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44

Input–Output Stability

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44.1 Introduction

A common task for an engineer is to design a system that reacts to stimuli in some specific and desirable way. One way to characterize appropriate behavior is through the formalism of input–output stability. In this setting a notion of well-behaved input and output signals is made precise and the question is posed: do well-behaved stimuli (inputs) produce well-behaved responses (outputs)?

General input–output stability analysis has its roots in the development of the electronic feedback amplifier of H.S. Black in 1927 and the subsequent development of classical feedback design tools for linear systems by H. Nyquist and H.W. Bode in the 1930s and 1940s, all at Bell Telephone Laboratories. These latter tools focused on determining input–output stability of linear feedback systems from the characteristics of the feedback components. Generalizations to nonlinear systems were made by several researchers in the late 1950s and early 1960s. The most notable contributions were those of G. Zames, then at M.I.T., I.W. Sandberg at Bell Telephone Laboratories, and V.M. Popov. Indeed, much of this chapter is based on the foundational ideas found in [5,7,10], with additional insights drawn from [6]. A thorough understanding of nonlinear systems from an input–output point of view is still an area of ongoing and intensive research.

The strength of input–output stability theory is that it provides a method for anticipating the qualitative behavior of a feedback system with only rough information about the feedback components. This, in turn, leads to notions of robustness of feedback stability and motivates many of the recent developments in modern control theory.

44.2 Systems and Stability

Throughout our discussion of input–output stability, a *signal* is a "reasonable" (e.g., piecewise continuous) function defined on a finite or semi-infinite time interval, i.e., an interval of the form $[0, T)$ where T is either a strictly positive real number or infinity. In general, a signal is vector-valued; its components typically represent actuator and sensor values. A *dynamical system* is an object which produces an output signal for each input signal.

To discuss stability of dynamical systems, we introduce the concept of a *norm function*, denoted $||\cdot||$, which captures the "size" of signals defined on the semi-infinite time interval. The significant properties of a norm function are that 1) the norm of a signal is zero if the signal is identically zero, and is a strictly positive number otherwise, 2) scaling a signal results in a corresponding scaling of the norm, and 3) the triangle inequality holds, i.e., $||u_1 + u_2|| \leq ||u_1|| + ||u_2||$. Examples of norm functions are the p-norms. For any positive real number $p \geq 1$, the p-norm is defined by

$$||u||_p := \left(\int_0^\infty |u(t)|^p\right)^{\frac{1}{p}} \tag{44.1}$$

where $|\cdot|$ represents the standard Euclidean norm, i.e., $|u| = \sqrt{\sum_{i=1}^{n} u_i^2}$. For $p = \infty$, we define

$$||u||_\infty := \sup_{t\geq 0} |u(t)|. \tag{44.2}$$

The ∞-norm is useful when amplitude constraints are imposed on a problem, and the 2-norm is of more interest in the context of energy constraints. The norm of a signal may very well be infinite. We will typically be interested in measuring signals which may only be defined on finite time intervals or measuring truncated versions of signals. To that end, given a signal u defined on $[0, T)$ and a strictly positive real number τ, we use u_τ to denote the *truncated signal* generated by extending u onto $[0, \infty)$ by defining $u(t) = 0$ for $t \geq T$, if necessary, and then truncating, i.e., u_τ is equal to the (extended) signal on the interval $[0, \tau]$ and is equal to zero on the interval (τ, ∞).

Informally, a system is *stable* in the input–output sense if small input signals produce correspondingly small output signals. To make this concept precise, we need a way to quantify the dependence of the norm of the output on the norm of the input applied to the system. To that end, we define a *gain function* as a function from the nonnegative real numbers to the nonnegative real numbers which is continuous, nondecreasing, and zero when its argument is zero. For notational convenience we will say that the "value" of a gain function at ∞ is ∞. A dynamical system is stable (with respect to the norm $||\cdot||$) if there is a gain function γ which gives a bound on the norm of truncated output signals as a function of the norm of truncated input signals, i.e.,

$$||y_\tau|| \leq \gamma(||u_\tau||), \qquad \text{for all } \tau. \tag{44.3}$$

In the very special case when the gain function is linear, i.e., there is at most an amplification by a constant factor, the dynamical system is *finite gain stable*. The notions of finite gain stability and closely related variants are central to much of classical input–output stability theory, but in recent years much progress has been made in understanding the role of more general (nonlinear) gains in system analysis.

The focus of this chapter will be on the stability analysis of interconnected dynamical systems as described in Figure 44.1. The composite system in Figure 44.1 will be called a *well-defined interconnection*

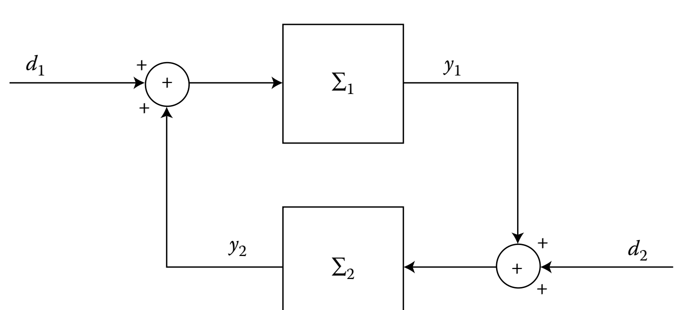

FIGURE 44.1 Standard feedback configuration.

if it is a dynamical system with $\begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$ as input and $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ as output, i.e., given an arbitrary input signal $\begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$, a signal $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ exists so that, for the dynamical system Σ_1, the input $d_1 + y_2$ produces the output y_1 and, for the dynamical system Σ_2, the input $d_2 + y_1$ produces the output y_2. To see that not every interconnection is well-defined, consider the case where both Σ_1 and Σ_2 are the identity mappings. In this case, the only input signals $\begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$ for which an output $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ can be found are those for which $d_1 + d_2 = 0$. The dynamical systems which make up a well-defined interconnection will be called its *feedback components*.

For stability of well-defined interconnections, it is not necessary for either of the feedback components to be stable nor is it sufficient for both of the feedback components to be stable. On the other hand, necessary and sufficient conditions for stability of a well-defined interconnection can be expressed in terms of the set of all possible input–output pairs for each feedback component. To be explicit, following are some definitions. For a given dynamical system Σ with input signals u and output signals y, the set of its ordered input–output pairs (u, y) is referred to as the *graph* of the dynamical system and is denoted $\mathcal{G}_\Sigma$. When the input and output are exchanged in the ordered pair, i.e., (y, u), the set is referred to as the *inverse graph* of the system and is denoted $\mathcal{G}^I_\Sigma$. Note that, for the system in Figure 44.1, the inverse graph of Σ_2 and the graph of Σ_1 lie in the same Cartesian product space called the *ambient space.* We will use as norm on the ambient space the sum of the norms of the coordinates.

The basic observation regarding input–output stability for a well-defined interconnection says, in informal terms, that if a signal in the inverse graph of Σ_2 is near any signal in the graph of Σ_1 then it must be small. To formalize this notion, we need the concept of the *distance* to the graph of Σ_1 from signals x in the ambient space. This (truncated) distance is defined by

$$d_\tau(x, \mathcal{G}_{\Sigma_1}) := \inf_{z \in \mathcal{G}_{\Sigma_1}} ||(x - z)_\tau||. \tag{44.4}$$

Theorem 44.1: Graph Separation Theorem

A well-defined interconnection is stable if, and only if, a gain function γ exists which gives a bound on the norm of truncated signals in the inverse graph of Σ_2 as a function of the truncated distance from the signals to the graph of Σ_1, i.e.,

$$x \in \mathcal{G}^I_{\Sigma_2} \implies ||x_\tau|| \le \gamma\left(d_\tau\left(x, \mathcal{G}_{\Sigma_1}\right)\right), \quad \text{for all } \tau. \tag{44.5}$$

In the special case where γ is a linear function, the well-defined interconnection is finite gain stable.

The idea behind this observation can be understood by considering the signals that arise in the closed loop which belong to the inverse graph of Σ_2, i.e., the signals $(y_2, y_1 + d_2)$. (Stability with these signals taken as output is equivalent to stability with the original outputs.) Notice that, for the system in Figure 44.1, signals in the graph of Σ_1 have the form $(y_2 + d_1, y_1)$. Consequently, signals $x \in \mathcal{G}^I_{\Sigma_2}$ and $z \in \mathcal{G}_{\Sigma_1}$, which satisfy the feedback equations, also satisfy

$$(x - z)_\tau = (d_1, -d_2)_\tau \tag{44.6}$$

and

$$||(x - z)_\tau|| = \|(d_1, d_2)_\tau\| \tag{44.7}$$

for truncations within the interval of definition. If there are signals x in the inverse graph of Σ_2 with large truncated norm but small truncated distance to the graph of Σ_1, i.e., there exists some $z \in \mathcal{G}_{\Sigma_1}$ and $\tau > 0$ such that $||(x - z)_\tau||$ is small, then we can choose (d_1, d_2) to satisfy Equation 44.6 giving, according to

Equation 44.7, a small input which produces a large output. This contradicts our definition of stability. Conversely, if there is no z which is close to x, then only large inputs can produce large x signals and thus the system is stable.

The distance observation presented above is the unifying idea behind the input–output stability criteria applied in practice. However, the observation is rarely applied directly because of the difficulties involved in exactly characterizing the graph of a dynamical system and measuring distances. Instead, various simpler conditions have been developed which constrain the graphs of the feedback components to guarantee that the graph of Σ_1 and the inverse graph of Σ_2 are sufficiently separated. There are many such sufficient conditions, and, in the remainder of this chapter, we will describe a few of them.

