Observers to the aid of "strictification" of Lyapunov functions (Long version)

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Abstract

We present a procedure for modifying a weak Lyapunov function V into a strict one. For this we augment the given function with an auxiliary function V_a obtained as a Lyapunov function associated with the error dynamics given by an observer designed for the system with the derivative of V as an output function.

Keywords: Strict Lyapunov function, observer, auxiliary function

1. Introduction

1.1. Problem statement

Let $\mathcal{O}$ be an open neighborhood of the origin in $\mathbb{R}^n$. We consider a system the dynamics of which on $\mathcal{O}$ are:

$$\dot{x} \;=\; f(x) \;, \tag{1}$$

with x in $\mathcal{O}$ and $f : \mathcal{O} \to \mathbb{R}^n$ a sufficiently many times differentiable function which is zero at the origin. We assume the knowledge of a C^1 positive definite[1] function $V : \mathcal{O} \to \mathbb{R}_{\geq 0}$ and we denote $L_f V(x)$ its Lie derivative, evaluated at x, along the vector field f. When $L_f V$ is a negative definite function, V is called a strict Lyapunov function, it is a weak Lyapunov function if $L_f V$ is only non negative. It follows from LaSalle invariance theorem or Barbashin-Krasovskii theorem that the origin is asymptotically stable when V is a weak Lyapunov function and, for any strictly positive real number v, the largest invariant set contained in:

$$\mathcal{Z}_v \;=\; \{x \in \mathcal{O} : \, L_f V(x) \;=\; 0 \;, \; V(x) = v\} \;, \tag{2}$$

is empty. In this case a strict Lyapunov function exists thanks to the Converse Lyapunov Theorem. We propose here a procedure which, starting from the knowledge of a weak Lyapunov function, builds a modification transforming it into a strict Lyapunov function with an as explicit as possible expression.

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[1] Assumption which can be relaxed.

1.2. Our approach and related results

The problem of "strictifying" a Lyapunov function has received a lot of attention, the results of which are partly reported in the monograph [1]. We address it by continuing with the idea, introduced in [2] and summarized in [1, §5.6], of exploiting output-to-state stability. More specifically, inspired by [3], given the function V as above, we are interested in finding functions $h : \mathcal{O} \to \mathbb{R}^p$, $V_a : \mathcal{O} \to \mathbb{R}_{\geq 0}$ and $W_a : \mathcal{O} \to \mathbb{R}_{\geq 0}$, a class $\mathcal{K}$ function α and a class $\mathcal{K}^\infty$ function γ satisfying:

$$L_f V(x) \;\leq\; -\alpha(|h(x|)) \;, \tag{3}$$

$$L_f V_a(x) \;\leq\; -W_a(x) + \gamma(|h(x)|) \tag{4}$$

and such that the function $x \mapsto W_a(x) + |h(x)|$ is positive definite. The motivation is that, in this case,

$$V_s(x) \;=\; \mu(V(x)) \;+\; \mu_a(V_a(x)) \;, \tag{5}$$

where μ and μ_a are appropriately chosen C^1 class $\mathcal{K}^\infty$ functions, is a good candidate for being a strict (global) Lyapunov function. Very advanced and sophisticated techniques for designing the functions μ and μ_a are available. We refer to [4, 1, 5] and many others for this and do not address this point in the paper besides what is in the appendix, extracted from these references, and formatted for our application.

In this context, V_a is called an auxiliary function. As noticed in [1, §5.6], the challenge is to find an expression for it. Below we propose a procedure to obtain it via the design of an observer.

According to [6, Section II], an observer is a dynamical system, with the given system output $h(x)$ as input, and the output of which estimates a function τ of the given system state x. Hence the idea of choosing V_a in (4) as a Lyapunov function quantifying the estimation error and the dynamic of which describes the observer convergence. This idea is not new. It has been exploited already for instance in [7, §V].

In section 2 we present our construction technique in the simplest possible way, concentrating on the idea of getting, via the introduction of an observer, an auxiliary function V_a. In section 3, we demonstrate, by means of examples, the possible interest and the many flexibilities of the technique.

In a complementary section at the end of this report, we give more advanced results on Examples 2 and 4.

Notation: In all what follows ω denotes a real number strictly larger than 1.

2. The observer design approach

2.1. The procedure

The starting point of the procedure is the function V with $L_f V$ non positive. We look for a function $h : \mathcal{O} \to \mathbb{R}^p$, called output function in our context, satisfying:

$$-L_f V(x) \;\geq\; \alpha(|h(x)|) \;, \tag{6}$$

where α is a class $\mathcal{K}$ function, and such that the system:

$$\dot{x} \;=\; f(x) \;,\; y \;=\; h(x)$$

is observable in such a way that we can design an observer. What we mean by *we can design an observer* is that there exist continuous functions $\tau : \mathcal{O} \to \mathbb{R}^m$ and $\varphi : h(\mathcal{O}) \times \mathbb{R}^m \to \mathbb{R}^m$ such that:

