Characterizations of global transversal exponential stability

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Abstract—**We study the relationship between the global exponential stability of an invariant manifold and the existence of a positive semi-definite Riemannian metric which is contracted by the flow. In particular, we investigate how the following properties are related to each other (in the global case): i). A manifold is globally "transversally" exponentially stable; ii). The corresponding variational system (c.f. (7) in Section II) admits the same property; iii). There exists a degenerate Riemannian metric which is contracted by the flow and can be used to construct a Lyapunov function. We show that the transverse contraction rate being larger than the expansion of the shadow on the manifold is a sufficient condition for the existence of such a Lyapunov function.**

An illustration of these tools is given in the context of global full-order observer design.

Keywords: Contraction, transversal exponential stability, exponentially attractive invariant manifold

I. Introduction

The use of Lyapunov functions has been instrumental in the (asymptotic) stability analysis of solutions or invariant sets of autonomous dynamical systems. It can be traced back to Lyapunov himself who has introduced this concept in his dissertation in 1892 (see [20] for an English translation). The seminal use of a Lyapunov function is for analyzing the asymptotic behavior of systems' trajectories and for studying the influence of systems' perturbations to the asymptotic stability property. In the past century, the applicability of Lyapunov stability theorems and functions has been extended beyond the field of dynamical systems and become one of the cornerstone tools in systems & control theory. It has become a very efficient tool for synthesizing stabilizing control laws, regulators and observers (see for example [14], [30], [16], [26]).

On the one hand, the study of converse Lyapunov theorems has received a considerable attention from the nonlinear control community (see, for example, [19], [25], [22], [21], [23], [17] for early results). Recent works on the various variations of converse Lyapunov theorems are, among many others, [31], [15].

On the other hand, instead of constructing Lyapunov functions (which can be non-trivial), it is also a common approach to use a first-order approximation for analyzing local stability of equilibrium points of nonlinear systems. Indeed, linearization has allowed one to apply directly tools for linear systems and it provides a simple way to construct local Lyapunov functions for the original nonlinear systems. Surprisingly this local approach can sometime also be employed to obtain the global properties and to construct global Lyapunov functions. Recent examples of the latter are the papers [9] or [1] that deal with contraction analysis and the paper [10] which deals with differential passivity property.

In this paper, we study the property of global exponential stability of an invariant manifold $\{(z,x)\in\mathbb{R}^{n_z+n_x} : z=0\}$, along some vector fields that can be decomposed as

$$\left\{\begin{array}{rcl}\dot z &=& F(z,x)\\ \dot x &=& G(z,x)\ ,\end{array}\right. \tag{1}$$

where z is in $\mathbb{R}^{n_z}$, x is in $\mathbb{R}^{n_x}$ and the functions $F : \mathbb{R}^{n_z}\times\mathbb{R}^{n_x}\to\mathbb{R}^{n_z}$ and $G:\mathbb{R}^{n_z}\times\mathbb{R}^{n_x}\to\mathbb{R}^{n_x}$ are C^2. In this systems' description, the z part of the state variables can refer to key state variables in various control problems. For instance, it can refer to the regulated output variables, or to the difference between two trajectories in an incrementally stable system, or to the error between the state and an estimate provided by an asymptotic observer.

Similar to the results in [2], we investigate how the global exponential stability property along solutions of some variational system is equivalent to the stability property of the invariant manifold on the system itself and how a Lyapunov function can be obtained from this stability property. In contrast to the local results as we have presented in [2], we investigate in this paper global properties.

In order to obtain this global characterization, we need to attach to each point (z,x) a bilinear map which defines a degenerate Riemannian metric. The first characterization is that the Lie derivative of this field of bilinear maps along the vector field has to be non positive. As will be shown, this characterization is valid as long as the expansion rate on the manifold is smaller than the attraction rate to the manifold. We note that this type of property has already been given in the literature. For instance this type of assumption implies the existence of an asymptotic phase [1] [12].

It is also the case in the literature related to the normal hyperbolic invariant manifold (see the work of Fenichel [6] or the books [13] and [32]). In our context, the manifold is a particular case of a normal hyperbolic invariant manifold.

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[1] It is the property that for each state trajectory there exists another trajectory inside an attractive and positively invariant manifold to which the state trajectory is converging to (also named shadow trajectory).

This has also been studied in [5] for the particular case of a compact invariant manifold. In their context, the global transverse exponential stability property can be rephrased in term of Normally Hyperbolic Invariant Manifold (NHIM)[2]. Note however that as opposed to these works on NHIM, we are not interested in the persistency or regularity properties. In this paper, we are interested to study equivalent conditions to the NHIM property and also a novel Lyapunov characterization for this property.

Notation : All along the paper, $|\cdot|$ is the Euclidean norm of vectors or matrices. For defining the dimension of the off-the-manifold, on-the-manifold and the complete system, we use three integers n_z, n_x and n_w such that $n_w = n_z + n_x$. We denote by $\mathcal{B}_z(a)$ the open ball of radius a centered at the origin in $\mathbb{R}^{n_z}$. The symbols I_z, I_x and I_w denote the identity matrices respectively in $\mathbb{R}^{n_z}$, $\mathbb{R}^{n_x}$ and $\mathbb{R}^{n_w}$. The first derivative of a function ϕ is denoted by ϕ'. Given a function $S : \mathbb{R}^n \mapsto \mathbb{R}^{n\times n}$ the values of which are bilinear maps, its Lie derivative along a vector field $\varphi : \mathbb{R}^n \mapsto \mathbb{R}^n$ is defined as

$$L_\varphi S(w) = \mathfrak{d}_\varphi S(w) + \left(\frac{\partial \varphi}{\partial w}(w)\right)^\top S(w) + S(w)\frac{\partial \varphi}{\partial w}(w) \ ,$$

where $\mathfrak{d}_\varphi S$ is the element wise upper right Dini derivative along the flow W of the vector field φ defined in the following sense

$$v^\top \mathfrak{d}_\varphi S(w) v \ := \ \limsup_{h\searrow 0} v^\top \frac{S(W(w,h)) - S(w)}{h} v \tag{2}$$

for all v in $\mathbb{R}^{n_w}$.

II. Transversally exponentially stable manifold

The system (1) may compactly be rewritten as

$$\dot{w} = \varphi(w) \ , \tag{3}$$

where $w = (z, x)$ is in $\mathbb{R}^{n_w}$ and $n_w = n_x + n_z$. We denote by $W(w_0, t) = (Z(w_0, t), X(w_0, t))$ the (unique) solution of (3) which goes through $w_0 = (z_0, x_0)$ in $\mathbb{R}^{n_w}$ at time $t = 0$. We assume throughout the paper the following assumptions.

Assumption 1: For all w_0 in $\mathbb{R}^{n_w}$ solutions $W(w_0, t)$ are defined for all positive times, i.e. the system (3) is *forward complete*.

Assumption 2: The manifold $\mathcal{Z} := \{w = (z, x) : z = 0\} \subset \mathbb{R}^{n_w}$ is invariant along the flow generated by (3) which is equivalent to

$$F(0, x) = 0 \qquad \forall x \in \mathbb{R}^{n_x} \ . \tag{4}$$

In our previous work [2], we have shown that the transverse uniform local exponential stability of the manifold $\mathcal{Z}$ can be fully characterized based on the stability property of the linearized dynamics of the z-subsystem. In this paper, following the approach taken in [1] we study a global version of this property, namely the global transverse exponential stability.

[2]A normally hyperbolic invariant manifold (NHIM) is a natural generalization of a hyperbolic fixed point. Roughly speaking, an invariant manifold is normally hyperbolic if, under the dynamics linearized about the manifold, the (positive or negative) growth rate of vectors transverse to the manifold dominates the growth rate of vectors tangent to the manifold. We refer to [32] for an exposition on this notion.

(Glob.)-TES *(Global transverse exponential stability)*
There exist a non decreasing continuous functions $k : \mathbb{R}_+ \to \mathbb{R}_+ \setminus \{0\}$ and a positive real number λ such that the inequality

$$|Z(w_0, t)| \leq k(|z_0|) \exp(-\lambda t) |z_0| \tag{5}$$

holds for all $w_0 = (z_0, x_0)$ in $\mathbb{R}^{n_w}$ and t in $\mathbb{R}_+$.

In other words, the manifold $\mathcal{Z}$ is globally exponentially stable for the system (1), uniformly in x.

Example 1: As a prototypical example in this paper, let us consider the following planar system defined on $\mathbb{R}^2$ which satisfies Assumptions 1 and 2:

$$\dot{z} = \phi(x) z \ , \ \dot{x} = \nu x \ , \ \phi(x) = -\lambda + x\sin(x) \ , \tag{6}$$

where $\lambda > 0$ and $\nu \in \mathbb{R}$. It can be checked, that its solutions are given by

$$W((z_0, x_0), t) = \left(e^{-\lambda t + \frac{\cos(x_0) - \cos(e^{\nu t} x_0)}{\nu}} z_0, e^{\nu t} x_0\right) \ \forall t \in \mathbb{R}.$$

This implies that Property (Glob.)-TES holds since we have for all (z_0, x_0) in $\mathbb{R}^2$

$$|Z((z_0, x_0), t)| \leq \exp\left(\frac{2}{\nu}\right) e^{-\lambda t} |z_0| \ .$$

△

In this paper, we show that the global transverse exponential stability can be characterized by the following two (almost) equivalent properties. Let us consider the following variational system obtained from (3) :

$$\dot{\widetilde{w}} = \frac{\partial \varphi}{\partial w}(w)\widetilde{w} \ , \ \dot{w} = \varphi(w) \ . \tag{7}$$

TES-VS *(Transverse exponential stability of the variational system (7))*
There exist a non decreasing function $\widetilde{k} : \mathbb{R}_+ \to \mathbb{R}_+ \setminus \{0\}$ and a non increasing function $\widetilde{\lambda} : \mathbb{R}_+ \to \mathbb{R}_+ \setminus \{0\}$ such that

$$\left|\widetilde{Z}(\widetilde{w}_0, w_0, t)\right| \leq \widetilde{k}(|z_0|) \exp\left(-\widetilde{\lambda}(|z_0|) t\right) |\widetilde{w}_0| \ , \tag{8}$$

for all $(\widetilde{w}_0, w_0)$ in $\mathbb{R}^{2n_w}$ where $\widetilde{Z}(\widetilde{w}_0, w_0, t)$ is the $\widetilde{z}$ component of the state $\widetilde{w} = (\widetilde{z}, \widetilde{x})$ of the variational system (7).

Namely the manifold $\tilde{\mathcal{Z}} := \{(\widetilde{z}, \widetilde{x}, z, x) : \widetilde{z} = 0\}$ is exponentially stable for the system (7) uniformly in x. Note however that the bound depends on $\widetilde{x}_0$ via $\widetilde{w}_0$ (and not only on $\widetilde{z}_0$).

Property TES-VS is a property on the variational system (7). It establishes that the $\widetilde{z}$ component converge exponentially toward zero uniformly with respect to x and it is independent of the dynamical behavior of $\widetilde{x}$.

(Glob.)-LMTE *(Global Lyapunov matrix transversal equation)*
There exist a non increasing continuous functions $\lambda_s : \mathbb{R}_+ \to \mathbb{R}_+$, a function $\underline{s} : \mathbb{R}^{n_w} \to \mathbb{R}_+ \setminus \{0\}$, a non

decreasing continuous function $\overline{s}:\mathbb{R}_+\to\mathbb{R}_+$ and a locally Lipschitz function $S:\mathbb{R}^{n_w}\to\mathbb{R}^{n_w\times n_w}$ such that

$$\underline{s}(w)\begin{bmatrix}\mathrm{I}_z & 0\\ 0 & 0\end{bmatrix}\leq S(w)\leq \overline{s}(|z|)\ \mathrm{I}_w\ , \tag{9}$$

and its Lie derivative along φ (see notations) exists and satisfies

$$L_\varphi S(w)\leq -\lambda_s(|z|)S(w)\ , \tag{10}$$

for all $w=(z,x)$ in $\mathbb{R}^{n_w}$.

