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Essential and Redundant Internal Models in Nonlinear Output Regulation

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Summary. This paper is focused on the problem of output regulation for nonlinear systems within the main framework developed in [23]. The main goal is to complement that theory with some new results showing how the dimension of the internal model-based regulator can be reduced by preserving the so-called internal model property. It is shown how the problem of reducing the regulator dimension can be approached by identifying "observability" parts of the so-called steady-state input generator system. A local analysis based on canonical geometric tools and local observability decomposition is also presented to identify lower bounds on the regulator dimension. Possible benefits in designing redundant internal models are also discussed.

This work is dedicated to Prof. Alberto Isidori
on the occasion of his 65th birthday.

1 Introduction

One of the main issue in control theory is in the ability to capture information about the plant to be supervised and the environment in which it operates and to employ such a knowledge in the design of the controller in order to achieve prescribed performances. A well-known control framework where such an issue is particularly emphasized, is the one of output regulation (see, besides others, [4], [19]) in which the problem is to design a regulator able to asymptotically offset the effect, on a controlled system, of persistent exogenous signals which are thought as generated by an autonomous system (the so-called exosystem) of known structure but unknown initial condition. Indeed, as pioneered in a linear setting in [12] and in a nonlinear setting in [18], the controller, to succeed in enforcing the desired asymptotic properties, is necessarily required to be designed by employing the a-priori knowledge of the environment in which the plant operates provided, in the classical framework, by the structure of the exo-system. This, in turn, has led to the fundamental concept of *internal* *model* and to the identification of design procedures for *internal model-based regulators*. To this respect the crucial property required to any regulator solving the problem is to be able to generate all possible *steady state*"feed-forward" control inputs needed to enforce an identically zero regulation error, namely the control inputs able to render invariant the so-called zero error manifold. This is what, in the important work [4], has been referred to as *internal model property*.

The design of regulators with the internal model property in a nonlinear context necessarily requires the ability to address two major points. The first regards the extension of the notion of steady state for nonlinear systems which, clearly, is instrumental to properly formulate the internal model property. The number of attempts along this direction which appeared in the related literature started with the work [18], in which the steady state has been characterized in terms of the solution of the celebrated *regulator equations* (somewhere also referred to as Francis-Isidori-Byrnes equations), and culminated with the notion recently given in [4]. In this work the authors showed how the right mathematical tool to look for is the omega limit set of a set and, upon this tool, they built up a non-equilibrium framework of output regulation.

The second critical point to be addressed consists of identifying methodologies to design regulators which on one hand posses the internal model property, and, on the other hand, enforce in the closed-loop system a steady state with zero regulated error. This double requirement justifies the usual regulator structure constituted by a first dynamical unit (the internal model), designed to provide the needed steady-state control action, and a second dynamical unit (the stabilizer), whose role is to effectively steer the closed-loop trajectories towards the desired steady-state. Of course the design of the two units are strongly interlaced in the sense that the ability of designing a stabilizer is affected by the specific structure of the internal model which, as a consequence, has to be identified with an eye to the available stabilization tools. The need of satisfying simultaneously the previous two properties motivated the requirement, characterizing all the frameworks appeared in literature, that the dynamical system defining all possible "feed-forward" inputs which force an identically zero regulation error be "immersed" into a system exhibiting certain structural properties. This requirement is what, in literature, is referred to as "immersion assumption". This is the side where, in the literature of the last fifteen years or so, the research attempts have mostly concentrated by attempting to weaken even more the immersion assumption. At the beginning, the system in question was assumed to be immersed into a linear *known* observable system (see [15], [21], [3], [24]). This assumption has been then weakened, in the framework of adaptive nonlinear regulation (see [25]), by asking immersion into a linear *un-known* (but linearly parameterized) system. Subsequent extensions have been presented in [6] (where immersion into a linear system having a nonlinear output map is assumed) and in [7] (where immersion into a nonlinear system linearizable by output injection is assumed). Finally the recent works in [5] and [8] (see also [9]) have definitely focused the attention on the design of nonlinear internal models requiring immersion into *nonlinear* systems described, respectively, in a canonical observability form and in a nonlinear adaptive observability form.

As clearly pointed out in [9], the inspiring idea in all the previous works was to adopt methodologies for the design of the internal model inherited by the design of observers. This perspective, along with the new theory to design nonlinear observers proposed in [20] and developed in [1], played a crucial role to completely drop the immersion assumption in the work [23]. In plain words the main achievement in [23] has been to show that the steady state input rendering invariant a compact attractor to be stabilized by output feedback can be dynamically generated, in a robust framework, by an appropriately designed regulator without any specific condition on this input (required, on the contrary, in the past through the immersion assumption).

This paper aims to extend [23] by exploring conditions under which the dimension of the controller can be decreased while preserving the internal model property and, on the other side, to show potential advantage in the regulator design resulting from a redundant implementation of the internal model. The major achievement in the reduction results is to show that the identification of an "essential" internal model is intimately related to the identification of "observability" parts of the so-called steady-state input generator system. Motivated by this result we show how a local analysis based on canonical geometric tools and local observability decomposition is useful to identify lower bounds on the regulator dimension. On the other side, it is presented a result showing that implementing a not essential internal model, in the sense better specified in the paper, leads to a simplification in the structure of the stabilizer which can be taken linear. Basically, the results presented in the paper reveal a trade-off between the redundancy of the internal model and the simplicity of the stabilizer.

The paper is organized as follows. In the next section we briefly review the framework of output regulation and the solution given in [23]. Section 3, articulated in two subsections, present the new results regarding essential regulators and the potential advantage coming from redundant internal models. Finally, Section 4 presents some concluding remarks.

2 The Framework of Output Regulation

2.1 The Class of Systems and the Problem

The typical setting where the problem of nonlinear output regulation is formulated is the one in which it is given a smooth nonlinear system described in the form[3]

[3] The form (1) is easily recognized to be the well-known *normal form* with relative degree 1 and unitary high-frequency gain (see [17]). As discussed in [23],

$$\begin{aligned}\dot z &= f(w,z,y)\\ \dot y &= q(w,z,y)+u\,,\end{aligned} \tag{1}$$

with state $(z,y) \in \mathbb{R}^n \times \mathbb{R}$ and *control* input $u \in \mathbb{R}$ and measurable output y, influenced by an exogenous input $w \in \mathbb{R}^s$ which is supposed to be generated by the smooth *exosystem*

$$\dot w = s(w) \tag{2}$$

whose initial state $w(0)$ is supposed to range on an *invariant* compact set $W \subset \mathbb{R}^s$. Depending on the control scenario, the variable w may assume different meanings. It may represent exogenous disturbances to be rejected and/or references to be tracked. It may also contain a set of (constant or time-varying) uncertain parameters affecting the controlled plant. Associated with (1) there is a *regulated* error $e \in \mathbb{R}$ expressed as

$$e = h(w,z,y) \tag{3}$$

in which $h : \mathbb{R}^s \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a smooth function.

For system (1)–(2)–(3) *the problem of semiglobal output regulation* is defined as follows. Given arbitrary compact sets $Z \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}$ find, if possible, an output feedback controller of the form

$$\begin{aligned}\dot\eta &= \varphi(\eta,y)\\ u &= \varrho(\eta,y)\end{aligned} \tag{4}$$

with state $\eta \in \mathbb{R}^\nu$ and a compact set $M \subset \mathbb{R}^\nu$ such that, in the associated closed-loop system (1), (2), (4) the positive orbit of $W \times Z \times Y \times M$ is bounded and, for each $w(0), z(0), y(0), \eta(0) \in W \times Z \times Y \times M$, $\lim_{t\to\infty} e(t) = 0$ uniformly in $w(0), z(0), y(0), \eta(0)$.

As in [23], we approach the solution of the problem at issue under the following assumption formulated on the zero dynamics (with respect to the input u and output y) of system (1), namely on the system

$$\begin{aligned}\dot w &= s(w)\\ \dot z &= f(w,z,0)\,.\end{aligned} \tag{5}$$

Note that, as a consequence of the fact that W is an *invariant* set for $\dot w = s(w)$, the closed cylinder $\mathcal{C} := W \times \mathbb{R}^n$ is locally invariant for (2)–(5) and thus it is natural to regard this system on $\mathcal{C}$ and endow the latter with the subset topology. This, indeed, is done in all the forthcoming analysis and, in particular, in the next assumption.

Assumption 1. *There exists a compact set $\mathcal{A} \subset \mathbb{R}^{s+n}$ which is locally asymptotically stable for (5) with a domain of attraction which contains the set of initial conditions $W \times Z$. Furthermore, $h(w,z,0) = 0$ for all $(w,z) \in \mathcal{A}$.*

Section 2.2, (see also [8]) the more general case (higher relative degree and not unitary high frequency gain) can be dealt with with simple modifications which, for sake of compactness, are not repeated here.

