n\right\rangle\left\langle n\right|}$ denotes the Dirac measure at $\left|n\right\rangle \left\langle n\right|$ and p_n is the probability of convergence towards $\left|n\right\rangle \left\langle n\right|$. In particular, we have $\mathbb{E}\left(\mathrm{Tr}\left(\boldsymbol{\rho}_k \left|n\right\rangle \left\langle n\right|\right)\right) \to p_n$. But $\mathrm{Tr}\left(\boldsymbol{\rho}_k \left|n\right\rangle \left\langle n\right|\right)$ is a martingale, thus $\mathbb{E}\left(\mathrm{Tr}\left(\boldsymbol{\rho}_k \left|n\right\rangle \left\langle n\right|\right)\right) = \mathbb{E}\left(\mathrm{Tr}\left(\boldsymbol{\rho}_0 \left|0\right\rangle \left\langle 0\right|\right)\right)$ and consequently $p_n = \langle n|\boldsymbol{\rho}_0|n\rangle$. □

1.6 QND measurements and quantum-state feedback

The Theorem 1 implies that the QND measurement of the Subsection 1.4 can be seen as a photon-number state preparation tool. However, this state preparation is non-deterministic as we can not be sure to converge towards a desired Fock state $|\bar{n}\rangle\langle\bar{n}|$. One way of removing this indeterminism is to repeat the QND measurement process by re-preparing the same initial state and re-launching the same measurement process up to reaching $|\bar{n}\rangle\langle\bar{n}|$. However this can be very time-consuming and perhaps inefficient when dealing with the measurement uncertainties and relaxations (to be studied later through these notes).

This non-deterministic preparation tool can be turned into a deterministic stabilization protocol with the addition of appropriate feedback strategies [13, 19]. We focus here on the feedback scheme experimentally tested in [19] (see Figure 1.6). This could be modeled through the following Markov chain:

$$\boldsymbol{\rho}_{k+1} = \mathbb{M}_{s_k,u_k}(\boldsymbol{\rho}_k)$$

where the control u_k at step k is chosen between three possible values:

1. $u_k = 0$ corresponds to dispersive interaction and QND atom with

$$\boldsymbol{M}_{g,0} = \cos(\varphi_0 + \boldsymbol{N}\vartheta), \quad \boldsymbol{M}_{e,0} = \sin(\varphi_0 + \boldsymbol{N}\vartheta);$$

2. $u_k = 1$ corresponds to resonant atom entering in $|e\rangle$ in cavity C ($\boldsymbol{U}_{R_1} = e^{-i\frac{\pi}{2}\boldsymbol{\sigma_y}}$) with resonant interaction $\boldsymbol{U}_C$ given by (10) and without post-cavity manipulation ($\boldsymbol{U}_{R_2} = \boldsymbol{I}$):

$$\boldsymbol{M}_{g,1} = \left(\frac{\sin\left(\frac{\Theta}{2}\sqrt{\boldsymbol{N}}\right)}{\sqrt{\boldsymbol{N}}}\right)\boldsymbol{a}^\dagger, \quad \boldsymbol{M}_{e,1} = \cos\left(\frac{\Theta}{2}\sqrt{\boldsymbol{N}+\boldsymbol{I}}\right);$$

3. $u_k = -1$ corresponds to resonant atom entering in $|g\rangle$ in cavity C ($\boldsymbol{U}_{R_1} = \boldsymbol{I}$) with resonant interaction $\boldsymbol{U}_C$ given by (10) and without post-cavity manipulation ($\boldsymbol{U}_{R_2} = \boldsymbol{I}$):

$$\boldsymbol{M}_{g,-1} = \cos\left(\frac{\Theta}{2}\sqrt{\boldsymbol{N}}\right), \quad \boldsymbol{M}_{e,-1} = \boldsymbol{a}\left(\frac{\sin\left(\frac{\Theta}{2}\sqrt{\boldsymbol{N}}\right)}{\sqrt{\boldsymbol{N}}}\right).$$

The idea is to construct a Lyapunov function $\overline{V}(\boldsymbol{\rho})$ similar to (15) but with a different weighting on various photon-number states to favor the convergence towards a particular Fock state with $\bar{n}$ photon (set-point),

$$\overline{V}(\boldsymbol{\rho}) = \sum_{n\geq 0} f(n)\,\mathrm{Tr}\left(\boldsymbol{\rho}\,|n\rangle\langle n|\right),$$

with $\mathbb{N} \ni n \mapsto f(n)$ being a real function, taking its minimum at $n = \bar{n}$, strictly decreasing (resp. increasing) for $n \in \{0, \dots \bar{n}\}$ (resp. $n \in \{\bar{n}, \dots + \infty\}$).

The control input will then be selected so that the function $\overline{V}(\boldsymbol{\rho}_k)$ becomes a super-martingale. This means that at each time-step k, the value u_k is the argument of the minimum of the conditional expectation of $\overline{V}(\boldsymbol{\rho}_{k+1})$ knowing the density operator at step k, $\boldsymbol{\rho}_k$, and the control input at step k, $u_k = u$ with $u \in \{-1, 0, 1\}$:

$$u_k := \underset{u\in\{-1,0,1\}}{\mathrm{argmin}} \left\{\mathbb{E}\left(\overline{V}(\rho_{k+1})|\rho_k, u_k = u\right)\right\}$$

where

$$\begin{aligned}\mathbb{E}\left(\overline{V}(\rho_{k+1})|\rho_k, u_k = u\right) \\ &= \mathrm{Tr}\left(\boldsymbol{M}_{g,u}\boldsymbol{\rho}_k\boldsymbol{M}_{g,u}\right)\overline{V}\Big(\mathbb{M}_{g,u}(\boldsymbol{\rho}_k)\Big) + \mathrm{Tr}\left(\boldsymbol{M}_{e,u}\boldsymbol{\rho}_k\boldsymbol{M}_{e,u}\right)\overline{V}\Big(\mathbb{M}_{e,u}(\boldsymbol{\rho}_k)\Big) \\ &= \sum_n f(n)\left\langle n\left|\boldsymbol{M}_{g,u}\boldsymbol{\rho}_k\boldsymbol{M}_{g,u} + \boldsymbol{M}_{e,u}\boldsymbol{\rho}_k\boldsymbol{M}_{e,u}\right|n\right\rangle.\end{aligned}$$

Thus u_k is a function of ρ_k, the quantum-state at step k. This kind of feedback law is called a *measurement-based feedback* since the controller is a classical controller based on the past measurement outcomes summarized in the present quantum state ρ_k.

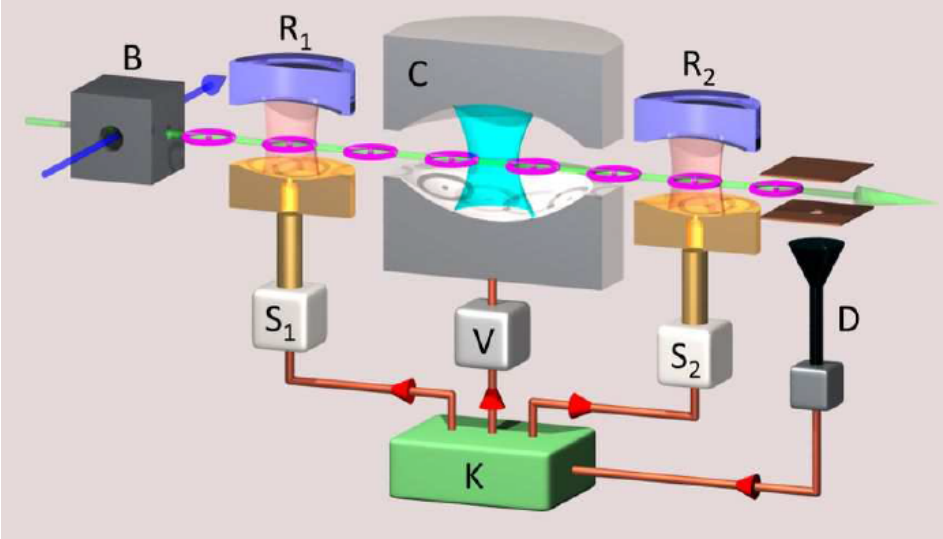

Figure 2: A schematic of the closed-loop system borrowed from [19]. The feedback control relies on three kinds of probe atoms (resonant atom prepared in $|g\rangle$ or $|e\rangle$ and dispersive atom prepared in $(|g\rangle + |e\rangle\,/sqrt2$.

Exercice 5 (Open-loop convergence in the resonant case)**.** *Consider the Markov chain* $\boldsymbol{\rho}_{k+1} = \mathbb{M}_{s_k}(\boldsymbol{\rho}_k)$ *where* $s_k = g$ *(resp.* $s_k = e$*) with probability* $p_{g,k} = \mathrm{Tr}\left(\boldsymbol{M}_g\boldsymbol{\rho}_k\boldsymbol{M}_g^\dagger\right)$ *(resp.* $p_{e,k} = \mathrm{Tr}\left(\boldsymbol{M}_e\boldsymbol{\rho}_k\boldsymbol{M}_e^\dagger\right)$*). The Kraus operator are given by* (13) *with* $\theta_1 = 0$*. Assume the initial state to be defined on the subspace* $\{|n\rangle\}_{n=0}^{n^{\max}}$ *and that the cavity state at step* k *is described by the density operator* $\boldsymbol{\rho}_k$*.*

1. *Show that*

$$\mathbb{E}\left(\mathrm{Tr}\left(\boldsymbol{N}\boldsymbol{\rho}_{k+1}\right)\ |\ \boldsymbol{\rho}_k\right) = \mathrm{Tr}\left(\boldsymbol{N}\boldsymbol{\rho}_k\right) - \mathrm{Tr}\left(\sin^2\left(\tfrac{\Theta}{2}\sqrt{\boldsymbol{N}}\right)\boldsymbol{\rho}_k\right).$$