Property (Glob.)-LMTE establishes the existence of a degenerate metric which is contracted by the flow. If, for all w in $\mathbb{R}^{n_w}$, S were positive definite and bounded then (10) implies that the flow generated by the system is contracting in the sense that the Riemannian distance associated to the metric S between any two trajectories decreases along the solutions. We note here that S may not be full rank everywhere and $\mathbb{R}^{n_w}$ endowed with this metric is not a Riemannian manifold. However, as will be shown later in Section VI, we can define a degenerate Riemannian metric which allows us to define a Lyapunov function characterizing the fact that the solution converges to the manifold $\mathcal{Z}$.

Example 1 (cont'd): When we consider the variational system obtained from (6) to the tangent bundle, we have

$$\dot{\widetilde{w}}=\begin{bmatrix}\phi(e^{\nu t}x_0) & \phi'(e^{\nu t}x_0)Z(w_0,t)\\ 0 & \nu\end{bmatrix}\widetilde{w}\ .$$

This implies that

$$\begin{aligned}&\widetilde{Z}(\widetilde{w}_0,w_0,t)\\ &=e^{\int_0^t\phi(e^{\nu s}x_0)ds}\widetilde{z}_0\\ &\qquad+\int_0^t e^{\int_s^t\phi(e^{\nu h}x_0)dh}\phi'(e^{\nu s}x_0)Z(w_0,s)e^{\nu s}\widetilde{x}_0ds\ ,\\ &=e^{\int_0^t\phi(e^{\nu s}x_0)ds}\left[\widetilde{z}_0+\int_0^t\phi'(e^{\nu s}x_0)e^{\nu s}z_0\widetilde{x}_0ds\right]\ .\end{aligned}$$

Hence if $x_0\neq 0$,

$$\widetilde{Z}(\widetilde{w}_0,w_0,t)=\\ e^{\int_0^t\phi(e^{\nu s}x_0)ds}\left[\widetilde{z}_0+\frac{\phi(e^{\nu t}x_0)-\phi(x_0)}{\nu}\frac{z_0\widetilde{x}_0}{x_0}\right].$$

Using the previously defined ϕ, it follows that

$$\widetilde{Z}(\widetilde{w}_0,w_0,t)=e^{\frac{\cos(x_0)-\cos(e^{\nu t}x_0)}{\nu}}\Big[e^{-\lambda t}\widetilde{z}_0\\ +\frac{e^{(\nu-\lambda)t}\sin(e^{\nu t}x_0)-e^{-\lambda t}\sin(x_0)}{\nu}z_0\widetilde{x}_0\Big]\ .$$

Two cases may be distinguished as follows.

- If $\lambda>\nu$ then $\widetilde{Z}(\widetilde{w}_0,w_0,t)$ converges exponentially toward zero for all $z_0,x_0,\widetilde{x}_0,\widetilde{z}_0$ and property TES-VS holds.
- if $\lambda\leq\nu$ then it can be checked that $\widetilde{Z}(\widetilde{w}_0,w_0,t)$ doesn't converge to zero. This is the case if $\lambda=\nu$. Moreover, when $\lambda<\nu$ it may be unbounded. For instance, when we take $z_0=1$, $x_0=1$, $\widetilde{x}_0=1$. Hence Property TES-VS doesn't hold.

The purpose of this paper is to show that what has been obtained in Example 1 is general. Indeed, it will be shown that these three properties are (almost) equivalent when the expansion rate in the $\mathcal{Z}$ manifold is smaller than the convergence rate to the $\mathcal{Z}$ manifold ($\lambda>\nu$ in the illustrative example). This together with some mild conditions on the bounds on the derivatives of the vector field φ, we establish that Property (Glob.)-TES implies TES-VS in Section IV. Section V is devoted to show that (Glob.)-TES and TES-VS imply (Glob.)-LMTE . Finally, Section VI contains the proof that property (Glob.)-LMTE implies the existence of Lyapunov function which characterizes the stability property (Glob.)-TES .

The following section discusses the relationship with existing results available in the literature.

III. Link with existing studies

A. Case in which there is no x dynamics

In the particular case in which there are no x-dynamics the system (1) becomes simply

$$\dot{z}=F(z)\ ,\ F(0)=0\ ,\ z\in\mathbb{R}^{n_z}\ . \tag{11}$$

In that case, the three properties introduced are drastically simplified and become :

- (Glob.)-TES becomes the local exponential stability and the global asymptotic stability of the origin.
- TES-VS becomes the fact that the $\widetilde{z}$ components of the solutions to following system
$$\dot{\widetilde{z}}=\frac{\partial F}{\partial z}(z)\widetilde{z}\ ,\ \dot{z}=F(z)\ ,$$
converge (exponentially) toward zero.
- (Glob.)-LMTE boils down to the existence of (non uniformly) contracting Riemannian metric.

It has been shown in [1] that in this particular case these 3 properties are equivalent. The results presented in this paper are direct extension of this work to the case in which the attractor is not an equilibrium but a (simple) linear manifold.

In this particular case, this equivalence property can also be obtained following the route of [8] on normal hyperbolicity. Indeed, assuming completeness of the trajectories in backward time, and employing [18], it is possible to show that a dynamical system admitting a locally exponentially stable and globally asymptotically stable equilibrium can be transformed via a global diffeomorphism into a linear system. With this, so in the case of no x-dynamics and backward completeness, equivalence of the three properties follows simply by applying Lyapunov methods for linear systems.

Actually global linearization by diffeomorphism is proved in [5] also for the case when there are x-dynamics or more precisely when the attractor is not a point but a compact manifold, but still requiring backward completeness. Unfortunately we do not know if it is possible or not to obtain this linearization result without compactness and (backward) completeness. These two assumptions are not made here and we follow a different route to establish the equivalence of the three properties (Glob.)-TES , TES-VS and (Glob.)-LMTE .

B. Relationship with the local properties in [2]

As briefly discussed before, we have presented local versions of the three properties given above in our previous work in [2].

For instance, instead of the global transverse exponential stability (i.e. Property (Glob.)-TES), we have considered the following property.

(Local) Transverse exponential stability *There exist positive real numbers $r > 0$, $k_0 > 0$ and $\lambda > 0$ such that we have, for all $w_0 = (z_0, x_0)$ in $\mathcal{B}_z(r) \times \mathbb{R}^{n_x}$ and for all t in $\mathbb{R}_+$,*

$$|Z(w_0,t)| \leq k_0 \exp(-\lambda t)|z_0| \ . \tag{12}$$

Using some technical assumptions (related to the bounds on derivatives of the vector field φ), we have shown in [2] that this property is equivalent to the following two properties.

ES-TLS *(Exponential stability for the locally transversally linearized system)*
There exist real numbers $\widetilde{k} > 0$ and $\tilde{\lambda} > 0$ such that, for the linear part of the variational system

$$\dot{\tilde{z}} = \frac{\partial F}{\partial z}(0,x)\widetilde{z} \ , \quad \dot{x} = G_0(x) := G(0,x) \ , \tag{13}$$

any solution $(\widetilde{Z}(\widetilde{z}_0, x_0, t), X_0(x_0,t))$ satisfies, for all $(\widetilde{z}_0, x_0, t)$ in $\mathbb{R}^{n_z} \times \mathbb{R}^{n_x} \times \mathbb{R}_+$,

$$|\widetilde{Z}(\widetilde{z}_0, x_0, t)| \leq \widetilde{k} \exp(-\tilde{\lambda} t)|\widetilde{z}_0| \ . \tag{14}$$

(Loc.)-LMTE *(Local Lyapunov matrix transversal equation)*
For all positive definite matrix Q, there exist a continuous function $P_\ell : \mathbb{R}^{n_x} \to \mathbb{R}^{n_z \times n_z}$ and positive real numbers $\underline{p}_0 > 0$ and $\overline{p}_0 > 0$ such that P_ℓ has a derivative $\mathfrak{d}_{G_0} P_\ell$ along the vector field G_0 in (13) and we have, for all x in $\mathbb{R}^{n_x}$,

$$\mathfrak{d}_{G_0}\{P_\ell(x)\} + P_\ell(x)\frac{\partial F}{\partial z}(0,x) + \frac{\partial F}{\partial z}(0,\widetilde{x})^\top P_\ell(x) \leq -Q \tag{15}$$

$$\underline{p}_0 I \leq P_\ell(x) \leq \overline{p}_0 I \ . \tag{16}$$

It is possible to show that each of these three properties are local version of the properties introduced in the previous section. In particular, the property ES-TLS (i.e. the exponential stability of the $\widetilde{z}$ component of system (13)) is induced by TES-VS (i.e. the exponential stability of the $\widetilde{z}$ component of system (7)). Indeed, consider solutions to system (7) with initial condition $(w_0, \widetilde{w}_0)$ with w_0 in $\mathcal{Z}$ and $\widetilde{x}_0 = 0$. Since $F(0,x) = 0$ for all x, it implies that $\frac{\partial F}{\partial x}(0,x) = 0$ and consequently, solutions of (7) initiated from $(w_0, \widetilde{w}_0)$ with w_0 in $\mathcal{Z}$ are solutions of (13). Consequently, for such solution with $\widetilde{x}_0 = 0$, (5) yields

$$\dot{\widetilde{Z}}(w_0, \widetilde{w}_0, t) \leq \widetilde{k}(0) \exp(-\widetilde{\lambda} t)|\widetilde{z}_0|$$

and consequently (14) holds with $\tilde{k}_0 = \tilde{k}(0)$, i.e. ES-TLS holds.

As shown in [2], (Loc.)-LMTE is a characterization of the transverse local exponential stability. Indeed, given P_ℓ solution to (15) it is shown in in [2] that the function $(z,x) \mapsto z^\top P_\ell(x) z$ is a local Lyapunov function. Moreover, it is possible to establish a direct link between P_ℓ, solution of (15) and the S solution of the global equation (10).

Proposition 1: Assume there exists a positive real number $\underline{p}$ such that Property (Glob.)-LMTE holds with S decomposed as

$$S(z,x) = \begin{bmatrix} P(z,x) & Q(z,x) \\ Q(z,x)^\top & R(z,x) \end{bmatrix} \ . \tag{17}$$

and with $\underline{p} \leq \underline{s}(0,x)$ then $P_\ell(x)$, the Schur complement of $S(0,x)$, i.e.

$$P_\ell(x) = P(0,x) - Q(0,x)R(0,x)^g Q(0,x)^\top \ ,$$

where $R(x)^g$ in $\mathbb{R}^{n_x \times n_x}$ is any symmetric generalized inverse matrix[3], defines a function $P_\ell : \mathbb{R}^{n_x} \to \mathbb{R}^{n_z \times n_z}$ which satisfies (16) for some positive real numbers $\underline{p_\ell}$ and $\overline{p_\ell}$. Moreover, P_ℓ has a derivative $\mathfrak{d}_{G_0} P_\ell$ along G_0 which satisfies (15).

In other words, putting aside the continuity requirement on the matrix function P_ℓ, (Glob.)-LMTE implies (Loc.)-LMTE as introduced in [2]. Note that in the particular case in which $Q(0,x) = 0$ for all x then $P_\ell(x) = P(0,x)$ and consequently P_ℓ has the same regularity as S. As pointed out later in Remark 2 this is typically the property which is obtained in the proof of Proposition 3 which establishes property (Glob.)-LMTE assuming (Glob.)-TES and the transverse exponential stability of the lifted system (i.e., TES-VS).

Proof : First of all, for any x in $\mathbb{R}^{n_x}$, the matrix $S(0,x)$ being positive semi-definite (for any generalized inverse $R^g(0,x)$) it can be shown that

$$P_\ell(x) \geq 0 \ , \ (\mathrm{I}_x - R(0,x)R(0,x)^g)Q(0,x)^\top = 0 \ . \tag{18}$$

Indeed, the second equation may be obtained as follows. First of all $S(0,x)$ being positive semi-definite, for each $\widetilde{x}$ in $\mathbb{R}^{n_x}$ such that $R\widetilde{x} = 0$, we have $Q(0,x)^\top \widetilde{x} = 0$. Otherwise, picking $\widetilde{z} = -\lambda Q(0,x)^\top \widetilde{x}$ and letting λ go to zero is such that $\begin{bmatrix} \widetilde{z}^\top & \widetilde{x}^\top \end{bmatrix} S(0,x) \begin{bmatrix} \widetilde{z} \\ \widetilde{x} \end{bmatrix} < 0$. Assume now that there exists $\widetilde{x}_2$ such that $Q(\mathrm{I}_x - R^g R)\widetilde{x}_2 \neq 0$. Note that from the previous statement, this implies that $R(\mathrm{I}_x - R^g R)\widetilde{x}_2 \neq 0$. However, since $RR^gR = R$, this is impossible. Consequently (18) holds true.