Following [4] this assumption can be regarded as a “weak” minimum phase assumption, with the adjective weak to highlight the fact that the “forced” zero dynamics of the plant $\dot z = f(w, z, 0)$ is not required to posses input-to-state stability (with respect to the input w) properties nor that the “unforced” $\dot z = f(0, z)$ dynamics exhibit equilibrium points with prescribed stability properties.

2.2 The Asymptotic Regulator in [23]

The regulator proposed in [23] to solve the problem at hand is a system of the form

$$\begin{aligned} \dot\eta &= F\eta + Gu \qquad \eta \in \mathbb{R}^m \\ u &= \gamma(\eta) + v \\ v &= -\kappa(y)\,, \end{aligned} \tag{6}$$

in which $m > 0$, $(F, G) \in \mathbb{R}^{m\times m} \times \mathbb{R}^m$ is a controllable pair with F Hurwitz and $\gamma : \mathbb{R}^m \to \mathbb{R}$ and $\kappa : \mathbb{R} \to \mathbb{R}$ are suitable continuous maps. The initial condition of (6) is supposed to be in an arbitrary compact set $M \subset \mathbb{R}^m$.

The key result proved in [23] is that, under the only assumption stated in Section 2.1, there exist a lower bound for m, a choice of the pair (F, G) and of the maps γ and κ such that the regulator (6) succeeds in solving the problem at hand. In this subsection we run very briefly over the key steps and ideas followed in [23] to prove this, which are instrumental for the forthcoming analysis in Section 3.

First of all, for sake of compactness, define $\mathbf{z} := \mathrm{col}(w, z)$ and rewrite system (5) as $\dot{\mathbf{z}} = \mathbf{f}_0(\mathbf{z})$ where

$$\mathbf{f}_0(\mathbf{z}) := \mathrm{col}(s(w), f(w, z, 0))\,. \tag{7}$$

Consistently set $\mathbf{q}_0(\mathbf{z}) := q(w, z, 0)$. A key role in the regulator (6) is played by the function $\gamma(\,\cdot\,)$ which is supposed to be an at least continuous function satisfying the design formula

$$\mathbf{q}_0(\mathbf{z}) + \gamma \circ \tau(\mathbf{z}) = 0 \qquad \forall\, \mathbf{z} \in \mathcal{A} \tag{8}$$

with the function $\tau : \mathcal{A} \to \mathbb{R}^m$ a continuous function satisfying

$$L_{\mathbf{f_0}}\tau(\mathbf{z}) = F\tau(\mathbf{z}) - G\mathbf{q}_0(\mathbf{z}) \qquad \forall\, \mathbf{z} \in \mathcal{A} \tag{9}$$

where $L_{\mathbf{f_0}}$ denotes the Lie derivative along $\mathbf{f_0}$.

In order to motivate the design formulas (8)–(9), consider the closed-loop system (1), (2), (6) given by

$$\begin{aligned} \dot w &= s(w) \\ \dot z &= f(w, z, y) \\ \dot\eta &= F\eta + G\gamma(\eta) + v \\ \dot y &= q(w, z, y) + v\,. \end{aligned} \tag{10}$$

The crucial property exhibited by this system is that, by the fact that the set $\mathcal{A}$ is forward invariant for (5) (as a consequence of the fact that $\mathcal{A}$ is locally asymptotically stable for (5)) and by (8), (9), the set

$$\mathrm{graph}(\tau) \times \{0\} = \{(\mathbf{z}, \eta, y) \in \mathcal{A} \times \mathbb{R}^m \times \mathbb{R} \ : \ \eta = \tau(\mathbf{z})\,, y = 0\} \tag{11}$$

is a forward invariant set for (10) (with $v \equiv 0$) on which, by assumption, the regulation error e is identically zero. This, in turn, makes it possible to consider the problem of output regulation as a *set stabilization problem* in which the issue is to design the function κ so that the set (11) is locally asymptotically stable for (10) with $v = \kappa(y)$ with a domain of attraction containing the set of initial conditions. Both the existence of a γ (and of the pair (F, G)) satisfying (8), (9) and the existence of κ so that the set (11) is locally asymptotically stable for (10) with $v = \kappa(y)$ are issues which have been investigated in [23] and [22]. In the remaining part of the section we present the main result along this direction. We start with a proposition presenting the main result as far as the existence of γ is concerned (see Propositions 2 and 3 in [23]).

Proposition 1. *Set*

$$m \geq 2(s + n) + 2\,.$$

There exist an $\ell > 0$ and a set $\mathcal{S} \subset \mathbb{C}$ of zero Lebesgue measure such that if $\sigma(F) \subset \{\lambda \in \mathbb{C} \ : \ Re\lambda < -\ell\} \setminus \mathcal{S}$, then there exists a function $\tau : \mathcal{A} \to \mathbb{R}^m$ solution of (9) which satisfies the partial injectivity condition

$$|\mathbf{q}_0(\mathbf{z}_1) - \mathbf{q}_0(\mathbf{z}_2)| \leq \varrho(|\tau(\mathbf{z}_1) - \tau(\mathbf{z}_2)|) \qquad \text{for all } \mathbf{z}_1, \mathbf{z}_2 \in \mathcal{A} \tag{12}$$

where ϱ is a class-$\mathcal{K}$ function. As a consequence of (12) there exists a continuous function γ satisfying (8).

On the other hand the problem of designing the function κ so that (11) is locally asymptotically stable for (10) with $v = \kappa(y)$ can be successfully handled by means of a generalization of the tools proposed in [27] (see also [17]) for stabilization of minimum-phase systems via high-gain output feedback. Here, in particular, is where the "weak" minimum phase assumption presented in Section 2.1 plays a role. The main result in this direction is presented in the following proposition collecting the main achievements of Theorems 1 and 2 and Proposition 1 in [23].

Proposition 2. *Let the pair (F, G) and the function γ be fixed according to Proposition 1. There exists a continuous κ such that the set* $\mathrm{graph}(\tau) \times \{0\}$ *is asymptotically stable for (10) with $v = -\kappa(y)$ with a domain of attraction containing $W \times Z \times X \times Y$.*
Furthermore, if γ is also locally Lipschitz and $\mathcal{A}$ is locally exponentially stable for (5) then there exists a $k^\star > 0$ such that for all $k \geq k^\star$, the set (11) is locally asymptotically stable for (10) with $v = -ky$.

The issue of providing an explicit expression of γ, whose existence is guaranteed by Proposition 1, has been dealt with, in an exact and approximated way, in the work [22]. For compactness we present only one of the two expressions of γ given in [22] to which the interested reader is referred for further details. In formulating the expression of γ it is argued that the class-$\mathcal{K}$ function ϱ in (12) satisfies

$$\varrho(|x_3 - x_1|) \;\leq\; \varrho(|x_3 - x_2|) \;+\; \varrho(|x_1 - x_2|) \qquad \forall\,(x_1, x_2, x_3) \in \mathbb{R}^{3m}\;. \tag{13}$$

This, indeed, can be assumed without loss of generality as shown in the proof of Proposition 3 of [22].

Proposition 3. *Let τ be fulfilling (12) with a function ϱ satisfying (13). Then the function $\gamma : \mathbb{R}^m \to \mathbb{R}$ defined by*

$$\gamma(x) \;=\; \inf_{\mathbf{z}\in\mathcal{A}} -\mathbf{q}_0(\mathbf{z}) \;+\; \min\{\varrho(|x - \tau(\mathbf{z})|), 2Q\} \tag{14}$$

where $Q \;=\; \sup_{\mathbf{z}\in\mathcal{A}} \mathbf{q}_0(\mathbf{z})$ satisfies (8).