44.3 Practical Conditions and Examples

44.3.1 The Classical Small Gain Theorem

One of the most commonly used sufficient conditions for graph separation constrains the graphs of the feedback components by assuming that each feedback component is finite gain stable. Then, the appropriate graphs will be separated if the product of the coefficients of the linear gain functions is sufficiently small. For this reason, the result based on this type of constraint has come to be known as the small gain theorem.

Theorem 44.2: Small Gain Theorem

If each feedback component is finite gain stable and the product of the gains (the coefficients of the linear gain functions) is less than one, then the well-defined interconnection is finite gain stable.

Figure 44.2 provides the intuition for the result. If we were to draw an analogy between a dynamical system and a static map whose graph is a set of points in the plane, the graph of Σ_1 would be constrained to the darkly shaded conic region by the finite gain stability assumption. Likewise, the inverse graph of Σ_2 would be constrained to the lightly shaded region. The fact that the product of the gains is less than one guarantees the positive aperture between the two regions and, in turn, that the graphs are separated sufficiently.

To apply the small gain theorem, we need a way to verify that the feedback components are finite gain stable (with respect to a particular norm) and to determine their gains. In particular, any linear dynamical system that can be represented with a real, rational transfer function $G(s)$ is finite gain stable in any of the p-norms if, and only if, all of the poles of the transfer function have negative real parts. A popular norm to work with is the 2-norm. It is associated with the energy of a signal. For a single-input, single-output (SISO) finite gain stable system modeled by a real, rational transfer function $G(s)$, the smallest possible coefficient for the stability gain function with respect to the 2-norm, is given by

$$\bar{\gamma} := \sup_{\omega} |G(j\omega)|. \tag{44.8}$$

For multi-input, multioutput systems, the magnitude in Equation 44.8 is replaced by the maximum singular value. In either case, this can be established using *Parseval's theorem*. For SISO systems, the quantity in Equation 44.8 can be obtained from a quick examination of the Bode plot or Nyquist plot for the transfer function. If the Nyquist plot of a stable SISO transfer function lies inside a circle of radius $\bar{\gamma}$ centered at the origin, then the coefficient of the 2-norm gain function for the system is less than or equal to $\bar{\gamma}$.

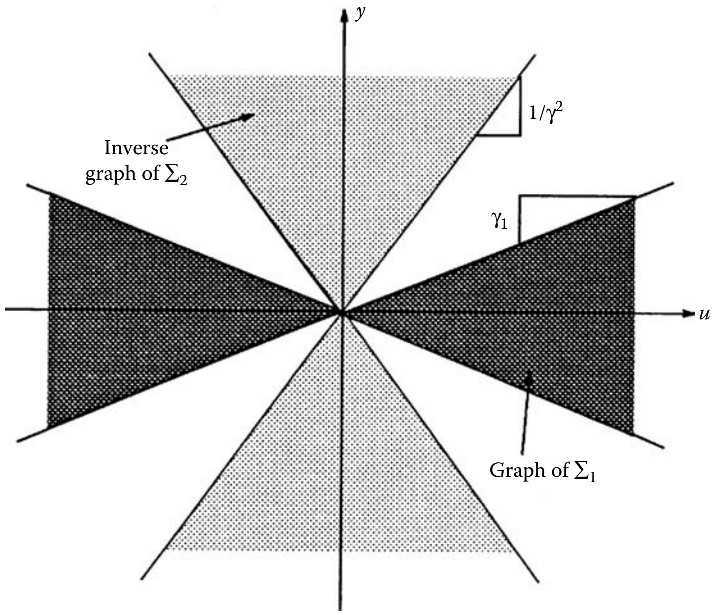

FIGURE 44.2 Classical small gain theorem.

More generally, consider a dynamical system that can be represented by a finite dimensional ordinary differential equation with zero initial state:

$$\dot{x} = f(x, u), \quad x(0) = 0 \quad \text{and} \quad y = h(x, u). \tag{44.9}$$

Suppose that f has globally bounded partial derivatives and that positive real numbers ℓ_1 and ℓ_2 exist so that

$$|h(x, u)| \leq \ell_1 |x| + \ell_2 |u|. \tag{44.10}$$

Under these conditions, if the trajectories of the unforced system with nonzero initial conditions,

$$\dot{x} = f(x, 0), \quad x(0) = x_\circ, \tag{44.11}$$

satisfy

$$|x(t)| \leq k \exp(-\lambda t)|x_\circ|, \tag{44.12}$$

for some positive real number k and λ and any $x_\circ \in \mathbb{R}^n$, then the system (Equation 44.9) is finite gain stable in any of the p-norms. This can be established using Lyapunov function arguments that apply to the system (Equation 44.11). The details can be found in the textbooks on nonlinear systems mentioned later.

Example 44.1:

Consider a nonlinear control system modeled by an ordinary differential equation with state $x \in \mathbb{R}^n$, input $v \in \mathbb{R}^m$ and disturbance $d_1 \in \mathbb{R}^m$:

$$\dot{x} = f(x, v + d_1). \tag{44.13}$$

Suppose that f has globally bounded partial derivatives and that a control $v = \alpha(x)$ can be found, also with a globally bounded partial derivative, so that the trajectories of the system

$$\dot{x} = f(x, \alpha(x)), \quad x(0) = x_\circ \tag{44.14}$$

satisfy the bound

$$|x(t)| \leq k \exp(-\lambda t)|x_\circ| \tag{44.15}$$

for some positive real numbers k and λ and for all $x_\circ \in \mathbb{R}^n$. As mentioned above, for any function h satisfying the type of bound in Equation 44.10, this implies that the system

$$\dot{x} = f(x, \alpha(x) + d_1), \quad x(0) = 0 \quad \text{and} \quad y = h(x, d_1) \tag{44.16}$$

has finite 2-norm gain from input d_1 to output y. We consider the output

$$y := \dot{\alpha} = \frac{\partial \alpha}{\partial x} f(x, \alpha(x) + d_1) \tag{44.17}$$

which satisfies the type of bound in Equation 44.10 because α and f both have globally bounded partial derivatives.

We will show, using the small gain theorem, that disturbances d_1 with finite 2-norm continue to produce outputs y with finite 2-norm even when the actual input v to the process is generated by the following fast dynamic version of the commanded input $\alpha(x)$:

$$\begin{aligned} \epsilon \dot{z} &= Az + B(\alpha(x)) + d_2, \quad z(0) = -A^{-1}B\alpha(x(0)) \\ v &= Cz. \end{aligned} \tag{44.18}$$

Here, ϵ is a small positive parameter, the eigenvalues of A all have strictly negative real part (thus A is invertible), and $-CA^{-1}B = I$. This system may represent unmodeled actuator dynamics.

To see the stability result, we will consider the composite system in the coordinates x and $\zeta = z + A^{-1}B\alpha(x)$. Using the notation from Figure 44.1,

$$\Sigma_1: \quad \begin{aligned} \dot{x} &= f(x, \alpha(x) + u_1), \quad x(0) = 0 \\ y_1 &= A^{-1}B\dot{\alpha}(x), \end{aligned} \tag{44.19}$$

and

$$\Sigma_2: \quad \begin{aligned} \dot{\zeta} &= \epsilon^{-1}A\zeta + u_2, \quad \zeta = 0 \\ y_2 &= C\zeta, \end{aligned} \tag{44.20}$$

with the interconnection conditions

$$u_1 = y_2 + d_1, \quad \text{and} \quad u_2 = y_1 + \epsilon^{-1}d_2. \tag{44.21}$$

Of course, if the system is finite gain stable with the inputs d_1 and $\epsilon^{-1}d_2$, then it is also finite gain stable with the inputs d_1 and d_2. We have already discussed that the system Σ_1 in Equation 44.19 has finite 2-norm gain, say γ_1. Now consider the system Σ_2 in Equation 44.20. It can be represented with the transfer function

$$\begin{aligned} G(s) &= C(sI - \epsilon^{-1}A)^{-1}, \\ &= \epsilon C(\epsilon sI - A)^{-1}, \\ &=: \epsilon\bar{G}(\epsilon s). \end{aligned} \tag{44.22}$$

Identifying $\bar{G}(s) = C(sI - A)^{-1}$, we see that, if

$$\gamma_2 := \sup_\omega \sigma(\bar{G}(j\omega)), \tag{44.23}$$

then

$$\sup_\omega \sigma(G(j\omega)) = \epsilon\gamma_2. \tag{44 24}$$

We conclude from the small gain theorem that, if $\epsilon < \frac{1}{\gamma_1\gamma_2}$, then the composite system (Equations 44.19 through 44 21), with inputs d_1 and d_2 and outputs $y_1 = A^{-1}B\dot{\alpha}(x)$ and $y_2 = C\zeta$, is finite gain stable.

44.3.2 The Classical Passivity Theorem

Another very popular condition used to guarantee graph separation is given in the *passivity theorem*. For the most straightforward passivity result, the number of input channels must equal the number of output channels for each feedback component. We then identify the relative location of the graphs of the feedback components using a condition involving the integral of the product of the input and the output signals. This operation is known as the *inner product*, denoted $\langle \cdot, \cdot \rangle$. In particular, for two signals u and y of the same dimension defined on the semi-infinite interval,

$$\langle u, y \rangle := \int_0^\infty u^T(t) y(t)\, dt. \tag{44.25}$$

Note that $\langle u, y \rangle = \langle y, u \rangle$ and $\langle u, u \rangle = ||u||_2^2$. A dynamical system is *passive* if, for each input–output pair (u, y) and each $\tau > 0$, $\langle u_\tau, y_\tau \rangle \geq 0$. The terminology used here comes from the special case where the input and output are a voltage and a current, respectively, and the energy absorbed by the dynamical system, which is the inner product of the input and output, is nonnegative.