Now, it can be shown that P_ℓ is uniquely defined (but it may be non continuous). Indeed, let R_1^g and R_2^g be two generalized inverse of R. Then, employing (18), this yields,

$$\begin{aligned} QR_1^g Q^\top &= QR_1^g RR_2^g Q^\top \\ &= QR_2^g Q^\top \end{aligned}$$

Hence, P_ℓ is uniquely defined.

Hence, we have that

$$P_\ell(x) \ = \ \begin{bmatrix} \mathrm{I}_z & -Q(0,x)R^g(0,x) \end{bmatrix} S(0,x) \begin{bmatrix} \mathrm{I}_z \\ -R^g(0,x)Q(0,x)^\top \end{bmatrix}, \tag{19}$$

which implies that

$$\begin{aligned} &\begin{bmatrix} \widetilde{z}^\top & \widetilde{x}^\top \end{bmatrix} S(0,x) \begin{bmatrix} \widetilde{z} \\ \widetilde{x} \end{bmatrix} \\ &\quad = \widetilde{z}^\top P_\ell(x)\widetilde{z} + \begin{bmatrix} \widetilde{x}^\top + \widetilde{z}^\top Q R^g \end{bmatrix} R \begin{bmatrix} \widetilde{x} + R^g Q^\top \widetilde{z} \end{bmatrix}, \end{aligned} \tag{20}$$

and

$$\widetilde{z}^\top P_\ell(x)\widetilde{z} = \inf_{\widetilde{x} \in \mathbb{R}^{n_x}} \left\{ \begin{bmatrix} \widetilde{z}^\top & \widetilde{x}^\top \end{bmatrix} S(0,x) \begin{bmatrix} \widetilde{z} \\ \widetilde{x} \end{bmatrix} \right\} \tag{21}$$

[3] Given a square matrix R, a generalized inverse is any matrix which satisfies $R^g R R^g = R^g$ and $RR^gR = R$.

hold for all $\widetilde{z}$. By letting

$$\widetilde{x} = -R(x)^g Q(x)\widetilde{z} \,, \tag{22}$$

it follows from (20) and (9) that

$$\underline{p} < \underline{s}(0,x)\, \mathrm{I}_z \leq P_\ell(x) \,.$$

In particular, when $\widetilde{x} = 0$, $P_\ell(x) \leq P(0,x) \leq \overline{s}(0)\, \mathrm{I}_z$. Hence, (16) holds with $\underline{p_\ell} = \underline{p}$ and $\overline{p_\ell} = \overline{s}(0)$.

Take any $\widetilde{z}$ in $\mathbb{R}^{n_z}$. We have by definition

$$\widetilde{z}^\top \mathfrak{d}_{G_0} P_0(x)\widetilde{z} = \limsup_{h\searrow 0} \frac{\widetilde{z}P_\ell(X_0(x,h))\widetilde{z} - \widetilde{z}P_\ell(x)\widetilde{z}}{h} \,.$$

But by letting $\widetilde{x}$ as in (22), equation (20) implies that

$$\widetilde{z}P_\ell(x)\widetilde{z} = \begin{bmatrix}\widetilde{z}^\top & \widetilde{x}^\top\end{bmatrix} S(0,x) \begin{bmatrix}\widetilde{z}\\ \widetilde{x}\end{bmatrix} \,, \tag{23}$$

whereas (21) gives

$$\widetilde{z}P_\ell(X_0(x,h))\widetilde{z} \leq \begin{bmatrix}\widetilde{z}^\top & \widetilde{x}^\top\end{bmatrix} S(0,X_0(x,h)) \begin{bmatrix}\widetilde{z}\\ \widetilde{x}\end{bmatrix} \,.$$

Together with (10), it follows then that

$$\begin{aligned}&\widetilde{z}^\top \mathfrak{d}_{G_0} P_\ell(x)\widetilde{z}\\ &\leq \begin{bmatrix}\widetilde{z}^\top & \widetilde{x}^\top\end{bmatrix} \limsup_{h\searrow 0} \frac{S(0,X_0(x,h)) - S(0,x)}{h} \begin{bmatrix}\widetilde{z}\\ \widetilde{x}\end{bmatrix} .\end{aligned} \tag{24}$$

Here we note that (4) implies that $Z((0,x),h) = 0$ for any $h \geq 0$ and therefore $(0, X_0(x,h)) = W((0,x),h)$. Thus the inequality (24) becomes

$$\begin{aligned}&\widetilde{z}^\top \mathfrak{d}_{G_0} P_\ell(x)\widetilde{z}\\ &\quad\leq \begin{bmatrix}\widetilde{z}^\top & \widetilde{x}^\top\end{bmatrix} \limsup_{h\searrow 0} \frac{S(W((0,x),h)) - S(0,x)}{h} \begin{bmatrix}\widetilde{z}\\ \widetilde{x}\end{bmatrix}\\ &\quad\leq \begin{bmatrix}\widetilde{z}^\top & \widetilde{x}^\top\end{bmatrix} \mathfrak{d}_\varphi S(0,x) \begin{bmatrix}\widetilde{z}\\ \widetilde{x}\end{bmatrix}\end{aligned}$$

On the other hand, equations (22) and (4) imply that

$$\begin{aligned}&\begin{bmatrix}\widetilde{z}^\top & \widetilde{x}^\top\end{bmatrix} S(0,x) \begin{bmatrix}\frac{\partial F}{\partial z}(0,x) & \frac{\partial F}{\partial x}(0,x)\\ \frac{\partial G}{\partial z}(0,x) & \frac{\partial G}{\partial x}(0,x)\end{bmatrix} \begin{bmatrix}\widetilde{z}\\ \widetilde{x}\end{bmatrix}\\ &\qquad= \begin{bmatrix}\widetilde{z}^\top P_\ell(x) & 0\end{bmatrix} \begin{bmatrix}\frac{\partial F}{\partial z}(0,x) & 0\\ \frac{\partial G}{\partial z}(0,x) & \frac{\partial G}{\partial x}(0,x)\end{bmatrix} \begin{bmatrix}\widetilde{z}\\ \widetilde{x}\end{bmatrix}\\ &\qquad= \widetilde{z}^\top P_\ell(x) \frac{\partial F}{\partial z}(0,x)\widetilde{z}\end{aligned}$$

It follows from this equality together with (10) and (23) that

$$\begin{aligned}&\widetilde{z}^\top \left(\mathfrak{d}_{G_0} P_\ell(x) + P_\ell(x)\frac{\partial F}{\partial z}(0,x) + \frac{\partial F}{\partial z}(0,x)^\top P_\ell(x)\right)\widetilde{z}\\ &\quad\leq \begin{bmatrix}\widetilde{z}^\top & \widetilde{x}^\top\end{bmatrix} L_\varphi S(0,x) \begin{bmatrix}\widetilde{z}\\ \widetilde{x}\end{bmatrix}\\ &\quad\leq -\lambda_s(0) \begin{bmatrix}\widetilde{z}^\top & \widetilde{x}^\top\end{bmatrix} S(0,x) \begin{bmatrix}\widetilde{z}\\ \widetilde{x}\end{bmatrix}\\ &\quad\leq -\lambda_s(0)\widetilde{z}^\top P_\ell(x)\widetilde{z} \,.\end{aligned}$$

This shows that (15) holds.

IV. (GLOB.)-TES "⇒" TES-VS

A. Statement of the result

In [2, Proposition 1], it was shown that the exponential stability of the locally transversally linearized system (Prop. ES-TLS) was implied by the local transverse exponential stability (i.e. equation (12)). In this section our aim is to show that this implication may also be true for the global version of these two properties.

In the spirit of Lyapunov first method, we have the following result

Proposition 2: If Property (Glob.)-TES holds and there exists a non decreasing function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ and a positive real number ν such that, for all $w = (z,x)$ in $\mathbb{R}^{n_w}$,

$$\left|\frac{\partial \varphi}{\partial w}(w)\right| \leq \mu(|z|) \;,\; \left|\frac{\partial^2 \varphi}{\partial w^2}(w)\right| \leq \mu(|z|) \tag{25}$$

and

$$\left|\frac{\partial G}{\partial x}(0,x)\right| < \nu < \lambda \tag{26}$$

hold then Property TES-VS holds.

The proof of this proposition is given in the next subsection. Let us first emphasize that the property (26) expresses a relationship between the expansion of the x component on the manifold $\{(z,x) \in \mathbb{R}^n, z = 0\}$ and the asymptotic convergence to zero of the z component. This is exactly the property which has been discussed in the illustrative Example 1. Indeed, λ is the convergence rate of the z component whereas ν expresses an estimation of the expansion rate on the manifold. More precisely, given $(0,x_0)$ an initial condition on the manifold, it follows that

$$\frac{d}{dt}\left|X((0,x_0),t)\right| \leq \nu \left|X((0,x_0),t)\right| + |G(0,0)|$$

which establishes that

$$\begin{aligned}|X((0,x_0),t)| &\leq \exp(\nu t)[|x_0| + \frac{1-\exp(-\nu t)}{\nu}|G(0,0)|] \,,\\ &\leq \exp(\nu t)\left[|x_0| + \frac{|G(0,0)|}{\nu}\right] \,.\end{aligned}$$

This assumption implies that the manifold $\{(z,x) \in \mathbb{R}^n, z = 0\}$ is normal hyperbolic (see also [6]).

Note that this restriction on the expansion in the manifold is trivially removed in the particular case in which $\frac{\partial F}{\partial z}(z,x) = 0$. Indeed, in this case, if Property(Glob.)-TES holds then Property TES-VS holds. This is trivially the case in the linear context since for linear systems Property (Glob.)-TES implies that F doesn't depend on x.

B. Proof of Proposition 2

The proof is decomposed in two steps. In the first step, we show that the fundamental matrix of the $\widetilde{z}$ component of a solution to the system

$$\dot{\widetilde{z}} = \frac{\partial F}{\partial z}(w)\widetilde{z} \;, \dot{w} = \varphi(w) \tag{27}$$

converges exponentially to zero. In the second step, we show the result by expressing solutions to system (7) employing the former fundamental matrix.

Proof : **First step: The fundamental matrix of the autonomous system is exponentially decreasing.**

We define the fundamental matrix of the $\tilde{z}$ component of a solution to the system (27) as the $\mathbb{R}^{n_z \times n_z}$ matrix function solution to

$$\frac{\partial \Phi_{\tilde{z}}}{\partial t}(w,t) = \frac{\partial F}{\partial z}(W(w,t))\, \Phi_{\tilde{z}}(w,t) \ , \qquad \Phi_{\tilde{z}}(w,0) = \mathrm{I}_z \ .$$

We want to evaluate $\sup_{|\tilde{z}_0| = \tilde{r}} \frac{|\Phi_{\tilde{z}}(z_0,x_0,t)\tilde{z}_0|}{|\tilde{z}_0|}$ where $\tilde{r} > 0$ is any positive real number. Since $\Phi_{\tilde{z}}(z_0,x_0,t)\tilde{z}_0$ is the $\tilde{z}$-component of a solution initiated from $(\tilde{z}_0, w_0) = (\tilde{z}_0, z_0, x_0)$ of the partially linear system (27), the idea is to approximate it with the z-component of $Z((\tilde{z}_0, x_0), t)$ of the nonlinear system (1). For this approximation to be appropriate, i.e. for the linearization to be close to the nonlinear function, $\tilde{z}_0$, i.e. $\tilde{r}$, should be small. Also such an approximation may not be good for all positive times t. To overcome this problem, after some time $\mathfrak{s}$, we reinitialize the solution of the non linear system at the current value of the linear one. Specifically, we approximate, $\Phi_{\tilde{z}}(z_0,x_0,t+i\mathfrak{s})\tilde{z}_0$, on the time interval $[0,\mathfrak{s})$, by $Z(\tilde{z}_i, x_i, t)$ where

$$\left.\begin{aligned} \tilde{z}_i &= \Phi_{\tilde{z}}(z_0,x_0,i\mathfrak{s})\tilde{z}_0 \\ (z_i,x_i) &= (Z(z_0,x_0,i\mathfrak{s})\,,\ X(z_0,x_0,i\mathfrak{s})) \end{aligned}\right\} \tag{28}$$

The expressions here make sense for any integer i because of the forward completeness assumption.