2.3 Comments on the Results

As clear by the previous analysis, the desired asymptotic behavior of the system (1) is the one in which the components (w, z) of the overall trajectory evolve on $\mathcal{A}$ and the y component is identically zero. This, in turn, guarantees, by the second part of the Assumption in Section 2.1, that the regulation error (3) is asymptotically vanishing. In order to have this asymptotic desired behavior enforced, a crucial property required to the regulator is to be able to generate any possible asymptotic control input which is needed to keep y identically zero while having (w, z) evolving on $\mathcal{A}$. This, in turn, is what in [4] has been referred to as *internal model property* (with respect to $\mathcal{A}$), namely the property, required to any regulator solving the problem at hand, of reproducing all the "steady state" control inputs needed to keep the regulated error to zero. By bearing in mind (10) and the notation around (8)–(9), it is not hard to see that, in our specific context, the regulator (6) posses the asymptotic internal model property with respect to $\mathcal{A}$ if for any initial condition $\mathbf{z}_0 \in \mathcal{A}$ of the system

$$\begin{aligned} \dot{\mathbf{z}} &= \mathbf{f}_0(\mathbf{z}) \\ y_{\mathbf{z}} &= -\mathbf{q}_0(\mathbf{z}) \end{aligned} \tag{15}$$

yielding a trajectory $\mathbf{z}(t)$, $t \geq 0$, there exists an initial condition $\eta_0 \in \mathbb{R}^m$ of the system

$$\begin{aligned} \dot{\eta} &= F\eta - G\mathbf{q_0}(\mathbf{z}(t)) \\ y_\eta &= \gamma(\eta) \end{aligned} \tag{16}$$

such that the corresponding two output trajectories $y_{\mathbf{z}}(t)$ and $y_\eta(t)$ are such that $y_{\mathbf{z}}(t) = y_\eta(t)$ for all $t \geq 0$. This, indeed, is what is guaranteed by the design formulas (8)–(9). As a matter of fact, by taking $\eta_0 = \tau(\mathbf{z}_0)$, the two formulas (8)–(9) along with the fact that $\mathcal{A}$ is forward invariant for (15), imply that the corresponding state trajectory $\eta(t)$ of (16) is such that $\eta(t) = \tau(\mathbf{z}(t))$ for all $t \geq 0$ and, by virtue of (8), that $y_{\mathbf{z}}(t) = y_\eta(t)$ for all $t \geq 0$. In these terms the triplet $(F, G, \gamma(\,\cdot\,))$ qualifies as an *internal model* able to reproduce all the asymptotic control inputs which are required to enforce a zero regulation error.

Seen from this perspective, Proposition 1 fixes precise conditions under which the asymptotic internal model property can be achieved by a regulator of the form (6). In particular it is interesting to note that, for the function γ to exist, the dimension m of the internal model is required to be sufficiently large with respect to the dimension $s+n$ of the dynamical system (15) whose output behaviors must be replied.

The result previously presented gains further interest in relation to the theory of nonlinear observers recently proposed in [20] and developed in [1], which has represented the main source of inspiration in [23]. In the observation framework of [20], systems (15), (16) are recognized to be the cascade of the "observed" system (15), with state $\mathbf{z}$ and output $y_{\mathbf{z}}$, driving the "observer" (16) whose output $\gamma(\eta)$ is designed to provide an asymptotic estimate of the observed state $\mathbf{z}$. To this purpose, in [1], the map $\gamma(\,\cdot\,)$ is computed as the *left-inverse* of $\tau(\,\cdot\,)$, i.e. such that $\gamma(\tau(\mathbf{z})) = \mathbf{z}$ for all $\mathbf{z} \in \mathcal{A}$, with τ solution of (9). Such a left-inverse, as shown in [1], always exists provided that the dimension of η is sufficiently large (precisely $\dim(\eta) \geq 2\dim(\mathbf{z}) + 2$ as in Proposition 1) and certain *observability conditions* for the system $(\mathbf{f}_0, \mathbf{q}_0)$ hold. To this regard it is interesting to note that, in the context of output regulation, the observability conditions are not needed as the design of $\gamma(\,\cdot\,)$, in order to achieve the internal model property, is done in order to reconstruct the output $\mathbf{q}_0(\mathbf{z})$ of the observed system and not the full state $\mathbf{z}$. This motivates the absence of observability conditions for the system $(\mathbf{f}_0, \mathbf{q}_0)$ in Proposition 1 and, in turn, the absence of *immersion conditions* in the above framework.

3 Essential and Redundant Internal Models

3.1 Essential Regulators

The goal of this part is to enrich the results previously recalled by exploring conditions under which the dimension m of the regulator (6) (fixed, according to Proposition 1, to be $2(s+n)+2$) can be reduced in order to obtain an *essential regulator* preserving the internal model property.

As discussed in Section 2.3, the crucial feature required to the regulator (6) in order to posses the internal model property with respect to $\mathcal{A}$ is that system (16) is able, through its output y_η, to reproduce all the possible output motions of the system (15) with initial conditions taken in the set $\mathcal{A}$, the latter being a compact set satisfying the basic assumption in Section 2.1. From this, it seems natural to approach the problem of identifying an essential regulator by addressing two subsequent issues. First, to address if there exists a *minimal* set $\mathcal{A}_0$ satisfying the basic assumption in Section 2.1. This would lead to identify *steady state trajectories* for (5) which originate essential output behaviors of (15) to be captured by the internal model. Second, to identify conditions under which all the output behaviors of (15) originating from initial conditions in $\mathcal{A}_0$ can be reproduced by the output of a system of the form (16) of minimal dimension (i.e. lower than $2(s+n)+2$). This would lead to identify an *essential internal model* $(F, G, \gamma(\,\cdot\,))$ possessing the internal model property with respect to $\mathcal{A}_0$ and thus suitable to obtain an essential regulator of the form (6).

In the next proposition we address the first of the previous issues, by showing the existence of a minimal set satisfying the assumption in Section 2.1 which turns out to be *(forward and backward) invariant* for (5) as precisely formulated in the following. The set in question turns out to be the ω-limit set of the set $W \times Z$ of system (5), denoted by $\omega(W \times Z)$ (see [13]), introduced in [4] in the context of output regulation.

Proposition 4. *Let $\mathcal{A}$ be a set satisfying the assumption in Section 2.1. Then the set $\mathcal{A}_0 := \omega(W \times Z)$ is the* unique invariant *set such that $\mathcal{A}_0 \subseteq \mathcal{A}$ which is asymptotically stable for (5) with a domain of attraction $W \times \mathcal{D}$ with $Z \subset \mathcal{D}$. Furthermore the set in question is minimal, that is there does not exist a compact set $\mathcal{A}_1 \subset \mathcal{A}_0$ which is asymptotically stable for (5) with a domain of attraction of the form $W \times \mathcal{D}$ with $Z \subset \mathcal{D}$.*

Proof. With the notation introduced around (7) in mind and by defining $\mathbf{Z} = W \times Z$, note that, as the positive flow of (5) is bounded, the omega limit set[4] $\omega(\mathbf{Z})$ of the set $\mathbf{Z}$ exists, is bounded and uniformly attracts the trajectories of (5) originating from $\mathbf{Z}$, namely for any $\epsilon > 0$ there exists a $t_\epsilon > 0$ such that $\mathrm{dist}(\mathbf{z}(t, \mathbf{z}), \omega(\mathbf{Z})) \leq \epsilon$ for all $t \geq t_\epsilon$ and $\mathbf{z} \in \mathbf{Z}$ where $\mathbf{z}(t, \mathbf{z})$ denotes the trajectory of $\dot{\mathbf{z}} = \mathbf{f}_0(\mathbf{z})$ at time t passing through $\mathbf{z}$ at time $t = 0$ (see [13]). Furthermore it is possible to prove that $\omega(\mathbf{Z}) \subseteq \mathcal{A}$. As a matter of fact suppose that it is not true, namely that there exists a $\bar{\mathbf{z}} \in \omega(\mathbf{Z})$ and an $\epsilon > 0$ such that $|\bar{\mathbf{z}}|_{\mathcal{A}} \geq \epsilon$. By definition of $\omega(\mathbf{Z})$, there exist sequences $\{\mathbf{z}_n\}_0^\infty$ and $\{t_n\}_0^\infty$, with $\mathbf{z}_n \in \mathbf{Z}$ and $\lim_{n\to\infty} t_n = \infty$, such that

$$\lim_{n\to\infty} \mathbf{z}(t_n, \mathbf{z}_n) = \bar{\mathbf{z}}\,.$$

This, in particular, implies that for any $\nu > 0$ there exists a $n_\nu > 0$ such that $|\mathbf{z}(t_n, \mathbf{z}_n) - \bar{\mathbf{z}}| \leq \nu$ for all $n \geq n_\nu$. But, by taking $\nu = \min\{\epsilon/2, \nu_1\}$ with ν_1 such that $t_n \geq t_{\epsilon/2}$ for all $n \geq n_{\nu_1}$, this contradicts that $\mathcal{A}$ uniformly attracts the trajectories of (5) from $\mathbf{Z}$ (which, in turn, is implied by asymptotic stability of $\mathcal{A}$ and compactness of $\mathbf{Z}$). This proves that $\omega(\mathbf{Z}) \subset \mathcal{A}$. From this, using the fact that $\mathcal{A}$ is asymptotically stable for $\dot{\mathbf{z}} = \mathbf{f}_0(\mathbf{z})$ and the definition of

[4] We recall that the *ω-limit set* of the set $\mathbf{Z}$, written $\omega(\mathbf{Z})$, is the totality of all points $\mathbf{z} \in \mathbb{R}^{n+s}$ for which there exists a sequence of pairs $(\mathbf{z}_k, t_k)$, with $\mathbf{z}_k \in \mathbf{Z}$ and $t_k \to \infty$ as $k \to \infty$, such that $\lim_{k\to\infty} \mathbf{z}(t_k, \mathbf{z}_k) = \mathbf{z}$.