Again by analogy to a static map whose graph lies in the plane, passivity of a dynamical system can be viewed as the condition that the graph is constrained to the darkly shaded region in Figure 44.3, i.e., the first and third quadrants of the plane. This graph and the inverse graph of a second system would be separated if, for example, the inverse graph of the second system were constrained to the lightly shaded region in Figure 44.3, i.e., the second and fourth quadrants but bounded away from the horizontal and vertical axes by an increasing and unbounded distance. But, this is the same as asking that the graph of the second system followed by the scaling "-1," i.e., all pairs $(u, -y)$, be constrained to the first and third quadrants, again bounded away from the axes by an increasing and unbounded distance, as in Figure 44.4a. For classical passivity theorems, this region is given a linear boundary as in Figure 44.4b. Notice that, for points $(u_\circ, y_\circ)$ in the plane, if $u_\circ \cdot y_\circ \geq \epsilon(u_\circ^2 + y_\circ^2)$ then $(u_\circ, y_\circ)$ is in the first or third quadrant, and $(\epsilon)^{-1}|u_\circ| \geq |y_\circ| \geq \epsilon|u_\circ|$ as in Figure 44.4b. This leads to

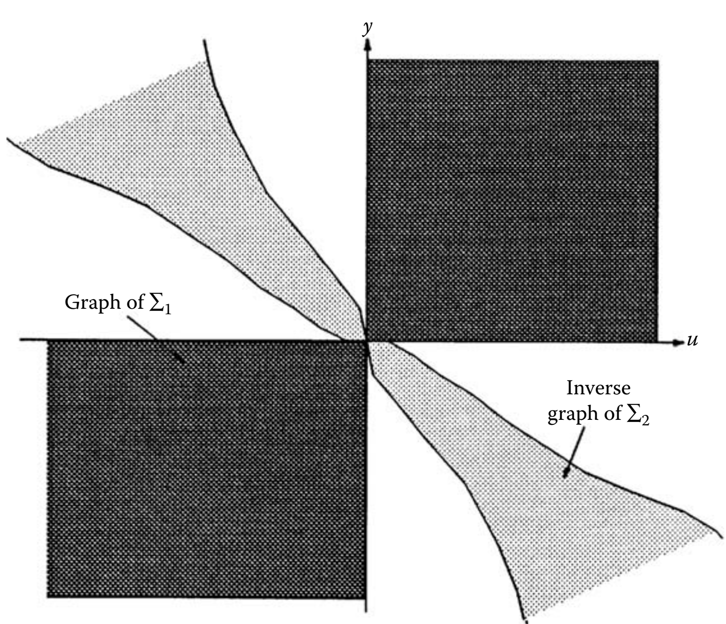

FIGURE 44.3 General passivity-based interconnection.

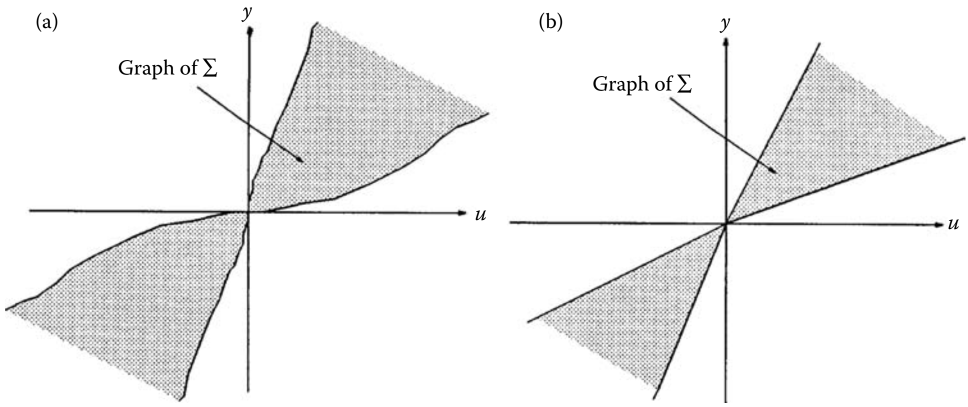

FIGURE 44.4 Different notions of input and output strict passivity.

the following stronger version of passivity. A dynamical system is *input and output strictly passive* if a strictly positive real number ϵ exists so that, for each input–output pair (u, y) and each $\tau > 0$, $\langle u_\tau, y_\tau \rangle \geq \epsilon \left(||u_\tau||_2^2 + ||y_\tau||_2^2 \right)$.

There are intermediate versions of passivity which are also useful. These correspond to asking for an increasing and unbounded distance from either the horizontal axis or the vertical axis but not both. For example, a dynamical system is *input strictly passive* if a strictly positive real number ϵ exists so that, for each input–output pair (u, y) and each $\tau > 0$, $\langle u_\tau, y_\tau \rangle \geq \epsilon ||u_\tau||_2^2$. Similarly, a dynamical system is *output strictly passive* if a strictly positive real number ϵ exists so that, for each input–output pair (u, y) and each $\tau > 0$, $\langle u_\tau, y_\tau \rangle \geq \epsilon ||y_\tau||_2^2$. It is worth noting that input and output strict passivity is equivalent to input strict passive plus finite gain stability. This can be shown with standard manipulations of the inner product. Also, the reader is warned that all three types of strict passivity mentioned above are frequently called "strict passivity" in the literature.

Again by thinking of a graph of a system as a set of points in the plane, output strict passivity is the condition that the graph is constrained to the darkly shaded region in Figure 44.5, i.e., the first and third quadrants with an increasing and unbounded distance from the vertical axis. To complement such a graph, consider a second dynamical system which, when followed by the scaling "-1," is also output strictly passive. Such a system has a graph (without the "-1" scaling) constrained to the second and fourth quadrants with an increasing and unbounded distance from the vertical axis. In other words, its inverse graph is constrained to the lightly shaded region of Figure 44.5, i.e., to the second and fourth quadrants but with an increasing and unbounded distance from the *horizontal* axis. The conclusions that we can then draw, using the graph separation theorem, are summarized in the following passivity theorem.

Theorem 44.3: Passivity Theorem

If one dynamical system and the other dynamical system followed by the scaling "-1" are

- *both input strictly passive, OR*
- *both output strictly passive, OR*
- *respectively, passive* and *input and output strictly passive,*

then the well-defined interconnection is finite gain stable in the 2-norm.

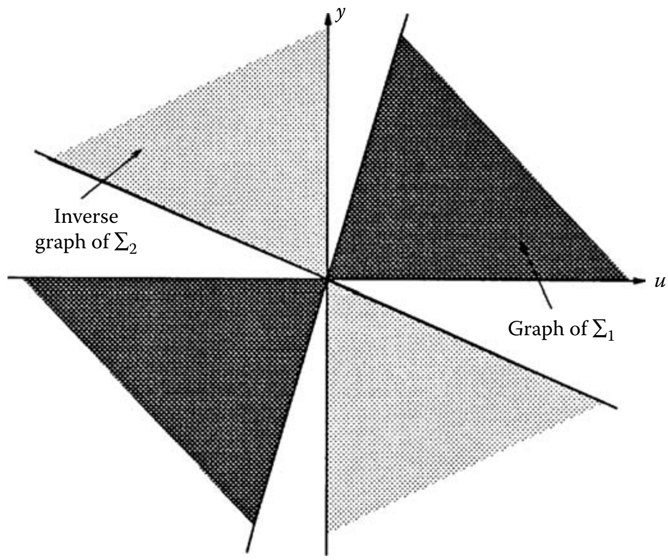

FIGURE 44.5 Interconnection of output strictly passive systems.

To apply this theorem, we need a way to verify that the (possibly scaled) feedback components are appropriately passive. For stable SISO systems with real, rational transfer function $G(s)$, it again follows from Parseval's theorem that, if

$$\text{Re } G(j\omega) \geq 0,$$

for all real values of ω, then the system is passive. If the quantity $\text{Re}\, G(j\omega)$ is positive and uniformly bounded away from zero for all real values of ω, then the linear system is input and output strictly passive. Similarly, if $\epsilon > 0$ exists so that, for all real values of ω,

$$\text{Re } G(j\omega - \epsilon) \geq 0, \tag{44.26}$$

then the linear system is output strictly passive. So, for SISO systems modeled with real, rational transfer functions, passivity and the various forms of strict passivity can again be easily checked by means of a graphical approach such as a Nyquist plot.

More generally, for any dynamical system that can be modeled with a smooth, finite dimensional ordinary differential equation,

$$\begin{aligned} \dot{x} &= f(x) + g(x)u, \quad x(0) = 0 \\ y &= h(x), \end{aligned} \tag{44.27}$$

if a strictly positive real number ϵ exists and a nonnegative function $V : \mathbb{R}^n \to \mathbb{R}\geq 0$ with $V(0) = 0$ satisfying

$$\frac{\partial V}{\partial x}(x)f(x) \leq -\epsilon h^T(x)h(x), \tag{44.28}$$

and

$$\frac{\partial V}{\partial x}(x)g(x) = h^T(x), \tag{44.29}$$

then the system is output strictly passive. With $\epsilon = 0$, the system is passive. Both of these results are established by integrating $\dot{V}$ over the semi-infinite interval.