2. *Assume that for any integer* n*,* $\Theta\sqrt{n}/\pi$ *is irrational. Then prove, using Theorem 3 of Appendix B, that almost surely* $\boldsymbol{\rho}_k$ *tends to the vacuum state* $|0\rangle\langle 0|$ *whatever its initial condition is.*

3. *When* $\Theta\sqrt{n}/\pi$ *is rational for some integer* n*, describes the set of asymptotic states for* $\boldsymbol{\rho}_k$*.*

1.7 Measurement uncertainties and Bayesian quantum filtering

This subsection is directly inspired from [5, 15]. Let us consider now the situation where the atom passes through the cavity but we do not detect it after the second Ramsey zone. To describe the cavity state we have to use mixed states and thus density matrix $\boldsymbol{\rho}$ and the operator $\mathbb{M}_g$ and $\mathbb{M}_e$ defined in (30). Having no knowledge on whether the atom ends up in the state $|g\rangle$ or $|e\rangle$, the best we can say about the cavity state (our knowledge of the system) after the passage of the atom is its expectation value:

$$\boldsymbol{\rho}_+ = p_g \mathbb{M}_g(\boldsymbol{\rho}) + p_e \mathbb{M}_e(\boldsymbol{\rho}) = \boldsymbol{M}_g \boldsymbol{\rho} \boldsymbol{M}_g^\dagger + \boldsymbol{M}_e \boldsymbol{\rho} \boldsymbol{M}_e^\dagger. \tag{16}$$

The above map, sending $\boldsymbol{\rho}$ to $\boldsymbol{\rho}_+$, defines the Kraus representation for a linear quantum operation (see Appendix A for a definition and properties of linear quantum operations).

Now consider the case where we realize the atom detection but we are uncertain about its result. Indeed, in practice, the detection process is not perfect and we need to take into account at least three kinds of uncertainties:

- the atom preparation process is itself a random process following a Poisson law; indeed the samples carrying the atoms that pass through the setup might be empty of atoms; we note the occupancy rate of the pulses by $\eta_a \in]0, 1]$ (η_a is about 0.4 for the LKB experimental setup);
- the atom detector is imperfect and can miss a certain percentage of the atoms; we denote the detector's efficiency by $\eta_d \in]0, 1]$ (η_d is about 0.8 for the LKB experimental setup);
- the atom detector is not fault-free and the result of the measurement (atom in the state $|g\rangle$ or $|e\rangle$) can be interchanged; we denote the fault rate by $\eta_f \in [0, 1/2)$ (η_f is about 0.1 for the LKB experimental setup).

Whenever realizing the atom detection, we can achieve three results: 1- the atom is in $|g\rangle$, 2- the atom in $|e\rangle$, 3-the detector does not detect any atom. For each situation we may have various possibilities:

Atom in $|g\rangle$: Either the atom is actually in the state $|e\rangle$ and the detector has made a mistake by detecting it in $|g\rangle$ (this happens with a probability p_g^f to be determined) or the atom is really in the state $|g\rangle$ (this happens with probability $1 - p_g^f$). Indeed, the conditional probability of having the atom in $|e\rangle$ while the detection result has been $|g\rangle$ may be computed through the Bayesian formula and is given by:

$$p_g^f = \frac{\eta_f p_e}{\eta_f p_e + (1 - \eta_f) p_g},$$

where $p_g = \mathrm{Tr}\left(\boldsymbol{M}_g \boldsymbol{\rho} \boldsymbol{M}_g^\dagger\right)$ and $p_e = \mathrm{Tr}\left(\boldsymbol{M}_e \boldsymbol{\rho} \boldsymbol{M}_e^\dagger\right)$.

Also, the conditional evolution of the density matrix (as our knowledge on the cavity state conditioned on the measurement result) is given as follows:

$$\begin{aligned}
\boldsymbol{\rho}_+ &= p_g^f \mathbb{M}_e(\boldsymbol{\rho}) + (1 - p_g^f) \mathbb{M}_g(\boldsymbol{\rho}) \\
&= \frac{\eta_f}{\eta_f p_e + (1 - \eta_f) p_g} \boldsymbol{M}_e \boldsymbol{\rho} \boldsymbol{M}_e^\dagger + \frac{1 - \eta_f}{\eta_f p_e + (1 - \eta_f) p_g} \boldsymbol{M}_g \boldsymbol{\rho} \boldsymbol{M}_g^\dagger \\
&= \frac{\eta_f \boldsymbol{M}_e \boldsymbol{\rho} \boldsymbol{M}_e^\dagger + (1 - \eta_f) \boldsymbol{M}_g \boldsymbol{\rho} \boldsymbol{M}_g^\dagger}{\mathrm{Tr}\left(\eta_f \boldsymbol{M}_e \boldsymbol{\rho} \boldsymbol{M}_e^\dagger + (1 - \eta_f) \boldsymbol{M}_g \boldsymbol{\rho} \boldsymbol{M}_g^\dagger\right)}.
\end{aligned}$$

Atom in $|e\rangle$: Exactly in the same way, the conditional evolution of the density matrix is given as follows:

$$\begin{aligned}\boldsymbol{\rho}_+ &= \frac{\eta_f}{\eta_f p_g + (1-\eta_f)p_e}\boldsymbol{M}_g\boldsymbol{\rho}\boldsymbol{M}_g^\dagger + \frac{1-\eta_f}{\eta_f p_g + (1-\eta_f)p_e}\boldsymbol{M}_e\boldsymbol{\rho}\boldsymbol{M}_e^\dagger \\ &= \frac{\eta_f \boldsymbol{M}_g\boldsymbol{\rho}\boldsymbol{M}_g^\dagger + (1-\eta_f)\boldsymbol{M}_e\boldsymbol{\rho}\boldsymbol{M}_e^\dagger}{\mathrm{Tr}\left(\eta_f \boldsymbol{M}_g\boldsymbol{\rho}\boldsymbol{M}_g^\dagger + (1-\eta_f)\boldsymbol{M}_e\boldsymbol{\rho}\boldsymbol{M}_e^\dagger\right)}.\end{aligned}$$

No atom detected: Either the pulse has been empty (this happens with a probability p_{na} to be determined) or there has been an atom which has not been detected by the detector (this happens with the probability $1-p_{\mathrm{na}}$). Indeed, the conditional probability of having an empty pulse while no atom has been detected by the detector can be computed through the Bayes rule and is given by:

$$p_{\mathrm{na}} = \frac{1-\eta_a}{\eta_a(1-\eta_d)+(1-\eta_a)} = \frac{1-\eta_a}{1-\eta_a\eta_d}.$$

In such case the density matrix remains untouched. The complementary situation corresponding to an undetected atom leads to an evolution of the density matrix through the Kraus map (16). Finally, the conditional evolution of the density matrix (conditioned on the result of the measurement indicating no detected atoms) is given as follows:

$$\begin{aligned}\boldsymbol{\rho}_+ &= p_{\mathrm{na}}\ \boldsymbol{\rho} + (1-p_{\mathrm{na}})(\boldsymbol{M}_g\boldsymbol{\rho}\boldsymbol{M}_g^\dagger + \boldsymbol{M}_e\boldsymbol{\rho}\boldsymbol{M}_e^\dagger) \\ &= \frac{1-\eta_a}{1-\eta_a\eta_d}\boldsymbol{\rho} + \frac{\eta_a(1-\eta_d)}{1-\eta_a\eta_d}(\boldsymbol{M}_g\boldsymbol{\rho}\boldsymbol{M}_g^\dagger + \boldsymbol{M}_e\boldsymbol{\rho}\boldsymbol{M}_e^\dagger) \\ &= \frac{(1-\eta_a)\boldsymbol{\rho} + \eta_a(1-\eta_d)\Big(\boldsymbol{M}_g\boldsymbol{\rho}\boldsymbol{M}_g^\dagger + \boldsymbol{M}_e\boldsymbol{\rho}\boldsymbol{M}_e^\dagger\Big)}{\mathrm{Tr}\left((1-\eta_a)\boldsymbol{\rho} + \eta_a(1-\eta_d)\Big(\boldsymbol{M}_g\boldsymbol{\rho}\boldsymbol{M}_g^\dagger + \boldsymbol{M}_e\boldsymbol{\rho}\boldsymbol{M}_e^\dagger\Big)\right)}\end{aligned}$$

Here, still, we have a Kraus representation for a linear quantum operation.

With the following quantum operations:

$$\begin{aligned}\mathbb{K}_g(\boldsymbol{\rho}) &= \eta_a\eta_d\Big(\eta_f\boldsymbol{M}_e\boldsymbol{\rho}\boldsymbol{M}_e^\dagger + (1-\eta_f)\boldsymbol{M}_g\boldsymbol{\rho}\boldsymbol{M}_g^\dagger\Big) \\ \mathbb{K}_e(\boldsymbol{\rho}) &= \eta_a\eta_d\Big(\eta_f\boldsymbol{M}_g\boldsymbol{\rho}\boldsymbol{M}_g^\dagger + (1-\eta_f)\boldsymbol{M}_e\boldsymbol{\rho}\boldsymbol{M}_e^\dagger\Big) \\ \mathbb{K}_o(\boldsymbol{\rho}) &= (1-\eta_a)\boldsymbol{\rho} + \eta_a(1-\eta_d)\Big(\boldsymbol{M}_g\boldsymbol{\rho}\boldsymbol{M}_g^\dagger + \boldsymbol{M}_e\boldsymbol{\rho}\boldsymbol{M}_e^\dagger\Big)\end{aligned}$$

the above computations define the following Markov chain describing the imperfect measurement process with three possible outcomes, one detection in g, one detection in e and zero detection:

$$\boldsymbol{\rho}_+ = \begin{cases} \frac{\mathbb{K}_g(\boldsymbol{\rho})}{\mathrm{Tr}(\mathbb{K}_g(\boldsymbol{\rho}))}, & \text{with probability } p_g = \mathrm{Tr}\left(\mathbb{K}_g(\boldsymbol{\rho})\right); \\ \frac{\mathbb{K}_e(\boldsymbol{\rho})}{\mathrm{Tr}(\mathbb{K}_e(\boldsymbol{\rho}))}, & \text{with probability } p_e = \mathrm{Tr}\left(\mathbb{K}_e(\boldsymbol{\rho})\right); \\ \frac{\mathbb{K}_o(\boldsymbol{\rho})}{\mathrm{Tr}(\mathbb{K}_o(\boldsymbol{\rho}))}, & \text{with probability } p_o = \mathrm{Tr}\left(\mathbb{K}_o(\boldsymbol{\rho})\right). \end{cases} \tag{17}$$

Notice that, since $\mathbb{K}_g(\boldsymbol{\rho}) + \mathbb{K}_e(\boldsymbol{\rho}) + \mathbb{K}_o(\boldsymbol{\rho}) = (1-\eta_a)\boldsymbol{\rho} + \eta_a\Big(\boldsymbol{M}_g\boldsymbol{\rho}\boldsymbol{M}_g^\dagger + \boldsymbol{M}_e\boldsymbol{\rho}\boldsymbol{M}_e^\dagger\Big)$ and $\boldsymbol{M}_g^\dagger\boldsymbol{M}_g + \boldsymbol{M}_e^\dagger\boldsymbol{M}_e = \boldsymbol{I}$, we have $p_g + p_e + p_o = 1$.