ω-limit set of the set $\mathbf{Z}$ (see [13]), it is possible to conclude that $\omega(\mathbf{Z})$ is also asymptotically stable and that the first statement of the proposition holds with $\mathcal{A}_0 = \omega(\mathbf{Z})$.

To prove the second statement of the proposition (namely that $\mathcal{A}_0$ is minimal) suppose that it is not true, that is there exists a closed set $\mathcal{A}_1 \subsetneq \mathcal{A}_0$ which is asymptotically stable with a domain of attraction containing $\mathbf{Z}$. Let $\bar{\mathbf{z}} \in \mathcal{A}_0$ and $\epsilon > 0$ such that $|\bar{\mathbf{z}}|_{\mathcal{A}_1} = 2\epsilon$. By assumption, $\mathcal{A}_1$ uniformly attracts trajectories of (5) originating from $\mathbf{Z}$ which implies that there exists a $t_\epsilon > 0$ such that $|\mathbf{z}(t,\mathbf{z})|_{\mathcal{A}_1} \leq \epsilon$ for any $\mathbf{z} \in \mathbf{Z}$ and for all $t \geq t_\epsilon$. Now set

$$\mathbf{z}^\star = \mathbf{z}(-(t_\epsilon + 1), \bar{\mathbf{z}})$$

and note that $\mathbf{z}^\star \in \mathcal{A}_0 \subseteq \mathbf{Z}$, as $\mathcal{A}_i$ is invariant, and $\bar{\mathbf{z}} = \mathbf{z}((t_\epsilon + 1), \mathbf{z}^\star)$ by uniqueness of trajectories. But the latter contradicts the fact that $\mathcal{A}_1$ uniformly attracts trajectories of (5) originating from $\mathbf{Z}$ and proves the claim. From this also uniqueness of the invariant set $\mathcal{A}_0$ immediately follows. □

Remark 1. By using the terminology introduced in [4], the set $\mathcal{A}_0 := \omega(W \times Z)$ is precisely the *steady state locus* of (5) with the trajectories of $\mathbf{f}_0|_{\mathcal{A}_0}$ being the *steady state trajectories* of (5). Furthermore, as shown in [4], the triangular structure of (5) leads to a specific structure of $\mathcal{A}_0$. In particular it has been shown in [4] that there exists a (possibly set-valued) upper semi-continuous map $\pi : \mathbb{R}^s \to \mathbb{R}^n$ such that the set $\mathcal{A}_0$ is described as

$$\mathcal{A}_0 = \{(w,z) \in W \times \mathbb{R}^n \;:\; z = \pi(w)\}\,. \tag{17}$$

◁

Remark 2. Note that, as $\mathcal{A}_0 \subseteq \mathcal{A}$ and $h(w,z,0) = 0$ for all $(w,z) \in \mathcal{A}$, it turns out that $h(w,z,0) = 0$ for all $(w,z) \in \mathcal{A}_0$. In particular, this and the claim of the previous proposition yield that the set $\mathcal{A}_0$ fulfills the assumption in Section 2.1. ◁

With this result at hand we pass now to consider the second issue pointed out before, namely the existence of an internal model $(F, G, \gamma(\,\cdot\,))$ of dimension lower than $2(s+n)+2$ having the internal model property with respect to $\mathcal{A}_0$. To this respect it is possible to prove that what determines the dimension of the essential internal model is not the dimension $(s+n)$ of (15) but rather the dimension of the lowest dimensional system able to reproduce, in an appropriate sense, the output behavior of (15). Details are as follows.

Assume the existence of an integer $r < n+s$, of a Riemannian differentiable manifold of dimension r of a compact subset $\mathcal{A}'_0$ of $\mathcal{M}$, of C^1 vector field $\mathbf{f}'_0 : \mathcal{M} \to T\mathcal{M}$ which leaves $\mathcal{A}'_0$ backward invariant and of a C^1 function $\mathbf{q}'_0 : \mathcal{M} \to \mathbb{R}$, such that for any $\mathbf{z}_0 \in \mathcal{A}_0$ there exists a $\mathbf{z}'_0 \in \mathcal{A}'_0$ satisfying

$$\mathbf{q}_0(\mathbf{z}(t,\mathbf{z}_0)) = \mathbf{q}'_0(\mathbf{z}'(t,\mathbf{z}'_0)) \qquad \forall t \leq 0.$$

If a triplet $(\mathbf{f}_0', \mathbf{q}_0', \mathcal{A}_0')$ satisfying the previous properties exists, it turns out that the internal model property with respect to $\mathcal{A}_0$ can be achieved by means of a regulator of dimension $m = 2r + 2$. More specifically, it can be proved that there exist an $\ell > 0$ and a set $\mathcal{S} \subset \mathbb{C}$ of zero Lebesgue measure, such that if $(F, G) \in \mathbb{R}^{m\times m} \times \mathbb{R}^{m\times 1}$, with $m = 2r + 2$, is a controllable pair with $\sigma(F) \subset \{\lambda \in \mathbb{C} \,:\, \mathrm{Re}\lambda < -\ell\} \setminus \mathcal{S}$ then there exist a continuous $\tau : \mathcal{A}_0 \to \mathbb{R}^m$ solution of

$$L_{\mathbf{f_0}}\tau(\mathbf{z}) = F\tau(\mathbf{z}) - G\mathbf{q}_0(\mathbf{z}) \qquad \forall\, \mathbf{z} \in \mathcal{A}_0 \tag{18}$$

and a continuous $\gamma : \mathbb{R}^m \to \mathbb{R}$ solution of

$$\mathbf{q}_0(\mathbf{z}) + \gamma \circ \tau(\mathbf{z}) = 0 \qquad \forall\, \mathbf{z} \in \mathcal{A}_0 \,. \tag{19}$$

The proof of this claim immediately follows by specializing Proposition 6 in Appendix A by taking $\mathcal{A}_1$, f_1, q_1, n_1 and $\mathcal{A}_2$, f_2, q_2, n_2 in the proposition respectively equal to $\mathcal{A}_0$, $\mathbf{f_0}$, $\mathbf{q}_0$, $n + s$ and $\mathcal{A}_0'$, $\mathbf{f_0}'$, $\mathbf{q}_0'$, r.

Remark 3. Note that the key feature required to the r-dimensional system $\dot{\mathbf{z}}' = \mathbf{f}_0'(\mathbf{z}')$ with output $y_{\mathbf{z}'} = -\mathbf{q}_0'(\mathbf{z}')$ with initial conditions taken in the set $\mathcal{A}_0'$ is to be able to reproduce all the output behaviors (backward in time) of the $(n + s)$-dimensional system (15) with output $y_{\mathbf{z}} = -\mathbf{q}_0(\mathbf{z}(t))$ originating from initial conditions in $\mathcal{A}_0$. ◁

Remark 4. Going throughout the proof of Proposition 6, it turns out that the continuous function γ solution of (18)–(19) coincides with the solution of the equation

$$\mathbf{q}_0'(\mathbf{z}') + \gamma \circ \tau'(\mathbf{z}') = 0 \qquad \forall\, \mathbf{z} \in \mathcal{A}_0' \tag{20}$$

with the function $\tau' : \mathcal{A}_0' \to \mathbb{R}^m$ satisfying

$$L_{\mathbf{f_0'}}\tau'(\mathbf{z}) = F\tau'(\mathbf{z}') - G\mathbf{q}_0'(\mathbf{z}') \qquad \forall\, \mathbf{z} \in \mathcal{A}_0' \,. \tag{21}$$

In other words the internal model (F, G, γ) can be tuned by considering, in the design formulas, the reduced-order triplet $(\mathbf{f_0'}, \mathbf{q}_0', \mathcal{A}_0')$. According to this, in the following, we will say that the triplet $(\mathbf{f_0'}, \mathbf{q}_0', \mathcal{A}_0')$ is *similar* (as far as the design of γ is concerned) to the triplet $(\mathbf{f_0}, \mathbf{q}_0, \mathcal{A}_0)$. ◁

It is interesting to note that a direct application of the previous considerations in conjunction with the results discussed at the end of Remark 1, immediately lead to a reduction of the regulator's dimension with respect to the one conjectured in Proposition 1 (equal to $2(s + n) + 2$). As a matter of fact, assume that the function π in (17) admits a C^2 *selection* $\pi_s(w)$. Set $r = s$ and let $\mathcal{A}_0' \subset \mathbb{R}^s$ be an arbitrary compact set containing W. Furthermore let $\mathbf{f}_0' : \mathbb{R}^s \to \mathbb{R}^s$ be any differentiable function which agrees with $s(\,\cdot\,)$ on W, and define $\mathbf{q}_0'(\,\cdot\,) := q(\,\cdot\,, \pi_s(\,\cdot\,), 0)$. By the structure of $\mathcal{A}_0$ in (17) (with π replaced by π_s) along with the fact that the set W is invariant for (2), it turns out that the triplets $(\mathbf{f}_0', \mathbf{q}_0', \mathcal{A}_0')$ and $(\mathbf{f}_0, \mathbf{q}_0, \mathcal{A}_0)$ are similar (see Remark 4) and thus that the internal model property with respect to $\mathcal{A}_0$ can be achieved by means of a regulator of dimension $m = 2s + 2$.

