An introduction to forwarding

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1 Introduction

The forwarding technique is a Lyapunov design of global asymptotic stabilizers to be used recursively for systems whose dynamics can be written as (see section 4.1 for a generalization) :

$$\left\{\begin{array}{rcl} \dot{x}_n &=& f_n(x_1,\ldots,x_{n-1}) \;+\; g_n(x_1,\ldots,x_n,u)\,u \\ &\vdots& \\ \dot{x}_2 &=& f_2(x_1) \;+\; g_1(x_1,x_2,u)\,u \\ \dot{x}_1 &=& f_1(x_1) \;+\; g_1(x_1,u)\,u \end{array}\right. \tag{1}$$

This form is called *feedforward form.* It is obtained by adding recursively an integrator x_{i+1} fed forward with functions of all the previously introduced state components $(x_1,\ldots,x_i)$.

The forwarding technique has been used in the design of controllers tested on real world applications. This is the case for instance for controlling down-range distance in guided atmospheric entry [11] or for swinging up 1D or 2D inverted pendulum [14].

For systems admitting a feedforward form, another technique, not presented here, has been developed in [15]. It is conceptualized around a small non linear gain Theorem. Also, we restrict our attention to global asymptotic stabilization. Other results are available, in particular for semi-global asymptotic stabilization (see [9]).

Glossary and Notations

- Class $\mathcal{K}^\infty$ function : a function $\alpha : [0,\infty) \to [0,\infty)$ is said of class $\mathcal{K}^\infty$ if it is continuous, strictly increasing, unbounded and $\alpha(0) = 0$.
- f' : for a C^1 function $f : \mathbb{R} \to \mathbb{R}$, we denote by f' its derivative.
- Lyapunov function : a function V with values in $[0,\infty)$ is said a C^r Lyapunov function if it is r times continuously differentiable, positive definite and proper.
- $\dot{\overbrace{V(x)}}$: Given a system $\dot{x} = f(x,u)$ and a C^1 function V, we denote :

$$\dot{\overbrace{V(x)}} \;=\; \frac{\partial V}{\partial x}(x)\,f(x,u)\ .$$

Note that $\overbrace{V(x)}^{\cdot}$ is actually a function of (x, u) and but not of the time t.

- $X(x; u, t)$: Given a system $\dot{x} = f(x, u)$ and a function $u(t)$, we denote by $X(x; u, t)$ a solution issued from x at $t = 0$.

2 C^1 Dissipative systems

Definition 2 *The system :*

$$\dot{x} \;=\; f(x) \;+\; g(x,u)\,u \tag{3}$$

is said C^1 dissipative if there exists a C^1 Lyapunov function V, called the storage function, *such that :*

$$\frac{\partial V}{\partial x}(x)\,f(x) \;\leq\; 0 \qquad \forall x \in \mathbb{R}^n\ . \tag{4}$$

Remark 5 *From this definition, for a C^1 dissipative system with its control at the origin, the origin is a globally stable solution. Unfortunately, the converse is not true (see [1, Example V.4.11]). It follows that, even if we know that the origin is globally stable, when u is at the origin, we still need to exhibit a C^1 Lyapunov function to establish C^1 dissipativity* •

Example 6 Consider the celebrated cart-pendulum system. Let :
– (M, x) be mass and position of the cart which is moving horizontally,
– (m, l, θ) be mass, length and angular deviation from the upward position for the pendulum which is pivoting around a point fixed on the cart,
– finally F be a horizontal force acting on the cart and considered here as control.
The associated kinetic and potential energies are :

$$E_k(\dot{x}, \dot{\theta}) \;=\; \tfrac{1}{2}\,(M+m)\dot{x}^2 \;+\; \tfrac{1}{2}\,m\,l^2\,\dot{\theta}^2 \;+\; m\,l\,\cos(\theta)\,\dot{x}\,\dot{\theta}\ . \tag{7}$$

$$E_p(\theta) \;=\; m\,l\,g\,(\cos(\theta)+1)\ . \tag{8}$$

It follows from the Euler-Lagrange equation that the dynamics are :

$$\left\{\begin{array}{rcl} (M+m)\ddot{x} + ml\cos(\theta)\ddot{\theta} &=& ml\dot{\theta}^2\sin(\theta) + F\ , \\ \ddot{x}\cos(\theta) + l\ddot{\theta} &=& g\sin(\theta)\ . \end{array}\right. \tag{9}$$

We restrict our attention to the three coordinates $(\theta, \dot{x}, \dot{\theta})$ leaving in the manifold $\mathbb{S}^1 \times \mathbb{R}^2$.

Consider the function :

$$\mathcal{V}(\theta, \dot{x}, \dot{\theta}) \;=\; E_k(\dot{x}, \dot{\theta}) \;+\; E_p(\theta) \tag{10}$$

i.e. the total energy. It is a C^1 Lyapunov function for the point $(\pi, 0, 0)$. Then, since we get :

$$\overbrace{\mathcal{V}(\theta, \dot{x}, \dot{\theta})}^{\cdot} \;=\; F\,\dot{x}\ , \tag{11}$$

the cart-pendulum system, restricted to the coordinates $(\theta, \dot{x}, \dot{\theta})$, is C^1 dissipative • ◇

Definition 12 *The system (3) with output function $h(x, u)$ is said zero-state observable if the origin is the only solution satisfying :*

$$h(X(x;0,t),0) \;=\; 0 \qquad \forall t \geq 0\ . \tag{13}$$

From its definition, a C^1 dissipative system is a passive system (see [16]) for the particular output function :

$$h(x,u) \;=\; \frac{\partial V}{\partial x}(x)\,g(x,u)\ . \tag{14}$$

If the system is also zero-state observable, it follows from passivity theory (see [16, Lemma 3.2.3] for instance) that global asymptotic stability is provided by the control obtained as solution of :

$$u \;=\; -h(x,u) \;=\; -\frac{\partial V}{\partial x}(x)\,g(x,u)\ , \tag{15}$$

when it makes sense. This result can be generalized as follows :

Theorem 16 ([3, Corollary 1.6]) *Assume the system (3) is C^1 dissipative and zero-state observable with output function $\left(W(x) \quad \frac{\partial V}{\partial x}(x) g(x,0) \right)$. Then, for any real number $\overline{u}$ in $(0,+\infty]$, there exists a continuous global asymptotic stabilizer strictly bounded in norm by $\overline{u}$.*

Remark 17

1. *The controller mentioned in this Theorem is any continuous function ϕ satisfying :*

$$|\phi(x)| \ < \ \overline{u} \qquad \forall x \in \mathbb{R}^n \ , \tag{18}$$

$$\left|\frac{\partial V}{\partial x}(x) g(x,0)\right| \ \neq \ 0 \quad \Rightarrow \quad \frac{\partial V}{\partial x}(x) g(x,\phi(x))\, \phi(x) \ < \ 0 \ . \tag{19}$$

For instance, when g does not depend on u, we can take :

$$\phi(x) \ = \ -\min \left\{ \frac{\overline{u}}{\left|\frac{\partial V}{\partial x}(x) g(x)\right|} \, , \, 1 \right\} \left(\frac{\partial V}{\partial x}(x) g(x) \right)^T \ . \tag{20}$$

When g depends on u, it is more difficult to give an expression, but instead we can propose the dynamic controller (see [10]) :

$$\dot{\varkappa} \ = \ - \left[1 - \frac{|\varkappa|^2}{\overline{u}^2} \right] \left[\frac{\partial V}{\partial x}(x)\, g(x,\varkappa) \right]^\top \ - \ \varkappa \quad , \qquad u \ = \ \varkappa \tag{21}$$

which can be seen to be appropriate by using the function :

$$V_\varkappa(x,\varkappa) \ = \ V(x) \ - \ \frac{\overline{u}^2}{2} \log \left(1 - \frac{|\varkappa|^2}{\overline{u}^2} \right) \ . \tag{22}$$