Example 44.2:

(This example is prompted by the work in Berghuis and Nijmeijer, *Syst. Control Lett.*, 1993, 21, 289–295.) Consider a "completely controlled dissipative Euler-Lagrange" system with generalized "forces" F, generalized coordinates q, uniformly positive definite "inertia" matrix $I(q)$, Rayleigh dissipation function $R(\dot{q})$ and, say positive, potential $V(q)$ starting from the position q_d. Let the dynamics of the system be given by the Euler-Lagrange-Rayleigh equations,

$$\overbrace{\frac{\partial L}{\partial \dot{q}}(q,\dot{q})}^{\cdot} = \frac{\partial L}{\partial q}(q,\dot{q}) + F^{\top} - \frac{\partial L}{\partial q}(\dot{q})$$
$$q(0) = qd, \quad \dot{q}(0) = 0, \tag{44.30}$$

where L is the Lagrangian

$$L(q,\dot{q}) = \frac{1}{2}\,\dot{q}^{\top} I(q)\,\dot{q} - V(q). \tag{44.31}$$

Along the solution of Equation 44.30,

$$\dot{L} = \frac{\partial L}{\partial \dot{q}}\ddot{q} + \frac{\partial L}{\partial q}\dot{q} = \frac{\partial L}{\partial \dot{q}}\ddot{q} + \left[\overbrace{\frac{\partial L}{\partial \dot{q}}}^{\cdot} - F^{\top} + \frac{\partial R}{\partial \dot{q}}\right]\dot{q}, \tag{44.32}$$

$$= \overbrace{\left(\frac{\partial L}{\partial \dot{q}}\dot{q}\right)}^{\cdot} - \left[F^{\top} - \frac{\partial R}{\partial \dot{q}}\right]\dot{q} = \overbrace{\left(\partial \dot{q}^{\top} I(q)\,\dot{q}\right)}^{\cdot} - \left[F^{\top} - \frac{\partial R}{\partial \dot{q}}\right]\dot{q}, \tag{44.33}$$

$$= 2\,\overbrace{L+V}^{\cdot} - \left[F^{\top} - \frac{\partial R}{\partial \dot{q}}\right]\dot{q} = -2\,\dot{V} + \left[F^{\top} - \frac{\partial R}{\partial \dot{q}}\right]\dot{q}. \tag{44 34}$$

We will suppose that $\epsilon > 0$ exists so that

$$\frac{\partial R}{\partial \dot{q}}(\dot{q})\,\dot{q} \;\geq\; \epsilon\,|\dot{q}|^2. \tag{44.35}$$

Now let V_d be a function so that the modified potential

$$V_m = V + V_d \tag{44.36}$$

has a global minimum at $q = q_d$, and let the generalized "force" be

$$F \;=\; -\frac{\partial V_d}{\partial q}(q) \;+\; F_m. \tag{44.37}$$

We can see that the system (Equation 44.30) combined with Equation 44.37, having input F_m and output $\dot{q}$, is output strictly passive by integrating the derivative of the defined Hamiltonian,

$$H = \frac{1}{2}\,\dot{q}^{\top} I(q)\,\dot{q} \;+\; V_m(q) \;=\; L \;+\; 2\,V + V_d. \tag{44.38}$$

Indeed the derivative is

$$\dot{H} = \left[F_m^{\top} - \frac{\partial R}{\partial \dot{q}}\right]\dot{q} \tag{44.39}$$

and, integrating, for each τ

$$\left\langle \left[F_{m_\tau} - \frac{\partial R}{\partial \dot{q}}(\dot{q}_\tau)^{\top}\right], \dot{q}_\tau \right\rangle \;=\; H(\tau) - H(0). \tag{44.40}$$

Since $H \geq 0$, $H(0) = 0$ and Equation 44.35 holds

$$\langle F_{m\tau}, \dot{q}_\tau \rangle \;\geq\; \epsilon \|\dot{q}_\tau\|_2^2. \tag{44.41}$$

Using the notation from Figure 44.1, let Σ_1 be the system (Equations 44.30 and 44.37) with input F_m and output $\dot{q}$. Let Σ_2 be any system that, when followed by the scaling "−1," is output strictly passive. Then, according to the passivity theorem, the composite feedback system as given in Figure 44.1 is finite gain stable using the 2-norm. One possibility for Σ_2 is minus the identity mapping. However, there is interest in choosing Σ_2 followed by the scaling "-1" as a linear, output strictly passive compensator which, in addition, has no direct feed-through term. The reason is that if, d_2 in Figure 44.1 is identically zero, we can implement Σ_2 with measurement only of q and without $\dot{q}$. In general,

$$G(s)\dot{q} = G(s)s\left(\frac{1}{s}\dot{q}\right) = G(s)s(q - q_d), \tag{44.42}$$

and the system $G(s)s$ is implementable if $G(s)$ has no direct feed-through terms. To design an output strictly passive linear system without direct feed-through, let A be a matrix having all eigenvalues with strictly negative real parts so that, by a well-known result in linear systems theory, a positive definite matrix P exists satisfying

$$A^T P + PA = -I. \tag{44.43}$$

Then, for any B matrix of appropriate dimensions, the system modeled by the transfer function,

$$G(s) = -B^T P(sI - A)^{-1} B, \tag{44.44}$$

followed by the scaling "-1," is output strictly passive. To see this, consider a state-space realization

$$\begin{aligned} \dot{x} &= Ax + Bu \quad x(0) = 0 \\ y &= B^T Px, \end{aligned} \tag{44.45}$$

and note that

$$\overbrace{x^\top P x}^{\cdot} = -x^\top x + 2x^\top PBu \tag{44.46}$$

$$= -x^\top x + 2y^\top u. \tag{44.47}$$

But, with Equation 44.45, for some strictly positive real number c,

$$2cy^\top y \le x^\top x. \tag{44.48}$$

So, integrating Equation 44.47 and with P positive definite, for all τ,

$$\langle y_\tau, u_\tau \rangle \ge c||y_\tau||_2^2. \tag{44.49}$$

As a point of interest, one could verify that

$$G(s)s = -B^T PA(sI - A)^{-1} B - B^T PB. \tag{44.50}$$

44.3.3 Simple Nonlinear Separation Theorems

In this section we illustrate how allowing regions with nonlinear boundaries in the small gain and passivity contexts may be useful. First we need a class of functions to describe nonlinear boundaries. A *proper separation function* is a function from the nonnegative real numbers to the nonnegative real numbers which is continuous, zero at zero, strictly increasing and unbounded. The main difference between a gain function and a proper separation function is that the latter is invertible, and the inverse is another proper separation function.

44.3.3.1 Nonlinear Passivity

We will briefly discuss a definition of nonlinear input and output strict passivity. To our knowledge, this idea has not been used much in the literature. The notion replaces the linear boundaries in the input and output strict passivity definition by nonlinear boundaries as in Figure 44.4a. A dynamical system is *nonlinearly input and output strictly passive* if a proper separation function ρ exists so that, for each input–output pair (u, y) and each $\tau > 0$, $\langle u_\tau, y_\tau \rangle \ge ||u_\tau||_2 \rho(||u_\tau||_2) + ||y_\tau||_2 \rho(||y_\tau||_2)$. (Note that in the classical definition of strict passivity, $\rho(\zeta) = \epsilon\zeta$ for all $\zeta \ge 0$.)

Theorem 44.4: Nonlinear Passivity Theorem

If one dynamical system is passive and the other dynamical system followed by the scaling "−1" is nonlinearly input and output strictly passive, then the well-defined interconnection is stable using the 2-norm.

Example 44.3:

Let Σ_1 be a single integrator system,

$$\begin{aligned} \dot{x}_1 &= u_1 \quad x_1(0) = 0 \\ y_1 &= x_1. \end{aligned} \tag{44.51}$$

This system is passive because

$$0 \leq \tfrac{1}{2}x_1(\tau)^2 = \int_0^\tau \frac{d}{dt}\tfrac{1}{2}x(t)^2\,dt = \int_0^\tau y_1(t)u_1(t)\,dt = \langle y_{1_\tau}, u_{1_\tau}\rangle. \tag{44.52}$$

Let Σ_2 be a system which scales the instantaneous value of the input according to the energy of the input:

$$\begin{aligned} \dot{x}_2 &= u_2^2 \quad x_2(0) = 0 \\ y_2 &= -u_2\left(\frac{1}{1+|x_2|^{0.25}}\right). \end{aligned} \tag{44.53}$$

This system followed by the scaling "−1" is nonlinearly strictly passive. To see this, first note that

$$x_2(t) = ||u_{2_t}||_2^2 \tag{44 54}$$

which is a nondecreasing function of t. So,

$$\begin{aligned} \langle -y_{2_\tau}, u_{2_\tau}\rangle &= \int_0^\tau u_2^2(t)\left(\frac{1}{1+|x_2(t)|^{0.25}}\right)dt, \\ &\geq \left(\frac{1}{1+|x_2(\tau)|^{0.25}}\right)\int_0^\tau u_2^2(t)\,dt, \\ &= \left(\frac{1}{1+||u_{2_\tau}||_2^{0.5}}\right)||u_{2_\tau}||_2^2. \end{aligned} \tag{44.55}$$

Now we can define

$$\rho(\zeta) := \frac{0.5\zeta}{1+\zeta^{0.5}}, \tag{44.56}$$

which is a proper separation function, so that

$$\langle -y_{2_\tau}, u_{2_\tau}\rangle \geq 2\rho(||u_{2_\tau}||_2)||u_{2_\tau}||_2. \tag{44.57}$$