It must be stressed, though, that the previous considerations highlight only one of the underlying aspects behind the reduction result previously illustrated, namely the fact that only the dimension of the *restricted* dynamics $\mathbf{f}_0|_{\mathcal{A}_0}$ (equal to s in the case the function $\pi(\,\cdot\,)$ in (17) is single valued), and not the full dimension of the dynamics (15) (equal to $n + s$), plays a role in determining the dimension of the regulator. The second fundamental aspect behind the reduction procedure is that possible dynamics of $\mathbf{f}_0|_{\mathcal{A}_0}$ which have no influence on the output behavior of system $(\mathbf{f}_0, \mathbf{q}_0)$ do not affect the dimension of the regulator. This feature can be further explored by making use of standard tools to study local observability decompositions of nonlinear systems as detailed in the following.

In particular assume that $\mathcal{A}_0$ is a smooth manifold (with boundary) of $\mathbb{R}^{n+s}$, denote by ρ its dimension (with $\rho = s$ if the map π in (17) is single valued), and denote by $< \mathbf{f}_0, d\mathbf{q}_0 >$ the minimal co-distribution defined on $\mathcal{A}_0$ which is invariant under $\mathbf{f}_0$ and which contains $d\mathbf{q}_0$, with the latter being the differential of $\mathbf{q}_0$ (see [16]). Furthermore let Q be the distribution defined as the annihilator of $< \mathbf{f}_0, d\mathbf{q}_0 >$, namely

$$Q :=< \mathbf{f}_0, d\mathbf{q}_0 >^{\perp} \ .$$

It is well-known (see [14],[16]) that if, at a point $\bar{\mathbf{z}} \in \mathcal{A}_0$, Q is not singular, it is possible to identify a local change of variables transforming system $(\mathbf{f}_0, \mathbf{q}_0)$ into a special "observability" form. More precisely there exist an open neighborhood $U_{\bar{\mathbf{z}}}$ of $\mathcal{A}_0$ containing $\bar{\mathbf{z}}$ and a (local) diffeomorphism $\Phi : U_{\bar{\mathbf{z}}} \to \mathbb{R}^{\rho}$ which transforms system (15) into the form

$$\begin{aligned} \dot{\chi}_1 &= f_{01}(\chi_1) & \chi_1 &\in \mathbb{R}^{\rho-\nu} \\ \dot{\chi}_2 &= f_{02}(\chi_1, \chi_2) & \chi_2 &\in \mathbb{R}^{\nu} \\ y &= q_{01}(\chi_1)\,, \end{aligned} \tag{22}$$

namely into a form in which only the first $(\rho - \nu)$ state variables influence the output. This representation clearly shows that, locally around $U_{\bar{\mathbf{z}}}$, all the output motions of system (15) can be generated by the system $\dot{\xi} = f_{01}(\xi)$ with output $y_{\xi} = q_{01}(\xi)$ with dimension $\rho - \nu$. In particular, according to the previous arguments, this suggests that the internal model property, *locally* with respect to $U_{\bar{\mathbf{z}}}$, is potentially achievable by a regulator of dimension $2(\rho - \nu) + 2$. Of course, the local nature of the previous tools prevents one to push further the above reasonings and to be conclusive with respect to the dimension of the regulator possessing the internal model property with respect to the whole $\mathcal{A}_0$. However, it is possible to employ the fact that the co-distribution $Q^{\perp}$ is minimal (which implies that the decomposition (22) is maximal in a proper sense, see [16]), to be conclusive about a *lower bound* on the dimension of any regulator possessing the internal model property with respect to $\mathcal{A}_0$. This is formalized in the next lemma in which we identify a lower bound on the dimension r of any triplet $(\mathbf{f}_0', \mathbf{q}_0', \mathcal{A}_0')$ similar (in the sense of Remark 4) to $(\mathbf{f}_0, \mathbf{q}_0, \mathcal{A}_0)$. The lemma is given under the assumption that there exists a *submersion* $\sigma : \mathcal{A}_0 \to \mathcal{M}$ satisfying

$$\begin{aligned} L_{\mathbf{f}_0}\sigma(\mathbf{z}) &= \mathbf{f}_0'(\sigma(\mathbf{z})) \\ \mathbf{q}_0(\mathbf{z}) &= \mathbf{q}_0'(\sigma(\mathbf{z})) \end{aligned} \tag{23}$$

for all $\mathbf{z} \in \mathcal{A}_0$

Lemma 1. *Let $\mathcal{A}_0$ be a smooth manifold with boundary of dimension ρ and assume the existence of a* regular *point $\bar{\mathbf{z}} \in \mathcal{A}_0$ of the distribution $Q =< \mathbf{f}_0, d\mathbf{q}_0 >^\perp$. Let $\nu < \rho$ be the dimension of Q at $\bar{\mathbf{z}}$. Assume, in addition, the existence of a positive $r \leq \rho$, of a smooth manifold $\mathcal{M}$ of dimension r, of smooth functions $\mathbf{f}_0' : \mathcal{M} \to T\mathcal{M}$ and $\mathbf{q}_0' : \mathcal{M} \to \mathbb{R}$, and of a* submersion *$\sigma : \mathcal{A}_0 \to \mathcal{M}$, which satisfy (23). Then necessarily $r \geq \rho - \nu$.*

Proof. The proof proceeds by contradiction. Suppose that the claim of the lemma is false namely that there exist a positive $r < \rho - \nu$, a triplet $(\mathbf{f_0}', \mathbf{q_0}', \mathcal{M})$ with $\mathcal{M}$ a smooth manifold of $\mathbb{R}^r$ and a submersion $\sigma : \mathcal{A}_0 \to \mathcal{M}$ such that (23) holds for all $\mathbf{z} \in \mathcal{A}_0$. As $\mathrm{rank}(d\sigma(\mathbf{z})/d\mathbf{z}|_{\bar{\mathbf{z}}}) = r$ (since σ is a submersion) it follows that it is always possible to identify a submersion $\lambda : \mathcal{A}_0 \to \mathcal{M}$ such that, by defining

$$\Phi'(\mathbf{z}) = \begin{pmatrix} \Phi_1'(\mathbf{z}) \\ \Phi_2'(\mathbf{z}) \end{pmatrix} := \begin{pmatrix} \sigma(\mathbf{z}) \\ \lambda(\mathbf{z}) \end{pmatrix} ,$$

$\mathrm{rank}(d\Phi'(\mathbf{z})/d\mathbf{z}|_{\bar{\mathbf{z}}}) = \rho$, namely Φ' qualifies as a local diffeomorphism at $\bar{\mathbf{z}}$. This, in view of (23), guarantees the existence of an open neighborhood $U_{\bar{\mathbf{z}}}'$ of $\mathcal{A}_0$ including $\bar{\mathbf{z}}$ such that system $(\mathbf{f}_0, \mathbf{q}_0)$ in the new coordinates reads locally at $U_{\bar{\mathbf{z}}}'$ as

$$\begin{aligned} \dot{\tilde{\chi}}_1 &= f_1(\tilde{\chi}_1) & \tilde{\chi}_1 &\in \mathbb{R}^r \\ \dot{\tilde{\chi}}_2 &= f_2(\tilde{\chi}_1, \tilde{\chi}_2) & \tilde{\chi}_2 &\in \mathbb{R}^{\rho - r} \\ y &= q_1(\tilde{\chi}_1) \,, \end{aligned} \tag{24}$$