2. *Many geometric sufficient conditions for the zero-state observability condition of Theorem 16 have been proposed (see [8] and the references therein)* •

Example 23 *Dealing with input constraints.* Consider the system :

$$\dot{x}_1 \ = \ x_2 \ , \quad \dot{x}_2 \ = \ -x_1 \ - \ x_2 \ + \ u \ , \quad \dot{x}_3 \ = \ (1 + x_2)\, u \ . \tag{24}$$

We look for a continuous global asymptotic stabilizer ϕ satisfying :

$$|\phi(x_1,x_2,x_3)| \ \leq \ 1 \ . \tag{25}$$

The system (24) is C^1 dissipative. Indeed the C^1 Lyapunov function :

$$V(x_1,x_2,x_3) \ = \ \left(x_2^2 + x_2 x_1 + x_1^2\right) \ + \ \tfrac{1}{2}\, x_3^2 \ . \tag{26}$$

gives :

$$\dot{\overbrace{V(x_1,x_2,x_3)}} = (2[1+x_2]x_3 + 2[2x_2 + x_1])\, u - \left(x_2^2 + x_2 x_1 + x_1^2\right) \ . \tag{27}$$

This derivative is made non positive by taking :

$$u \ = \ \phi(x_1,x_2,x_3) \ = \ -\mathtt{sat}\left(2[1+x_2]x_3 + 2[2x_2 + x_1]\right) \tag{28}$$

where `sat` is the standard saturation function :

$$\mathtt{sat}(s) = \max\{-1\, , \, \min\{1, s\}\} \ . \tag{29}$$

This control law satisfies the constraint (25) and, by making $\dot{\overbrace{V(x_1,x_2,x_3)}}$ negative definite, guarantees the global asymptotic stability of the origin • ◇

Example 30 *Orbit transfer with weak but continuous thrust.* The Gauss equations describe the dynamics of a point mass satellite subject to a thrust. In appropriate coordinates, these equations are for the case of a two dimensional thrust (see [2]) :

$$\left\{\begin{array}{rcl} \dot{p} & = & 2\,p\,S \\ \dot{\varepsilon} & = & -j\,\varpi(p,\varepsilon)\,\varepsilon + [\varepsilon + (2+\mathrm{Re}(\varepsilon))]\,S \\ \dot{\eta} & = & -j\,[\varpi(p,\varepsilon) - \mathrm{Im}(\eta)W]\,\eta + \frac{1}{2}(1+|\eta|^2)\,W \end{array}\right. \tag{31}$$

where :

– the state variables (p,ε,η) in $\mathbb{R}\times\mathbb{C}^2$, called the orbital parameters, are, with $j^2=-1$,

$$\left\{\begin{array}{rcl} p & = & a\,(1-e^2)\ , \quad L \ = \ \omega \ + \ \Omega \ + \ v\ , \\ \varepsilon & = & e\,[\cos(\omega+\Omega) + j\sin(\omega+\Omega)]\,[\cos(L) - j\sin(L)]\ , \\ \eta & = & \tan(i/2)\,[\cos(\Omega)+j\sin(\Omega)]\,[\cos(L)-j\sin(L)]\ , \end{array}\right. \tag{32}$$

where a is the semi-major axis, e is the eccentricity, i is the inclination to the equator, Ω is the right ascension of the ascending node, ω is the angle between the ascending node and the perigee, v is the true anomaly. Note that, from their definitions, p and ε satisfy :

$$|\varepsilon| < 1 \quad , \qquad p > 0\ . \tag{33}$$

– S is the component of the thrust, colinear with the kinetic momentum, and W is the component orthogonal to S and to the earth-satellite axis.

– Re and Im denote the real and imaginary part and $\varpi(p,\varepsilon)$ is a C^1 function.

We look for a continuous control law for (S,W) satisfying :

$$S^2 \ + \ W^2 \ \leq \ \gamma_{\max}^2\ , \tag{34}$$

for a given strictly positive real number $\gamma_{\max}$, and such that the orbit whose parameters are $(\bar{p},0,0)$ is made an asymptotically stable attractor.

To get a solution, we note that, by definition, without any thrust, the orbit and therefore each orbital parameter is unchanged. So an appropriate storage function is given by the sum of functions of one parameter only. With in mind the constraints (33), we let :

$$V(p,\varepsilon,\eta) \ = \ \tfrac{1}{2}\left[\log\left(\frac{p}{\bar{p}}\right)\right]^2 \ - \ \tfrac{1}{2}\log\left(1-|\varepsilon|^2\right) \ + \ |\eta|^2\ . \tag{35}$$

This yields :

$$\dot{\overbrace{V(p,\varepsilon,\eta)}} \ = \ \mathrm{Re}(\eta)\,(1+|\eta|^2)\,W \ + \ \left(2\,\log\left(\frac{p}{\bar{p}}\right) + \frac{|\varepsilon|^2 + \mathrm{Re}(\varepsilon)(2+\mathrm{Re}(\varepsilon))}{1-|\varepsilon|^2}\right)\,S \tag{36}$$

which establishes the C^1 dissipativity property. Also $\dot{\overbrace{V(p,\varepsilon,\eta)}}$ is made non negative by picking the vector $(S\ W)$ as a Lipschitz continuous function of the state, colinear with the vector :

$$-\left(\ 2\,\log\left(\tfrac{p}{\bar{p}}\right) + \tfrac{|\varepsilon|^2+\mathrm{Re}(\varepsilon)(2+\mathrm{Re}(\varepsilon))}{1-|\varepsilon|^2} \qquad \mathrm{Re}(\eta)\,(1+|\eta|^2)\ \right)$$

and with any non zero norm, satisfying (34), as long as this latter vector is non zero. With such a control and the Lipschitz continuity property, asymptotic stability of the desired orbital parameters can be checked by applying, to the full order system (31), the invariance principle as stated for instance in [12, Theorem 2.21]. Actually, by studying the linearization of the closed-loop system, we can check that local exponential stability holds also • ◇

3 C^1 dissipative systems via reduction or extension

Consider a system whose dynamics can be written in the triangular form :

$$\dot{y} \ = \ h(x) \ + \ h_u(x,y,u)\,u \quad , \qquad \dot{x} \ = \ f(x) \ + \ f_u(x,u)\,u \tag{37}$$

with x in $\mathbb{R}^n$ and y in $\mathbb{R}^q$ and where f and h are C^1 and zero at the origin. We want to study when the C^1 dissipativity of the full order system (37) implies the C^1 dissipativity of the reduced order x subsystem, and conversely.

3.0.1 Reduction

Assume that the system (37) is C^1 dissipative, i.e. we have a C^1 Lyapunov function V satisfying, when u is at the origin,

$$\dot{\overbrace{V(x,y)}} \;\leq\; 0 \; . \tag{38}$$

Since V is a C^1 Lyapunov function, for each given x, it has a global minimum reached at say $y = \mathcal{M}(x)$. Note that we have :

$$\mathcal{M}(0) \;=\; 0 \; . \tag{39}$$

Lemma 40 *If V is C^2 and $\mathcal{M}(x)$ is locally Hölder continuous of order strictly larger than $\frac{1}{2}$, then the x subsystem of (37) is C^1 dissipative with storage function :*

$$V_x(x) \;=\; V(x, \mathcal{M}(x)) \; . \tag{41}$$

Moreover, if the function $\mathcal{M}$ is C^1 and, for each x in $\mathbb{R}^n$, each unit vector v in $\mathbb{R}^q$, and each positive real number k, there exists y in $\mathbb{R}^q$ such that we have :

$$k\left|\frac{\partial V}{\partial x}(x,y) + \frac{\partial V}{\partial y}(x,y)\frac{\partial \mathcal{M}}{\partial x}(x)\right| \;<\; \frac{\partial V}{\partial y}(x,y)\, v \; , \tag{42}$$

then we have :

$$\dot{\overbrace{\mathcal{M}(x)}} \;=\; \frac{\partial \mathcal{M}}{\partial x}(x)\, f(x) \;=\; h(x) \tag{43}$$

and $V_x(x) + \frac{1}{2}|y - \mathcal{M}(x)|^2$ is another storage function for the full order system (37).