To study the relation between these solutions, we start with some estimations. Given arbitrary $\tilde{z}$ in $\mathbb{R}^{n_z}$, (z_a,x_a) in $\mathbb{R}^{n_w}$ and (z_b,x_b) in $\mathbb{R}^{n_w}$ and let $y = z_b - \tilde{z}$, we have

$$F(z_b,x_b) - \frac{\partial F}{\partial z}(z_a,x_a)\tilde{z} = \frac{\partial F}{\partial z}(z_a,x_a)[z_b - \tilde{z}] + \Delta(z_a,x_a,z_b,x_b)$$

where

$$\begin{aligned} \Delta(z_a,x_a,z_b,x_b) &= F(z_b,x_b) - \frac{\partial F}{\partial z}(z_a,x_a)z_b \\ &= [F(z_b,x_b) - F(z_b,x_a)] \\ &\quad + \left[F(z_b,x_a) - \frac{\partial F}{\partial z}(0,x_a)z_b\right] \\ &\quad + \left[\frac{\partial F}{\partial z}(0,x_a) - \frac{\partial F}{\partial z}(z_a,x_a)\right] z_b \end{aligned}$$

Since $F(0,x) = 0$ and with the Hadamard's lemma (see [24, Page 17]), (4) and (25), we obtain the existence of a non decreasing function $c : \mathbb{R}_+ \to \mathbb{R}_+$ (depending on the function μ)[4]such that, for all (z_a,x_a) in $\mathbb{R}^{n_w}$ and (z_b,x_b) in $\mathbb{R}^{n_w}$,

$$\begin{aligned} &|\Delta(z_a,x_a,z_b,x_b)| \\ &\qquad \le c(|z_a|+|z_b|)\left[|z_b|^2 + |z_b||z_a| + |z_b||x_a - x_b|\right] \ . \end{aligned}$$

This together with (25) implies that

$$\begin{aligned} &\left|F(z_b,x_b) - \frac{\partial F}{\partial z}(z_a,x_a)\tilde{z}\right| \le c(|z_a|+|z_b|) \\ &\qquad \times \left[|z_b - \tilde{z}| + |z_b|^2 + |z_b||z_a| + |z_b||x_a - x_b|\right] \ , \end{aligned} \tag{29}$$

holds for all (z_a,x_a) in $\mathbb{R}^{n_w}$ and (z_b,x_b) in $\mathbb{R}^{n_w}$. Similarly we obtain also that

$$\begin{aligned} &|G(z_a,x_a) - G(z_b,x_b)| \\ &\le |G(z_a,x_b) - G(z_b,x_b)| \ + \ |G(z_a,x_a) - G(z_a,x_b)| \ , \\ &\le c(|z_a|+|z_b|)|z_a - z_b| + c(|z_a|)|x_a - x_b| \ , \\ &\le c(|z_a|+|z_b|)\left[|z_a| + |z_b| + |x_a - x_b|\right] \ , \end{aligned} \tag{30}$$

[4]In the following the notation c is used generically without distinction.

holds for all (z_a,x_a) in $\mathbb{R}^{n_w}$ and (z_b,x_b) in $\mathbb{R}^{n_w}$. In the following we will substitute (z_a,x_a) by $(Z(\tilde{z}_i,x_i,s), X(\tilde{z}_i,x_i,s))$ and similarly $(Z(z_i,x_i,s), X(z_i,x_i,s))$ will replace (z_b,x_b). Note at this point that, because of (5), if $\tilde{z}_i$ is in $B_z(|z_0|)$, then $Z(\tilde{z}_i,x_i,s)$ is in $B_z(k(|z_0|)|z_0|)$ for all positive times s. In this case, as (28) implies that

$$Z(z_i,x_i,s) = Z(z_0,x_0,s+i\mathfrak{s})$$

we have that

$$c(|Z(\tilde{z}_i,x_i,s)| + |Z(z_i,x_i,s)|) \le c(2k(|z_0|)|z_0|) := \tilde{c}(|z_0|)$$

holds for all $\mathfrak{s}$, s and i. Now, for each integer i, we define the following functions on $[0,\mathfrak{s}]$

$$\begin{aligned} Y_i(s) &= |Z(\tilde{z}_i,x_i,s) - \Phi_{\tilde{z}}(z_0,x_0,s+i\mathfrak{s})\tilde{z}_0| \ , \\ D_i(s) &= |X(\tilde{z}_i,x_i,s) - X(z_0,x_0,s+i\mathfrak{s})| \ , \\ &= |X(\tilde{z}_i,x_i,s) - X(z_i,x_i,s)| \ . \end{aligned}$$

Note that we have $Y_i(0) = D_i(0) = 0$. By integration of the differential inequality obtained from (30), and using (5), we get, for each integer i such that $\tilde{z}_i$ is in $B_z(|z_0|)$ and for all s in $[0,\mathfrak{s}]$, that

$$\begin{aligned} &D_i(s) \le \tilde{c}(|z_0|) \int_0^s e^{\tilde{c}(|z_0|)(s-\sigma)} \\ &\times \Big[|Z(\tilde{z}_i,x_i,\sigma)| + |Z(z_i,x_i,\sigma)|\Big] d\sigma \ . \\ &\qquad \le \tilde{c}(|z_0|) \int_0^s e^{\tilde{c}(|z_0|)(s-\sigma)} \\ &\times \Big[k(|\tilde{z}_i|)\exp(-\lambda\sigma)|\tilde{z}_i| + k(|z_0|)\exp(-\lambda(i\mathfrak{s}+\sigma))|z_0|\Big] d\sigma \ , \\ &\qquad \le \tilde{c}(|z_0|) e^{\tilde{c}(|z_0|)s} \frac{1 - \exp(-(\tilde{c}(|z_0|)+\lambda)s)}{\tilde{c}(|z_0|)+\lambda} k(|z_0|) \\ &\times \Big[|\tilde{z}_i| + \exp(-\lambda i\mathfrak{s})|z_0|\Big] \ . \end{aligned}$$

Similarly, using (29),

$$\begin{aligned} Y_i(s) \le\ & \tilde{c}(|z_0|) \int_0^s \exp(\tilde{c}(|z_0|)(s-\sigma)) \\ & \times \Big(|Z(\tilde{z}_i,x_i,\sigma)|^2 + |Z(\tilde{z}_i,x_i,\sigma)||Z(z_i,x_i,\sigma)| \\ & \qquad + |Z(\tilde{z}_i,x_i,\sigma)|D_i(\sigma)\Big) d\sigma \ , \\ \le\ & \gamma(s,|z_0|)\ \left[|\tilde{z}_i|^2 + \exp(-\lambda i\mathfrak{s})|z_0||\tilde{z}_i|\right] \ , \end{aligned}$$

holds for all s in $[0,\mathfrak{s}]$ where

$$\begin{aligned} &\gamma(s,z_0) = \tilde{c}(|z_0|)k(|z_0|)^2 \int_0^s e^{\tilde{c}(|z_0|)(s-\sigma)} e^{-2\lambda\sigma} \times \\ &\times \left(1 + \tilde{c}(|z_0|) e^{(\tilde{c}(|z_0|)+\lambda\sigma} \frac{1 - e^{-(\tilde{c}(|z_0|)+\lambda)\sigma}}{\tilde{c}(|z_0|)+\lambda}\right) d\sigma \ . \end{aligned}$$

Hence it follows then that together with (5) and the definition of $Y_i(s)$, we have obtained that, for all i, if we have $\tilde{z}_\ell$ in $B_z(|z_0|)$ for all ℓ in $\{0,\ldots,i\}$, then, we have also, for all s in $[0,\mathfrak{s}]$,

$$\begin{aligned} |\Phi_{\tilde{z}}(z_0,x_0,s+i\mathfrak{s})\tilde{z}_0| &\le |Z(\tilde{z}_i,x_i,s)| + |Y_i(s)| \ , \\ &\le k(|z_0|)e^{-\lambda s}|\tilde{z}_i| + \\ &\qquad \gamma(s,|z_0|)\left[|\tilde{z}_i| + e^{-\lambda i\mathfrak{s}}|z_0|\right]|\tilde{z}_i| \ . \end{aligned} \tag{31}$$

On the other hand, (25) implies

$$\frac{\mathsf{d}}{\mathsf{d}t}|\tilde{z}| \le \tilde{c}(|z_0|)|\tilde{z}| \ .$$

Therefore, for all i and with no restriction on $\tilde{z}_\ell$ for ℓ in $\{0,\ldots,i\}$, we have for all s in $[0,\mathfrak{s}]$

$$|\Phi_{\tilde{z}}(z_0,x_0,s+i\mathfrak{s})\tilde{z}_0| \le e^{\tilde{c}(|z_0|)s}|\tilde{z}_i| \le e^{\tilde{c}(|z_0|)[s+i\mathfrak{s}]}|\tilde{z}_0| \ . \tag{32}$$

Let us take $\mathfrak{s}$ large enough such that

$$\mathfrak{s} > \frac{\log(3k(|z_0|))}{\lambda} .$$

Then by letting $\lambda_1 = \lambda - \frac{\log(3k(|z_0|))}{\mathfrak{s}}$, we have that

$$k(|z_0|)e^{-\lambda \mathfrak{s}} \le \frac{e^{-\lambda_1 \mathfrak{s}}}{3}.$$

Subsequently, we take a positive integer j_0 large enough so that

$$\gamma(\mathfrak{s}, |z_0|)e^{-\lambda j_0 \mathfrak{s}}|z_0| \le \frac{e^{-\lambda_1 \mathfrak{s}}}{3} .$$

Finally we pick $\widetilde{r}$ small enough such that

$$e^{\tilde{c}(|z_0|)j_0\mathfrak{s}}\widetilde{r} \le |z_0| \quad \text{and} \quad \gamma(\mathfrak{s}, |z_0|) \le \frac{e^{-\lambda_1 \mathfrak{s}}}{3}\frac{1}{\widetilde{r}e^{\tilde{c}(|z_0|)j_0\mathfrak{s}}} .$$

hold where $\mathfrak{s}$, $\widetilde{r}$ and j_0 depend on $|z_0|$. In this case, it follows from (32) with $s = 0$ that

$$|\widetilde{z}_\ell| \le \left[e^{\tilde{c}(|z_0|)j_0\mathfrak{s}}\widetilde{r}\right]\frac{|\widetilde{z}_0|}{\widetilde{r}} \qquad \forall \ell \le j_0.$$

Similarly, it follows from (31) that

$$\begin{aligned} |\widetilde{z}_{i+1}| &= |\Phi_{\widetilde{z}}(z_0, x_0, [i+1]\mathfrak{s})\widetilde{z}_0| && (33)\\ &\le \frac{e^{-\lambda_1 \mathfrak{s}}}{3}\left[1 + \frac{|\widetilde{z}_i|}{\widetilde{r}e^{\tilde{c}(|z_0|)j_0\mathfrak{s}}} + e^{-\lambda[i-j_0]\mathfrak{s}}\right]|\widetilde{z}_i| && (34)\end{aligned}$$

holds for all $i \ge j_0$ for which we have $\widetilde{z}_\ell$ in $B_z(|z_0|)$ for all ℓ in $\{0, \ldots, i\}$. Finally, using (32), we can deduce that

$$|\Phi_{\widetilde{z}}(z_0, x_0, s + i\mathfrak{s})\widetilde{z}_0| \le e^{\tilde{c}(|z_0|)\mathfrak{s}}|\widetilde{z}_i|. \qquad (35)$$