1. the function $x \mapsto (h(x), \tau(x))$ is injective on $\mathcal{O}$.
2. We have:
$$L_f\tau(x) \;=\; \varphi(h(x), \tau(x)) \;, \quad \tau(0) \;=\; 0 \,. \tag{7}$$
3. For the augmented system:
$$\dot{x} = f(x) \quad , \qquad \dot{z} \;=\; \varphi(h(x), z) \;, \tag{8}$$
we know a C^1 function $\mathcal{V} : \mathcal{O}\times\mathbb{R}^m \to \mathbb{R}_{\geq 0}$ satisfying:
$$\mathcal{V}(x, \tau(x)) \;=\; 0$$
and, for all (x, z) such that $z \neq \tau(x)$,
$$\mathcal{V}(x, z) \;>\; 0 \;,$$
$$\frac{\partial \mathcal{V}}{\partial x}(x, z) f(x) + \frac{\partial \mathcal{V}}{\partial z}(x, z)\varphi(h(x), z) = -\mathcal{W}(x, z) < 0 \,. \tag{9}$$

This does give an observer since, on one hand, when z is in the image $\tau(\mathcal{O})$, and the value $h(x)$ is known, there is a unique x satisfying:

$$z \;=\; \tau(x) \;,$$

and, on the other hand, the solution $(X(x, t), Z((x, z), t))$ of (8), issued from (x, z) at time 0, is such that $Z((x, z), t)$ converges to $\tau(X(x, t))$ as t goes to infinity. See [8, Theorem 1] to find a more precise result.

By evaluating at $(0, \tau(x))$ the functions involved in (9), we obtain:

$$\frac{\partial \mathcal{V}}{\partial z}(0, \tau(x))\varphi(0, \tau(x)) = -\mathcal{W}(0, \tau(x)) < 0 \qquad \forall x \in \mathcal{O} : \; \tau(x) \neq 0 \;. \tag{10}$$

This motivates us to define the auxiliary function V_a as:

$$V_a(x) \;=\; \mathcal{V}(0, \tau(x)) \;.$$

In view of (7), we have the decomposition:

$$L_f V_a(x) \;=\; -\mathcal{W}(0, \tau(x)) + \frac{\partial \mathcal{V}}{\partial z}(0, \tau(x)) \left[\varphi(h(x), \tau(x)) - \varphi(0, \tau(x))\right] \;.$$

Here, $h(x)$ plays the role of a disturbance. For this to be useful in our context, we assume we are in an ISS-like context, i.e. there exist a class $\mathcal{K}^\infty$ function γ and a continuous positive definite function $\alpha_a : \tau(\mathcal{O}) \to \mathbb{R}_{\geq 0}$ satisfying:

$$\frac{\partial \mathcal{V}}{\partial z}(0, \tau(x))\, \varphi(h(x), \tau(x)) \;\leq\; -\omega\, \alpha_a(V_a(x)) \;+\; \gamma(|h(x)|) \;. \tag{11}$$

In this case, (4) holds and, with the help of Lemma 1 in appendix, (5) gives an expression of a strict Lyapunov function. Fortunately, (11) holds always when $|\tau(x)|$ and $|h(x)|$ are small enough, since the function $z \mapsto \mathcal{W}(0, z)$ is positive definite, and we have (10) and continuity.

At the end, we keep only the functions τ and $\mathcal{V}$ from the procedure and we need explicit expressions for them. The other points are useful only to make sure that these functions do have the required properties.

In nonlinear system theory there are different approaches for observer design, hence as many variants for the construction of τ and $\mathcal{V}$. In the next Section, we shall restrict our attention to high gain observers and nonlinear Luenberger observers. For the former, an expression of τ is given by successive differentiations of the output function and $\mathcal{V}$ is a quadratic form known up to a scalar constant gain ℓ. In the latter, $\mathcal{V}$ is a known quadratic form, but an expression has to be found for τ solving (7).

For the time being, to help the reader in getting a better grip on the procedure, we propose the following example.

Example 1 ([1, Section 5.5.1]). Consider the system:

$$\dot{x}_1 = -(x_1 + x_2)(x_1 + a) \quad , \qquad \dot{x}_2 \;=\; b x_1 (x_2 + c) \;,$$

where a, b and c are strictly positive real numbers and (x_1, x_2) lives in the open rectangle:

$$\mathcal{O} \;=\; (-a, +\infty) \times (-c, +\infty) \;.$$

Let V be the function defined as:

$$V(x) = b\left[x_1 - a \ln\left(\frac{x_1 + a}{a}\right)\right] + x_2 - c \ln\left(\frac{x_2 + c}{c}\right) \;.$$

It is positive definite, proper and C^1 on $\mathcal{O}$ and satisfies:

$$\overbrace{V(x)}^{\cdot} = -\frac{x_1}{(x_2+c)}\dot{x}_2 = -b\,x_1^2 \ .$$

This is (3), with:

$$h(x) = x_1 \ , \quad \alpha(s) = bs^2 \ .$$

It follows that the origin is stable and that x_1 converges to 0 along any solution. So, if by "measuring" (=knowing) that x_1 goes to 0, we can "estimate" (=deduce) that x_2 goes also to 0, we are done. For this "measure-estimate" process, we look for an observer for the system:

$$\dot{x}_1 = -(x_1+x_2)(x_1+a) \ , \ \dot{x}_2 = bx_1(x_2+c) \ , \ y = x_1 \ .$$