Remark 44 *We observe that (43) implies that for each solution $X(x;0,t)$ of the reduced order x subsystem with u at the origin, we have :*

$$\mathcal{M}(X(x;0,T)) \;-\; \mathcal{M}(x) \;=\; \int_0^T h(X(x;0,t))dt \qquad \forall T \geq 0 \; . \tag{45}$$

So if the origin is a globally asymptotically stable solution of the reduced order x subsystem with u at the origin, we get, with (39),

$$\mathcal{M}(x) \;=\; -\lim_{T\to+\infty} \int_0^T h(X(x;0,t))dt \;\; \bullet \tag{46}$$

Lemma 40 states that, for a system in the triangular form (37), modulo an extra smoothness properties on $\mathcal{M}$, the C^1 dissipativity of the full order system implies the dissipativity of the reduced order x subsystem.

3.0.2 Extension

Let us study now how to establish C^1 dissipativity by extension. The idea to tackle with this problem is to find conditions under which Lemma 40 applies. So we impose that, when u is at the origin, the origin is a globally asymptotically stable solution of the reduced order x subsystem of the system (37). From [13, Proposition 7], there exist two class $\mathcal{K}^\infty$ functions α_1 and α_2 such that, for any solution $X(x;0,t)$ of the x subsystem with u at the origin, we have :

$$\alpha_1(|X(x;0,t)|) \;\leq\; \alpha_2(|x|)\, \exp(-t) \qquad \forall t \geq 0 \; . \tag{47}$$

Now in order to guarantee the existence of the limit in (46), we assume also that the function h is sufficiently "flat" around the origin so that we have[1] :

$$\limsup_{|x|\to 0} \frac{|h(x)|}{\alpha_1(|x|)} \;<\; +\infty \; . \tag{48}$$

[1] In the case where the origin is locally exponentially stable, we can always get $\alpha_1(s) = s$ for s small, so that (48) holds always.

Indeed, with (47), this inequality implies the existence of a class $\mathcal{K}^\infty$ function α_3 such that we have :

$$\int_0^\infty |h(X(x;0,t))|dt \ \leq \ \alpha_3(|x|) \qquad \forall x \in \mathbb{R}^n \ . \tag{49}$$

Example 50 Consider the solutions :

$$X(x;t) \ = \ \frac{x}{\sqrt{1+2tx^2}} \tag{51}$$

of the system $\dot{x} = -x^3$. When the function h is $h(x) = x^2$, we get :

$$\int_0^T h(X(x;t))dt \ = \ \tfrac{1}{2} \log(1+2Tx^2) \ . \tag{52}$$

As T goes to infinity, this integral goes to $+\infty$ for all non zero x. On the other hand, when we have the "flatter" function $h(x) = x^4$, we get :

$$\int_0^T h(X(x;t))dt \ = \ \frac{x^2}{2}\left(1 - \frac{1}{1+2Tx^2}\right) \ . \tag{53}$$

This integral converges to $\frac{x^2}{2}$. We conclude that, for the system $\dot{x} = -x^3$, the function $h(x) = x^2$ is not "flat" enough whereas the function $h(x) = x^4$ is "flat" enough. It is interesting to relate this fact with the following ones :

- The origin of the system :

$$\dot{y} \ = \ x^2 \ + \ u \quad , \qquad \dot{x} \ = \ -x^3 \ + \ u \tag{54}$$

  is not asymptotically stabilizable by a continuous controller.

- The origin of the system :

$$\dot{y} \ = \ x^4 \ + \ u \quad , \qquad \dot{x} \ = \ -x^3 \ + \ u \tag{55}$$

  is asymptotically stabilizable by the continuous controller :

$$u \ = \ -x \ - \ (1+x)\left(y + \tfrac{1}{2}x^2\right) \tag{56}$$

  associated with the Lyapunov function :

$$V(x,y) \ = \ \tfrac{1}{2}\,x^2 \ + \ \tfrac{1}{2}\ \left(y+\tfrac{1}{2}x^2\right)^2 \ \bullet \tag{57}$$

$\diamond$

When (49) holds, from [4, Théorème (3.149)] for instance, we know that the following function $\mathcal{M}$ is well defined and continuous on $\mathbb{R}^n$:

$$\mathcal{M}(x) \ = \ -\int_0^\infty h(X(x;0,t))dt \ . \tag{58}$$

And, as it can be checked "by hand", we have :

$$\dot{\overbrace{\mathcal{M}(x)}} \ = \ h(x) \qquad \forall x \in \mathbb{R}^n \ . \tag{59}$$

With this function, the system (37) gives, when u is at the origin,

$$\dot{\overbrace{y - \mathcal{M}(x)}} \ = \ 0 \quad , \qquad \dot{x} \ = \ f(x) \ . \tag{60}$$

It follows that the origin is a globally stable solution of (37) when u is at the origin. If $\mathcal{M}$ is not only continuous but also C^1, then the system (37) is C^1 dissipative. Indeed, from [7], there exists a C^1 Lyapunov function V_x such that the function :

$$\frac{\partial V_x}{\partial x}(x)\, f(x) \;<\; 0 \qquad \forall x \in \mathbb{R}^n \setminus \{0\} \tag{61}$$

is continuous and positive definite. Then by letting :

$$V(x,y) \;=\; V_x(x) \;+\; \tfrac{1}{2}\, |y - \mathcal{M}(x)|^2 \;, \tag{62}$$

we get, for the system (37), when u is at the origin,

$$\dot{\overbrace{V(x,y)}} \;<\; 0 \qquad \forall (x,y) \in \mathbb{R}^n \setminus \{0\} \times \mathbb{R}^q \;. \tag{63}$$

This shows that V is a storage function.

To summarize the possibility of going from global asymptotic stability to C^1 dissipativity while extending the reduced order x subsystem into the full order system (37) relies on the following two properties :

1. The function h is "flat" enough so that (48) holds. This guarantees the existence and the continuity of a function $\mathcal{M}$ satisfying (59).
2. The function $\mathcal{M}$ is actually C^1. This guarantees that the system (37) is C^1 dissipative.

3.0.3 Application

With the storage function (62), we are in position to apply Theorem 16 and possibly get a global asymptotic stabilizer if the zero-state observability assumption holds for the system (37). So with the two properties above, this observability condition and the stabilizer we can get, we are with the closed loop full order system (37) in exactly the same situation as we were with the open loop reduced order x subsystem. This means that we are ready to deal with the further extended system :

$$\left\{\begin{array}{rcl} \dot z &=& k(x,y) \;+\; k_u(x,y,z,u)\, u \\ \dot y &=& h(x) \;+\; h_u(x,y,u)\, u \\ \dot x &=& f(x) \;+\; f_u(x,u)\, u \end{array}\right. \tag{64}$$

So we may be able by recursion to do a Lyapunov design of global asymptotic stabilizers for systems in the feedforward form (1). The corresponding design technique is called forwarding. In the following paragraphs, we study this technique in more details. But before closing this section, an important remark has to be made.