These transition rules provide simple update rules of $\boldsymbol{\rho}_{k+1}$ depending on $\boldsymbol{\rho}_k$ and the detection outcomes at step k belonging to $\{g, e, o\}$. The resulting quantum state $\boldsymbol{\rho}_k$ depends thus on the initial state $\boldsymbol{\rho}_0$ and the measurement outcomes between 0 and $k-1$. In other words, the quantum state obeys to a filtering process of state ρ with the measurement outcomes as input, a so called *quantum filter*.

1.8 Relaxation as an unread measurement

Additionally to the above uncertainties in the measurement process, one needs to consider the relaxation of the system due to its coupling to the environment to obtain a complete model for the open system. Two main sources of relaxation can be considered here. A first source concerns the photon loss phenomenon caused by their absorption by the environment (the mirrors in particular). The second source concerns the photon gain phenomenon due to the coupling of the field with a reservoir of non-zero temperature ($T \approx 0.8K$). Denoting by κ_- and by κ_+, respectively the photon loss and the photon gain rate, and assuming that the environment is in thermal equilibrium at temperature T, we have (k_b denoting the Boltzmann constant and ω_c the cavity's resonance frequency),

$$\kappa_+ = \kappa_- e^{-\frac{\hbar\omega_c}{k_b T}}.$$

We refer to [6, Chapter 4, Page 187] for more details. By defining n_{th} as the average number of thermal photons per mode at frequency ω_c, given by Planck's law:

$$n_{\text{th}} = \frac{1}{e^{\frac{\hbar\omega_c}{k_b T}} - 1},$$

we can express both κ_- and κ_+ in term of unique cavity rate κ:

$$\kappa_- = \kappa(1 + n_{\text{th}}), \qquad \kappa_+ = \kappa n_{\text{th}}.$$

Note that, here the dominant phenomenon is the photon loss as we work in low temperature regime and therefore $n_{\text{th}} \ll 1$ ($n_{\text{th}} \approx 0.05$ for the LKB experiment). We start therefore by investigating the relaxation caused by the photon loss, which can be modeled through a measurement operator $\boldsymbol{M}_{\text{loss}}$, proportional to the photon annihilation operator $\boldsymbol{a}$. Indeed, considering τ_a the duration of a pulse (time interval between the passage of the two atoms), this measurement operator $\boldsymbol{M}_{\text{loss}}$ can be written as $\sqrt{\kappa_- \tau_a}\boldsymbol{a}$ so that the probability of losing a photon during the current pulse is given by (we neglect the possibility of losing many photons at a same pulse as it admits a very small probability)

$$P_{\text{loss}} = \text{Tr}\left(\boldsymbol{M}_{\text{loss}}^\dagger \boldsymbol{M}_{\text{loss}} \boldsymbol{\rho}\right) = \kappa_- \tau_a \,\text{Tr}\left(\boldsymbol{a}^\dagger \boldsymbol{a} \boldsymbol{\rho}\right) = \kappa_- \tau_a \,\text{Tr}\left(\boldsymbol{N}\boldsymbol{\rho}\right).$$

This natural expression indicates that the probability of the photon loss is proportional to the duration of the pulse and to the mean number of photons in the cavity. Here, we assume moreover that the pulse duration is much smaller than the cavity decay time $T_{\text{cav}} = 1/\kappa$ ($\tau_a \ll T_{\text{cav}}$). For the LKB experimental setup, the pulse duration τ_a is about 85.10^{-6} s and T_{cav} is about 13.10^{-2} s and therefore this assumption is clearly satisfied.

Let us assume now that we dispose of an instrument allowing us to the detect this photon loss when it happens. As soon as we detect a photon loss, the cavity density matrix $\boldsymbol{\rho}$ evolves

drastically as follows:

$$\boldsymbol{\rho}_+ = \frac{\boldsymbol{M}_{\text{loss}}\boldsymbol{\rho}\boldsymbol{M}_{\text{loss}}^\dagger}{\text{Tr}\left(\boldsymbol{M}_{\text{loss}}\boldsymbol{\rho}\boldsymbol{M}_{\text{loss}}^\dagger\right)} = \frac{\boldsymbol{a}\boldsymbol{\rho}\boldsymbol{a}^\dagger}{\text{Tr}\left(\boldsymbol{N}\boldsymbol{\rho}\right)},$$

recalling that this loss happens with a small probability of $\text{Tr}\left(\boldsymbol{N}\boldsymbol{\rho}\right)(1+n_{\text{th}})\tau_a/\tau_{\text{cav}}$. Now, let us consider the situation where we do not detect any photon loss. A first impression would be that the density matrix should not change. This is not correct and the fact that we do not detect any photon, actually, updates our information on the system as it privileges the probability of having a fewer number of photons in the cavity. In order to have a more clear idea of the situation, let us assume that, similarly to the photon loss case, we associate a measurement operator $\boldsymbol{M}_{\text{no-loss}}$ to the phenomenon of not detecting a photon loss. Let us now find this jump operator.

In order to have a well-defined POVM, we need to have

$$\boldsymbol{M}_{\text{loss}}^\dagger\boldsymbol{M}_{\text{loss}} + \boldsymbol{M}_{\text{no-loss}}^\dagger\boldsymbol{M}_{\text{no-loss}} = \boldsymbol{I}. \tag{18}$$

This, in particular, forbids the possibility of having $\boldsymbol{M}_{\text{no-loss}} = \boldsymbol{I}$. Indeed, a possible solution, up to the first order in τ_a/T_{cav}, is given by:

$$\boldsymbol{M}_{\text{no-loss}} = \boldsymbol{I} - (1+n_{\text{th}})\tau_a\frac{\tau_a}{2T_{\text{cav}}}\boldsymbol{a}^\dagger\boldsymbol{a}.$$

Noting now that, we actually do not dispose of a measurement instrument indicating the loss of the photons, the evolution of the density matrix is given by the following Kraus representation:

$$\boldsymbol{\rho}_+ = \boldsymbol{M}_{\text{loss}}\boldsymbol{\rho}\boldsymbol{M}_{\text{loss}}^\dagger + \boldsymbol{M}_{\text{no-loss}}\boldsymbol{\rho}\boldsymbol{M}_{\text{no-loss}}^\dagger = \boldsymbol{\rho} + (1+n_{\text{th}})\frac{\tau_a}{T_{\text{cav}}}\left(\boldsymbol{a}\boldsymbol{\rho}\boldsymbol{a}^\dagger - \tfrac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}\boldsymbol{\rho} - \tfrac{1}{2}\boldsymbol{\rho}\boldsymbol{a}^\dagger\boldsymbol{a}\right),$$

where we have still neglected the second order terms in τ_a/T_{cav}.

The photon gain phenomenon can be treated exactly in the same way and through the measurement operator $\boldsymbol{M}_{\text{gain}} = \sqrt{\kappa_+\tau_a}\boldsymbol{a}^\dagger$ proportional to the photon creation operator. The total evolution can be therefore written as follows:

$$\boldsymbol{\rho}_+ = \boldsymbol{M}_{\text{loss}}\boldsymbol{\rho}\boldsymbol{M}_{\text{loss}}^\dagger + \boldsymbol{M}_{\text{gain}}\boldsymbol{\rho}\boldsymbol{M}_{\text{gain}}^\dagger + \boldsymbol{M}_{\text{no}}\boldsymbol{\rho}\boldsymbol{M}_{\text{no}}^\dagger$$

where the operator $\boldsymbol{M}_{\text{no}}$ closed to $\boldsymbol{I}$ and corresponding to no-loss and no-gain has to satisfy

$$\boldsymbol{M}_{\text{loss}}^\dagger\boldsymbol{M}_{\text{loss}} + \boldsymbol{M}_{\text{gain}}^\dagger\boldsymbol{M}_{\text{gain}} + \boldsymbol{M}_{\text{no}}^\dagger\boldsymbol{M}_{\text{no}} = \boldsymbol{I}.$$

Up to second order terms versus τ_a/T_{cav} we have

$$\boldsymbol{M}_{\text{no}} = \boldsymbol{I} - (1+n_{\text{th}})\tau_a\frac{\tau_a}{2T_{\text{cav}}}\boldsymbol{a}^\dagger\boldsymbol{a} - n_{\text{th}}\tau_a\frac{\tau_a}{2T_{\text{cav}}}\boldsymbol{a}\boldsymbol{a}^\dagger.$$