Now partition the change of variables Φ as $\Phi(\mathbf{z}) = \mathrm{col}(\Phi_1(\mathbf{z}), \Phi_2(\mathbf{z}))$ according to (22) and let $\mathbf{z}'$ be a point of $U_{\bar{\mathbf{z}}} \bigcap U_{\bar{\mathbf{z}}}'$ such that $\Phi_1'(\mathbf{z}') = \Phi_1'(\bar{\mathbf{z}})$ and $\Phi_1(\mathbf{z}') \neq \Phi_1(\bar{\mathbf{z}})$ (which is possible as $r < \rho - \nu$). By (24) it turns out that, as long as the trajectories $\mathbf{z}(t, \mathbf{z}')$ and $\bar{\mathbf{z}}(t, \mathbf{z})$ belongs to $U_{\bar{\mathbf{z}}} \bigcap U_{\bar{\mathbf{z}}}'$, the corresponding outputs coincides. This, by minimality of the co-distribution $Q^\perp$ implies that $\Phi_1(\mathbf{z}') = \Phi_1(\bar{\mathbf{z}})$ (see Theorem 1.9.7 in [16]) which is a contradiction. □

3.2 The Potential Advantage of Redundant Regulators

The fact of fulfilling the internal model property with respect to a generic set $\mathcal{A}$ (satisfying the main assumption in Section 2.1), not necessarily coincident with the essential steady state set $\mathcal{A}_0$, inevitably leads to design a regulator (6) which is redundant, namely whose dimension is larger than that is strictly necessary. In more meaningful terms, by bearing in mind the discussion in Section 2.3, the redundancy shows up in the fact that system (16), with $(F, G, \gamma(\cdot))$ having the internal model property with respect to $\mathcal{A} \supset \mathcal{A}_0$, posses the ability of reproducing the output behaviors of (15) generated by trajectories in $\mathcal{A} \backslash \mathcal{A}_0$ which are not, strictly speaking, steady state trajectories.

It's legitimate to wonder what, if there, is the advantage of designing a redundant regulator. The answer to this is given in the next proposition, of interest by its own, in which it is claimed that any "redundant" set is always exponentially stable for (5). The result of this proposition, proved in Appendix B, gains interest in conjunction with Proposition 2 as discussed after the statement.

Proposition 5. *Any compact set $\mathcal{A}$ which is asymptotically stable for (5) and such that $\mathcal{A}_0 \subset \mathrm{int}\mathcal{A}$ is also locally exponentially stable for (5).*

In terms of the framework presented in Section 2, the previous result gains interest in conjunction with Proposition 2 which, besides others, claims that a linear stabilizer $\kappa(\cdot)$ can be obtained if the set $\mathcal{A}$ is locally exponentially stable for[5] (5). In other words the results of Propositions 5 and 2 in relation to the results of Proposition 4 and Remark 2, reveal a trade-off between the simplicity of the stabilizer $\kappa(\cdot)$ and the dimension of the regulator (6). As a matter of fact in Proposition 4 it is claimed that the set $\mathcal{A}$ can be always "shrunk" to obtain a minimal invariant set $\mathcal{A}_0$ instrumental to obtain an essential (low-order) internal model as detailed in the previous section. The possible drawback in this, is that the set $\mathcal{A}_0$ is not guaranteed to be exponentially stable if the set $\mathcal{A}$ is such. This means that a reduced order regulator can be obtained by possibly complicating the function $\kappa(\cdot)$ in (6). On the other hand Proposition 5 asserts that exponential stability can be gained by enlarging a bit the set $\mathcal{A}_0$ but, so doing, necessarily loosing backward invariance as claimed in Proposition 4. This means that a linear function $\kappa(\cdot)$ can be possibly obtained by necessarily accepting a not essential regulator.

The previous considerations highlight a possible benefit in the design of the stabilizer $\kappa(\cdot)$ coming from the redundancy of the regulator, where the redundancy comes from the fact of considering, in the design of the triplet $(F, G, \gamma(\cdot))$, a redundant set $\mathcal{A} \supset \mathcal{A}_0$ instead of the steady state set $\mathcal{A}_0$. At this point one would be tempted to wonder if the redundancy of the regulator can be employed also to obtain a benefit in the design of the function $\gamma(\cdot)$ which, according to the previous arguments, is the true bottleneck in the design procedure of the regulator. A possible answer to this point will be given in the following in which it is assumed fixed a compact set $\mathcal{A} \supseteq \mathcal{A}_0$ which is locally exponentially stable for (5) and it is assumed that the triplet $(\mathbf{f}_0, \mathbf{q}_0, \mathcal{A})$ is similar, in the sense specified below, to a linear system.

[5] Indeed, the extra condition required in Proposition 2 is that the function $\gamma(\cdot)$ is locally Lipschitz. In this paper we do not address this issue and assume it is satisfied.

In the case $\mathcal{A}$ is backward invariant, in the following we simply let $\mathcal{A}_1 = \mathcal{A}$, $f_1 = \mathbf{f}_0$, and $q_1 = \mathbf{q}_0$. In the case $\mathcal{A}$ is not backward invariant, let $\mathcal{A}_1$ be an arbitrary compact set such that $\mathcal{A} \subset \mathtt{Int}(\mathcal{A}_1)$ and $f_1 : \mathbb{R}^{n+s} \to \mathbb{R}^{n+s}$ be an arbitrary differentiable function such that f_1 agrees with $\mathbf{f}_0$ on $\mathcal{A}$ and $f_1(z_1) = 0$ for all $z_1 \in \mathbb{R}^{n+s} \setminus \mathcal{A}_1$. Furthermore, set $q_1 = \mathbf{q}_0$ and note that, by construction, the set $\mathcal{A}_1$ is invariant for $\dot{z}_1 = f_1(z_1)$. Assume now the existence of an integer $r \geq n+s$, of a linear pair $(F_2, Q_2) \in \mathbb{R}^{r\times r} \times \mathbb{R}^{r\times 1}$ and of a compact set $\mathcal{A}_2 \in \mathbb{R}^r$ such that the triplet $(F_2, Q_2, \mathcal{A}_2)$ is similar (in the sense of Remark 4) to the triplet $(f_1, q_1, \mathcal{A}_1)$, that is for all $z_{10} \in \mathcal{A}_1$ there exists a $z_{20} \in \mathcal{A}_2$ such that[6]

$$q_1(z_1(t, z_{10})) = Q_2 e^{F_2 t} z_{20} \qquad \forall\, t \leq 0\,.$$

In this setting Proposition 6 in Appendix A, with n_1 and n_2 respectively set to $n+s$ and r, immediately yields that the internal model property with respect to $\mathcal{A}$ can be achieved by means of a *linear* internal model. More specifically, setting $m \geq 2r+2$, Proposition 4 yields that there exists an $\ell > 0$ and a set $\mathcal{S} \subset \mathbb{C}$ of zero Lebesgue measure such that if $(F, G) \in \mathbb{R}^{m\times m} \times \mathbb{R}^m$ is a controllable pair with $\sigma(F) \subset \{\lambda \in \mathbb{C} \,:\, \mathrm{Re}\lambda < -\ell\} \setminus \mathcal{S}$, then there exists a continuous function $\tau_1 : \mathcal{A}_1 \to \mathbb{R}^m$ solution of

$$L_{f_1}\tau_1(z_1) = F\tau_1(z_1) - Gq_1(z_1) \qquad \forall\, z_1 \in \mathcal{A}_1 \tag{25}$$

and a linear function $\Gamma : \mathbb{R}^m \to \mathbb{R}$ satisfying

$$\Gamma\tau_1(z_1) + q_1(z_1) = 0 \qquad \forall\, z_1 \in \mathcal{A}_1\,. \tag{26}$$

From this, using the fact that f_1 agrees with $\mathbf{f}_0$ on $\mathcal{A}$ and that $q_1 = \mathbf{q_0}$, it turns out that equations (9) and (8) are satisfied with $\tau = \tau_1|_{\mathcal{A}}$ and with $\gamma = \Gamma$. This implies that the regulator (4) having the internal model property with respect to $\mathcal{A}$ can be taken linear.

Remark 5. The previous conditions can be interpreted as an *immersion* of the system $(f_1, q_1, \mathcal{A}_1)$ into a linear system $(F_2, Q_2, \mathcal{A}_2)$ in the sense specified before. In particular the theory in [11] can be used to identify sufficient conditions under which such a immersion exists. It is worth also noting that, in the way in which it is formulated, the existence of such a immersion is affected by the choice of the set $\mathcal{A}_1$ and of the function f_1 which can be arbitrarily chosen as indicated above. This is not the case if the set $\mathcal{A}$ is backward invariant for $\dot{\mathbf{z}} = \mathbf{f}_0(\mathbf{z})$ as the previous considerations are done with $\mathcal{A}_1 = \mathcal{A}$, $f_1 = \mathbf{f}_0$, and $q_1 = \mathbf{q}_0$, and the immersion conditions can be formulated by referring to the "original" triplet $(\mathbf{f}_0, \mathbf{q}_0, \mathcal{A})$. ◁

Remark 6. It is interesting to note that the computation of the (linear) internal model (F, G, Γ) does not require the knowledge of the immersing linear system

[6] Note how the condition in question is satisfied if Assumption 2 of [4] holds (see Lemma 7.1 of [4]).