Remark 65

1. *The Lyapunov function V defined in (62) has received an interpretation in terms of a new of coordinate. This has been introduced and developed in [10]. The change of coordinate is :*

$$\mathfrak{y} \;=\; y \;-\; \mathcal{M}(x) \;. \tag{66}$$

Its existence relies on the existence of the vector valued function $\mathcal{M}$. It yields to the Lyapunov function :

$$V_{\mathfrak{y}}(x,\mathfrak{y}) \;=\; V(x,y) \;=\; V_x(x) \;+\; \tfrac{1}{2}\, |\mathfrak{y}|^2 \;. \tag{67}$$

2. *The Lyapunov function V can also be written as :*

$$V(x,y) \;=\; V_x(x) \;+\; \tfrac{1}{2}\, |y|^2 \;+\; S(x,y) \tag{68}$$

with :

$$S(x,y) \;=\; -y^T \mathcal{M}(x) \;+\; \tfrac{1}{2}\, |\mathcal{M}(x)|^2 \;. \tag{69}$$

Namely V is made of the sum of three terms :
– *the Lyapunov function for the reduced order x subsystem,*
– *the Lyapunov function for the extending y subsystem, which would be appropriate if x were at the origin,*
– *a cross term S.*

This point of view with a cross term has been introduced and developed in [5]. It applies to a broader class of systems than the change of coordinate, the existence of a C^1 scalar cross term S holding under weaker conditions than a C^1 vector function $\mathcal{M}$ (see [12]). In the following, we deal only with the change of coordinate, leaving to the reader to consult [12] to get more information on the cross term technique •

4 The forwarding technique with an exact change of coordinates

4.1 The technique

The forwarding technique with an exact change of coordinates applies in fact to a larger class of systems than (37) (and therefore, by recursion, larger than (1)). It is :

$$\left\{ \begin{array}{rcl} \dot{y} &=& h_y(y) \;+\; h_x(x,y)\,x \;+\; h_u(x,y,u)\,u \\ \dot{x} &=& f(x) \;+\; f_u(x,y,u)\,u \end{array} \right. \tag{70}$$

where all the functions are C^1 and with still the assumption of global asymptotic stability of the origin of the x subsystem when u is at the origin. More precisely :

H1 : There exists a C^1 Lyapunov function V_x such that :

$$W_x(x) \;=\; -\frac{\partial V_x}{\partial x}(x)\,f(x) \;>\; 0 \qquad \forall x \in \mathbb{R}^n \setminus \{0\}\ . \tag{71}$$

The difference between (37) and (70) is in the fact that the function h can actually depend on y and is then decomposed into the sum $h_y + h_x x$. However this dependence on y is restricted by the following assumption :

H2 : There exists a C^1 Lyapunov function V_y such that :

$$W_y(y) \;=\; -\frac{\partial V_y}{\partial y}(y)\,h_y(y) \;\geq\; 0 \qquad \forall y \in \mathbb{R}^q\ . \tag{72}$$

This implies that, when u and x are at the origin, the origin is a globally stable solution of the y subsystem.

For the system (70), we assume the knowledge of a function Ψ which is C^1 and satisfies the properties :
P1 : We have $\Psi(0,y) \;=\; y$.
P2 : The set $\{y \;:\; \exists x \in \mathbb{R}^n \;:\; |x| \leq c\,,\; |\Psi(x,y)| \leq c\}$ is bounded whatever the non negative real number c is.
P3 : When u is at the origin, $\overset{\frown}{\Psi(x,y)}$ does not depend on x.
A direct consequence of the properties P1 and P3 is that Ψ is a solution of the partial differential equation :

$$0 = \tfrac{\partial \Psi}{\partial x}(x,y)\,f(x) + \tfrac{\partial \Psi}{\partial y}(x,y)\,(h_y(y) + h_x(x,y)x) - h_y(\Psi(x,y))\ . \tag{73}$$

This shows that finding an expression for Ψ may not be an easy task. But assuming we have it satisfying the properties P1, P2 and P3, we introduce a new "coordinate"[2] :

$$\mathfrak{y} \;=\; \Psi(x,y)\ . \tag{74}$$

Then, in the $(x,\mathfrak{y})$ "coordinates", the system (70) rewrites as :

$$\left\{ \begin{array}{rcl} \dot{\mathfrak{y}} &=& h_y(\mathfrak{y}) \;+\; \left(\tfrac{\partial \Psi}{\partial x}(x,y)f_u(x,y,u) + \tfrac{\partial \Psi}{\partial y}(x,y)h_u(x,y,u)\right) u \\ \dot{x} &=& f(x) \;+\; f_u(x,y,u)\,u \end{array} \right. \tag{75}$$

We have (see [10]) :

Theorem 76 *Under the assumptions H1 and H2, if there exists a function Ψ satisfying the properties P1, P2 and P3, the system (70) is C^1 dissipative with storage function (see Remark 78.1 below) :*

$$V_{\mathfrak{y}}(x,\mathfrak{y}) \;=\; V_x(x) \;+\; V_y(\mathfrak{y})\ . \tag{77}$$

Moreover, for any real number $\overline{u}$ in $(0,+\infty]$, there exists a continuous global asymptotic stabilizer strictly bounded in norm by $\overline{u}$ when the system (70) is zero-state observable with output function :

$$\left(\begin{array}{ccc} \begin{array}{l} \tfrac{\partial V_x}{\partial x}(x)f_u(x,y,0) + \tfrac{\partial V_y}{\partial y}(\Psi(x,y)) \quad \times \\ \quad \times \left(\tfrac{\partial \Psi}{\partial x}(x,y)f_u(x,y,0) + \tfrac{\partial \Psi}{\partial y}(x,y)h_u(x,y,0)\right) \end{array} & W_x(x) & W_y(\Psi(x,y)) \end{array} \right)$$

or, in the case where W_x is positive definite, the y subsystem is zero-state observable with input (x,u) and output function :

$$\left(\begin{array}{cc} \tfrac{\partial V_y}{\partial y}(y)\tfrac{\partial \Psi}{\partial x}(0,y)f_u(0,y,0) + \tfrac{\partial V_y}{\partial y}(y)h_u(0,y,0) & W_y(y) \end{array} \right)$$

[2] $\mathfrak{y}$ is abusively called a coordinate since we do not impose a bijection between (x,y) and $(x,\mathfrak{y})$. In fact the analysis goes by picking a solution in the (x,y) coordinates, study its properties with $(x,\mathfrak{y})$ and infer properties in the (x,y) coordinates.

Remark 78

1. *For the storage function (77), actually V_x needs only to be a C^1 Lyapunov function such that W_x given by (71) is non negative (and not positive definite as imposed by H1).*

2. *An expression of the stabilizer can be obtained as explained in Remark 17.1. It does require an expression for the function Ψ* •

Example 79 Consider the system :

$$\dot{y} \;=\; x_1 \;+\; x_2^2 \quad , \qquad \dot{x}_1 \;=\; x_2 \quad , \qquad \dot{x}_2 \;=\; -x_1 \;-\; x_2 \;+\; u \; . \tag{80}$$

Assumption H1 holds with the functions :

$$V_x(x_1,x_2) \;=\; x_2^2 \;+\; x_2\, x_1 \;+\; x_1^2 \quad , \qquad W_x(x_1,x_2) \;=\; V_x(x_1,x_2) \; . \tag{81}$$

Also, the origin is an exponentially stable solution of the (x_1,x_2) subsystem when u is zero.
Assumption H2 holds with the functions :

$$V_y(y) \;=\; \tfrac{1}{2}\, y^2 \quad , \qquad W_y(y) \;=\; 0 \; . \tag{82}$$

Since the solutions $(X_1(x_1,x_2;t), X_2(x_1,x_2;t))$ of the (x_1,x_2) subsystem are exponentially converging to zero when u is zero, the function :

$$\mathcal{M}(x_1,x_2) \;=\; -\int_0^\infty \left[X_1(x_1,x_2;s) + X_2(x_1,x_2;s)^2\right] ds \; , \tag{83}$$

is well defined, C^1 (see [4, Théorème 3.150]) and, as can be checked "by hand" satisfies :

$$\dot{\overbrace{\mathcal{M}(x_1,x_2)}} \;=\; x_1 \;+\; x_2^2 \quad , \qquad \mathcal{M}(0,0) \;=\; 0 \; . \tag{84}$$