Finally, note that

$$||y_{2_\tau}||_2^2 = \int_0^\tau u_2^2(t)\frac{1}{\left(1+x_2^{0.25}(t)\right)^2}\,dt \leq ||u_{2_\tau}||_2^2, \tag{44.58}$$

so that

$$\langle -y_{2_\tau}, u_{2_\tau}\rangle \geq \rho(||u_{2_\tau}||_2)||u_{2_\tau}||_2 + \rho(||y_{2_\tau}||_2)||y_{2_\tau}||_2. \tag{44.59}$$

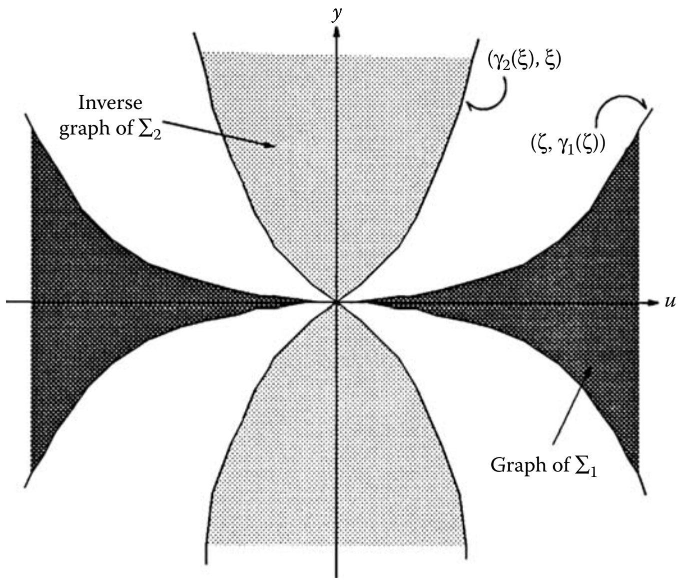

FIGURE 44.6 Nonlinear small gain theorem.

The conclusion that we can then draw from the nonlinear passivity theorem is that the interconnection of these two systems:

$$\begin{aligned} \dot{x}_1 &= -(x_1 + d_2)\left(\frac{1}{1 + |x_2|^{0.25}}\right) + d_1, \quad \dot{x}_2 = (x_1 + d_2)^2, \\ y_1 &= x_1, \quad \text{and} \quad y_2 = -(x_1 + d_2)\left(\frac{1}{1 + |x_2|^{0.25}}\right) \end{aligned} \tag{44.60}$$

is stable when measuring input (d_1, d_2) and output (y_1, y_2) using the 2-norm.

44.3.3.2 Nonlinear Small Gain

Just as with passivity, the idea behind the small gain theorem does not require the use of linear boundaries. Consider a well-defined interconnection where each feedback component is stable but not necessarily finite gain stable. Let γ_1 be a stability gain function for Σ_1 and let γ_2 be a stability gain function for Σ_2. Then the graph separation condition will be satisfied if the distance between the curves $(\zeta, \gamma_1(\zeta))$ and $(\gamma_2(\xi), \xi)$ grows without bound as in Figure 44.6. This is equivalent to asking whether it is possible to add to the curve $(\zeta, \gamma_1(\zeta))$ in the vertical direction and to the curve $(\gamma_2(\xi), \xi)$ in the horizontal direction, by an increasing and unbounded amount, to obtain new curves $\big(\zeta, \gamma_1(\zeta) + \rho(\zeta)\big)$ and $\big(\gamma_2(\xi) + \rho(\xi), \xi\big)$ where ρ is a proper separation function, so that the modified first curve is never above the modified second curve. If this is possible, we will say that the composition of the functions γ_1 and γ_2 is a *strict contraction*. To say that a curve $(\zeta, \tilde{\gamma}_1(\zeta))$ is never above a second curve $(\tilde{\gamma}_2(\xi), \xi)$ is equivalent to saying that $\tilde{\gamma}_1(\tilde{\gamma}_2(\zeta)) \leq \zeta$ or $\tilde{\gamma}_2(\tilde{\gamma}_1(\zeta)) \leq \zeta$ for all $\zeta \geq 0$. (Equivalently, we will write $\tilde{\gamma}_1 \circ \tilde{\gamma}_2 \leq \mathrm{Id}$ or $\tilde{\gamma}_2 \circ \tilde{\gamma}_1 \leq \mathrm{Id}$.) So, requiring that the composition of γ_1 and γ_2 is a strict contraction is equivalent to requiring that a strictly proper separation function ρ exists so that $(\gamma_1 + \rho) \circ (\gamma_2 + \rho) \leq \mathrm{Id}$ (equivalently $(\gamma_2 + \rho) \circ (\gamma_1 + \rho) \leq \mathrm{Id}$). This condition was made precise in [3]. (See also [2].) Note that it is not enough to add to just one curve because it is possible for the vertical or horizontal distance to grow without bound while the total distance remains bounded. Finally, note that, if the gain functions are linear, the condition is the same as the condition that the product of the gains is less than one.

Theorem 44.5: Nonlinear Small Gain Theorem

If each feedback component is stable (with gain functions γ_1 and γ_2) and the composition of the gains is a strict contraction, then the well-defined interconnection is stable.

To apply the nonlinear small gain theorem, we need a way to verify that the feedback components are stable. To date, the most common setting for using the nonlinear small gain theorem is when measuring the input and output using the ∞-norm. For a nonlinear system which can be represented by a smooth, ordinary differential equation,

$$\dot{x} = f(x,u), \quad x(0) = 0, \quad \text{and} \quad y = h(x,u), \tag{44.61}$$

where $h(0,0) = 0$, the system is stable (with respect to the ∞-norm) if there exist a positive definite and radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}{\geq}\, 0$, a proper separation function ψ, and a gain function $\tilde{\gamma}$ so that

$$\frac{\partial V}{\partial x} f(x,u) \leq -\psi(|x|) + \tilde{\gamma}(|u|). \tag{44.62}$$

Since V is positive definite and radially unbounded, additional proper separation functions $\underline{\alpha}$ and $\bar{\alpha}$ exist so that

$$\underline{\alpha}(|x|) \leq V(x) \leq \bar{\alpha}(|x|). \tag{44.63}$$

Also, because h is continuous and zero at zero, gain functions ϕ_x and ϕ_u exist so that

$$|h(x,u)| \leq \phi_x(|x|) + \phi_u(|u|). \tag{44.64}$$

Given all of these functions, a stability gain function can be computed as

$$\gamma = \phi_x \circ \underline{\alpha}^{-1} \circ \bar{\alpha} \circ \psi^{-1} \circ \tilde{\gamma} + \phi_u. \tag{44.65}$$

For more details, the reader is directed to [8].

Example 44.4:

Consider the composite system,

$$\begin{aligned} \dot{x} &= Ax + B\,\mathrm{sat}(z + d_1), \quad x(0) = 0 \\ \dot{z} &= -z + \epsilon(\exp(|x| + d_2) - 1), \quad z(0) = 0, \end{aligned} \tag{44.66}$$

where $x \in \mathbb{R}^n$, $z \in \mathbb{R}$, the eigenvalues of A all have strictly negative real part, ϵ is a small parameter, and $\mathrm{sat}(s) = \mathrm{sgn}(s)\min\{|s|, 1\}$. This composite system is a well-defined interconnection of the subsystems

$$\Sigma_1 : \begin{aligned} \dot{x} &= Ax + B\,\mathrm{sat}(u_1), \quad x(0) = 0 \\ y_1 &= |x| \end{aligned} \tag{44.67}$$

and

$$\Sigma_2 : \begin{aligned} \dot{z} &= -z + \epsilon\left(\exp(u_2) - 1\right), \quad z(0) = 0 \\ y_2 &= z. \end{aligned} \tag{44.68}$$

A gain function for the Σ_1 system is the product of the ∞-gain for the linear system

$$\begin{aligned} \dot{x} &= Ax + Bu, \quad x(0) = 0 \\ y &= x, \end{aligned} \tag{44.69}$$

which we will call $\bar{\gamma}_1$, with the function sat(s), i.e., for the system Σ_1,

$$||y||_\infty \leq \bar{\gamma}_1 \,\mathrm{sat}(||u_1||_\infty). \tag{44.70}$$

For the system Σ_2,

$$||z||_\infty \leq |\epsilon| \left(\exp\left(||u_2||_\infty\right) - 1\right). \tag{44.71}$$

The distance between the curves $(\zeta, \bar{\gamma}_1 \mathrm{sat}(\zeta))$ and $(|\epsilon| \left(\exp\left(\xi\right) - 1\right), \xi)$ must grow without bound. Graphically, one can see that a necessary and sufficient condition for this is that

$$|\epsilon| < \frac{1}{\exp(\bar{\gamma}_1) - 1}. \tag{44.72}$$

44.3.4 General Conic Regions

There are many different ways to partition the ambient space to establish the graph separation condition in Equation 44.5. So far we have looked at only two very specific sufficient conditions, the small gain theorem and the passivity theorem. The general idea in these theorems is to constrain signals in the graph of Σ_1 within some conic region, and signals in the inverse graph of Σ_2 outside of this conic region. Conic regions more general than those used for the small gain and passivity theorems can be generated by using operators on the input–output pairs of the feedback components.