In the case where $|\widetilde{z}_0|$ is smaller than $\widetilde{r}$, we have obtained successively that

1. $\widetilde{z}_\ell$ is smaller than $|z_0|$ for each ℓ smaller than j_0,
2. $|\widetilde{z}_{j_0}| \le \widetilde{r}e^{\tilde{c}(|z_0|)j_0\mathfrak{s}} \le |z_0|$,
3. $|\widetilde{z}_{j_0+1}| \le e^{-\lambda_1\mathfrak{s}}|\widetilde{z}_{j_0}| \le \widetilde{r}e^{\tilde{c}(|z_0|)j_0\mathfrak{s}} \le |z_0|$.

which, by induction, we get

$$|\widetilde{z}_i| \le \widetilde{r}e^{\tilde{c}(|z_0|)j_0\mathfrak{s}} \le |z_0| \qquad \forall i .$$

Hence (35) implies that

$$|\Phi_{\widetilde{z}}(z_0, x_0, t)\widetilde{z}_0| \le e^{\tilde{c}(|z_0|)[j_0+1]\mathfrak{s}}|\widetilde{z}_0| \qquad \forall t \in [0, j_0\mathfrak{s}] .$$

Also (33) and (35) imply that

$$|\Phi_{\widetilde{z}}(z_0, x_0, t)\widetilde{z}_0| \le e^{[\tilde{c}(|z_0|)+\lambda_1]\mathfrak{s}}e^{-\lambda_1 t}|\widetilde{z}_0| \qquad \forall t \ge j_0\mathfrak{s} .$$

Consequently, we obtain that

$$|\Phi_{\widetilde{z}}(z_0, x_0, t)| \le \widetilde{k}_a(|z_0|)e^{-\lambda_1 t} \qquad (36)$$

with $\widetilde{k}_a(|z_0|) = e^{[\tilde{c}(|z_0|)+\lambda_1](j_0+1)\mathfrak{s}}$.

Second step: Showing the upper bound. Let $\widetilde{w}_0 = (\widetilde{z}_0, \widetilde{x}_0)$ be in $\mathbb{R}^{n_w}$. Note that for all $t \ge 0$, we have

$$\begin{aligned}\widetilde{Z}(\widetilde{w}_0, w_0, t) &= \Phi_{\widetilde{z}}(w_0, t)\widetilde{z}_0\\ &+ \int_0^t \Phi_{\widetilde{z}}(w_0, t-s)\frac{\partial F}{\partial x}(W(w_0, s))\widetilde{X}(\widetilde{w}_0, w_0, s)ds . \quad (37)\end{aligned}$$

Using Hadamard's lemma (see [24, Page 17]) and using equations (4) and (25), there exists a non decreasing continuous function $c : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\left|\frac{\partial F}{\partial x}(z, x)\right| \le c(|z|)|z|$ for all (z, x). Since $Z(z_0, x_0, t)$ is in $\mathcal{B}_z(k(|z_0|)|z_0|)$ together with (5), we have that[5], for all $s \ge 0$,

$$\left|\frac{\partial F}{\partial x}(W(w_0, s))\right| \le c(|z_0|)e^{-\lambda s}|z_0| .$$

Using (36), it follows also that

$$\begin{aligned}\left|\widetilde{Z}(\widetilde{w}_0, w_0, t)\right| &\le \widetilde{k}_a(|z_0|)\exp(-\lambda_1 t)|\widetilde{z}_0|\\ + c(|z_0|)|z_0| &\int_0^t \exp(-\lambda_1(t-s) - \lambda s)\left|\widetilde{X}(\widetilde{w}_0, w_0, s)\right| ds , \quad (38)\end{aligned}$$

holds for all $t \ge 0$ and all $\widetilde{w}_0$ in $\mathbb{R}^{n_w}$. On the other hand, we have for all (z, x) in $\mathbb{R}^{n_w}$ with (26)

$$\begin{aligned}\left|\frac{\partial G}{\partial x}(z, x)\right| &\le \left|\frac{\partial G}{\partial x}(z, x) - \frac{\partial G}{\partial x}(0, x)\right| + \left|\frac{\partial G}{\partial x}(0, x)\right|\\ &\le c(|z|)|z| + \nu .\end{aligned}$$

This inequality in combination with (5) for all $s \ge 0$ and all $\widetilde{w}_0$ in $\mathbb{R}^{n_w}$

$$\begin{aligned}\left|\widetilde{X}(\widetilde{w}_0, w_0, s)\right| &\le \exp\left(c(|z_0|)\frac{1 - e^{-\lambda s}}{\lambda} + \nu s\right)|\widetilde{x}_0| +\\ &\int_0^s \exp\left(c(|z_0|)\frac{e^{-\lambda\ell} - e^{-\lambda s}}{\lambda} + \nu(s - \ell)\right)\\ &\times c(|z_0|)\left|\widetilde{Z}(\widetilde{w}_0, \widetilde{x}_0, \ell)\right| d\ell .\end{aligned}$$

By rearranging this inequality (and changing again the c function), we obtain

$$\left|\widetilde{X}(\widetilde{w}_0, w_0, s)\right| \le c(|z_0|)\exp(\nu s) \times \left[|\widetilde{x}_0| + \int_0^s \left|\widetilde{Z}(\widetilde{w}_0, \widetilde{x}_0, \ell)\right| d\ell\right] .$$

Hence inequality (38) becomes (changing one more time the c function)

$$\begin{aligned}\left|\widetilde{Z}(\widetilde{w}_0, w_0, t)\right| &\le \widetilde{k}_a(|z_0|)e^{-\lambda_1 t}|\widetilde{z}_0| +\\ &c(|z_0|)|z_0|\left[e^{[\nu-\lambda]t} + e^{-\lambda_1 t}\right]\\ &\times\left[|\widetilde{x}_0| + \int_0^t \left|\widetilde{Z}(\widetilde{w}_0, \widetilde{x}_0, \ell)\right| d\ell\right]\end{aligned}$$

By assumption we have that $\nu < \lambda$ and by setting $u(t) = \exp(\lambda_2 t)\widetilde{Z}(\widetilde{w}_0, w_0, t)$, where $\lambda_2 = \min\{\lambda_1, \nu - \lambda\}$, it implies

$$u(t) \le \alpha + \int_0^t \beta(\ell)u(\ell)d\ell$$

with

$$\begin{aligned}\alpha &= \widetilde{k}_a(|z_0|)|\widetilde{z}_0| + 2c(|z_0|)|z_0||\widetilde{x}_0| ,\\ \beta(\ell) &= 2c(|z_0|)|z_0|\exp(-\lambda_2\ell).\end{aligned}$$

Employing Grönwall Lemma (see [11, p. 36]), it follows that

$$\begin{aligned}\left|\widetilde{Z}(\widetilde{w}_0, w_0, t)\right| &\le \exp(-\lambda_2 t)\alpha\exp\left(\int_0^t \beta(\ell)d\ell\right) ,\\ &\le \exp(-\lambda_2 t)\alpha\exp\left(\frac{1 - e^{-\lambda_2 t}}{\lambda_2}2c(|z_0|)|z_0|\right) .\end{aligned}$$

[5] c is again a generic notation.

V. (Glob.)-TES and TES-VS $\Rightarrow$ (Glob.)-LMTE

In the spirit of Lyapunov matrix equation, it was shown in [2, Proposition 2] that the exponential stability of the locally transversally linearized system (Prop. ES-TLS) can be characterized by a local Lyapunov matrix transversal equation (i.e. Property (Loc.)-LMTE). Again, in this section, it is shown that this implication holds true in the global context as given in the following proposition.

Proposition 3: If Properties (Glob.)-TES and TES-VS hold then Property (Glob.)-LMTE holds.

The proof of this theorem is decomposed in three steps. In the first step we introduce a candidate matrix function S. In the second step we show that this matrix function satisfies the lower and upper bound given in (9). The final step is devoted to show that the field of bilinear maps we have obtained admits a Lie derivative along the vector field φ which satisfies (10).

Proof : **First step: Introduction of S.** Consider the following nonlinear dynamical system where the state of which is a matrix in $\mathbb{R}^{n_w\times n_w}$:

$$\dot{\Phi}(w,t) = \frac{\partial\varphi}{\partial w}(W(w,t))\Phi(w,t)\ ,\ \Phi(w,0) = \mathrm{I}_w\ . \tag{39}$$

It follows trivially that

$$\widetilde{W}(\widetilde{w},w,t) = \Phi(w,t)\widetilde{w}\ .$$

Using the property (Glob.)-TES , (i.e. uniform exponential convergence of $\widetilde{z}$ as given in (8)), we have for all $(\widetilde{w},w,t)$ in $\mathbb{R}^{2n_w}\times\mathbb{R}_{\geq 0}$ that

$$\left|\tilde{Z}(w,\widetilde{w},t)\right| \leq \widetilde{k}(|z|)e^{-\widetilde{\lambda}(|z|)t}|\widetilde{w}|\ .$$

Therefore if we denote

$$\Phi_1(w,t) = \begin{bmatrix}\mathrm{I}_z & 0\end{bmatrix}\Phi(w,t)\ ,$$

we have that

$$|\Phi_1(w,t)| \leq \widetilde{k}(|z|)\, e^{-\widetilde{\lambda}(|z|)t}\ , \tag{40}$$

holds for all w in $\mathbb{R}^{n_w}$ and all t in $\mathbb{R}_+$. Let $\widetilde{\lambda}_s$ be a non increasing function such that $\widetilde{\lambda}_s\left(k(|z|)\,|z|\right) \leq \widetilde{\lambda}(|z|)$. Consider the function $S:\mathbb{R}^{n_w}\to\mathbb{R}^{n_w\times n_w}$ defined by

$$S(w) = \lim_{T\to+\infty}\int_0^T e^{\int_0^s\widetilde{\lambda}_s(|Z(w,\ell)|)d\ell}\Phi_1(w,s)^\top\Phi_1(w,s)ds\ . \tag{41}$$

Second step: Lower and upper bound on S. With (Glob.)-TES property we have that (5) holds and

$$\widetilde{\lambda}_s(|Z(w,\ell)|) \leq \widetilde{\lambda}_s\left(k(|z|)\,e^{-\lambda\ell}\,|z|\right) \leq \widetilde{\lambda}(|z|)\ ,\ \forall\ell.$$

Therefore

$$S(z,x) \leq \lim_{T\to+\infty}\int_0^T e^{\widetilde{\lambda}_s(k(|z|)|z|)s}\Phi_1(w,s)^\top\Phi_1(w,s)ds\ ,$$
$$\leq \lim_{T\to+\infty}\int_0^T e^{\widetilde{\lambda}(|z|)s}\Phi_1(w,s)^\top\Phi_1(w,s)ds\ .$$

Hence, it is well-defined, continuous and satisfies

$$\lambda_{\max}\{S(z,x)\} \leq \frac{\widetilde{k}(|z|)^2}{\widetilde{\lambda}(|z|)} := \bar{s}(|z|) \qquad \forall(z,x)\in\mathbb{R}^{n_w}.$$

To obtain the lower bound on the matrix function S, we decompose it in blocks, as we have done before in Proposition 1,

$$S(w) = \begin{bmatrix}P(w) & Q(w)\\ Q(w)^\top & R(w)\end{bmatrix}\ .$$

Let P_s be the Schur complement of S given by

$$P_s(w) = P(w) - Q(w)R(w)^gQ(w)^\top\ ,$$

where $R(w)^g$ is any symmetric generalized inverse matrix of $R(w)$. Following similar argumentation as in Proposition 1, it can be shown that P_s is uniquely defined since S is positive semi definite for all w, and writing P_s as a solution to a minimization procedure. Moreover, employing the fact that $(\mathrm{I}_x - R(w)^gR(w))Q(w)^\top = 0$, we know that for all $\widetilde{w} = (\widetilde{z},\widetilde{x})$,

$$\widetilde{w}^\top S(w)\widetilde{w} = \widetilde{z}^\top P_s(w)\widetilde{z}$$
$$+\,(\widetilde{x} + R(w)^gQ(w)\widetilde{z})^\top R(w)(\widetilde{x} + R(w)^gQ(w)\widetilde{z})\ .$$

It can be shown that P_s is a positive definite matrix. Indeed, assume that $\widetilde{z}\neq 0$ is such that $P_s\widetilde{z} = 0$. Let $\widetilde{x} = -R(w)^gQ(w)\widetilde{z}$. Then this implies

$$\widetilde{w}^\top S(w)\widetilde{w} = 0\ .$$

However, according to the definition of S, where $e^{\int_0^s\widetilde{\lambda}_s(|Z(w,\ell)|)d\ell} \geq 1$ we have that

$$\lim_{T\to+\infty}\int_0^T|\Phi_1(w,s)\widetilde{z}|^2\,ds = \lim_{T\to+\infty}\int_0^T\left|\widetilde{Z}(w,\widetilde{w},s)\right|^2ds = 0$$

which is impossible since $\widetilde{Z}(w,\widetilde{w},0) = \widetilde{z} \neq 0$ and $s\mapsto\widetilde{Z}(w,\widetilde{w},s)$ is a C^1 function. Consequently, for all w, $P_s(w)$ is a positive definite matrix. Let

$$\underline{s}(w) = \lambda_{\min}\{P_s(w)\}\ .$$

This function takes positive value and

$$\underline{s}(w)\begin{bmatrix}\mathrm{I}_z & 0\\ 0 & 0\end{bmatrix} \leq S(w)\ .$$

Hence, (9) is obtained.