Then, the function Ψ defined as :

$$\Psi(x_1,x_2,y) \;=\; y \;-\; \mathcal{M}(x_1,x_2) \tag{85}$$

satisfies P1, P2 and P3.
To get an expression for $\mathcal{M}$, and therefore Ψ, we note that it is a solution of :

$$\frac{\partial \mathcal{M}}{\partial x_1}(x_1,x_2)\, x_2 \;-\; \frac{\partial \mathcal{M}}{\partial x_2}(x_1,x_2)\,(x_1+x_2) \;=\; \left(x_1 \;+\; x_2^2\right) \; . \tag{86}$$

By taking a solution in the form of a polynomial of order 2 in (x_1,x_2) and by equating the coefficients, we get :

$$\mathcal{M}(x_1,x_2) \;=\; -\left(x_1 + x_2 + \tfrac{1}{2}[x_1^2 + x_2^2]\right) \; . \tag{87}$$

With this expression at hand, we introduce the new coordinate :

$$\mathfrak{y} \;=\; \Psi(x,y) \;=\; y \;+\; \left(x_1 + x_2 + \tfrac{1}{2}[x_1^2 + x_2^2]\right) \; . \tag{88}$$

The system (80) rewrites as the system (24) (with $x_3 = \mathfrak{y}$). So we know that a global asymptotic stabilizer is :

$$\begin{aligned} \phi(x_1,x_2,\mathfrak{y}) \;&=\; -\left(2[1+x_2]\mathfrak{y} + 2[2x_2 + x_1]\right) \; , && (89)\\ &=\; -\left(2[1+x_2]\left[y + \left(x_1 + x_2 + \tfrac{1}{2}[x_1^2 + x_2^2]\right)\right] + 2[2x_2 + x_1]\right) \;\; \bullet && (90) \end{aligned}$$

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To summarize, the forwarding technique with an exact change of coordinates relies mainly on the existence and the knowledge of an expression of the C^1 function Ψ satisfying the three properties P1, P2 and P3.

4.2 About the change of "coordinates"

To get a better idea of what this function Ψ is, we set u at the origin and we observe that the mapping between the solutions $(X(x,y;t),Y(x,y;t))$ of the system (70) and those $\mathfrak{Y}(\mathfrak{y};t)$ of the system (75) yields :

$$\Psi(X(x,y;t),Y(x,y;t)) \;=\; \mathfrak{Y}(\Psi(x,y);t) \qquad \forall t \geq 0 \;. \tag{91}$$

Since h_y is C^1, the solutions $\mathfrak{Y}(\mathfrak{y};t)$ are unique and we have :

$$\mathfrak{Y}(\mathfrak{Y}(\mathfrak{y};t);-t) \;=\; \mathfrak{y} \;. \tag{92}$$

So (91) rewrites :

$$\Psi(x,y) \;=\; \mathfrak{Y}(\Psi(X(x,y;t),Y(x,y;t));-t) \;. \qquad \forall t \geq 0 \;. \tag{93}$$

This implies :

$$\Psi(x,y) \;=\; \lim_{t\to+\infty} \mathfrak{Y}(\Psi(X(x,y;t),Y(x,y;t));-t) \;. \tag{94}$$

Now, we remark that we have, from H1 and P1,

$$\lim_{t\to+\infty} X(x,y;t) \;=\; 0 \quad , \qquad \Psi(0,y) \;=\; y \;. \tag{95}$$

So if "everything works fine" when the limit "enters" the function $\mathfrak{Y}$ in (93), the function Ψ is given by :

$$\Psi(x,y) \;=\; \lim_{t\to+\infty} \mathfrak{Y}(Y(x,y;t);-t) \;. \tag{96}$$

In this general framework it is difficult to figure out when this definition makes sense and gives the right answer. In the case where the function h_y is linear, i.e. :

$$h_y(y) \;=\; H\,y \;, \tag{97}$$

we are able to conclude. Indeed, we get :

$$\begin{aligned} \mathfrak{Y}(\mathfrak{y};t) \;&=\; \exp(Ht)\,\mathfrak{y} \;, && (98)\\ Y(x,y;t) \;&=\; \exp(Ht)\,y \;+\; \int_0^t \exp(H(t-s))\,h_x(X(x,y;s),Y(x,y;s))\,X(x,y;s)\,ds \;. && (99)\end{aligned}$$

In this case (96) rewrites :

$$\Psi(x,y) \;=\; y \;+\; \int_0^\infty \exp(-Hs)\,h_x(X(x,y;s),Y(x,y;s))\,X(x,y;s)\,ds \;. \tag{100}$$

By mimicking the arguments of [12, Sections 5.2.1 and 5.2.2], we get the following existence result :

Lemma 101 *Assume H1 and H2 hold with $h_y(y) = Hy$. If we have :*

1. $$\max\left\{\mathrm{Re}\left(\textit{eig. val.}\;\left(\frac{\partial f}{\partial x}(0)\right)\right)\right\} < \min\left\{0\,,\,\mathrm{Re}\left(\textit{eig. val.}\;(H)\right)\right\} \tag{102}$$

2. *the function (of x) $\sup_y \frac{|h_x(x,y)|}{1+|y|}$ is locally bounded,*

then the function Ψ given by (100) is well defined, C^1, satisfies the properties P1, P2 and P3 and is a solution of (73) which is in the present case :

$$0 \;=\; \frac{\partial \Psi}{\partial x}(x,y)\,f(x) \;+\; \frac{\partial \Psi}{\partial y}(x,y)\,[Hy + h_x(x,y)x] \;-\; H\,\Psi(x,y) \;. \tag{103}$$

Remark 104 *Since the condition (102) implies the local exponential stability of the origin of the x subsystem of (70), the "flatness" constraint discussed in section 3 is always satisfied.*

Knowing with this Lemma that the function Ψ exists, as already observed in Example 79, we can get an expression for it either by solving the partial differential equation (103) or, for each (x, y), by computing the solutions $(X(x, y; t), Y(x, y; t))$ of the system (70) with u at the origin, those $\mathfrak{Y}(\mathfrak{y}; t)$ of the system (75) and then by evaluating the limit (96) (or the integral (100)) (method of characteristics). For any real world application, this program seems to be out of range. Nevertheless, it may still be possible to get an expression for Ψ. The idea for this is that, by definition, we should have, when $h_y(y) = 0$:

$$\dot{\overbrace{\Psi(x, y) - y}} \;=\; h_x(x, y)\, x \; . \tag{105}$$

This says that we look for a function of (x, y) whose derivative is $h_x(x, y)x$. To make such a search fruitful, it may be opportune to get prepared while dealing before with the x subsystem.

Example 106 *The cart pendulum system (continued) (see [14]).* Let us come back to the cart-pendulum of Example 6. Our ultimate goal is to make the upward position of the pendulum and the zero position of the cart asymptotically stable with a basin of attraction as large as possible. As a step toward this goal, we study here the possibility of asymptotically stabilizing the homoclinic orbit[3] of the pendulum and the zero position of the cart. The motivation is that, if such an objective is met, all the solutions arrive in finite time in the neighborhood of the point to be made asymptotically stable. In this situation, we can switch the controller to a linear controller locally stabilizing this point.