Let $\mathbf{C}$ and $\mathbf{R}$ be operators on truncated ordered pairs in the ambient space, and let γ be a gain function. We say that the graph of Σ_1 is inside Cone$(\mathbf{C}, \mathbf{R}, \gamma)$ if, for each $(u, y) =: z$ belonging to the graph of Σ_1,

$$||\mathbf{C}(z_\tau)|| \leq \gamma(||\mathbf{R}(z_\tau)||), \quad \text{for all } \tau. \tag{44.73}$$

On the other hand, we say that the inverse graph of Σ_2 is strictly outside Cone$(\mathbf{C}, \mathbf{R}, \gamma)$ if a proper separation function ρ exists so that, for each $(y, u) =: x$ belonging to the inverse graph of Σ_2,

$$||\mathbf{C}(x_\tau)|| \geq \gamma \circ (\mathrm{Id} + \rho)(||\mathbf{R}(x_\tau)||) + \rho(||x_\tau||), \quad \text{for all } \tau. \tag{44.74}$$

We will only consider the case where the maps $\mathbf{C}$ and $\mathbf{R}$ are incrementally stable, i.e., a gain function $\bar{\gamma}$ exists so that, for each x_1 and x_2 in the ambient space and all τ,

$$\begin{aligned} ||\mathbf{C}(x_{1_\tau}) - \mathbf{C}(x_{2_\tau})|| &\leq \bar{\gamma}(||x_{1_\tau} - x_{2_\tau}||) \\ ||\mathbf{R}(x_{1_\tau}) - \mathbf{R}(x_{2_\tau})|| &\leq \bar{\gamma}(||x_{1_\tau} - x_{2_\tau}||). \end{aligned} \tag{44.75}$$

In this case, the following result holds.

Theorem 44.6: Nonlinear Conic Sector Theorem

If the graph of Σ_1 is inside CONE$(\mathbf{C}, \mathbf{R}, \gamma)$ *and the inverse graph of Σ_2 is strictly outside* CONE$(\mathbf{C}, \mathbf{R}, \gamma)$*, then the well-defined interconnection is stable.*

When γ and ρ are linear functions, the well-defined interconnection is finite gain stable.

The small gain and passivity theorems we have discussed can be interpreted in the framework of the nonlinear conic sector theorem. For example, for the nonlinear small gain theorem, the operator $\mathbf{C}$ is a projection onto the second coordinate in the ambient space, and $\mathbf{R}$ is a projection onto the first coordinate;

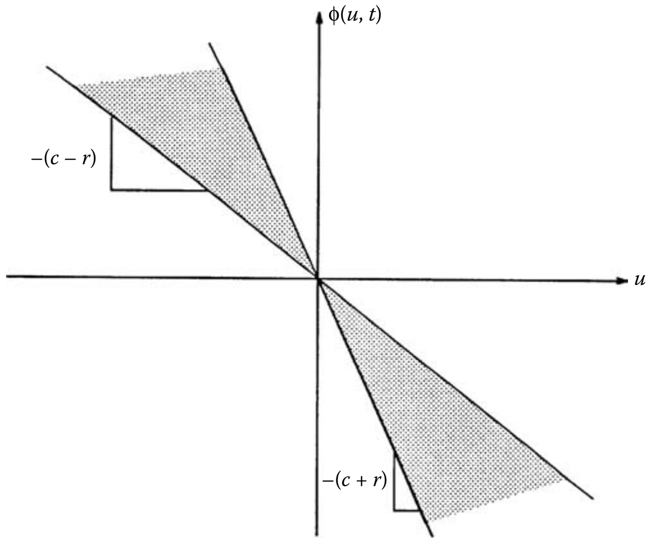

FIGURE 44.7 Instantaneous sector.

γ is the gain function γ_1, and the small gain condition guarantees that the inverse graph of Σ_2 is strictly outside of the cone specified by this **C**, **R** and γ.

In the remaining subsections, we will discuss other useful choices for the operators **C** and **R**.

44.3.4.1 The Classical Conic Sector (Circle) Theorem

For linear SISO systems connected to memoryless nonlinearities, there is an additional classical result, known as the circle theorem, which follows from the nonlinear conic sector theorem using the 2-norm and taking

$$\begin{aligned} \mathbf{C}\left(u,y\right) &= y + cu \\ \mathbf{R}\left(u,y\right) &= ru \quad r \geq 0 \\ \gamma(\zeta) &= \zeta. \end{aligned} \tag{44.76}$$

Suppose ϕ is a memoryless nonlinearity which satisfies

$$|\phi(u,t) + cu| \leq |ru| \quad \text{for all } t, u. \tag{44.77}$$

Graphically, the constraint on ϕ is shown in Figure 44.7. (In the case shown, $c > r > 0$.) We will use the notation $\text{Sector}[-(c+r), -(c-r)]$ for the memoryless nonlinearity. It is also clear that the graph of ϕ lies in the $\text{Cone}(\mathbf{C}, \mathbf{R}, \gamma)$ with $\mathbf{C}, \mathbf{R}, \gamma$ defined in Equation 44.76. For a linear, time invariant, finite dimensional SISO system, whether its inverse graph is strictly outside of this cone can be determined by examining the Nyquist plot of its transfer function. The condition on the Nyquist plot is expressed relative to a disk $\mathcal{D}_{c,r}$ in the complex plane centered on the real axis passing through the points on the real axis with real parts $-1/(c+r)$ and $-1/(c-r)$ as shown in Figure 44.8.

Theorem 44.7: Circle Theorem

Let $r \geq 0$, *and consider a well-defined interconnection of a memoryless nonlinearity belonging to* $\text{Sector}[-(c+r), -(c-r)]$ *with a SISO system having a real, rational transfer function* $G(s)$. *If*

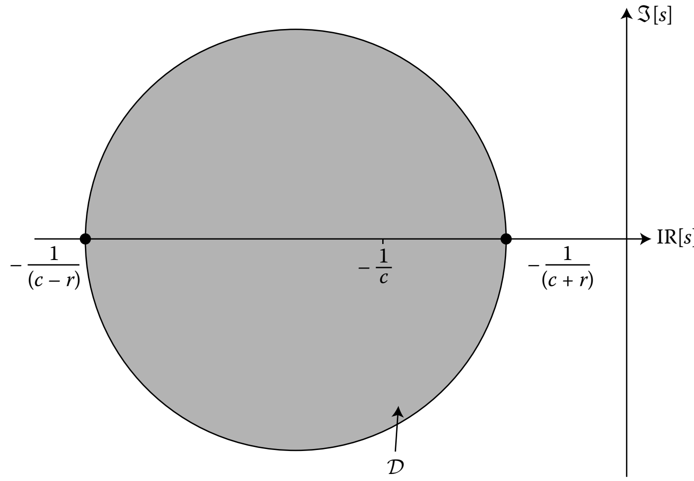

FIGURE 44.8 A disc in the complex plane.

- *$r > c$, $G(s)$ is stable and the Nyquist plot of $G(s)$ lies in the interior of the disc $\mathcal{D}_{c,r}$, or*
- *$r = c$, $G(s)$ is stable and the Nyquist plot of $G(s)$ is bounded away and to the right of the vertical line passing through the real axis at the value $-1/(c+r)$, or*
- *$r < c$, the Nyquist plot of $G(s)$ (with Nyquist path indented into the right-half plane) is outside of and bounded away from the disc $\mathcal{D}_{c,r}$, and the number of times the plot encircles this disc in the counterclockwise direction is equal to the number of poles of $G(s)$ with strictly positive real parts,*

then the interconnection is finite gain stable.

Case 1 is similar to the small gain theorem, and case 2 is similar to the passivity theorem. We will now explain case 3 in more detail. Let $n(s)$ and $d(s)$ represent, respectively, the numerator and denominator polynomials of $G(s)$. Since the point $(-1/c, 0)$ is inside the disc $\mathcal{D}_{c,r}$, it follows, from the assumption of the theorem together with the well-known Nyquist stability condition, that all of the roots of the polynomial $d(s) + cn(s)$ have negative real parts. Then $y = G(s)u = N(s)D(s)^{-1}u$ where

$$D(s) := \frac{d(s)}{d(s) + cn(s)}, \quad \text{and} \quad N(s) := \frac{n(s)}{d(s) + cn(s)}, \tag{44.78}$$

and, by taking $z = D(s)^{-1}u$, we can describe all of the possible input–output pairs as

$$(u, y) = \big(D(s)z, \quad N(s)z\big). \tag{44.79}$$

Notice that $D(s) + cN(s) = 1$, so that

$$||u + cy||_2 = ||z||_2. \tag{44.80}$$

To put a lower bound on this expression in terms of $||u||_2$ and $||y||_2$, to show that the graph is strictly outside of the cone defined in Equation 44.76, we will need the 2-norm gains for systems modeled by the transfer functions $N(s)$ and $D(s)$. We will use the symbols γ_N and γ_D for these gains. The condition of the circle theorem guarantees that $\gamma_N < r^{-1}$. To see this, note that

$$N(s) = \frac{G(s)}{1 + cG(s)} \tag{44.81}$$

implying

$$\gamma_N := \sup_{\omega\in\mathbb{R}} \left| \frac{G(j\omega)}{1+cG(j\omega)} \right|. \tag{44.82}$$

But

$$\begin{aligned} &|1+c\ G(j\omega)|^2 - r^2|G(j\omega)|^2 \\ &\quad = \left(c\ \mathrm{Re}\left\{G(j\omega)\right\}+1\right)^2 + c^2\ \mathrm{Im}^2\left\{G(j\omega)\right\} - r^2\ \mathrm{Re}^2\left\{G(j\omega)\right\} - r^2\ \mathrm{Im}^2\left\{G(j\omega)\right\}, \\ &\quad = (c^2-r^2)\ \left(\mathrm{Re}\left\{G(j\omega)\right\} + \frac{c}{c^2-r^2}\right)^2 + (c^2-r^2)\ \mathrm{Im}^2\left\{G(j\omega)\right\} - \frac{r^2}{c^2-r^2}. \end{aligned} \tag{44.83}$$

Setting the latter expression to zero defines the boundary of the disc $\mathcal{D}_{c,r}$. Since the expression is positive outside of this disc, it follows that $\gamma_N < r^{-1}$.