The associated Kraus map reads then

$$\boldsymbol{\rho}_+ = \boldsymbol{\rho} + (1+n_{\text{th}})\frac{\tau_a}{T_{\text{cav}}}\left(\boldsymbol{a}\boldsymbol{\rho}\boldsymbol{a}^\dagger - \tfrac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}\boldsymbol{\rho} - \tfrac{1}{2}\boldsymbol{\rho}\boldsymbol{a}^\dagger\boldsymbol{a}\right) + n_{\text{th}}\frac{\tau_a}{T_{\text{cav}}}\left(\boldsymbol{a}^\dagger\boldsymbol{\rho}\boldsymbol{a} - \tfrac{1}{2}\boldsymbol{a}\boldsymbol{a}^\dagger\boldsymbol{\rho} - \tfrac{1}{2}\boldsymbol{\rho}\boldsymbol{a}\boldsymbol{a}^\dagger\right).$$

Exploiting the fact that $\tau_a \ll T_{\text{cav}}$ (small sampling period τ_a) this equation becomes a differential equation:

$$\frac{\boldsymbol{\rho}_+ - \boldsymbol{\rho}}{\tau_a} \approx \frac{d}{dt}\boldsymbol{\rho} = (1+n_{\text{th}})\kappa\left(\boldsymbol{a}\boldsymbol{\rho}\boldsymbol{a}^\dagger - \tfrac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}\boldsymbol{\rho} - \tfrac{1}{2}\boldsymbol{\rho}\boldsymbol{a}^\dagger\boldsymbol{a}\right) + n_{\text{th}}\kappa\left(\boldsymbol{a}^\dagger\boldsymbol{\rho}\boldsymbol{a} - \tfrac{1}{2}\boldsymbol{a}\boldsymbol{a}^\dagger\boldsymbol{\rho} - \tfrac{1}{2}\boldsymbol{\rho}\boldsymbol{a}\boldsymbol{a}^\dagger\right). \tag{19}$$

This kind of equation will be investigated in next chapter on continuous-time open quantum systems.

2 Structure of discrete-time open quantum systems

The theory of open quantum systems starts with the contributions of Davies [4]. The goal of this section is first to present in an elementary way the general structure of the Markov models describing such systems. Throughout this section, $\mathcal{H}$ is an Hilbert space; for each time-step $k \in \mathbb{N}$, $\boldsymbol{\rho}_k$ denotes the density operator describing the state of the quantum Markov process; for all k, $\boldsymbol{\rho}_k$ is a trace class operator on $\mathcal{H}$, Hermitian and of trace one.

2.1 Markov models

These models generalize the models developed for the photon box (17) merging quantum measurement and probability theory with classical probability through Bayesian estimation. Take a positive integer m and consider a finite set $(\boldsymbol{M}_\mu)_{\mu\in\{1,\dots,m\}}$ of operators on $\mathcal{H}$ such that

$$\boldsymbol{I} = \sum_{\mu=1}^{m} \boldsymbol{M}_\mu^\dagger \boldsymbol{M}_\mu \tag{20}$$

Then each $\boldsymbol{M}_\mu \in \mathcal{L}(\mathcal{H})$. Take another positive integer m' and consider a left stochastic $m' \times m$-matrix $(\eta_{\mu'\mu})$: its entries are non-negative and $\forall \mu \in \{1,\dots,m\}$, $\sum_{\mu'=1}^{m'} \eta_{\mu'\mu} = 1$. Consider the Markov process of state $\boldsymbol{\rho}$ and output $y \in \{1,\dots,m'\}$ (measurement outcome) defined via the transition rule

$$\boldsymbol{\rho}_{k+1} = \frac{\sum_\mu \eta_{\mu'\mu} \boldsymbol{M}_\mu \boldsymbol{\rho}_k \boldsymbol{M}_\mu^\dagger}{\mathrm{Tr}\left(\sum_\mu \eta_{\mu'\mu} \boldsymbol{M}_\mu \boldsymbol{\rho}_k \boldsymbol{M}_\mu^\dagger\right)}, \quad y_k = \mu' \text{ with probability } p_{\mu'}(\boldsymbol{\rho}_k) \tag{21}$$

where $p_{\mu'}(\boldsymbol{\rho}) = \mathrm{Tr}\left(\sum_\mu \eta_{\mu'\mu} \boldsymbol{M}_\mu \boldsymbol{\rho} \boldsymbol{M}_\mu^\dagger\right)$. The left stochastic matrix η yields to the decomposition of the Kraus map $\mathbb{K}$ into the sum of m' partial Kraus maps $(\mathbb{K}_{\mu'})_{\mu'\in\{1,\dots,m'\}}$:

$$\mathbb{K}(\boldsymbol{\rho}) = \sum_{\mu'=1}^{m'} \mathbb{K}_{\mu'}(\boldsymbol{\rho}) \quad \text{with } \mathbb{K}_{\mu'}(\boldsymbol{\rho}) = \sum_\mu \eta_{\mu'\mu} \boldsymbol{M}_\mu \boldsymbol{\rho} \boldsymbol{M}_\mu^\dagger. \tag{22}$$

The Markov chain (21) reads:

$$\boldsymbol{\rho}_{k+1} = \frac{\mathbb{K}_{\mu'}(\boldsymbol{\rho}_k)}{\mathrm{Tr}\left(\mathbb{K}_{\mu'}(\boldsymbol{\rho}_k)\right)}, \quad y_k = \mu' \text{ with probability } p_{\mu'}(\boldsymbol{\rho}_k) = \mathrm{Tr}\left(\mathbb{K}_{\mu'}(\boldsymbol{\rho}_k)\right). \tag{23}$$

2.2 Kraus and unital maps

The Kraus map $\mathbb{K}$ corresponds to the master equation of (21). It is given by the expectation value of $\boldsymbol{\rho}_{k+1}$ knowing $\boldsymbol{\rho}_k$:

$$\mathbb{K}(\boldsymbol{\rho}) \triangleq \sum_\mu \boldsymbol{M}_\mu \boldsymbol{\rho} \boldsymbol{M}_\mu^\dagger = \mathbb{E}\left(\boldsymbol{\rho}_{k+1} \;/\; \boldsymbol{\rho}_k = \boldsymbol{\rho}\right). \tag{24}$$

In quantum information [9] such Kraus maps describe quantum channels. They admit many interesting properties. In particular, they are contractions for many metrics (see [10] for the

characterization, in finite dimension, of metrics for which any Kraus map is a contraction). We just recall below two such metrics. For any density operators $\boldsymbol{\rho}$ and $\boldsymbol{\rho}'$ we have

$$D(\mathbb{K}(\boldsymbol{\rho}), \mathbb{K}(\boldsymbol{\rho}')) \leq D(\boldsymbol{\rho}, \boldsymbol{\rho}') \text{ and } F(\mathbb{K}(\boldsymbol{\rho}), \mathbb{K}(\boldsymbol{\rho}')) \geq F(\boldsymbol{\rho}, \boldsymbol{\rho}') \tag{25}$$

where the trace distance D and fidelity F are given by

$$D(\boldsymbol{\rho}, \boldsymbol{\rho}') \triangleq \mathrm{Tr}\left(|\boldsymbol{\rho} - \boldsymbol{\rho}'|\right) \text{ and } F(\boldsymbol{\rho}, \boldsymbol{\rho}') \triangleq \mathrm{Tr}^2\left(\sqrt{\sqrt{\boldsymbol{\rho}}\boldsymbol{\rho}'\sqrt{\boldsymbol{\rho}}}\right). \tag{26}$$

Fidelity is between 0 and 1: $F(\boldsymbol{\rho}, \boldsymbol{\rho}') = 1$ if and only if, $\boldsymbol{\rho} = \boldsymbol{\rho}'$. Moreover $F(\boldsymbol{\rho}, \boldsymbol{\rho}') = F(\boldsymbol{\rho}', \boldsymbol{\rho})$. If $\boldsymbol{\rho}' = |\psi\rangle\langle\psi|$ is a pure state ($|\psi\rangle$ element of $\mathcal{H}$ of length one), $F(\boldsymbol{\rho}, \boldsymbol{\rho}')$ coincides with the Frobenius product: $F(\boldsymbol{\rho}, |\psi\rangle\langle\psi|) \equiv \mathrm{Tr}\left(\boldsymbol{\rho}\,|\psi\rangle\langle\psi|\right) = \langle\psi|\,\boldsymbol{\rho}\,|\psi\rangle$. Kraus maps provide the evolution of open quantum systems from an initial state $\boldsymbol{\rho}_0$ without information coming from the measurements (see [6, chapter 4: the environment is watching]):

$$\boldsymbol{\rho}_{k+1} = \mathbb{K}(\boldsymbol{\rho}_k) \text{ for } k = 0, 1, \ldots, . \tag{27}$$

This corresponds to the "Schrödinger description" of the dynamics.

The "Heisenberg description" is given by the dual map $\mathbb{K}^*$. It is characterized by $\mathrm{Tr}\left(\boldsymbol{A}\mathbb{K}(\boldsymbol{\rho})\right) = \mathrm{Tr}\left(\mathbb{K}^*(\boldsymbol{A})\boldsymbol{\rho}\right)$ and defined for any bounded operator $\boldsymbol{A}$ on $\mathcal{H}$ by

$$\mathbb{K}^*(\boldsymbol{A}) = \sum_{\mu} \boldsymbol{M}_{\mu}^{\dagger}\boldsymbol{A}\boldsymbol{M}_{\mu}.$$

Technical conditions on $\boldsymbol{A}$ are required when $\mathcal{H}$ is of infinite dimension, they are not given here (see, e.g., [4]). The map $\mathbb{K}^*$ is unital since (20) reads $\mathbb{K}^*(\boldsymbol{I}) = \boldsymbol{I}$. As $\mathbb{K}$, the dual map $\mathbb{K}^*$ admits a lot of interesting properties. It is noticed in [14] that, based on a theorem due of Birkhoff [3], such unital maps are contractions on the cone of non-negative Hermitian operators equipped with the Hilbert's projective metric. In particular, when $\mathcal{H}$ is of finite dimension, we have, for any Hermitian operator $\boldsymbol{A}$:

$$\lambda_{min}(\boldsymbol{A}) \leq \lambda_{min}(\mathbb{K}^*(\boldsymbol{A})) \leq \lambda_{max}(\mathbb{K}^*(\boldsymbol{A})) \leq \lambda_{max}(\boldsymbol{A})$$

where λ_{min} and λ_{max} correspond to the smallest and largest eigenvalues. As shown in [11], such contraction properties based on Hilbert's projective metric have important implications in quantum information theory.