To meet our objective and simplify the computations, we modify the cart-pendulum system into another one by changing the control into :

$$F \;=\; m\,\sin(\theta)\left(g\cos(\theta) - l\dot{\theta}^2\right) \;+\; \frac{g}{l}\;\left(M + m\sin(\theta)^2\right) u \; , \tag{107}$$

with u as new control. Also, in order to simplify the equations, we change the coordinates and time as follows :

$$x = \frac{1}{l}\, \mathcal{x} \; , \quad v = \frac{1}{\sqrt{gl}}\, \dot{\mathcal{x}} \; , \quad \theta = \theta \; , \quad \omega = \sqrt{\frac{l}{g}}\, \dot{\theta} \; , \quad \tau = \sqrt{\frac{g}{l}}\, t \; . \tag{108}$$

Then still denoting by " $\dot{}$ " the derivation with respect to the new time τ, the new system is :

$$\dot{x} \;=\; v \; , \quad \dot{v} \;=\; u \; , \quad \dot{\theta} \;=\; \omega \; , \quad \dot{\omega} \;=\; \sin(\theta) \;-\; u\,\cos(\theta) \; . \tag{109}$$

We view this system as made of a first subsystem with state variables (v, θ, ω) which is extended by adding the integrator giving the position. So we proceed in two steps :

Step 1 : We consider the (v, θ, ω) subsystem. We look for a controller making asymptotically stable the following set with an as large as possible domain of attraction :

$$\mathcal{S} \;=\; \left\{(v, \theta, \omega) : \; E(\omega, \theta) = \tfrac{1}{2}\,\omega^2 + \cos(\theta) = 1 \; , \; v = 0\right\} \; . \tag{110}$$

E is actually the total mechanical energy of the pendulum alone. When $E = 1$, the pendulum is on its homoclinic orbit. To meet the stabilization objective of this step, it is sufficient to find a C^1 positive definite and proper function V in the variable $(E - 1, v)$ and to make its derivative non positive on an as large as possible domain. Since we have :

$$\dot{E} \;=\; -\cos(\theta)\,\omega\, u \quad , \qquad \dot{v} \;=\; u \; , \tag{111}$$

a good candidate for V is :

$$V(E - 1, v) \;=\; V_E(E - 1) \;+\; \tfrac{k_v}{2}\, v^2 \; , \tag{112}$$

where k_v is a strictly positive real number and V_E is a C^2 function defined at least on $[-2, +\infty)$ where it is proper and satisfies[4] :

$$\{V_E(s) = 0 \text{ or } V_E'(s) = 0\} \;\Rightarrow\; s = 0 \quad , \qquad \lim_{s \to +\infty} V_E(s) = +\infty \tag{113}$$

At this stage we do not need nor want to specify what V_E is. We shall use this flexibility for handling the second step. We get :

$$\dot{V} \;=\; [-V_E'(E - 1)\,\omega\,\cos(\theta) \;+\; k_v\, v]\, u \; . \tag{114}$$

[3]The one which makes just one turn in infinite time.
[4]The last property and the fact that θ lives in $\mathbb{S}^1$ imply that if, E is bounded, so is ω.

Hence an appropriate controller is :

$$u = \phi(v,\theta,\omega) = m(v,\theta,\omega)\,(V_E'(E-1)\,\omega\,\cos(\theta) - k_v\,v) \tag{115}$$

where m is any strictly positive and Lipschitz continuous function.

By successive derivations, we can check that any solution of the closed loop system which satisfies :

$$-V_E'(E-1)\,\omega\,\cos(\theta) + k_v\,v = 0 \tag{116}$$

satisfies also either :

$$E = 1 \quad , \qquad v = 0\,, \tag{117}$$

or :

$$\theta \in \{0,\pi\} \quad , \qquad \omega = v = 0\,. \tag{118}$$

From the invariance principle as stated for instance in [12, Theorem 2.21], we conclude that all the solutions of the closed loop system converge either to the desired set $\mathcal{S}$ in (110) or one of the two the equilibrium points ($\theta \in \{0,\pi\}, \omega = v = 0$). By looking at the linearization of the dynamics at these points, it can be seen that they have a stable manifold and an unstable manifold. From [6], we conclude that the set of points in $\mathbb{R}\times\mathbb{S}^1\times\mathbb{R}$ and not belonging to the domain of attraction of $\mathcal{S}$ is of measure zero.

Step 2 : We consider now the full order system (108). To meet our stabilization objective it remains to asymptotically stabilize x at zero. We observe that the system (108) is obtained from the (v,θ,ω) subsystem by adding the integrator :

$$\dot{x} = v\,. \tag{119}$$

So we apply the forwarding technique with an exact change of coordinates. It leads us to look for a C^1 function $\mathcal{M}(v,\theta,\omega)$ such that, when u is given by (115), we have :

$$\dot{\overbrace{\mathcal{M}(v,\theta,\omega)}} = v\,. \tag{120}$$

To find an expression for this function, we try to express v as the derivative of a function of (v,θ,ω). We remark that, with (115), the $\dot{v}$ equation in (109) rewrites as :

$$u = \dot{v} = m(v,\theta,\omega)\,(V_E'(E-1)\,\omega\,\cos(\theta) - k_v\,v)\;. \tag{121}$$

But, with the help of (111), we get :

$$\begin{aligned} V_E'(E-1)\,\omega\,\cos(\theta) &= \dot{\overbrace{V_E'(E-1)\,\sin(\theta)}} + V_E''(E-1)\,\sin(\theta)\,\omega\,\cos(\theta)\,u\,, &(122)\\ &= \dot{\overbrace{V_E'(E-1)\,\sin(\theta)}} + V_E''(E-1)\,\sin(\theta)\,\omega\,\cos(\theta)\,\dot{v}\,. &(123)\end{aligned}$$

Also, for any C^1 function q, we have :

$$\dot{\overbrace{q(v)}} = q'(v)\,\dot{v}\,. \tag{124}$$

Collecting all the above relations, we get :

$$k_v\,v = -\dot{\overbrace{q(v)}} + \dot{\overbrace{V_E'(E-1)\sin(\theta)}} - \frac{1 - m(v,\theta,\omega)\,[V_E''(E-1)\,\sin(\theta)\,\omega\,\cos(\theta) + q'(v)]}{m(v,\theta,\omega)}\,\dot{v}\,. \tag{125}$$

This shows that, if we choose the function m satisfying :

$$\frac{1 - m(v,\theta,\omega)\,[V_E''(E-1)\,\sin(\theta)\,\omega\,\cos(\theta) + q'(v)]}{m(v,\theta,\omega)} = 1\,, \tag{126}$$

or in other words :

$$m(v,\theta,\omega) = \frac{1}{1 + q'(v) + V_E''(E-1)\sin(\theta)\cos(\theta)\omega}\,, \tag{127}$$

then we have simply :

$$-\dot{\overbrace{q(v)}} + \dot{\overbrace{V_E'(E-1)\sin(\theta)}} - \dot{v} = k_v\, v\ . \tag{128}$$

By comparing to (120), we see that we have obtained the expression we were looking for :

$$\mathcal{M}(v,\theta,\omega) = \frac{-q(v) + V_E'(E-1)\sin(\theta) - v}{k_v}\ . \tag{129}$$

Before going on, we have to make sure that the function m given by (127) is appropriate, i.e., for all (v,θ,ω), we have :

$$1 + q'(v) + V_E''(E-1)\sin(\theta)\cos(\theta)\omega > 0\ . \tag{130}$$

But, since the definition of E in (110) gives :

$$|\omega| \le \sqrt{2(E+1)}\ , \tag{131}$$

we conclude that it is sufficient to impose that $q'(v)$ is non negative for all v and :

$$V_E''(s)\sqrt{2(s+2)} \le \eta < 1 \qquad \forall s \in [-2, +\infty)\ . \tag{132}$$

Let us note also that if the functions[5] $|V_E'(s)|\sqrt{2(s+2)}$ and $\frac{|v|}{q'(v)}$ are bounded then so is the control ϕ in (115).