Returning to the calculation initiated in Equation 44.80, note that $\gamma_N < r^{-1}$ implies that a strictly positive real number ϵ exists so that

$$(1-\epsilon\gamma_D)\gamma_N^{-1} \geq r + 2\epsilon. \tag{44.84}$$

So,

$$\begin{aligned} ||u+cy||_2 = ||z||_2 &= (1-\epsilon\gamma_D)||z||_2 + \epsilon\gamma_D||z||_2, \\ &\geq (1-\epsilon\gamma_D)\gamma_N^{-1}||y||_2 + \epsilon||u||_2, \\ &\geq (r+\epsilon)||y||_2 + \epsilon(||u||_2 + ||y||_2). \end{aligned} \tag{44.85}$$

We conclude that the inverse graph of the linear system is strictly outside of the CONE($\mathbf{C}, \mathbf{R}, \gamma$) as defined in Equation 44.76.

Note, incidentally, that $N(s)$ is the closed loop transfer function from d_1 to y_1 for the special case where the memoryless nonlinearity satisfies $\phi(u) = -cu$. This suggests another way of determining stability: first make a preliminary loop transformation with the feedback $-cu$, changing the original linear system into the system with transfer function $N(s)$ and changing the nonlinearity into a new nonlinearity $\tilde{\phi}$ satisfying $|\tilde{\phi}(u,t)| \leq r|u|$. Then apply the classical small gain theorem to the resulting feedback system.

Example 44.5:

Let

$$G(s) = \frac{175}{(s-1)(s+4)^2}. \tag{44.86}$$

The Nyquist plot of $G(s)$ is shown in Figure 44.9. Because G(s) has one pole with positive real part, only the third condition of the circle theorem can apply. A disc centered at -8.1 on the real axis and with radius 2.2 can be placed inside the left loop of the Nyquist plot. Such a disc corresponds to the values $c = 0.293$ and $r = 0.079$. Because the Nyquist plot encircles this disc once in the counterclockwise direction, it follows that the standard feedback connection with the feedback components $G(s)$ and a memoryless nonlinearity constrained to the SECTOR[-0.372,-0.214] is stable using the 2-norm.

44.3.4.2 Coprime Fractions

Typical input–output stability results based on stable coprime fractions are corollaries of the conic sector theorem. For example, suppose both Σ_1 and Σ_2 are modeled by transfer functions $G_1(s)$ and $G_2(s)$. Moreover, assume stable (in any p-norm) transfer functions $N_1, D_1, \tilde{N}_1, \tilde{D}_1$, N_2 and D_2 exist so that D_1,

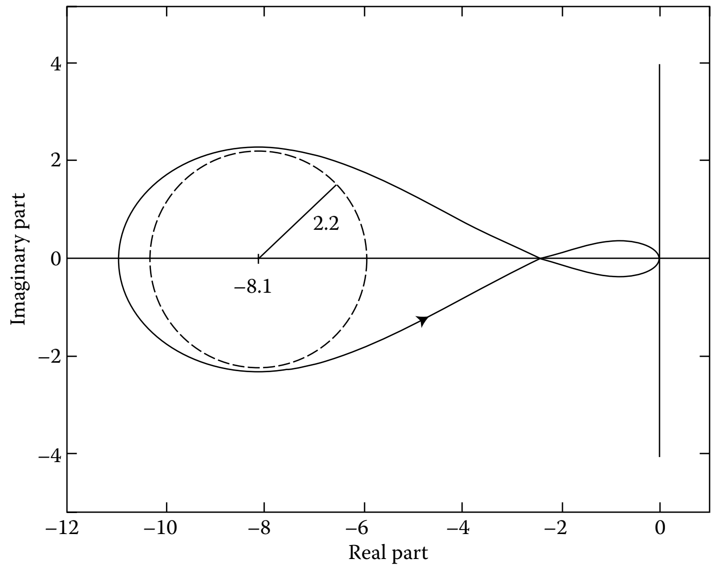

FIGURE 44.9 The Nyquist plot for $G(s)$ in Example 44.5.

D_2 and $\tilde{D}_1$ are invertible, and

$$\begin{aligned} G_1 &= N_1 D_1^{-1} = \tilde{D}_1^{-1}\tilde{N}_1 \\ G_2 &= N_2 D_2^{-1} \\ \mathrm{Id} &= \tilde{D}_1 D_2 - \tilde{N}_1 N_2. \end{aligned} \tag{44.87}$$

Let $\mathbf{C}(u,y) = \tilde{D}_1(s)y - \tilde{N}_1(s)u$, which is incrementally stable in any p-norm, let $\mathbf{R}(u,y) = 0$, and let $\gamma \equiv 0$. Then, the graph of Σ_1 is inside and the inverse graph of Σ_2 is strictly outside $\text{Cone}(\mathbf{C},\mathbf{R},\gamma)$ and thus the feedback loop is finite gain stable in any p-norm.

To verify these claims about the properties of the graphs, first recognize that the graph of Σ_i can be represented as

$$\mathcal{G}_{\Sigma_i} = \begin{pmatrix} D_i(s)z, & N_i(s)z \end{pmatrix} \tag{44.88}$$

where z represents any reasonable signal. Then, for signals in the graph of Σ_1,

$$\mathbf{C}\left(D_1(s)z_\tau, N_1(s)z_\tau\right) = \tilde{D}_1(s)N_1(s)z_\tau - \tilde{N}_1(s)D_1(s)z_\tau \equiv 0. \tag{44.89}$$

Conversely, for signals in the inverse graph of Σ_2,

$$\begin{aligned} \left\|\mathbf{C}\left(N_2(s)z_\tau, D_2(s)z_\tau\right)\right\| &= \left\|\tilde{D}_1(s)D_2(s)z_\tau - \tilde{N}_1(s)N_2(s)z_\tau\right\| \\ &= \|z_\tau\| \geq \epsilon \left\|\left(N_2(s)z_\tau, D_2(s)z_\tau\right)\right\| \end{aligned} \tag{44.90}$$

for some strictly positive real number ϵ. The last inequality follows from the fact that D_2 and N_2 are finite gain stable.

Example 44.6:

(This example is drawn from the work in Potvin, M-J., Jeswiet, J., and Piedboeuf, J.-C. , *Trans. NAMRI/SME 1994, XXII,* pp 373–377.) Let Σ_1 represent the fractional Voigt–Kelvin model for the relation between stress and strain in structures displaying plasticity. For suitable values of Young's modulus, damping magnitude, and order of derivative for the strain, the transfer function of Σ_1 is

$$g_1(s) = \frac{1}{1+\sqrt{s}}.$$

Integral feedback control, $g_2(s) = -\frac{1}{s}$, may be used for asymptotic tracking. Here

$$\begin{aligned} N_1(s) &= \frac{1}{s+1}, & D_1(s) &= \frac{1+\sqrt{s}}{s+1}, \\ N_2(s) &= -\frac{s+1}{1+s(1+\sqrt{s})}, & D_2(s) &= \frac{s(s+1)}{1+s(1+\sqrt{s})}. \end{aligned} \tag{44.91}$$

It can be shown that these fractions are stable linear operators, and thereby incrementally stable in the 2-norm. (This fact is equivalent to proving nominal stability and can be shown using Nyquist theory.) Moreover, it is easy to see that $D_1D_2 - N_1N_2 = 1$ so that the feedback loop is stable and finite gain stable.

44.3.4.3 Robustness of Stability and the Gap Metric

It is clear from the original graph separation theorem that, if a well-defined interconnection is stable, i.e., the appropriate graphs are separated in distance, then modifications of the feedback components will not destroy stability if the modified graphs are close to the original graphs.

Given two systems Σ_1 and Σ, define $\vec{\delta}(\Sigma_1, \Sigma) = \alpha$ if α is the smallest number for which

$$x \in \mathcal{G}_\Sigma, \implies d_\tau(x, \mathcal{G}_{\Sigma_1}) \leq \alpha \|x\|_\tau \quad \text{for all } \tau.$$

The quantity $\vec{\delta}(\cdot,\cdot)$ is called the "directed gap" between the two systems and characterizes basic neighborhoods where stability as well as closed-loop properties are preserved under small perturbations from the nominal system Σ_1 to a nearby system Σ.

More specifically, if the interconnection of (Σ_1, Σ_2) is finite gain stable, we define the gain $\beta_{\Sigma_1,\Sigma_2}$ as the smallest real number so that

$$\left\| \begin{pmatrix} d_1 + y_2 \\ y_1 \end{pmatrix} \right\|_\tau \leq \beta_{\Sigma_1,\Sigma_2} \left\| \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \right\|_\tau, \quad \text{for all } \tau.$$

If Σ is such that

$$\vec{\delta}(\Sigma_1, \Sigma)\beta_{\Sigma_1,\Sigma_2} < 1,$$

then the interconnection of (Σ, Σ_2) is also finite gain stable.