Third step : Lie derivative of S. To get (10), let us exploit the semi-group property of the solutions of the variational system (7), i.e.

$$\begin{aligned}
W(w,h+s) &= W(W(w,h),s)\\
\Phi(w,h+s)\tilde{w} &= \tilde{W}\left((\tilde{w},w),h+s\right)\\
&= \tilde{W}\left(\left(\tilde{W}(\tilde{w},w,h)\ ,\ W(w,h)\right),s\right)\\
&= \Phi(W(w,h),s)\tilde{W}(\tilde{w},w,h)\\
&= \Phi(W(w,h),s)\Phi(w,h)\tilde{w}
\end{aligned}$$

where $\tilde{w}$ is arbitrary. In particular for $s=-h$, we have $I = \Phi(W(w,h),-h)\Phi(w,h)$. Thus $\Phi(w,h+s)\Phi(W(w,h),-h)\Phi(w,h) = \Phi(W(w,h),s)\Phi(w,h)$ where $\Phi(w,h)$ is invertible. Consequently, $S(W(w,h))$ being well defined for sufficiently small h implies that

$$\begin{aligned}
&S(W(w,h))\\
&= \lim_{T\to+\infty}\int_0^T e^{\int_0^s\widetilde{\lambda}_s(|Z(W(w,h),\ell)|)d\ell}\\
&\quad\times\Phi(W(w,h),s)^\top\begin{bmatrix}\mathrm{I}_z & 0\\ 0 & 0\end{bmatrix}\Phi(W(w,h),s)ds\\
&= \lim_{T\to+\infty}\Phi(W(w,h),-h)^\top\\
&\quad\times\int_0^T e^{\int_0^s\widetilde{\lambda}_s(|Z(w,h+\ell)|)d\ell}\Phi_1(w,s+h)^\top\Phi_1(w,s+h)ds\\
&\qquad\times\Phi(W(w,h),-h)\\
&= \lim_{T\to+\infty}\left(\Phi(W(w,h),-h)^\top\right)^\top\\
&\quad\times\int_h^{T+h} e^{\int_h^s\widetilde{\lambda}_s(|Z(w,\ell)|)d\ell}\Phi_1(w,s)^\top\Phi_1(w,s)ds\\
&\qquad\times\Phi(W(w,h),-h)
\end{aligned} \tag{42}$$

We note that

$$\lim_{h\to 0}\frac{\Phi(W(w,h),-h)-\mathrm{I}}{h}=-\frac{\partial\varphi}{\partial w}(w)\ ,$$

and

$$\lim_{h\to 0}\frac{e^{\int_h^s \widetilde{\lambda}_s(|Z(w,\ell)|)d\ell}-e^{\int_0^s \widetilde{\lambda}_s(|Z(w,\ell)|)d\ell}}{h} = -\widetilde{\lambda}_s(|z|)e^{\int_0^s \widetilde{\lambda}_s(|Z(w,\ell)|)d\ell}\ .$$

This implies that

$$\mathfrak{d}_\varphi S(w)=-\widetilde{\lambda}_s(|z|)S(w)-\begin{bmatrix}\mathrm{I}_z & 0\\ 0 & 0\end{bmatrix} - \frac{\partial\varphi}{\partial w}(w)^\top S(w)-S(w)\frac{\partial\varphi}{\partial w}(w)\ .$$

We conclude then that the derivative (2) does exist and we get

$$L_\varphi S(w)\leq -\widetilde{\lambda}_s(|z|)S(w)\ .$$

Remark 1: Note that with Propositions 2 and 3, (Glob.)-LMTE holds if Property TES-VS and the bounds (25) and (26) hold.

Remark 2: The function S introduced in (41) for proving the theorem takes only positive semi definite values. For instance, for all $w=(0,x)$ (i.e. in $\mathcal{Z}$),

$$\Phi_1(w,t)=\begin{bmatrix}\Phi_{\tilde{z}}((0,x),t) & 0\end{bmatrix}\ ,$$

where $\Phi_{\tilde{z}}$ is the fundamental matrix of the autonomous part of the $\tilde{z}$ dynamics which is defined in (27). This implies that for all $w=(0,x)$ in $\mathcal{Z}$ the function S introduced in (41) has the following structure

$$S(0,x)=\begin{bmatrix}P(0,x) & 0\\ 0 & 0\end{bmatrix}\ , \tag{43}$$

whose rank is n_z.

Example 1 (cont'd): Going back to the illustrative system (6) for which it is assumed that $\lambda>\nu$, the function (39) is given as :

$$\Phi(w,t)=\begin{bmatrix}\kappa(x,t) & \kappa_x(x,t)z\\ 0 & e^{\nu t}\end{bmatrix}\ ,\forall(x,t), \tag{44}$$

where

$$\kappa(x,t)=e^{-\lambda t+\frac{\cos(x)-\cos(e^{\nu t}x)}{\nu}}, \tag{45}$$

and,

$$\kappa_x(x,t)=\frac{\partial\kappa}{\partial x}(x,t) \tag{46}$$

$$=e^{-\lambda t+\frac{\cos(x)-\cos(e^{\nu t}x)}{\nu}}\frac{e^{\nu t}\sin(e^{\nu t}x)-\sin(x)}{\nu}, \tag{47}$$

Consequently

$$\Phi_1(w,t)^\top\Phi_1(w,t)=\begin{bmatrix}1 & 0\\ 0 & z\end{bmatrix}M(x,t)\begin{bmatrix}1 & 0\\ 0 & z\end{bmatrix}, \tag{48}$$

where

$$M(x,t)=\begin{bmatrix}\kappa(x,t)^2 & \kappa(x,t)\kappa_x(x,t)\\ \kappa(x,t)\kappa_x(x,t) & \kappa_x(x,t)^2\end{bmatrix}. \tag{49}$$

Note that if $\lambda>\nu>0$, M is bounded and converge exponentially to zero as time goes to infinity. Consider the matrix

$$S(w)=\begin{bmatrix}1 & 0\\ 0 & z\end{bmatrix}N(x)\begin{bmatrix}1 & 0\\ 0 & z\end{bmatrix} \tag{50}$$

where,

$$N(x)=\lim_{T\to+\infty}\int_0^T e^{\lambda_s s}M(x,s)ds \tag{51}$$

It is well defined for $0<\lambda_s<\lambda-\nu$ and following Step 3 in the proof, it satisfies (10). Moreover, it verifies a bound in the form (9) since N is bounded.

VI. (Glob.)-LMTE "⇒" (Glob.)-TES and Lyapunov function construction

The matrix function S obtained from the property (Glob.)-LMTE may be used to define a degenerate Riemannian metric on $\mathbb{R}^{n_w}$. More precisely, on each point w in $\mathbb{R}^{n_w}$, $S(w)$ defines a quadratic form $\widetilde{w}^\top S(w)\widetilde{w}$ which is semi positive definite and may admit a kernel of dimension n_x. From this, it is possible to define a degenerate Riemannian metric on $\mathbb{R}^{n_w}$ as for instance done in [4].

In the following we consider the associated energy integral as a Lyapunov function. More precisely, if S is a function with values that are symmetric semi definite matrices satisfying (9) then energy of any piece-wise C^1 path $\gamma:[0,1]\to\mathbb{R}^{n_w}$ between two arbitrary points $w_1=\gamma(0)$ and $w_2=\gamma(1)$ in $\mathbb{R}^{n_w}$ is

$$E(\gamma)=\int_0^1\frac{d\gamma}{d\tau}(\sigma)^\top S(\gamma(\sigma))\frac{d\gamma}{d\tau}(\sigma)\,d\sigma\ . \tag{52}$$

Note that in contrast with the case in which S defines a usual Riemannian metric, it is possible to consider two points w_1 and w_2 and a C^1 path the energy integral of which is zero. This is for instance the case when considering for instance $w_1=(0,x_1)$ and $w_2=(0,x_2)$ for the matrix function defined in (41) (see Remark 2) and a path such that $\gamma_z(s)=0$ for all s in $[0,1]$ where γ_z is the z-component of γ.

Given $w=(z,x)$ in $\mathbb{R}^{n_w}$, we define the set $\Omega(w)$ as the set of all C^2 paths γ such that

$$\gamma(0)=w\ ,\ \gamma_z(1)=0\ .$$

We can now define a candidate Lyapunov function, denoted by V, as the infimum energy of all paths between w and points in the manifold, i.e.

$$V(w)=\inf_{\gamma\in\Omega(w)}E(\gamma)\ . \tag{53}$$

In the following proposition we show that this is indeed a good Lyapunov function candidate and moreover it admits a negative definite upper right Dini derivative along the solution of system (3). Note however that to get the result we need a uniform lower bound of the matrix function S.

Proposition 4: Assume that Property (Glob.)-LMTE holds. Assume moreover that there exists a positive real number $\underline{s}_0$ such that

$$\underline{s}_0\leq\underline{s}(w)\ ,\ \forall w\in\mathbb{R}^{n_w}\ . \tag{54}$$

Then V, defined in (53), satisfies

$$\underline{s}_0|z|^2 \leq V(z,x) \leq \overline{s}(|z|)|z|^2 \ , \tag{55}$$

and admits an upper right Dini derivative along the solutions of system (3) defined as

$$D_\varphi^+ V(w) := \limsup_{h \searrow 0} \frac{V(W(w,h)) - V(w)}{h} \ ,$$

which satisfies

$$D_\varphi^+ V(w) \leq -\lambda_v(|z|)V(w) \ . \tag{56}$$

Hence Property (Glob.)-TES holds.

Example 1 (cont'd): Going back to the illustrative system (6) for which it is assumed that $\lambda > \nu$, due to the particular structure of the function S it is possible to show that $\underline{s}$ can be taken independent of w in term of s_0. Indeed, note that for each x in $\mathbb{R}$ $\widetilde{x} \mapsto \begin{bmatrix} 1 & \widetilde{x} \end{bmatrix} N(x) \begin{bmatrix} 1 \\ \widetilde{x} \end{bmatrix}$ is a non negative polynomial of degree 2 with continuous coefficient in x. Hence, it reaches a minimum value denote $c(x)$ which is continuous and we have,

$$\begin{bmatrix} 1 & \widetilde{x} \end{bmatrix} S(1,x) \begin{bmatrix} 1 \\ \widetilde{x} \end{bmatrix} = \begin{bmatrix} 1 & \widetilde{x} \end{bmatrix} N(x) \begin{bmatrix} 1 \\ \widetilde{x} \end{bmatrix} \geq c(x) \tag{57}$$

It can be shown after lengthy computation that $c(x)$ is lower bounded (see [3, Appendix]). Hence,

$$\inf_{x\in\mathbb{R}} c(x) = c(x^*) \geq \underline{s}(1,x^*) := \underline{s}_0 > 0 \tag{58}$$

This implies,

$$\begin{bmatrix} \widetilde{z} & \widetilde{x} \end{bmatrix} S(z,x) \begin{bmatrix} \widetilde{z} \\ \widetilde{x} \end{bmatrix} = \widetilde{z}^2 \begin{bmatrix} 1 & \frac{\widetilde{x}}{\widetilde{z}} \end{bmatrix} S(z,x) \begin{bmatrix} 1 \\ \frac{\widetilde{x}}{\widetilde{z}} \end{bmatrix} \tag{59}$$

$$= \widetilde{z}^2 \begin{bmatrix} 1 & \frac{\widetilde{x}z}{\widetilde{z}} \end{bmatrix} S(1,x) \begin{bmatrix} 1 \\ \frac{\widetilde{x}z}{\widetilde{z}} \end{bmatrix} \tag{60}$$

$$\geq \widetilde{z}^2 \underline{s}_0 \tag{61}$$

And consequently, on this example, the lower bound on $\underline{s}$ is obtained and the function V defined in (53) can be employed as a Lyapunov function.