Let us now come back to our design. We follow the forwarding technique with an exact change of coordinate and let :

$$\mathfrak{y} = x - \frac{q(v) - V_E'(E-1)\sin(\theta) - v}{k_v}\ , \tag{133}$$

$$u = u_{\mathfrak{y}} + \frac{V_E'(E-1)\omega\cos(\theta) - k_v v}{1 + q'(v) + V_E''(E-1)\sin(\theta)\cos(\theta)\omega}\ . \tag{134}$$

This yields :

$$\dot{\mathfrak{y}} = \frac{1 + q'(v) + V_E''(E-1)\sin(\theta)\cos(\theta)\omega}{k_v}\, u_{\mathfrak{y}}\ . \tag{135}$$

Then, with (112), we take :

$$V(x,v,\theta,\omega) = V_E(E-1) + \frac{k_v}{2}v^2 + V_{\mathfrak{y}}(\mathfrak{y}) \tag{136}$$

where $V_{\mathfrak{y}}$ is any C^1 Lyapunov function. The stationary points of V are all on the homoclinic orbit we want to asymptotically stabilize. We get :

$$\dot{V} = [-V_E'\omega\cos(\theta) + k_v v]\left[u_{\mathfrak{y}} + \frac{V_E'\omega\cos(\theta) - k_v v}{1 + q' + V_E''\sin(\theta)\cos(\theta)\omega}\right] + \frac{1 + q' + V_E''\sin(\theta)\cos(\theta)\omega}{k_v} V_{\mathfrak{y}}'\, u_{\mathfrak{y}}\ . \tag{137}$$

This yields to the possible choice :

$$u_{\mathfrak{y}} = -[-V_E'(E-1)\omega\cos(\theta) + k_v v] - \frac{1 + q'(v) + V_E''(E-1)\sin(\theta)\cos(\theta)\omega}{k_v} V_{\mathfrak{y}}'(\mathfrak{y})\ . \tag{138}$$

Actually we can choose $u_{\mathfrak{y}}$, V_E, q and $V_{\mathfrak{y}}$ in such a way that :
– the final controller is bounded by any a priori given bound,
– we have :

$$\dot{V} = 0 \quad \Rightarrow \quad [V_E'(E-1)\omega\cos(\theta) + k_v v] = \mathfrak{y} = 0\ ,\ u = 0. \tag{139}$$

This allows us to conclude that the set of points in $\mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{R}$ and not belonging to the domain of attraction of the desired homoclinic orbit is a set of measure zero. And the solutions issued from such points all converge to the equilibrium points $(\theta = k\pi, x = v = \omega = 0)$ •

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[5]We can take for instance : $q(v) = v|v|$ et $V_E(s) = (1+s^2)^{\frac{1}{4}}$.

5 The forwarding technique with an approximate change of coordinates

We have mentioned that, from a practical point of view, the main difficulty in applying the forwarding technique with an exact change of coordinates is to find an expression for the function Ψ. So one may ask if we could use an approximation. We address this question now (see [10]).

Let Ψ_a be an approximation of Ψ to which we impose to be C^1, and to satisfy P1 and :

P2' : For each x, $\Psi_a(x,\cdot)$ is a global diffeomorphism and in particular is proper.

With this condition, we introduce the new (now true) coordinate :

$$\mathfrak{y} \;=\; \Psi_a(x,y) \; . \tag{140}$$

With it and the properties of Ψ_a, the system (70) rewrites as :

$$\left\{\begin{array}{rcl} \dot{\mathfrak{y}} & = & h_y(\mathfrak{y}) \;+\; \mathfrak{h}_x(x,\mathfrak{y})\,x \;+\; \mathfrak{h}_u(x,\mathfrak{y},u)\,u \\ \dot{x} & = & f(x) \;+\; \mathfrak{f}_u(x,\mathfrak{y},u)\,u \end{array}\right. \tag{141}$$

where in particular the function $\mathfrak{h}_x$ is given by :

$$\mathfrak{h}_x(x,\Psi_a(x,y))\,x \;=\; \frac{\partial \Psi_a}{\partial x}(x,y)\,f(x) \;+\; \frac{\partial \Psi_a}{\partial y}(x,y)\,(h_y(y)+h_x(x,y)x) \;-\; h_y(\Psi_a(x,y)) \; . \tag{142}$$

As opposed to (73), we see that, Ψ_a being only an approximation, the right hand side of (142) is not zero. Concerning f and h_y, we keep the assumptions H1 and H2. Concerning $\mathfrak{h}_x x$, we assume[6] :

P3' : There exists a continuous function ℓ which is proper and with a continuous strictly positive derivative ℓ' defined on $(0,+\infty)$ such that $\ell'(V_x(x))\frac{\partial V_x}{\partial x}(x)$ has a continuous extension at the origin and we have :

$$\left|\frac{\partial V_y}{\partial y}(\mathfrak{y})\,\mathfrak{h}_x(x,\mathfrak{y})\,x\right| \;\leq\; \ell'(V_x(x))\,W_x(x)\,(1+V_y(\mathfrak{y})) \tag{143}$$

for all $(x,\mathfrak{y})$. In the case where the origin is locally exponentially stable for the x subsystem of (141) with u at the origin, the above inequality reduces to :

$$\left|\frac{\partial V_y}{\partial y}(\mathfrak{y})\,\mathfrak{h}_x(x,\mathfrak{y})\right| \;\leq\; |x|\,\gamma(|x|)\,(1+V_y(\mathfrak{y})) \qquad \forall (x,\mathfrak{y}) \tag{144}$$

with γ some non decreasing, non negative continuous function.

While P3 was leading to the fact that, with the exact change of "coordinate", $\mathfrak{h}_x x$ was zero, P3' imposes only a magnitude limitation on this term for x small and $\mathfrak{y}$ large. W_x in the right hand side of (143) quantifies how much $\mathfrak{h}_x x$ should be "flat" for x close to the origin with respect to the strength of attractiveness of the origin of the x subsystem. In particular, in a generic situation, (143) or (144) implies :

$$\mathfrak{h}_x(0,\mathfrak{y}) \;=\; 0 \; . \tag{145}$$

Then, with P1, (142) gives :

$$0 \;=\; \frac{\partial \Psi_a}{\partial x}(0,y)\,\frac{\partial f}{\partial x}(0) \;+\; \frac{\partial^2 \Psi_a}{\partial x \partial y}(0,y) \odot h_y(y) \;+\; h_x(0,y) \;-\; \frac{\partial h_y}{\partial y}(y)\frac{\partial \Psi_a}{\partial x}(0,y) \; . \tag{146}$$

So, instead of the partial differential equation (73) in the (x,y) variables, we have now a partial differential equation in the y variable only (see Example 150).

We have :

[6] The meaning of the inequalities (143) or (144) is explained in [10].

Theorem 147 ([10]) *Under the assumptions H1 and H2, if there exists a function Ψ_a satisfying the properties P1, P2' and P3', the system (70) is C^1 dissipative with storage function :*

$$V_{\mathfrak{y}}(\mathfrak{y}) \;=\; 2\,\ell(V_x(x)) \;+\; \log(1+V_y(\mathfrak{y}))\ . \tag{148}$$

Moreover, for any real number $\overline{u}$ in $(0,+\infty]$, there exists a continuous global asymptotic stabilizer strictly bounded in norm by $\overline{u}$ when the y subsystem of (70) is zero-state observable with input (x,u) and output function :

$$\left(\begin{array}{cc} \frac{\partial V_y}{\partial x}(y)\frac{\partial \Psi_a}{\partial x}(0,y)f_u(0,y,0)+\frac{\partial V_y}{\partial y}(y)h_u(0,y,0) & W_y(y)\end{array}\right)$$

Remark 149

1. *As opposed to (77), in (148), V_x must be the function given by assumption H1 with corresponding W_x positive definite (see Remark 78.1).*
2. *In (148), if not given, the function ℓ is to be designed such that (143) holds* •