(F_2, Q_2) but only the knowledge of its dimension r. As a matter of fact, as clear by the previous analysis, the computation of Γ can be carried out in terms of the immersed triplet $(f_1, q_1, \mathcal{A}_1)$ by means of the design formulas (25)–(26) which are known to have a linear solution Γ if $m \geq 2r+2$. $\triangleleft$

4 Conclusions

In this paper we presented some complementary results of [23] in the context of output regulation for nonlinear systems. Specifically, we presented and discussed results on how to identify internal model-based regulators of minimal dimension preserving the so-called internal model property. The reduction tools consisted in the identification of "essential" steady state dynamics of the regulated plant and on the identification of an "essential" internal model-based regulators. Regarding the first aspect, it has been shown that the crucial tool is the concept of omega-limit set of a set pioneered in [4] in the context of output regulation. As far as the second aspect is concerned, we showed how the crucial step is the identification of observability parts of the steady state-input generator system. The usefulness of "redundant" regulators have been also investigated in terms of design features of the high-gain stabilizer which characterizes the proposed regulator.

A A Reduction Result

Proposition 6. *Let $f_1 : \mathbb{R}^{n_1} \to \mathbb{R}^{n_1}$ and $q_1 : \mathbb{R}^{n_1} \to \mathbb{R}$ be locally Lipschitz functions and let $\mathcal{A}_1$ be a compact backward invariant set for $\dot{z}_1 = f_1(z_1)$. Let $\mathcal{M}$ be a Riemannian differentiable manifold[7] of dimension n_2 and $\mathcal{A}_2$ and $\mathcal{A}_{2e}$ be compact subsets of $\mathcal{M}$ with $\mathcal{A}_2$ subset of the interior $\mathtt{Int}(\mathcal{A}_{2e})$ of $\mathcal{A}_{2e}$. Let $f_2 : \mathcal{M} \to T\mathcal{M}$ be a C^1 vector field which leaves $\mathcal{A}_{2e}$ backward invariant and $q_2 : \mathcal{M} \to \mathbb{R}$ a C^1 function. Assume that, for all $z_1 \in \mathcal{A}_1$, there exists $z_2 \in \mathcal{A}_2$ such that*

$$q_1(\zeta_1(t, z_1)) \equiv q_2(\zeta_2(t, z_2)) \qquad \text{for all } t \leq 0 \ .$$

Set $m = 2n_2 + 2$. There exist an $\ell > 0$ and a set $\mathcal{S} \subset \mathbb{C}$ of zero Lebesgue measure such that, if $(F, G) \in \mathbb{R}^{m\times m} \times \mathbb{R}^m$ is a controllable pair with $\sigma(F) \subset \{\lambda \in \mathbb{C} : \ Re\lambda < -\ell\} \backslash \mathcal{S}$, then there exists a continuous function $\tau_1 : \mathcal{A}_1 \to \mathbb{R}^m$ solution of

$$L_{f_1}\tau_1(z_1) = F\tau_1(z_1) - Gq_1(z_1) \qquad \forall\, z_1 \in \mathcal{A}_1 \tag{27}$$

and a continuous function $\gamma : \mathbb{R}^m \to \mathbb{R}$ satisfying

$$\gamma \circ \tau_1(z_1) + q_1(z_1) = 0 \qquad \forall\, z_1 \in \mathcal{A}_1 \ .$$

[7] We adopt here the definition [2, Definition III.1.2].

Moreover, if $\mathcal{M} = \mathbb{R}^{n_2}$ *and* f_2 *and* q_2 *are linear, the above result holds and* γ *can be chosen linear.*

Proof. Because $\mathcal{A}_1$ is compact and backward invariant and q_1 is continuous, we can show, by following the same steps as the ones of the proof of Proposition 1 of [23], that if F is Hurwitz then the function $\tau : \mathcal{A}_1 \mapsto \mathbb{R}^m$ defined as

$$\tau_1(z_1) = \int_{-\infty}^{0} e^{Fs}\, G\, q_1(\zeta_1(s, z_1))ds \tag{28}$$

is well-defined, continuous and solution of (27).

Also our assumptions imply that, for any $z_2 \in \mathcal{A}_{2e}$, the solution $t \in (-\infty, 0] \to \zeta_2(t, z_2) \in \mathcal{A}_{2e}$ is well-defined and $t \in (-\infty, 0] \to q_2(\zeta_2(t, z_2))$ is a bounded function. So, $\tau_2 : \mathcal{A}_{2e} \mapsto \mathbb{R}^m$ defined as

$$\tau_2(\zeta_2) \;=\; \int_{-\infty}^{0} e^{Fs}\, G\, q_2(\zeta_2(s, z_2))ds \tag{29}$$

is well-defined.

If $\mathcal{M} = \mathbb{R}^{n_2}$ and f_2 and q_2 are linear, there exists $\ell > 0$ such that if $\sigma(F) \subset \{\lambda \in \mathbb{C} \;:\; \mathrm{Re}\lambda < -\ell\}$, then this expression makes sense and gives a linear function.

In the case where $\mathcal{M}$ is a more general Riemannian differentiable manifold, we need some more involved steps to show that τ_2 is C^1 on $\mathtt{Int}(\mathcal{A}_{2e})$. To lighten their presentation we replace ζ_2 by ζ, z_2 by z, f_2 by f and q_2 by q. Since q is C^1, it defines a C^0 covector denoted dq satisfying (see [2, Example V.1.4] or [26, p. 150])

$$dq_z(v) \;=\; L_v q(z) \qquad \forall v \in T_z M \;,\; \forall z \in M \;.$$

Here dq_z denote the evaluation of dq at z and $dq_z(v)$ is the real number given by the evaluation of the linear form dq_z at the vector v. Then, let $\underline{\Psi}$ be the contravariant tensor field of order 2 (i.e. the bilinear map) given by the Riemannian metric. Since it is non-degenerate, it defines a covariant tensor field $\overline{\Psi}$ of order 2 (See [2, Exercice V.5.5]) or [10, §3.19] or [26, pp. 414-416]) such that we have the following Cauchy-Schwarz inequality

$$|dq_z(v)| \;\leq\; \underline{\Psi}_z(v, v)\, \overline{\Psi}_z(dq_z, dq_z) \qquad \forall v \in T_z M \;,\; \forall z \in M \;. \tag{30}$$

Also, $\mathcal{A}_{2e}$ being compact, there exists a real number Q such that we have

$$0 \;\leq\; \overline{\Psi}_z(dq_z, dq_z) \;\leq\; Q \qquad \forall z \in \mathcal{A}_{2e} \;. \tag{31}$$

Finally, we note that the one-parameter group action $z \mapsto \zeta(t, z)$ defines the induced one-parameter pushforward map $d\zeta : TM \to TM$, mapping for each t, vectors in $T_z M$ into vectors in $T_{\zeta(t,z)} M$ (see [2, Theorem IV.1.2] or [26, pp. 88-89 and Theorem 3.1]).

With this at hand, by following the arguments in the proof of Proposition 2 of [23], we can prove that τ is C^1 provided there exist real numbers a and ℓ such that we have

$$dq_{\zeta(t,z)}\left(d\zeta_{\zeta(t,z)}v\right) \;\leq\; a\,\exp(\ell|t|)\,\sqrt{\underline{\Psi}_z(v,v)} \qquad \forall v\in T_zM,\ \forall t\leq 0,\ \forall z\in\mathcal{A}_{2e}\,.$$

With (30) and (31), this holds if we have

$$\underline{\Psi}_{\zeta(t,z)}\left(d\zeta_{\zeta(t,z)}v, d\zeta_{\zeta(t,z)}v\right)\leq \exp(\ell|t|)\sqrt{\underline{\Psi}_z(v,v)}\ \ \forall v\in T_zM\,,\forall t\leq 0\,,\forall z\in\mathcal{A}_{2e}. \tag{32}$$

This leads us to evaluate the Lie derivative along f of the contravariant tensor field of order 2 given at the point $\zeta(t,z)$ by $\underline{\Psi}_{\zeta(t,z)}\left(d\zeta_{\zeta(t,z)}\cdot, d\zeta_{\zeta(t,z)}\cdot\right)$ (See [2, Exercise V.2.8] or [10, §3.23.4] or [26, Problem 5.14]). This Lie derivative is a contravariant tensor field of order 2 and therefore, $\underline{\Psi}$ being non-degenerate and $\mathcal{A}_{2e}$ being compact, there exists a positive real number ℓ such that we have

$$-2\ell\,\underline{\Psi}_z\left(d\zeta_z\cdot, d\zeta_z\cdot\right) \;\leq\; L_f\underline{\Psi}_z\left(d\zeta_z\cdot, d\zeta_z\cdot\right) \qquad \forall z\in\mathcal{A}_{2e}\ .$$

From this (32) follows readily and hence τ_2 is C^1 on $\mathtt{Int}(\mathcal{A}_{2e})$ if $\sigma(F)\subset\{\lambda\in\mathbb{C}\,:\,\mathrm{Re}\lambda<-\ell\}$.