Proof : Let us first show that V, defined in (53), satisfies (55) for all w in $\mathbb{R}^{n_w}$ and all paths γ in $\Omega(w)$. Using (54), we have

$$E(\gamma) = \int_0^1 \frac{d\gamma}{ds}(s)^\top S(\gamma(s)) \frac{d\gamma}{ds}(s) d\tau$$
$$\geq \ \underline{s}_0 \int_0^1 \frac{d\gamma_z}{ds}(s)^\top \frac{d\gamma_z}{ds}(s) d\tau \ , \tag{62}$$

where γ_z is the z-component of γ. An energy integral minimizer for an Euclidean metric is a straight line $\tau \mapsto (1-\tau)z$. Indeed, consider a path $\tau \mapsto \delta(\tau) \in \mathbb{R}^{n_z}$ such that $\delta(0) = 0$ and $\delta(1) = 0$. Consider the function

$$\ell(h) = \int_0^1 (\dot{\gamma}_z(s) + h\dot{\delta}(s))^\top (\dot{\gamma}_z(s) + h\dot{\delta}(s)) ds$$

Note that ℓ reaches its minimum value at 0 then

$$0 = \ell'(0) = 2\int_0^1 \dot{\delta}(s)^\top \dot{\gamma}_z(s) d\tau = -2\int_0^1 \delta(s)^\top \ddot{\gamma}_z(s) d\tau$$

for all δ. This implies that $\ddot{\gamma}_z(s) = 0$ and γ_z is affine. Hence,

$$\int_0^1 \frac{d\gamma_z}{d\tau}(\tau)^\top \frac{d\gamma_z}{d\tau}(\tau) d\tau \geq z^\top z \int_0^1 ds = |z|^2 \ .$$

By the definition of the function V, for all k there exists a path γ_k in $\Omega(w)$ such that

$$\underline{s}_0|z|^2 \leq E(\gamma_k) \leq V(w) + \frac{1}{k} \ .$$

This implies that the left inequality in (55) holds.

On the other hand the particular path $\gamma_* : \tau \to ((1-\tau)z, x)$ is in $\Omega(w)$. Hence,

$$V(z,x) \leq E(\gamma_*) = \int_0^1 \begin{bmatrix} z^\top & 0 \end{bmatrix} S((1-s)z, x) \begin{bmatrix} z \\ 0 \end{bmatrix} ds$$
$$\leq \overline{s}(|z|) \int_0^1 z^\top z ds$$
$$\leq \overline{s}(|z|)|z|^2 \ .$$

So the left inequality in (55) holds also.

Let us now establish (56). We start by studying the evolution with time of the energy integral of an arbitrary path. Let w be an arbitrary point in $\mathbb{R}^{n_w}$ and γ be an arbitrary point in $\Omega(w)$. Let also $\tau > 0$ be a positive real number. We denote $\Gamma : [0,1] \times [-\tau,\tau] \to \mathbb{R}^{n_w}$ the function defined by

$$\Gamma(s,t) \ = \ W(\gamma(s),t) \ .$$

Following the properties of γ and W, the function Γ is C^2, bounded and is a path between $W(w,t)$ and $W(\gamma(1),t)$ with $\gamma(1)$ belonging to $\mathcal{Z}$. Hence, $\Gamma(\cdot,t)$ is in $\Omega(W(w,t))$. Let us denote $\vartheta_{\Gamma(\cdot,t)}(W(w,t))$ the energy integral of this path. By the definition of V, we have

$$V(W(w,t)) \leq \vartheta_{\Gamma(\cdot,t)}(W(w,t)) \qquad \forall t \geq 0 \ . \tag{63}$$

Let us evaluate the derivative of the function $t \mapsto \vartheta_{\Gamma(\cdot,t)}(W(w,t))$. We define the function ℓ by

$$\ell(s,t) = \frac{\partial\Gamma}{\partial s}(s,t)^\top S(\Gamma(s,t)) \frac{\partial\Gamma}{\partial s}(s,t) \ .$$

It is continuous and satisfies

$$\vartheta_{\Gamma(\cdot,t)}(W(w,t)) \ = \ \int_0^1 \ell(s,t) ds \ . \tag{64}$$

Also it satisfies, for any s in $[0,1]$, t in $(-\tau,\tau)$ and h sufficiently small,

$$\ell(s,t+h) - \ell(s,t)$$
$$= \left[\frac{\partial\Gamma}{\partial s}(s,t+h) - \frac{\partial\Gamma}{\partial s}(s,t)\right]^\top S(\Gamma(s,t+h)) \frac{\partial\Gamma}{\partial s}(s,t+h)$$
$$+ \frac{\partial\Gamma}{\partial s}(s,t)^\top \left[S(\Gamma(s,t+h)) - S(\Gamma(s,t))\right] \frac{\partial\Gamma}{\partial s}(s,t+h)$$
$$+ \frac{\partial\Gamma}{\partial s}(s,t)]^\top S(\Gamma(s,t)) \left[\frac{\partial\Gamma}{\partial s}(s,t+h) - \frac{\partial\Gamma}{\partial s}(s,t)\right] \ .$$

Here the function $t \mapsto \frac{\partial\Gamma}{\partial s}(s,t)$ is continuously differentiable with

$$\frac{\partial^2\Gamma}{\partial s\partial t}(s,t) \ = \ \frac{\partial\varphi}{\partial w}(\Gamma(s,t)) \frac{\partial\Gamma}{\partial s}(s,t)$$

and, S admitting a Lie derivative satisfies (c.f. (2)),

$$\lim_{h\to 0} \frac{[S(\Gamma(s,t+h)) - S(\Gamma(s,t))]}{h} \ = \ \mathfrak{d}_\varphi S(\Gamma(s,t)) \ .$$

With (10), it follows that

$$\frac{\partial\ell}{\partial t}(s,t)$$
$$= \frac{d\Gamma}{ds}(s,t)^\top \frac{\partial\varphi}{\partial w}(\Gamma(s,t))^\top S(\Gamma(s,t)) \frac{d\gamma}{ds}(s)$$
$$+ \frac{\partial\Gamma}{\partial s}(s,t)^\top \mathfrak{d}_\varphi S(\Gamma(s,t)) \frac{\partial\Gamma}{\partial s}(s,t)$$
$$+ \frac{\partial\Gamma}{\partial s}(s,t)^\top S(|\Gamma(s,t)|) \frac{\partial\varphi}{\partial w}(\Gamma(s,t)) \frac{\partial\Gamma}{\partial s}(s,t)$$

$$
\begin{aligned}
&= \frac{\partial \Gamma}{\partial s}(s,t)^\top L_\varphi S(\Gamma(s,t)) \frac{\partial \Gamma}{\partial s}(s,t) \\
&\leq -\lambda_s(|\Gamma_z(s,t)|) \frac{\partial \Gamma}{\partial s}(s,t)^\top S(\Gamma(s,t)) \frac{\partial \Gamma}{\partial s}(s,t) \\
&\leq -\lambda_s(|\Gamma_z(s,t)|) \ell(s,t)
\end{aligned}
$$

Thus for each s in $[0,1]$, the function $t \in [-\tau, \tau] \mapsto \ell(s,t)$ is continuous and admits a derivative (not necessarily continuous) satisfying the above differential inequality. It follows (see [27, Theorem A.1.2.1] for example) that for all $(s,t) \in [0,1] \times [-\tau, \tau]$

$$
\ell(s,t) \;\leq\; e^{-\int_0^t \lambda_s(|\Gamma_z(s,r)|)dr} \ell(s,0) \ .
$$

Let us find an s-independent upperbound of $\lambda_s(|\Gamma_z(s,r)|)$, appearing in the integral. Since $\Gamma_z(1,r) = 0$, we have

$$
\Gamma_z(s,r) = \int_1^s \frac{\partial \Gamma_z}{\partial s}(\sigma, r) d\sigma \ .
$$

Subsequently, together with (54) and (62), we have that

$$
\begin{aligned}
|\Gamma_z(s,r)| &\leq \int_1^s \frac{\partial \Gamma_z}{\partial s}(\sigma, r) d\sigma \\
&\leq \int_0^1 \left| \frac{\partial \Gamma_z}{\partial s}(\sigma, r) \right| d\sigma \\
&\leq \int_0^1 \left| \frac{\partial \Gamma_z}{\partial s}(\sigma, r) \right|^2 d\sigma, \\
&\leq \frac{1}{\underline{s}_0} \vartheta_{\Gamma(\cdot,r)}(W(w,r)) \ ,
\end{aligned}
$$

holds for all s in $[0,1]$. The function λ_s being non increasing implies that

$$
\lambda_s(|\Gamma_z(s,r)|) \;\geq\; \lambda_s \left(\frac{1}{\underline{s}_0} \vartheta_{\Gamma(\cdot,r)}(W(w,r)) \right)
$$

and therefore for all $(s,t) \in [0,1] \times [0,\tau]$

$$
\ell(s,t) \;\leq\; e^{-\int_0^t \lambda_s\left(\frac{1}{\underline{s}_0} \vartheta_{\Gamma(\cdot,r)}(W(w,r))\right) dr} \ell(s,0) \ .
$$

By integrating in s and using (64), we obtain for all $t \in [-\tau, \tau]$

$$
\vartheta_{\Gamma(\cdot,t)}(W(w,t)) \;\leq\; e^{-\int_0^t \lambda_s\left(\frac{1}{\underline{s}_0} \vartheta_{\Gamma(\cdot,r)}(W(w,r))\right) dr} \vartheta_\gamma(w)
$$

This means that

$$
\begin{aligned}
D^+_\varphi \vartheta_{\gamma,x_p}(w) \;=\; & \limsup_{t \searrow 0} \frac{\vartheta_{\Gamma(\cdot,t)}(W(w,t)) - \vartheta_\gamma(w)}{t} \\
& \leq -\lambda_s \left(\frac{1}{\underline{s}_0} \vartheta_\gamma(w) \right) \vartheta_\gamma(w) \ .
\end{aligned}
$$

In other words, $\vartheta_\gamma(w)$ is non increasing along the solutions and

$$
\vartheta_{\Gamma(\cdot,t)}(W(w,t)) \;\leq\; e^{-\lambda_s\left(\frac{1}{\underline{s}_0} \vartheta_\gamma(w)\right)t} \vartheta_\gamma(w) \qquad \forall t \in [0,\tau] \ .
$$

By the definition of V as a minimum, for any path γ in $\Omega(w)$ and any t in $[0,\tau]$ we have

$$
V(W(w,t)) \;\leq\; e^{-\lambda_s\left(\frac{1}{\underline{s}_0} \vartheta_\gamma(w)\right)t} \vartheta_\gamma(w)
$$

However we know that for any k, there exists a path γ_k in $\Omega(w)$ satisfying

$$
V(w) \;\leq\; \vartheta_{\gamma_k}(w) \;\leq\; V(w) + \frac{1}{k} \ .
$$

Hence

$$
V(W(w,t)) \;\leq\; e^{-\lambda_s\left(\frac{1}{\underline{s}_0}[V(w)+\frac{1}{k}]\right)t} \left[V(w) + \frac{1}{k} \right]
$$

and therefore

$$
V(W(w,t)) \;\leq\; e^{-\lambda_s\left(\frac{1}{\underline{s}_0} V(w)\right)t} V(w) \ . \tag{65}
$$

Together with (55), we obtain finally

$$
|Z(w,t)| \leq \sqrt{\frac{\overline{s}(|z|)}{\underline{s}_0}} \exp\left(-\frac{\lambda_s\left(\frac{\overline{s}(|z|)|z|^2}{\underline{s}_0}\right)}{2} t \right) |z| \ .
$$

Using (65), we get also that

$$
\begin{aligned}
D^+_\varphi V(w) &= \limsup_{t \searrow} \frac{V(W(w,t)) - V(w)}{t} \\
&\leq \limsup_{t \searrow 0} \frac{\left(e^{-\lambda_s\left(\frac{1}{\underline{s}_0} V(w)\right)t} V(w) \right) - V(w)}{t} \\
&\leq -\lambda_s \left(\frac{1}{\underline{s}_0} V(w) \right) V(w) \ .
\end{aligned}
$$

This ends the proof.