To emphasize the difference between the "Schrödinger description" and the 'Heisenberg description" of the dynamics, let us translate convergence issues from the "Schrödinger description" to the "Heisenberg one". Assume, for clarity's sake, that $\mathcal{H}$ is of finite dimension. Suppose also that $\mathbb{K}$ admits the density operator $\bar{\boldsymbol{\rho}}$ as unique fixed point and that, for any initial density operator $\boldsymbol{\rho}_0$, the density operator at step k, $\boldsymbol{\rho}_k$, defined by k iterations of $\mathbb{K}$, converges towards $\bar{\boldsymbol{\rho}}$ when k tends to ∞. Then $k \mapsto D(\boldsymbol{\rho}_k, \bar{\boldsymbol{\rho}})$ is decreasing and converges to 0 whereas $k \mapsto F(\boldsymbol{\rho}_k, \bar{\boldsymbol{\rho}})$ is increasing and converges to 1.

The translation of this convergence in the "Heisenberg description" is the following: for any initial operator $\boldsymbol{A}_0$, its k iterates via $\mathbb{K}^*$, $\boldsymbol{A}_k$, converge towards $\mathrm{Tr}\left(\boldsymbol{A}_0\bar{\boldsymbol{\rho}}\right)\boldsymbol{I}$. Moreover when $\boldsymbol{A}_0$ is Hermitian, $k \mapsto \lambda_{min}(\boldsymbol{A}_k)$ and $k \mapsto \lambda_{max}(\boldsymbol{A}_k)$ are respectively increasing and decreasing and both converge to $\mathrm{Tr}\left(\boldsymbol{A}_0\bar{\boldsymbol{\rho}}\right)$.

Notice finally that any bounded operator $\overline{\boldsymbol{A}}$ that is a fixed point of $\mathbb{K}^*$, $\mathbb{K}^*(\overline{\boldsymbol{A}}) = \overline{\boldsymbol{A}}$, yields to a constant of motion for $\boldsymbol{\rho}_{k+1} = \mathbb{K}(\boldsymbol{\rho}_k)$:

$$\mathrm{Tr}\left(\overline{\boldsymbol{A}}\boldsymbol{\rho}_k\right) = \mathrm{Tr}\left(\overline{\boldsymbol{A}}\boldsymbol{\rho}_0\right).$$

This means that, for any unraveling Markov process of the form (22), the stochastic variable $\mathrm{Tr}\left(\overline{\boldsymbol{A}}\boldsymbol{\rho}_k\right)$ is a martingale and is attached to fundamental properties of the dynamics (analogue of a priori estimates and first integral for (partial) differential equations).

2.3 Quantum filtering

Quantum filtering has its origin in Belavkin's work [2] on continuous-time open quantum systems (see next chapter). We just give here a discrete-time version. The state $\boldsymbol{\rho}_k$ of (23) is not directly measured: open quantum systems are governed by hidden-state Markov model. Quantum filtering provides an estimate $\boldsymbol{\rho}_k^{\mathrm{est}}$ of $\boldsymbol{\rho}_k$ based on an initial guess $\boldsymbol{\rho}_0^{\mathrm{est}}$ (possibly different from $\boldsymbol{\rho}_0$) and the measurement outcomes y_l between 0 and $k-1$:

$$\boldsymbol{\rho}_{l+1}^{\mathrm{est}} = \frac{\mathbb{K}_{y_l}(\boldsymbol{\rho}_l^{\mathrm{est}})}{\mathrm{Tr}\left(\mathbb{K}_{y_l}(\boldsymbol{\rho}_l^{\mathrm{est}})\right)}, \quad l \in \{0, \ldots, k-1\}. \tag{28}$$

Thus $(\boldsymbol{\rho}, \boldsymbol{\rho}^{\mathrm{est}})$ is the state of an extended Markov process governed by the following rule

$$\boldsymbol{\rho}_{k+1} = \frac{\mathbb{K}_{\mu'}(\boldsymbol{\rho}_k)}{\mathrm{Tr}\left(\mathbb{K}_{\mu'}(\boldsymbol{\rho}_k)\right)} \text{ and } \boldsymbol{\rho}_{k+1}^{\mathrm{est}} = \frac{\mathbb{K}_{\mu'}(\boldsymbol{\rho}_k^{\mathrm{est}})}{\mathrm{Tr}\left(\mathbb{K}_{\mu'}(\boldsymbol{\rho}_k^{\mathrm{est}})\right)}$$

with transition probability $p_{\mu'}(\boldsymbol{\rho}_k) = \mathrm{Tr}\left(\mathbb{K}_{\mu'}(\boldsymbol{\rho}_k)\right)$ depending only on $\boldsymbol{\rho}_k$.

When $\mathcal{H}$ is of finite dimension, it is shown in [15] with an inequality proved in [12] that such discrete-time quantum filters are always stable in the following sense: the fidelity between $\boldsymbol{\rho}$ and its estimate $\boldsymbol{\rho}^{\mathrm{est}}$ is a sub-martingale for any initial condition $\boldsymbol{\rho}_0$ and $\boldsymbol{\rho}_0^{\mathrm{est}}$: $\mathbb{E}\left(F(\boldsymbol{\rho}_{k+1}, \boldsymbol{\rho}_{k+1}^{\mathrm{est}}) \mid (\boldsymbol{\rho}_k, \boldsymbol{\rho}_k^{\mathrm{est}})\right) \geq F(\boldsymbol{\rho}_k, \boldsymbol{\rho}_k^{\mathrm{est}})$. This result does not guaranty that $\boldsymbol{\rho}_k^{\mathrm{est}}$ converges to $\boldsymbol{\rho}_k$ when k tends to infinity. The convergence characterization of $\boldsymbol{\rho}^{\mathrm{est}}$ towards $\boldsymbol{\rho}$ via checkable conditions on the partial Kraus maps $(\mathbb{K}_{\mu'})$ remains an open problem [16, 17]. Characterization of asymptotic almost-sure convergence is an open-problem with recent progresses in [1].

3 Quantum measurement

Whenever talking about the quantum state of a system, we refer to an observer's knowledge about a system. More precisely, it is the knowledge of the observer about the outcome of the future measurements on the system.

Such information theoretical definition of the state of a physical system may appear unfamiliar and uncomfortable as for instance, the observers with different knowledge may assign different states, simultaneously, to a single system. The most natural way to talk about the consistency of these assigned states is to define a *common state of maximal knowledge* as a common pure state. So far through these lecture notes, we have only considered such a *common state of maximal knowledge* and its evolution for a closed quantum system where no measurement is performed on the system. This pure state is well represented by a wave function $|\psi\rangle$ and its evolution is given by a Schrödinger equation as discussed through the

previous chapters. The rest of these notes, however, is devoted to the study of the case where the quantum system is measured by an observer and in such a case, one needs to consider a wider formulation of the quantum state called the density operator (or density matrix in the case of finite dimensional quantum system). A density operator $\boldsymbol{\rho}$ is a Hermitian, semi-definite positive, trace-class operator (we refer to the Appendix **??** for an introduction to operator spaces) defined on the Hilbert space of the quantum system. Moreover its trace is constant and equals unity during the evolution of the system. Such a density operator represents the knowledge of an observer about the quantum system.

Considering the collection $\{\boldsymbol{\rho}_j\}$ of different density matrices assigned by different observers to a same physical system, the *common state of maximal knowledge* is a pure state defined by a wave function $|\psi\rangle$ such that there exists an $\epsilon_j > 0$ for which, $\boldsymbol{\rho}_j - \epsilon_j |\psi\rangle \langle\psi|$ is a positive operator, i.e. $\boldsymbol{\rho}_j$ is the mixture of $|\psi\rangle$ with some other states. From a system theoretical point of view, we can think of this common state of maximal knowledge as the actual state of the system and the density matrix $\boldsymbol{\rho}_j$ is the filtering state encoding the information gained by an observer j.

Another consequence of such definition of the quantum state is that any measurement of the system, which leads to obtaining information on the system, necessarily changes the state of the system. This is known as the projection postulate. Through this section, we provide a brief overview of important measurement paradigms for quantum systems and the two next sections are devoted to some concrete examples. This chapter is strongly inspired from [6] and [18].

3.1 Projective measurement

The projective measurement is the traditional description of measurement in quantum mechanics. Indeed, assume the measurement of a physical quantity O to which we can assign a Hermitian operator (observable) $\boldsymbol{O}$ defined on $\mathcal{H}$ the Hilbert space of the system. We start by diagonalizing the operator as

$$\boldsymbol{O} = \sum_{\nu} \lambda_\nu \boldsymbol{P}_\nu,$$

where λ_ν's are the eigenvalues of $\boldsymbol{O}$, which are all real and different, and $\boldsymbol{P}_\nu$ the projection operator over the associated eigenspace. Note that, in general, the spectrum of the operator $\boldsymbol{O}$ can be degenerate and therefore the projection operator $\boldsymbol{P}_\nu$ is not necessarily a rank-1 operator.

When we measure $\boldsymbol{O}$, the result will be necessarily one of the eigenvalues λ_ν. Moreover, an outcome λ_ν of the measurement implies an instantaneous projection of the state of our knowledge through the associated projection operator. We also talk of the conditional state of the system as it is conditioned on the measurement outcome. Indeed, assuming that our state of knowledge at time t is given by the density matrix[2] $\boldsymbol{\rho}$, measurement of the physical observable $\boldsymbol{O}$ at time t can be formulated as below:

1. The probability of obtaining the value λ_ν is given by $p_\nu = \mathrm{Tr}\,(\boldsymbol{\rho}\boldsymbol{P}_\nu)$; note that $\sum_\nu p_\nu = 1$ as $\sum_\nu \boldsymbol{P}_\nu = \boldsymbol{I}_{\mathcal{H}}$ ($\boldsymbol{I}_{\mathcal{H}}$ represents the identity operator of $\mathcal{H}$).