As a special case, let $\Sigma, \Sigma_1, \Sigma_2$ represent linear systems acting on finite energy signals. Further, assume that stable transfer functions N, D exist where D is invertible, $G_1 = ND^{-1}$, and N and D are normalized so that $D^T(-s)D(s) + N^T(-s)N(s) = \text{Id}$. Then, the class of systems in a ball with radius $\gamma \geq 0$, measured in the directed gap, is given by $\textsc{Cone}(\mathbf{C}, \mathbf{R}, \gamma)$, where $\mathbf{R} = \text{Id}$ and

$$\mathbf{C} = \text{Id} - \begin{pmatrix} D(s) \\ N(s) \end{pmatrix} \mathbf{P}_+(D^T(-s), N^T(-s))$$

where $\mathbf{P}_+$ designates the truncation of the Laplace transform of finite energy signals to the part with poles in the left half plane. At the same time, if $\beta_{\Sigma_1,\Sigma_2} < 1/\gamma$, then it can be shown that Σ_2 is strictly outside the cone $\textsc{Cone}(\mathbf{C}, \mathbf{R}, \gamma)$ and, therefore, stability of the interconnection of Σ with Σ_2 is guaranteed for any Σ inside $\textsc{Cone}(\mathbf{C}, \mathbf{R}, \gamma)$.

Given Σ and Σ_1, the computation of the directed gap reduces to a standard $\mathcal{H}_\infty$-optimization problem (see [1]). Also, given Σ_1, the computation of a controller Σ_2, which stabilizes a maximal cone around

Σ_1, reduces to a standard $\mathcal{H}_\infty$-optimization problem [1] and forms the basis of the $\mathcal{H}_\infty$-loop shaping procedure for linear systems introduced in [4].

A second key result which prompted introducing the gap metric is the claim that the behavior of the feedback interconnection of Σ and Σ_2 is "similar" to that of the interconnection of Σ_1 and Σ_2 if, and only if, the distance between Σ and Σ_1, measured using the gap metric, is small (i.e., Σ lies within a "small aperture" cone around Σ_1). The "gap" function is defined as

$$\delta(\Sigma_1, \Sigma) = \max\{\vec{\delta}(\Sigma_1, \Sigma), \vec{\delta}(\Sigma, \Sigma_1)\}$$

to "symmetrize" the distance function $\vec{\delta}(\cdot, \cdot)$ with respect to the order of the arguments. Then, the above claim can be stated more precisely as follows: for each $\epsilon > 0$, a $\zeta(\epsilon) > 0$ exists so that

$$\delta(\Sigma_1, \Sigma) < \zeta(\epsilon) \implies \|x - x_1\|_\tau < \epsilon\|d\|_\tau$$

where $d = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$ is an arbitrary signal in the ambient space and x (resp. x_1) represents the response $\begin{pmatrix} d_1 + y_2 \\ y_1 \end{pmatrix}$ of the feedback interconnection of (Σ, Σ_2) (resp. (Σ_1, Σ_2)). Conversely, if $\|x - x_1\|_\tau < \epsilon\|d\|_\tau$ for all d and τ, then $\delta(\Sigma_1, \Sigma) \leq \epsilon$.

Defining Terms

Ambient space: the Cartesian product space containing the inverse graph of Σ_2 and the graph of Σ_1.

Distance (from a signal to a set): measured using a norm function; the infimum, over all signals in the set, of the norm of the difference between the signal and a signal in the set; see Equation 44.4; used to characterize necessary and sufficient conditions for input–output stability; see Section 44.2.

Dynamical system: an object which produces an output signal for each input signal.

Feedback components: the dynamical systems which make up a well-defined interconnection.

Finite gain stable system: a dynamical system is finite gain stable if a nonnegative constant exists so that, for each input–output pair, the norm of the output is bounded by the norm of the input times the constant.

Gain function: a function from the nonnegative real numbers to the nonnegative real numbers which is continuous, nondecreasing and zero when its argument is zero; used to characterize stability; see Section 44.2; some form of the symbol γ is usually used.

Graph (of a dynamical system): the set of ordered input–output pairs (u, y).

Inner product: defined for signals of the same dimension defined on the semi-infinite interval; the integral from zero to infinity of the component-wise product of the two signals.

Inside (or strictly outside) CONE(**C**, **R**, γ)**:** used to characterize the graph or inverse graph of a system; determined by whether or not signals in the graph or inverse graph satisfy certain inequalities involving the operators **C** and **R** and the gain function γ; see Equations 44.73 and 44.74; used in the conic sector theorem.

Inverse graph (of a dynamical system): the set of ordered output-input pairs (y, u).

Norm function ($||\cdot||$)**:** used to measure the size of signals defined on the semi-infinite interval; examples are the p-norms $p \in [1, \infty]$ (see Equations 44.1 and 44.2).

Parseval's theorem: used to make connections between properties of graphs for SISO systems modeled with real, rational transfer functions and frequency domain characteristics of their transfer functions; Parseval's theorem relates the inner product of signals to their Fourier transforms if they exist. For example, it states that, if two scalar signals u and y, assumed to be zero for negative values of time, have Fourier transforms $\hat{u}(j\omega)$ and $\hat{y}(j\omega)$ then

$$\langle u, y\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{y}^*(j\omega)\hat{u}(j\omega)\, d\omega.$$

Passive: terminology resulting from electrical network theory; a dynamical system is passive if the inner product of each input–output pair is nonnegative.

Proper separation function: a function from the nonnegative real numbers to the nonnegative real numbers which is continuous, zero at zero, strictly increasing and unbounded; such functions are invertible on the nonnegative real numbers; used to characterize nonlinear separation theorems; some form of the symbol ρ is usually used.

Semi-infinite interval: the time interval $[0, \infty)$.

Signal: a "reasonable" vector-valued function defined on a finite or semi-infinite time interval; by "reasonable" we mean piecewise continuous or measurable.

SISO systems: an abbreviation for single input, single output systems.

Stable system: a dynamical system is stable if a gain function exists so that, for each input–output pair, the norm of the output is bounded by the gain function evaluated at the norm of the input.

Strict contraction: the composition of two gain functions γ_1 and γ_2 is a strict contraction if a proper separation function ρ exists so that $(\gamma_1 + \rho) \circ (\gamma_2 + \rho) \leq \mathrm{Id}$, where $\mathrm{Id}(\zeta) = \zeta$ and $\tilde{\gamma}_1 \circ \tilde{\gamma}_2(\zeta) = \tilde{\gamma}_1(\tilde{\gamma}_2(\zeta))$. Graphically, this is the equivalent to the curve $(\zeta, \gamma_1(\zeta) + \rho(\zeta))$ never being above the curve $(\gamma_2(\xi) + \rho(\xi), \xi)$. This concept is used to state the nonlinear small gain theorem.

Strictly passive: We have used various notions of strictly passive including input-, output-, input and output-, and nonlinear input and output-strictly passive. All notions strengthen the requirement that the inner product of the input–output pairs be positive by requiring a positive lower bound that depends on the 2-norm of the input and/or output.

Truncated signal: A signal defined on the semi-infinite interval which is derived from another signal (not necessarily defined on the semi-infinite interval) by first appending zeros to extend the signal onto the semi-infinite interval and then keeping the first part of the signal and setting the rest of the signal to zero. Used to measure the size of finite portions of signals.

Well-defined interconnection: An interconnection of two dynamical systems in the configuration of Figure 44.1 which results in another dynamical system, i.e., one in which an output signal is produced for each input signal.

References

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Further Reading

As mentioned at the outset, the material presented in this chapter is based on the results in [6,7,9,10]. In the latter, a more general feedback interconnection structure is considered where nonzero initial conditions can also be consider as inputs.

Other excellent references on input–output stability include *The Analysis of Feedback Systems*, 1971, by J.C. Willems and *Feedback Systems: Input–Output Properties*, 1975, by C. Desoer and M. Vidyasagar. A nice text addressing the factorization method in linear systems control design is *Control Systems Synthesis: A Factorization Approach*, 1985, by M. Vidyasagar. A treatment of input–output stability for linear, infinite dimensional systems can be found in Chapter 6 of *Nonlinear Systems Analysis*, 1993, by M. Vidyasagar. That chapter also discusses many of the connections between input–output stability and state-space (Lyapunov) stability. Another excellent reference is *Nonlinear Systems*, 1992, by H. Khalil.

There are results similar to the circle theorem that we have not discussed. They go under the heading of "multiplier" results and apply to feedback loops with a linear element and a memoryless, nonlinear element with extra restrictions such as time invariance and constrained slope. Special cases are the well-known Popov and off-axis circle criterion. Some of these results can be recovered using the general conic sector theorem although we have not taken the time to do this. Other results, like the Popov criterion, impose extra smoothness conditions on the external inputs which are not found in the standard problem. References for these problems are *Hyperstability of Control Systems*, 1973, V.M. Popov, the English translation of a book originally published in 1966, and *Frequency Domain Criteria for Absolute Stability*, 1973, by K.S. Narendra and J.H. Taylor.

Another topic closely related to these multiplier results is the structured small gain theorem for linear systems which lends to much of the μ-synthesis control design methodology. This is described, for example, in *μ-Analysis and Synthesis Toolbox*, 1991, by G. Balas, J. Doyle, K. Glover, A. Packard and R. Smith.

There are many advanced topics concerning input–output stability that we have not addressed. These include the study of small-signal stability, well-posedness of feedback loops, and control design based on input–output stability principles. Many articles on these topics frequently appear in control and systems theory journals such as *IEEE Transactions on Automatic Control*, *Automatica*, *International Journal of Control*, *Systems and Control Letters*, *Mathematics of Control, Signals, and Systems*, to name a few.