Now, coming back to our initial notations, as $\mathcal{M}$ is a differentiable manifold and $\mathcal{A}_2$ is compact, it is possible to cover $\mathcal{A}_2$ with a finite set $\mathcal{I}$ of open sets $\mathcal{O}_i$ each diffeomorphic to $\mathbb{R}^{n_2}$ (see [2, Theorem I.3.6]).

By using off-the-shelf the arguments in the proof of Proposition 2 of [23], it is possible to claim for each of the open set $\mathcal{O}_i$ the existence of a set $\mathcal{S}_i\subset\mathbb{C}$ of zero Lebesgue measure such that, if $\sigma(F)\subset\{\lambda\in\mathbb{C}\,:\,\mathrm{Re}\lambda<-\ell\}\setminus\mathcal{S}_i$, then we have

$$\tau_2(z_{2a})=\tau_2(z_{2b}) \quad\Rightarrow\quad q_2(z_{2a})=q_2(z_{2b}) \qquad \forall\, z_{2a},z_{2b}\in\mathcal{O}_i\,. \tag{33}$$

Since $\mathcal{I}$ is finite, the set $\mathcal{S}=\bigcup_{i\in\mathcal{I}}\mathcal{S}_i$ is of measure zero and, if $\sigma(F)\subset\{\lambda\in\mathbb{C}\,:\,\mathrm{Re}\lambda<-\ell\}\setminus\mathcal{S}$, then we have

$$\tau_2(z_{2a})=\tau_2(z_{2b}) \quad\Rightarrow\quad q_2(z_{2a})=q_2(z_{2b}) \qquad \forall\, z_{2a},z_{2b}\in\mathcal{A}_2\,. \tag{34}$$

With the above and by following the same arguments as the ones used at the end of Proposition 2 of [23] which apply since $\mathcal{A}_2$ is compact, there exists a continuous function $\gamma:\tau_2(\mathcal{A}_2)\subset\mathbb{R}^m\to\mathbb{R}$ satisfying

$$\gamma\circ\tau_2(z_2)+q_2(z_2)=0 \qquad \forall\, z_2\in\mathcal{A}_2\,.$$

As in the proof of Proposition 3 of [23], this function can be extended to all $\mathbb{R}^m$. Clearly if τ_2 and q_2 are linear, then γ is linear.

Finally, pick any $z_1\in\mathcal{A}_1$. By condition (ii) there exist $z_2\in\mathcal{A}_2$ such that $q_1(\zeta_1(t,z_1))=q_2(\zeta_2(t,z_2))$ for all $t\leq 0$ and therefore $\tau_1(z_1)=\tau_2(z_2)$ and $q_1(z_1)=q_2(z_2)$. This implies

$$\gamma\circ\tau_1(z_1)=\gamma\circ\tau_2(z_2)=-q_2(z_2)=-q_1(z_1)$$

which concludes the proof. □

B Proof of Proposition 5

The proof strongly relies on notations and results used in the proof of Theorem 4 in [23] which, for compactness, are not repeated here. We prove the proposition by showing that there exists a compact set $\mathcal{A}_e$ satisfying $\mathcal{A}_0 \subseteq \mathcal{A}_e \subseteq \mathcal{A}$ which is locally exponentially stable for (5). First of all note that, by assumption, there exists a locally Lipschitz Lyapunov function V satisfying the properties of Theorem 4 (with the set $\mathcal{B}$ replaced by $\mathcal{A}_0$) in [23] and in particular

$$\underline{a}(|\mathbf{z}|_{\mathcal{A}_0}) \;\leq\; V(\mathbf{z}).$$

Now let

$$r = \min_{\mathbf{z}\in\mathbb{R}^{n+s}\setminus\mathcal{A}} |\mathbf{z}|_{\mathcal{A}_0} \;>\; 0 \qquad \text{and} \qquad c_1 = \frac{1}{2}\underline{a}(r) \;>\; 0\,,$$

fix

$$\mathcal{A}_e = V^{-1}([0, c_1])\,,$$

and note that $\mathcal{A}_e$ is forward invariant. Moreover $\mathcal{A}_0 \subset \mathcal{A}_e$. We prove now that $\mathcal{A}_e$ is locally exponentially stable by proving the following two facts.

Fact #1 there exists a time T such that $|\mathbf{z}(t,\mathbf{z}_0)|_{\mathcal{A}_e} = 0$ for all $\mathbf{z}_0 \in \mathbf{Z} := W \times Z$ and for all $t \geq T$ (*finite time convergence*).

Fact #2 there exists a constant $L > 0$ such that $|\mathbf{z}(t,\mathbf{z}_0)|_{\mathcal{A}_e} \leq L|\mathbf{z}_0|_{\mathcal{A}_e}$ for all $\mathbf{z}_0 \in \mathbf{Z}$ and for all $t \geq 0$.

To prove fact #1 note that, by property (a) in Theorem 4 of [23] there exists an $a \geq c_1$ such that $\mathbf{Z} \subset V^{-1}([0,a])$ and, by property (c) in the same theorem, there exists $c > 0$ such that $D^+V(\mathbf{z}(t,\mathbf{z}_0)) \leq -cV(\mathbf{z}(t,\mathbf{z}_0))$ for all $\mathbf{z}_0 \in \mathbf{Z}$. By this, using the appropriate comparison lemma, it turns out that

$$V(\mathbf{z}(t,\mathbf{z}_0)) \leq e^{-ct}V(\mathbf{z}_0) \leq e^{-ct}a \qquad \text{for all } t \geq 0 \;\; \mathbf{z}_0 \in \mathbf{Z}$$

by which standard arguments can be used to prove that $\mathcal{A}_e$ is a forward invariant set and that fact #1 holds with $T = 1/c \ln a/c_1$.

To prove fact #2, since $V^{-1}([0,a])$ is a compact set, we can let

$$F = \max_{\mathbf{z}\in V^{-1}([0,a])} |\partial \mathbf{f_0}(\mathbf{z})/\partial \mathbf{z}|\,.$$

Note that, since $V(\mathbf{z}(t,\mathbf{z}_0))$ is non increasing in t, for all $\mathbf{z}_1, \mathbf{z}_2 \in \mathbf{Z}$ and for all $t \geq 0$

$$|\mathbf{z}(t,\mathbf{z}_1) - \mathbf{z}(t,\mathbf{z}_2)| \leq e^{Ft}|\mathbf{z}_1 - \mathbf{z}_2|\,.$$

Now fix $\mathbf{z}_1 \in \mathbf{Z}$ and let $\mathbf{z}_2 \in \mathcal{A}_e$ be such that $|\mathbf{z}_1 - \mathbf{z}_2| = |\mathbf{z}_1|_{\mathcal{A}_e}$. As $\mathcal{A}_e$ is forward invariant, it turns out that $\mathbf{z}(t,\mathbf{z}_2) \in \mathcal{A}_e$ for all $t \geq 0$. Moreover, by fact #1, $\mathbf{z}(t,\mathbf{z}_1) \in \mathcal{A}_e$ for all $t \geq T$. From this $|\mathbf{z}(t,\mathbf{z}_1)|_{\mathcal{A}_e} = 0$ for all $t \geq T$ and

$$|\mathbf{z}(t,\mathbf{z}_1)|_{\mathcal{A}_e} \leq |\mathbf{z}(t,\mathbf{z}_1) - \mathbf{z}(t,\mathbf{z}_2)| \leq e^{FT}|\mathbf{z}_1 - \mathbf{z}_2| \leq e^{FT}|\mathbf{z}_1|_{\mathcal{A}_e}\,.$$

This concludes the proof of fact #2 (taking $L = e^{FT}$) and of the proposition.

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