[2] $\boldsymbol{\rho}$ is a Hermitian, semi-definite positive, trace-class operator on $\mathcal{H}$ of trace 1. Thus $\mathrm{Tr}\left(\boldsymbol{\rho}^2\right) \leq 1$ with equality only when $\boldsymbol{\rho}$ is an orthogonal projector on some pure quantum state $|\psi\rangle$, i.e., $\boldsymbol{\rho} = |\psi\rangle \langle\psi|$.

2. After the measurement, the conditional (a posteriori) state of the system given the outcome λ_ν is

$$\boldsymbol{\rho}_+ = \frac{\boldsymbol{P}_\nu\ \boldsymbol{\rho}\ \boldsymbol{P}_\nu}{p_\nu}.$$

Here, $\boldsymbol{\rho}_+$ denotes the state of the system just after the measurement. Furthermore, we have assumed that the evolution, from other causes, of the system during the measurement process is not significant and can be neglected.

A particular feature of the projective measurement is that, if the same measurement is immediately repeated, then the same result is guaranteed. Indeed, the probability of obtaining the same result λ_ν for the second measurement of the observable $\boldsymbol{O}$ is given by

$$\mathrm{Tr}\left(\boldsymbol{P}_\nu \boldsymbol{\rho}_+\right) = \mathrm{Tr}\left(\boldsymbol{P}_\nu\ \boldsymbol{\rho}\ \boldsymbol{P}_\nu\right)/p_\nu = 1,$$

where we have applied the fact that $\boldsymbol{P}_\nu \boldsymbol{P}_\nu = \boldsymbol{P}_\nu$.

For pure states (encoding the common state of maximal knowledge), $\boldsymbol{\rho} = |\psi\rangle\langle\psi|$, the projective measurement can be more simply expressed as

$$p_\nu = \langle\psi|\,\boldsymbol{P}_\nu\,|\psi\rangle\,,$$
$$\psi_+ = \frac{\boldsymbol{P}_\nu \psi}{\sqrt{p_\nu}}.$$

Finally, the particular case of a projective measurement where the eigenvalues $\{\lambda_\nu\}$ are non-degenerate, and therefore the eigenprojections $\boldsymbol{P}_\nu$ are rank-1 operators, is called a *von Neumann measurement*.

3.2 Positive Operator-Valued Measure (POVM)

The projective measurements are, generally, inadequate for describing real measurements, as the experimenter never directly measures the system of interest. In fact, the system of interest (for instance an atom or a quantized electromagnetic field) interacts with its environment (electromagnetic field or a probe atom), and the experimenter observes the effect of the system on the environment (the radiated field or the probe atom).

In order to formulate such measurement paradigm, we need to consider the quantum state in a larger Hilbert space consisting of the system and the measurement apparatus (also called the meter). Indeed, we consider a total initial state (before the measurement process) for the system together with the meter, which is given by a separable wavefunction

$$|\Psi\rangle = |\psi_S\rangle \otimes |\theta_M\rangle$$

living on the total Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_M$. The measurement process consists in a unitary evolution of the whole state (leading to a non-separable -entangled- state) followed by a projective von Neumann measurement of the measurement apparatus. Let us denote by $\boldsymbol{U}_{S,M}$ the unitary evolution entangling the state of the system to that of the meter, and by $\boldsymbol{O}_M = \boldsymbol{I}_S \otimes \left(\sum_\nu \lambda_\nu \boldsymbol{P}_\nu\right)$ the measured observable for the meter. Here, the projection operator $\boldsymbol{P}_\nu$ is a rank-1 projection in $\mathcal{H}_M$ over the eigenstate $|\lambda_\nu\rangle \in \mathcal{H}_M$: $\boldsymbol{P}_\nu = |\lambda_\nu\rangle\langle\lambda_\nu|$. The measurement procedure can be formulated as below

1. The probability of obtaining the value λ_ν is given by $p_\nu = \langle\psi_S|\, \boldsymbol{M}_\nu^\dagger \boldsymbol{M}_\nu\, |\psi_S\rangle$ where $\boldsymbol{M}_\nu$ is an operator defined on $\mathcal{H}_S$, the Hilbert space of the system, by

$$\big(\boldsymbol{M}_\nu\, |\psi_S\rangle\,\big) \otimes |\lambda_\nu\rangle = \big(\boldsymbol{I}_S \otimes \boldsymbol{P}_\nu\big)\boldsymbol{U}_{S,M}\big(\, |\psi_S\rangle \otimes |\theta_M\rangle\,\big).$$

Thus we have

$$\boldsymbol{U}_{S,M}\big(\, |\psi_S\rangle \otimes |\theta_M\rangle\,\big) = \sum_\nu \big(\boldsymbol{M}_\nu\, |\psi_S\rangle\,\big) \otimes |\lambda_\nu\rangle\,.$$

Note that $\sum_\nu p_\nu = 1$ as

$$\sum_\nu \langle\psi_S|\, \boldsymbol{M}_\nu^\dagger \boldsymbol{M}_\nu\, |\psi_S\rangle = \big(\, |\psi_S\rangle \otimes |\theta_M\rangle\,\big)^\dagger \boldsymbol{U}_{S,M}^\dagger \left(\sum_\nu \boldsymbol{I}_H \otimes \boldsymbol{P}_\nu\right) \boldsymbol{U}_{S,M}\big(\, |\psi_S\rangle \otimes |\theta_M\rangle\,\big) = 1, \quad (29)$$

where we have used $\sum_\nu |\lambda_\nu\rangle\, \langle\lambda_\nu| = \boldsymbol{I}_M$ and $\boldsymbol{U}_{S,M}^\dagger \boldsymbol{U}_{S,M} = \boldsymbol{I}_{SM}$.

2. After the measurement, the conditional (a posteriori) state of the system given the outcome λ_ν is

$$|\psi_S\rangle_+ = \frac{\boldsymbol{M}_\nu\, |\psi_S\rangle}{\sqrt{p_\nu}}.$$

The operators $\boldsymbol{M}_\nu$ are called the *measurement operators* (see appendix A).

This can also be extended to the case of a mixed state where the probability of obtaining the value λ_ν is simply given by $p_\nu = \mathrm{Tr}\left(\boldsymbol{M}_\nu \boldsymbol{\rho} \boldsymbol{M}_\nu^\dagger\right)$ and the conditional state given the outcome λ_ν is

$$\boldsymbol{\rho}_+ = \mathbb{M}_\nu(\boldsymbol{\rho}) := \frac{\boldsymbol{M}_\nu \boldsymbol{\rho} \boldsymbol{M}_\nu^\dagger}{\mathrm{Tr}\left(\boldsymbol{M}_\nu \boldsymbol{\rho} \boldsymbol{M}_\nu^\dagger\right)}, \quad (30)$$

with $\mathbb{M}_\nu$ a nonlinear superoperator (it sends an operator to an operator) on $\mathcal{H}_S$. Indeed, through the computations of (29), $\sum_\nu \boldsymbol{M}_\nu^\dagger \boldsymbol{M}_\nu = \boldsymbol{I}_S$ and this, together with the positiveness of the operators $\boldsymbol{M}_\nu^\dagger \boldsymbol{M}_\nu$, are the only conditions for the set $\{\boldsymbol{M}_\nu\}$ to define a *Positive Operator-Valued Measure* (POVM).

Also, one can define the *Generalized POVM* as the case where the initial state of the meter is not a pure state or that the projective measurement of the meter is not a von Neumann measurement (see [18, chapter 1] for a tutorial exposure to quantum measurements).

3.3 Quantum Non-Demolition (QND) measurement

Before anything, we need that the measurement of the meter observable $\boldsymbol{O}_M$ after the interaction between the system and the meter encodes some information on the system S itself. This imposes some constraints on unitary transformation $\boldsymbol{U}_{S,M}$ considered in the previous subsection:

$$\boldsymbol{U}_{S,M}\, |\Psi\rangle = \boldsymbol{U}_{S,M}\big(\, |\psi_S\rangle \otimes |\theta_M\rangle\,\big).$$

Assume that such unitary transformation $\boldsymbol{U}_{S,M}$ results from a Hamiltonian $\boldsymbol{H} = \boldsymbol{H}_S + \boldsymbol{H}_M + \boldsymbol{H}_{SM}$ where $\boldsymbol{H}_S$ and $\boldsymbol{H}_M$ describe, respectively, the evolutions of the system and the meter and $\boldsymbol{H}_{SM}$ denotes the system-meter interaction Hamiltonian. Then $\boldsymbol{U}_{S,M}$ is the propagator generated by $\boldsymbol{H}$ during the interaction interval of length τ between S and M (for time-invariant $\boldsymbol{H}$, we have $\boldsymbol{U}_{S,M} = e^{-i\tau \boldsymbol{H}}$). It is clear that a necessary condition for the influence of S on $\boldsymbol{O}_M$ just after the interaction is that $[\boldsymbol{H}, \boldsymbol{O}_M] \neq 0$. Otherwise $\boldsymbol{O}_M \boldsymbol{U}_{S,M} = \boldsymbol{U}_{S,M} \boldsymbol{O}_M$.