Example 150 Let us come back to the system (80) and work with an approximate change of coordinate. For this, we restrict our attention to an approximating function Ψ_a of the form :

$$\Psi_a(x_1,x_2,y) \;=\; y \;-\; \mathcal{M}_a(x_1,x_2) \tag{151}$$

where $\mathcal{M}_a$ is to be designed so that :

$$\mathcal{M}_a(0,0) \;=\; 0\ . \tag{152}$$

In this case, we get, from (142) and (80),

$$\mathfrak{h}_x\,(x_1,x_2,y-\mathcal{M}_a(x_1,x_2))\begin{pmatrix} x_1\\ x_2\end{pmatrix} \;=\; -\frac{\partial \mathcal{M}_a}{\partial x_1}(x_1,x_2)\,x_2 \;+\; \frac{\partial \mathcal{M}_a}{\partial x_2}(x_1,x_2)\,(x_1+x_2) \;+\; (x_1+x_2^2)\,. \tag{153}$$

Then, with $V_y(y)=\frac{1}{2}y^2$, the condition (143) of P3' is equivalent to :

$$\begin{aligned}|y-\mathcal{M}_a(x_1,x_2)|\left|-\tfrac{\partial \mathcal{M}_a}{\partial x_1}(x_1,x_2)\,x_2+\tfrac{\partial \mathcal{M}_a}{\partial x_2}(x_1,x_2)\,(x_1+x_2)+(x_1+x_2^2)\right| \\ \le\; (x_1^2+x_2^2)\,\gamma(\sqrt{x_1^2+x_2^2})\,\left(1+\tfrac{1}{2}\,|y-\mathcal{M}_a(x_1,x_2)|^2\right)\end{aligned} \tag{154}$$

for all (x_1,x_2,y). This is implied in particular by :

$$\left|-\frac{\partial \mathcal{M}_a}{\partial x_1}(x_1,x_2)\,x_2+\frac{\partial \mathcal{M}_a}{\partial x_2}(x_1,x_2)\,(x_1+x_2)+(x_1+x_2^2)\right| \;\le\; (x_1^2+x_2^2)\,\gamma(\sqrt{x_1^2+x_2^2}) \tag{155}$$

for all (x_1,x_2). In its turn this condition says that the left hand side should be of order two at the origin and therefore implies :

$$-\frac{\partial \mathcal{M}_a}{\partial x_1}(0,0)\,x_2 \;+\; \frac{\partial \mathcal{M}_a}{\partial x_2}(0,0)\,(x_1+x_2) \;+\; x_1 \;=\; 0\ . \tag{156}$$

We get directly :

$$\frac{\partial \mathcal{M}_a}{\partial x_1}(0,0) \;=\; \frac{\partial \mathcal{M}_a}{\partial x_2}(0,0) \;=\; -1\ . \tag{157}$$

Having obtained a constraint only on the derivatives of $\mathcal{M}_a$ at the origin. Let us try if a function $\mathcal{M}_a$ simply linear would be appropriate. We pick :

$$\mathcal{M}_a(x_1,x_2) \;=\; -x_1 \;-\; x_2\ . \tag{158}$$

We get that (155) and therefore (154) hold, with $\gamma(s)=1$.

With the function $\mathcal{M}_a$ we have found, the change of coordinate is :

$$\mathfrak{y} \;=\; y \;+\; x_1 \;+\; x_2\ , \tag{159}$$

So the system (80) rewrites :

$$\dot{\mathfrak{y}} \;=\; x_2^2 \;+\; u\ ,\quad \dot{y}_1 \;=\; x_2\ ,\quad \dot{y}_2 \;=\; -x_1 \;-\; x_2 \;+\; u\ . \tag{160}$$

As expected, the term of second order x_2^2 as not been removed from the $\dot{\mathfrak{y}}$ equation (compare with (24)).

Following (148) in Theorem 147, we let[7] :

$$V_{\mathfrak{y}}(x_1,x_2,\mathfrak{y}) \;=\; 2\,\ell(x_2^2+x_2x_1+x_1^2) \;+\; \log\left(1+\tfrac{1}{2}\mathfrak{y}^2\right) \;, \tag{161}$$

with a function ℓ to be designed. This yields :

$$\overbrace{V_{\mathfrak{y}}(x_1,x_2,\mathfrak{y})}^{\cdot} \;=\; \frac{\mathfrak{y}}{1+\frac{1}{2}\mathfrak{y}^2}\,(x_2^2+u) \;+\; 2\,\ell'(x_2^2+x_2x_1+x_1^2)\,\left[-(x_2^2+x_2x_1+x_1^2)+(2x_2+x_1)u\right] \;. \tag{162}$$

Since we have[8] :

$$\frac{\mathfrak{y}}{1+\frac{1}{2}\mathfrak{y}^2}\,x_2^2 \;\leq\; \sqrt{2}\,(x_2^2+x_2x_1+x_1^2) \qquad \forall(\mathfrak{y},x_1,x_2)\;, \tag{163}$$

we choose $\ell(s)=\frac{1+\sqrt{2}}{2}\,s$. Indeed, this yields :

$$\overbrace{V_{\mathfrak{y}}(x_1,x_2,\mathfrak{y})}^{\cdot} \;=\; -(x_2^2+x_2x_1+x_1^2)+\left[\left(1+\sqrt{2}\right)(2x_2+x_1)+\frac{\mathfrak{y}}{\left(1+\frac{1}{2}\mathfrak{y}^2\right)}\right]u\;. \tag{164}$$

A candidate for a stabilizer is therefore :

$$\phi_a(x_1,x_2,\mathfrak{y}) \;=\; -\left[\left(1+\sqrt{2}\right)(2x_2+x_1)+\frac{\mathfrak{y}}{\left(1+\frac{1}{2}\mathfrak{y}^2\right)}\right]\;. \tag{165}$$

It gives :

$$\overbrace{V_{\mathfrak{y}}(x_1,x_2,\mathfrak{y})}^{\cdot} \;=\; -(x_2^2+x_2x_1+x_1^2)-\left[\left(1+\sqrt{2}\right)(2x_2+x_1)+\frac{\mathfrak{y}}{\left(1+\frac{1}{2}\mathfrak{y}^2\right)}\right]^2\;. \tag{166}$$

This implies global asymptotic stability of the origin.

To summarize, the key points of this example are :

1. By comparing (86) and (155), we see that we are asking to the approximation $\mathcal{M}_a$ of $\mathcal{M}$ to solve the partial differential equation (86) only up to the first order around the origin. Namely, we have transformed the problem of solving the partial differential equation (86) into the one of solving the linear system (156).
2. It is important to compare the new stabilizer ϕ_a in (165), obtained with the approximate change of coordinate, with ϕ in (90) obtained with the exact change of coordinate. In particular, we see that, for (x_1,x_2) fixed, ϕ_a is a bounded function of y, although we were not looking for this property. On the contrary, ϕ is not a bounded function of y. Not being able to remove the terms of higher order in (x_1,x_2), the strategy for the new stabilizer is to privilege the (x_1,x_2) components of the reduced order x subsystem at times where they are large without paying attention to what the y component of the integrator is doing at those times •

$\diamond$

Let us recapitulate on the forwarding technique with an approximate change of coordinates :

- The main benefit is that, instead of solving exactly a partial differential equation in (x,y) like (103), it is sufficient to approximate its solution up to the first order in x, for x at the origin. As a consequence, typically, we are left with solving a partial differential equation in y only.
- The losses are :
  1. Instead of a function V_x with a non negative function W_x, we need now an expression of a function V_x with a positive definite W_x. This may generate difficulties when the forwarding technique with an approximate change of coordinates is applied recursively. However, the problem can be overcome some how as shown in [10, Proposition III.3]
  2. We have to design the function ℓ by manipulating inequalities.
  3. The class of stabilizers that we can reach is poorer. In particular they are typically bounded in the state component of the integrator.

[7] Another appropriate choice is $V_{\mathfrak{y}}(x_1,x_2,\mathfrak{y})=\ell(x_2^2+x_2x_1+x_1^2)+\left[\sqrt{1+\xi^2}-1\right]$.

[8] It is to get such an inequality with the right hand side not depending on $\mathfrak{y}$ that the log is introduced in (161).

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