VII. Application to the design of global full-order observer

A. Analysis based on former results

In order to illustrate some of the propositions that have been given in this paper, we consider the observer design context in this section. Consider an autonomous dynamical system given by

$$
\dot{x} = f(x) \ , \ y = h(x) \ , \tag{66}
$$

where x in $\mathbb{R}^n$ is the state and y in $\mathbb{R}$ is a measured output. The vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ and the function $h : \mathbb{R}^n \to \mathbb{R}$ are both C^1. Solutions initiated from x_0 in $\mathbb{R}^n$ is denoted by $X(x_0,t)$ and is assumed to be defined for all positive time.

Let us consider the problem of a (full order) observer design for this system. In other word, we are looking for a necessary and sufficient condition for the existence of a vector field $K : \mathbb{R}^n \to \mathbb{R}^n$ such that for the dynamical system

$$
\begin{cases}
\dot{\hat{x}} = f(\hat{x}) + K(\hat{x})(y - h(\hat{x})) \\
\dot{x} = f(x) \ , \ y = h(x)
\end{cases} \tag{67}
$$

we have an attractive invariant manifold $\{(x,\hat{x}) \in \mathbb{R}^{2n}, x = \hat{x}\}$ which is globally asymptotically stable and uniformly locally exponentially stable. More precisely, we introduce the following property.

Definition 1 (Global full-order observer): The system (66) is said to admit a global full-order observer with rate of convergence $\lambda > 0$ if there exist a vector field K and a non decreasing continuous function k such that for all $(x_0, \hat{x}_0)$ in $\mathbb{R}^{2n}$, the solutions of (67) satisfy

$$
\left| \hat{X}(x_0, \hat{x}_0, t) - X(x_0, t) \right| \leq k(|x_0 - \hat{x}_0|) \exp(-\lambda t) |x_0 - \hat{x}_0| \ , \tag{68}
$$

where $\hat{X}(x_0, \hat{x}_0, t)$ is the $\hat{x}$ component of the solution of (67) initiated from $(x_0, \hat{x}_0)$.

By letting $z = \hat{x} - x$, this system is in the form of system (5) with $\varphi(z,x) = (F(z,x), G(z,x))$ and

$$\begin{cases} F(z,x) &= f(x+z) - f(x) + K(x+z)(h(x) - h(x+z)) \ , \\ G(z,x) &= f(x) \ . \end{cases}$$

The asymptotic stability property previously mentioned can be rephrased as the property (Glob.)-TES for a non decreasing continuous function k and a positive real number λ.

Using Proposition 3, a necessary condition for a global full order observer to exist can be given as follow.

Proposition 5: Assume that there exists ν such that

$$\left|\frac{\partial f}{\partial x}(x)\right| \leq \nu \ , \ \forall x \in \mathbb{R}^n \ .$$

Assume that system (66) admits the existence of a global full order observer with rate of convergence λ. Assume moreover that K, f and h are such that (25) holds and

$$\nu < \lambda \ .$$

Then there exist a non increasing continuous functions $\lambda_s : \mathbb{R}_+ \to \mathbb{R}_+$, a function $\underline{p} : \mathbb{R}^{2n} \to \mathbb{R}_+ \setminus \{0\}$ and a non decreasing continuous function $\overline{p} : \mathbb{R}_+ \to \mathbb{R}_+$ and a locally Lipschitz function $P : \mathbb{R}^{2n} \to \mathbb{R}^{n\times n}$ such that

$$\mathfrak{d}_\varphi P(z,x) + P(z,x)\frac{\partial F}{\partial z}(z,x) + \frac{\partial F}{\partial z}(z,x)^\top P(z,x) \leq -\lambda_p(|z|)P(z,x) \ , \quad (69)$$

and,

$$\underline{p}(w)\,\mathrm{I}_n \leq P(z,x) \leq \overline{p}(|z|)\,\mathrm{I}_n \ . \quad (70)$$

Proof : The proof of this proposition can be directly deduced from Propositions 2 and 3. Indeed, from these propositions, it follows that (Glob.)-LMTE holds for this system. Hence, there exist a non increasing continuous functions $\lambda_s : \mathbb{R}_+ \to \mathbb{R}_+$, a function $\underline{s} : \mathbb{R}^{2n} \to \mathbb{R}_+ \setminus \{0\}$ and a non decreasing continuous function $\overline{s} : \mathbb{R}_+ \to \mathbb{R}_+$ and a locally Lipschitz function $S : \mathbb{R}^{2n} \to \mathbb{R}^{2n\times 2n}$ such that (10) and (9) hold. Restricting (10) to this context, we have that $S = \begin{bmatrix} P & Q \\ Q^\top & R \end{bmatrix}$ satisfies

$$\mathfrak{d}_\varphi S(w) + \begin{bmatrix} P(w) & Q(w) \\ Q^\top(w) & R(w) \end{bmatrix} \varphi_w(w) + \varphi_w(w)^\top \begin{bmatrix} P(w) & Q(w) \\ Q^\top(w) & R(w) \end{bmatrix} \leq -\lambda_s \begin{bmatrix} P(w) & Q(w) \\ Q^\top(w) & R(w) \end{bmatrix} \ , \quad (71)$$

where

$$\varphi_w(w) = \begin{bmatrix} F_z(z,x) & F_x(z,x) \\ 0 & f_x(x) \end{bmatrix}$$

and

$$F_z(z,x) = \frac{\partial f}{\partial x}(x+z) + \frac{\partial K}{\partial x}(x+z)(h(x) - h(x+z)) - K(x+z)\frac{\partial h}{\partial x}(x+z))$$

$$F_x(z,x) = \frac{\partial f}{\partial x}(x+z) - \frac{\partial f}{\partial x}(x) + \frac{\partial K}{\partial x}(x+z)(h(x) - h(x+z)) + K(x+z)(\frac{\partial h}{\partial x}(x) - \frac{\partial h}{\partial x}(x+z)).$$

The result is directly obtained restricting these matrix inequalities to the upper left elements.

B. A design procedure inspired from the necessary condition

Even if most of the results exposed so far are analysis tools, it is possible to deduce synthesis tools from them. Indeed, Equations (69) and (70) establish that for all x_0 the dynamics

$$\dot{z} = F(z, X(x_0,t))$$

is a contraction when $z \mapsto P(z, X(x_0,t))$ is considered as a (parameterized and time varying) Riemannian metric on $\mathbb{R}^n$. Assuming that $\underline{p}$ is a constant (and not a function), it yields that the metric induced by P is complete. In that case, (69) is a sufficient condition for z to converge toward zero and so for the observer to converge.

Finding P and K such that (69) and (70) hold is not an easy task. A context in which this is possible is when considering a correction term P in the form of the (Riemannian) gradient of the output map h for a metric P arising from a detectability property. This is the path which is followed in [28] (see also [29]) to obtain semi-global or local result. In our context, assuming stronger assumptions, and considering the same gradient based observer a global result may be obtained.

Proposition 6 (Sufficient condition for observer): Assume that

1) The system (66) is infinitesimally detectable (see [28]). In other words, there exists a function $P_\ell : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ and positive real numbers $(\gamma, \lambda, \underline{p_\ell}, \overline{p_\ell})$ such that the following property holds for all x in $\mathbb{R}^n$.

$$\underline{p_\ell}\,\mathrm{I} \leq P_\ell(x) \leq \overline{p_\ell}\,\mathrm{I} \ , \quad (72)$$

$$L_f P_\ell(x) \leq -\lambda P_\ell(x) + \gamma \frac{\partial h}{\partial x}(x)^\top \frac{\partial h}{\partial x}(x) \ . \quad (73)$$

2) The vector field $\kappa : \mathbb{R}^n \to \mathbb{R}^n$ defined as $\kappa(x) = {P_\ell}^{-1}(x)\frac{\partial h}{\partial x}(x)^\top$ is a Killing vector field[6] for P_ℓ. In other words,

$$L_\kappa P_\ell(x) = 0 \ , \ \forall x \in \mathbb{R}^n \ . \quad (74)$$

Then picking $P(w) = P_\ell(x+z)$, and $K(x) = \gamma\kappa(x)$ inequalities (69) and (70) hold and consequently the first subsystem in (67) is a global full-order observer.

Proof : Note that for all v in $\mathbb{R}^n$ we have,

$$v^\top \mathfrak{d}_\varphi P(z,x) v = \frac{\partial v^\top P_\ell(z+x)v}{\partial z} F(z,x) + \frac{\partial v^\top P_\ell(z+x)v}{\partial x} G(z,x). \quad (75)$$

With the particular structure of F, G and P it implies

$$\mathfrak{d}_\varphi P(z,x) = \mathfrak{d}_f P_\ell(z+x) + \gamma(h(x) - h(x+z))\,\mathfrak{d}_\ell P_\ell(z+x) \ . \quad (76)$$

[6] Killing vector field: $\forall x \in \mathbb{R}^n$, $v^\top L_\kappa P_\ell(x) v = 0$, $\forall v$ such that $\frac{\partial h}{\partial x}(x)v = 0$.

On the other hand,

$$P(z,x)\frac{\partial F}{\partial z}(z,x) = P_\ell(z+x)\frac{\partial f}{\partial x}(x+z) - \gamma\frac{\partial h}{\partial x}(x+z)\frac{\partial h}{\partial x}(x+z) + \gamma(h(x)-h(x+z))P_\ell(z+x)\frac{\partial \ell}{\partial x}(x+z) \quad (77)$$

Hence, (69) becomes

$$\mathfrak{d}_\varphi P(z,x) + P(z,x)\frac{\partial F}{\partial z}(z,x) + \frac{\partial F}{\partial z}(z,x)^\top P(z,x) = L_f P_\ell(x+z) - 2\gamma\frac{\partial h}{\partial x}(x+z)\frac{\partial h}{\partial x}(x+z) + \gamma(h(x)-h(x+z))L_\ell P_\ell(x+z)\ . \quad (78)$$

With (73) and (74), it implies that

$$\mathfrak{d}_\varphi P(z,x) + P(z,x)\frac{\partial F}{\partial z}(z,x) + \frac{\partial F}{\partial z}(z,x)^\top P(z,x) \leq -\lambda P(z,x)\ . \quad (79)$$

Then (69) and (70) hold. This property implies that the observer dynamics is a (global) contraction and with (72) ensures that the estimation error z converges to zero.

Assuming the gradient is a Killing vector field is equivalent to assume that the level sets of h are totally geodesic and h is a Riemannian submersion. See [7, Theorem 8.1 on 3 $\Leftrightarrow$ 5]. In [28] the later condition is not needed. But without loss of generality, it can be imposed via a modification of P.

VIII. Conclusion

In this paper we have given a characterization of the property of global exponential stability of an invariant manifold in terms of property on the variational system. This framework allows the construction of new kind of Lyapunov function to characterize this property. Note however that for this type of Lyapunov function to be constructed it is required that the convergence rate toward the manifold is larger then an expansion rate in the manifold. The obtained Lyapunov function is a degenerate Riemannian energy integral to the manifold.

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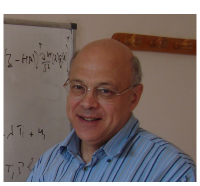

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