Using the spectral decomposition $\boldsymbol{O}_M = \sum_\nu \lambda_\nu \boldsymbol{I}_S \otimes |\lambda_\nu\rangle$ (see previous subsection), we have for any ν,

$$\boldsymbol{O}_M \boldsymbol{U}_{S,M}\big(\,|\psi_S\rangle \otimes |\lambda_\nu\rangle\,\big) = \boldsymbol{U}_{S,M}\boldsymbol{O}_M\big(\,|\psi_S\rangle \otimes |\lambda_\nu\rangle\,\big) = \lambda_\nu \boldsymbol{U}_{S,M}\big(\,|\psi_S\rangle \otimes |\theta_M\rangle\,\big).$$

Thus, necessarily $\boldsymbol{U}_{S,M}\big(\,|\psi_S\rangle\otimes|\lambda_\nu\rangle\,\big) = \big(\boldsymbol{U}_\nu\,|\psi_S\rangle\,\big)\otimes|\lambda_\nu\rangle$ where $\boldsymbol{U}_\nu$ is a unitary transformation on $\mathcal{H}_S$ only. With $|\theta_M\rangle = \sum_\nu \theta_\nu\,|\lambda_\nu\rangle$, we get, for any $|\psi_S\rangle$,

$$\boldsymbol{U}_{S,M}\big(\,|\psi_S\rangle \otimes |\theta_M\rangle\,\big) = \sum_\nu \theta_\nu \big(\boldsymbol{U}_\nu\,|\psi_S\rangle\,\big) \otimes |\lambda_\nu\rangle$$

Then measurement operators $\boldsymbol{M}_\nu$ are equal to $\theta_\nu \boldsymbol{U}_\nu$. The probability to get measurement outcome ν, $\langle\psi_S|\boldsymbol{M}_\nu^\dagger \boldsymbol{M}_\nu|\psi_S\rangle = |\theta_\nu|^2$, is completely independent of systems state $|\psi_S\rangle$. This means that the measurement statistics for the meter observable $\boldsymbol{O}_M$ does not encode any information on the system S and therefore $[\boldsymbol{H}, \boldsymbol{O}_M]$ must not vanish. When $\boldsymbol{H}_M = 0$, this necessary condition reads $[\boldsymbol{H}_{SM}, \boldsymbol{O}_M] \neq 0$.

Let us consider the measurement of a physical observable $\boldsymbol{O}_S$ defined for the system S, through its coupling with a meter M with a von Neumann measurements of an observable $\boldsymbol{O}_M$ on the meter. The essential condition for a measurement process of $\boldsymbol{O}_S$ to be quantum non-demolition (abbreviated as QND) is that the measurement should not affect the eigenstates of $\boldsymbol{O}_S$ when $\boldsymbol{O}_S$ admits a non degenerate spectrum (other-wise we have to consider the eigenspace instead of the eigenstate). A sufficient but not necessary condition for this is

$$[\boldsymbol{H}, \boldsymbol{O}_S] = 0$$

Under this condition $\boldsymbol{O}_S$ and $\boldsymbol{U}_{S,M}$ commute. For eigenstate $|\mu\rangle$ of $\boldsymbol{O}_S$ associated to eigenvalue μ, we have

$$\boldsymbol{O}_S \boldsymbol{U}_{S,M}\big(\,|\mu\rangle \otimes |\theta_M\rangle\,\big) = \boldsymbol{U}_{S,M}\boldsymbol{O}_S\big(\,|\mu\rangle \otimes |\theta_M\rangle\,\big) = \mu \boldsymbol{U}_{S,M}\big(\,|\mu\rangle \otimes |\theta_M\rangle\,\big).$$

Exercice 6. *Prove that the above formula implies* $\boldsymbol{U}_{S,M}\big(\,|\mu\rangle \otimes |\theta_M\rangle\,\big) = |\mu\rangle \otimes \big(\boldsymbol{U}_\mu\,|\theta_M\rangle\,\big)$ *where* $\boldsymbol{U}_\mu$ *is a unitary operator on* $\mathcal{H}_M$ *only:* $\boldsymbol{U}_{S,M}$ *does not entangle eigenstates of* $\boldsymbol{O}_S$ *with the meter.*

With the measurement operators $\boldsymbol{M}_\nu$, we also have

$$\boldsymbol{U}_{S,M}\big(\,|\mu\rangle \otimes |\theta_M\rangle\,\big) = \sum_\nu \boldsymbol{M}_\nu\,|\mu\rangle \otimes |\lambda_\nu\rangle\,.$$

Thus necessarily, using exercise 6 each $\boldsymbol{M}_\nu\,|\mu\rangle$ is colinear to $|\mu\rangle$. Whatever the measurement outcome ν is, the conditional state provided by (30) remains unchanged: $\boldsymbol{\rho}_+ = \mathbb{M}_\nu(\boldsymbol{\rho})$ when $\boldsymbol{\rho} = |\mu\rangle\langle\mu|$. When the spectrum of $\boldsymbol{O}_S$ is degenerate and $\boldsymbol{P}_\mu$ is the projector on the eigenspace associated to the eigenvalue μ of $\boldsymbol{O}_S$, this invariance reads: for all ν, $\boldsymbol{M}_\nu \boldsymbol{P}_\mu = \boldsymbol{P}_\mu \boldsymbol{M}_\nu$. Any eigenspace of $\boldsymbol{O}_S$ is invariant with respect to all the $\boldsymbol{M}_\nu$'s.

3.4 Stochastic process attached to a POVM

To any POVM defined by a set of measurement operators $(\boldsymbol{M}_\nu)$ on $\mathcal{H}_S$, is attached a stochastic process. This process admits the set $\{\boldsymbol{\rho}\}$ of density operators on $\mathcal{H}_S$ as state space. It is defined by the transition rules:

$$\boldsymbol{\rho}_+ = \frac{\boldsymbol{M}_\nu \boldsymbol{\rho} \boldsymbol{M}_\nu^\dagger}{\mathrm{Tr}\left(\boldsymbol{M}_\nu \boldsymbol{\rho} \boldsymbol{M}_\nu^\dagger\right)} \text{ with probability } p_\nu = \mathrm{Tr}\left(\boldsymbol{M}_\nu \boldsymbol{\rho} \boldsymbol{M}_\nu^\dagger\right). \tag{31}$$

For any observable $\boldsymbol{A}$ on $\mathcal{H}_S$, its conditional expectation value after the transition knowing the state $\boldsymbol{\rho}$ just before the transition is given by

$$\mathbb{E}\left(\mathrm{Tr}\left(\boldsymbol{A}\boldsymbol{\rho}_+\right)/\boldsymbol{\rho}\right) = \mathrm{Tr}\left(\boldsymbol{A}\mathbb{K}(\boldsymbol{\rho})\right) \tag{32}$$

where the linear map $\mathbb{K}(\boldsymbol{\rho}) = \sum_\nu \boldsymbol{M}_\nu \boldsymbol{\rho} \boldsymbol{M}_\nu^\dagger$ is a Kraus map (see appendix A).

Assume that this POVM provides a QND measurement of an observable $\boldsymbol{O}_S$ on $\mathcal{H}_S$. Then the orthogonal projector $\boldsymbol{P}_{\boldsymbol{O}_S}$ on any eigenspace of $\boldsymbol{O}_S$, yields to a martingale[3] $\mathrm{Tr}\left(\boldsymbol{\rho}\boldsymbol{P}_{\boldsymbol{O}_S}\right)$:

$$\mathbb{E}\left(\mathrm{Tr}\left(\boldsymbol{P}_{\boldsymbol{O}_S}\boldsymbol{\rho}_+\right)/\boldsymbol{\rho}\right) = \mathrm{Tr}\left(\boldsymbol{P}_{\boldsymbol{O}_S}\boldsymbol{\rho}\right)$$

since $\boldsymbol{P}_{\boldsymbol{O}_S}$ is a stationary point of the dual Kraus map $\mathbb{K}^*$: $\mathbb{K}^*(\boldsymbol{P}_{\boldsymbol{O}_S}) = \sum_\nu \boldsymbol{M}_\nu^\dagger \boldsymbol{P}_{\boldsymbol{O}_S} \boldsymbol{M}_\nu = \boldsymbol{P}_{\boldsymbol{O}_S}$. Moreover, if $\boldsymbol{P}_{\boldsymbol{O}_S}$ is of rank one, then it corresponds to a stationary state $\bar{\boldsymbol{\rho}} = \boldsymbol{P}_{\boldsymbol{O}_S}$ of the Markov process (31): for all ν, $\boldsymbol{M}_\nu \bar{\boldsymbol{\rho}} \boldsymbol{M}_\nu^\dagger = \mathrm{Tr}\left(\boldsymbol{M}_\nu \bar{\boldsymbol{\rho}} \boldsymbol{M}_\nu^\dagger\right) \bar{\boldsymbol{\rho}}$.

Exercice 7. *Prove that for a QND measurement of a system observable $\boldsymbol{O}_S$, the random process $\mathrm{Tr}\left(\boldsymbol{\rho}\boldsymbol{O}_S\right)$ is also a martingale.*

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A Linear quantum operations

A linear quantum operation $\mathbb{K}$ is a linear superoperator acting on the space of the density matrices in the system's Hilbert space $\mathcal{S}$, and satisfying the following properties:

- $\mathbb{K}$ is trace-preserving or decreasing. This is, $0 \leq \mathrm{Tr}\left(\mathbb{K}(\rho)\right) \leq 1$ for any density matrix ρ.
- $\mathbb{K}$ is completely positive. That is, not only does $\mathbb{K}$ map positive operators to positive operators in the system's Hilbert space $\mathcal{S}$, but so does $\boldsymbol{I}_{\mathcal{H}} \otimes \mathbb{K}$ for positive operators in $\mathcal{H} \otimes \mathcal{S}$. Here $\mathcal{H}$ is the Hilbert space of a second arbitrary system and $\boldsymbol{I}_{\mathcal{H}}$ is its identity operator.

Concerning the last property, it may seem that positivity of a superoperator would be sufficient to represent a physical process. However, in practice, the considered system can be entangled to another system before the physical process acts on it. It must still be the case that the total state of both systems remains a physical state with a positive density operator. This justifies the last property.

We have the following theorem called the Kraus representation theorem (see [9, page 368] for a proof):

Theorem 2. *Any linear quantum operation satisfying the above conditions, can be expressed in the form*

$$\mathbb{K}(\rho) = \sum_j \boldsymbol{M}_j \rho \boldsymbol{M}_j^\dagger$$

with

$$\boldsymbol{I}_{\mathcal{S}} - \sum_j \boldsymbol{M}_j^\dagger \boldsymbol{M}_j \geq 0.$$

The above formula is known as the Kraus representation or the operator-sum representation of the linear quantum operation and the operators $\boldsymbol{M}_j$ are known as the measurement operators. Moreover, $\mathbb{K}$ is trace-preserving ($\mathrm{Tr}\left(\mathbb{K}(\rho)\right) = \mathrm{Tr}\left(\rho\right)$ for any density operator ρ) if, and only if, $\sum_j \boldsymbol{M}_j^\dagger \boldsymbol{M}_j = \boldsymbol{I}_{\mathcal{S}}$.

Linear quantum operations are also called *Kraus maps* or *quantum channels*. When not trace preserving, they are also called *partial Kraus maps*.

As soon as we make the additional assumption of a trace-preserving quantum operation, we can also prove some contraction properties. In this aim, we first define the quantum fidelity and quantum trace distance between two density matrices:

Definition 1. *Consider two well-defined density matrices $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$. The quantum trace distance $D(\boldsymbol{\rho}, \boldsymbol{\sigma})$ and the quantum fidelity $F(\boldsymbol{\rho}, \boldsymbol{\sigma})$ are then defined as follows:*

$$D(\boldsymbol{\rho}, \boldsymbol{\sigma}) = \frac{1}{2}\mathrm{Tr}\left(|\boldsymbol{\rho} - \boldsymbol{\sigma}|\right), \qquad F(\boldsymbol{\rho}, \boldsymbol{\sigma}) = \mathrm{Tr}\left(\sqrt{\boldsymbol{\rho}^{1/2}\boldsymbol{\sigma}\boldsymbol{\rho}^{1/2}}\right),$$

where $|\boldsymbol{A}| \equiv \sqrt{\boldsymbol{A}^\dagger \boldsymbol{A}}$ is the positive square root of $\boldsymbol{A}^\dagger \boldsymbol{A}$.

Remark 1. *One can prove that (see [9, Chapter 9]) as soon as one of the density matrices is a projector state $\boldsymbol{\sigma} = |\psi\rangle\langle\psi|$, the fidelity between $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$ is given by the standard form*

$$F(\boldsymbol{\rho}, \boldsymbol{\sigma}) = \sqrt{\mathrm{Tr}\left(\boldsymbol{\rho}\boldsymbol{\sigma}\right)} = \sqrt{\langle\psi|\,\boldsymbol{\rho}\,|\psi\rangle}.$$

We have the following contraction properties for trace-preserving quantum operations:

Theorem 3. *Suppose that $\mathbb{K}$ is a trace-preserving quantum operation. Let $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$ be two well-defined density operators. Then*

$$D(\mathbb{K}(\boldsymbol{\rho}), \mathbb{K}(\boldsymbol{\sigma})) \leq D(\boldsymbol{\rho}, \boldsymbol{\sigma}) \quad \textit{and} \quad F(\mathbb{K}(\boldsymbol{\rho}), \mathbb{K}(\boldsymbol{\sigma})) \geq F(\boldsymbol{\rho}, \boldsymbol{\sigma}).$$

The proof of this theorem is beyond the scope of these notes and we refer to [9, Chapter 9] for a rigorous proof.

B Markov chains, martingales and convergence theorems

This Appendix has for aim to give a very brief overview of some definitions and some theorems in the theory of random processes. The stability Theorems 1, 2 and 3 can be seen as stochastic analogues of deterministic Lyapunov function techniques.

We start the appendix by defining three types of convergence for random processes:

Definition 2. *Consider (X_n) a sequence of random variables defined on the probability space $(\Omega, \mathcal{F}, p)$ and taking values in a metric space $\mathcal{X}$. The random process X_n is said to,*

- *converge in probability towards the random variable X if for all $\epsilon > 0$,*

$$\lim_{n\to\infty} p\left(|X_n - X| > \epsilon\right) = \lim_{n\to\infty} p\left(\omega \in \Omega \mid |X_n(\omega) - X(\omega)| > \epsilon\right) = 0;$$

- *converge almost surely towards the random variable X if*

$$p\left(\lim_{n\to\infty} X_n = X\right) = p\left(\omega \in \Omega \mid \lim_{n\to\infty} X_n(\omega) = X(\omega)\right) = 1;$$

- *converge in mean towards the random variable X if*

$$\lim_{n\to\infty} \mathbb{E}\left(|X_n - X|\right) = 0.$$

We can prove that the almost sure convergence and the convergence in mean imply the convergence in probability. However no such relation can be proved between the convergence in mean and the almost sure convergence in general.

Before defining the Markov processes, martingales, and discussing their convergence theorems, we provide two useful results from probability theory that are used for the proof of convergence of QND measurement process. The first result is the Markov's inequality

Lemma 1 (Markov's inequality)**.** *If $X \geq 0$ is a random variable and $\epsilon > 0$, we have*

$$\mathbb{P}\left(X \geq \epsilon\right) \leq \frac{\mathbb{E}\left(X\right)}{\epsilon}.$$

The second result is the Borel-Cantelli lemma about sequences of events in the σ-algebra $\mathcal{F}$.

Lemma 2 (Borel-Cantelli lemma)**.** *Let $E_k \in \mathcal{F}$ be a sequence of events in the probability space $(\Omega, \mathcal{F}, p)$. Assuming*

$$\sum_{n=1}^{\infty} p(E_n) < \infty,$$

we have

$$p\left(\limsup_{n\to\infty} E_n\right) = p\left(\cap_{n=1}^{\infty} \cup_{k=n}^{\infty} E_k\right) = 0.$$

Let $(\Omega, \mathcal{F}, p)$ be a probability space, and let $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots \subseteq \mathcal{F}$ be a nondecreasing family of sub-σ-algebras. We have the following definitions

Definition 3. *The sequence $(X_n, \mathcal{F}_n)_{n=1}^{\infty}$ is called a Markov process with respect to $F = (\mathcal{F}_n)_{n=1}^{\infty}$, if for $n' > n$ and any measurable function $f(x)$ with $\sup_x |f(x)| < \infty$,*

$$\mathbb{E}\left(f(X_{n'}) \mid \mathcal{F}_n\right) = \mathbb{E}\left(f(X_{n'}) \mid X_n\right).$$

Definition 4. *The sequence $(X_n, \mathcal{F}_n)_{n=1}^{\infty}$ is called respectively a supermartingale, a submartingale or a martingale, if $\mathbb{E}(|X_n|) < \infty$ for $n = 1, 2, \cdots$, and*

$$\mathbb{E}(X_n \mid \mathcal{F}_m) \leq X_m \qquad (p \text{ almost surely}), \qquad n \geq m,$$

or

$$\mathbb{E}(X_n \mid \mathcal{F}_m) \geq X_m \qquad (p \text{ almost surely}), \qquad n \geq m,$$

or finally,

$$\mathbb{E}(X_n \mid \mathcal{F}_m) = X_m \qquad (p \text{ almost surely}), \qquad n \geq m.$$

Remark 2. *A time-continuous version of the above definitions can also be considered for $(X_t, \mathcal{F}_t)_{t\geq 0}$, where $F = (\mathcal{F}_t)_{t\geq 0}$, is non decreasing family of sub-σ-alegbras of $\mathcal{F}$.*

The following theorem characterizes the convergence of bounded martingales:

Theorem 1 (Doob's first martingale convergence theorem)**.** *Let $(X_n, \mathcal{F}_n)_{n<\infty}$ be a submartingale such that (x^+ is the positive part of x)*

$$\sup_n \mathbb{E}\left(X_n^+\right) < \infty.$$

Then $\lim_n X_n$ $(= X_\infty)$ exists with probability 1, and $\mathbb{E}(X_\infty^+) < \infty$.

For a proof we refer to [8, Chapter 2, Page 43].

Here, we recall two results that are often referred as the stochastic versions of the Lyapunov stability theory and the LaSalle's invariance principle. For detailed discussions and proofs we refer to [7, Sections 8.4 and 8.5]. The first theorem is the following:

Theorem 2 (Doob's Inequality)**.** *Let $\{X_n\}$ be a Markov chain on state space $\mathcal{X}$. Suppose that there is a non-negative function $V(x)$ satisfying $\mathbb{E}(V(X_1) \mid X_0 = x) - V(x) = -k(x)$, where $k(x) \geq 0$ on the set $\{x : V(x) < \lambda\} \equiv Q_\lambda$. Then*

$$p\left(\sup_{\infty > n \geq 0} V(X_n) \geq \lambda \mid X_0 = x\right) \leq \frac{V(x)}{\lambda}.$$

Corollary 1. *Consider the same assumptions as in Theorem 2. Assume moreover that there exists $\bar{x} \in \mathcal{X}$ such that $V(\bar{x}) = 0$ and that $V(x) \neq 0$ for all x different from $\bar{x}$. Then the Theorem 2 implies that the Markov process X_n is **stable in probability** around $\bar{x}$, i.e.*

$$\lim_{x \to \bar{x}} p\left(\sup_n \|X_n - \bar{x}\| \geq \epsilon \mid X_0 = x\right) = 0, \qquad \forall \epsilon > 0.$$

Theorem 3. *Let $\{X_n\}$ be a Markov chain on the compact state space S. Suppose that there exists a non-negative function $V(x)$ satisfying $\mathbb{E}(V(X_{n+1}) \mid X_n = x) - V(x) = -k(x)$, where $k(x) \geq 0$ is a positive continuous function of x. Then the ω-limit set (in the sense of almost sure convergence) of X_n is included in the following set*

$$I = \{X \mid k(X) = 0\}.$$

Trivially, the same result holds true for the case where $\mathbb{E}(V(X_{n+1}) \mid X_n = x) - V(x) = k(x)$ ($V(X_n)$ is a submartingale and not a supermartingale), with $k(x) \geq 0$ and $V(x)$ bounded from above.

The proof is just an application of the Theorem 1 in [7, Ch. 8], which shows that $k(X_n)$ converges to zero for almost all paths. It is clear that the continuity of $k(x)$ with respect to x and the compactness of S implies that the ω-limit set of X_n is necessarily included in the set I.