We propose:

$$\dot{z} = -ab\,z - bx_1\,(x_1 - c + (1-b)a) \ .$$

We note that:

$$\tau(x) = x_2 + bx_1$$

is a solution of (7) which reads here as:

$$\begin{aligned} -\frac{\partial \tau}{\partial x_1}(x_1,x_2)(x_1+x_2)(x_1+a) \\ +\frac{\partial \tau}{\partial x_2}(x_1,x_2)bx_1(x_2+c) \\ = -ab\,\tau(x) - bx_1\,(x_1 - c + (1-b)a) \ . \end{aligned}$$

We define the function $\mathcal{V}$ as:

$$\mathcal{V}(x,z) = \frac{(\tau(x)-z)^2}{2} = \frac{(x_2+bx_1-z)^2}{2} \ .$$

It satisfies:

$$\overbrace{\mathcal{V}(x,z)}^{\cdot} = -ab\,(x_2+bx_1-z)^2 = -ab\,\mathcal{V}(x,z) \ .$$

Hence we have a convergent observer providing an estimate of $\tau(x)$. Since the data of $h(x) = x_1$ and $\tau(x) = x_2 + bx_1$ defines $x = (x_1, x_2)$ uniquely, we have actually an estimate of x.

From all this, we propose:

$$V_a(x) = \mathcal{V}(0,\tau(x)) = \frac{(x_2+bx_1)^2}{2}$$

as an auxiliary function. It satisfies:

$$\begin{aligned} \overbrace{V_a(x)}^{\cdot} &= -ab\left(x_2+bx_1\right)^2 \\ &\qquad - bx_1\left(x_2+bx_1\right)\left(x_1-c+(1-b)a\right), \\ &\le -\frac{ab}{2}\left(x_2+bx_1\right)^2 + \frac{b}{2a}x_1^2\left(x_1-c+(1-b)a\right)^2. \end{aligned}$$

This is (11) with:

$$\gamma(s) = \frac{b}{2a}s^2\left|s + |(1-b)-c|\right|^2 \ .$$

So, from Lemma 1 in appendix, with:

$$\alpha_a(s) = \frac{abs}{2\omega} \ , \ \kappa(s) = a[1+\ln(3)]\max\left\{\frac{s}{ab}, \sqrt{\frac{s}{ab}}\right\},$$

we can get an expression of a strict Lyapunov function. More directly, since there exists[2] a continuous non decreasing function $\kappa : \mathbb{R} \to \mathbb{R}_{>0}$ satisfying:

$$\kappa\left(x_1 - a\ln\left(\frac{x_1+a}{a}\right)\right) \ge 1 + \frac{1}{a}\left(x_1-c+(1-b)a\right)^2,$$

for all x_1 in the open interval $(-a, +\infty)$, the following function is a strict Lyapunov function:

$$V_s(x) = \int_0^{V(x)} \kappa(s)ds + V_a(x) \ .$$

2.2. Solution with a high gain observer

To ease the presentation in this paragraph, we restrict our attention to the case where the dimension p of $h(x)$ is 1 and (6) reads:

$$-L_f V(x) \ge |h(x)|^2 \qquad \forall x \in \mathcal{O} \ . \tag{12}$$

Assume the system (1) with h as an output function is strongly differentially observable of order m. This means that there exists an integer m such that the function τ defined as:

$$\tau(x) = \begin{pmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{m-1}h(x) \end{pmatrix} \tag{13}$$

is an injective immersion on $\mathcal{O}$. In this case there exists a C^1 function φ_m which is zero at the origin and satisfies:

$$\varphi_m(\tau(x)) = L_f^m h(x) \qquad \forall x \in \mathcal{O}$$

and such that the components z_i of $z = \tau(x)$ satisfy:

$$\dot{z}_1 = z_2 \ , \quad \dots \ , \quad \dot{z}_{m-1} = z_m \ , \quad \dot{z}_m = \varphi_m(z) \ .$$

We rewrite these equations more compactly as:

$$\dot{z} = A\,z + B\,\varphi_m(z) \quad , \qquad z_1 = C\,z \ .$$

We choose a compact subset $\mathfrak{C}$ of $\mathcal{O}$ which is forward invariant for (1), e.g. a bounded sublevel set of V contained in $\mathcal{O}$. We consider the Kalman observer-like for $\tau(x)$:

$$\dot{z} = A\,z + B\,\varphi_m(\mathsf{sat}(z)) + K\,(h(x) - Cz) \ ,$$

where sat is a bounded function satisfying:

$$\mathsf{sat}(\tau(x)) = \tau(x) \qquad \forall x \in \mathfrak{C} \ ,$$

[2] e.g. $\kappa(s) = 1 + \sup_{x>-a:\, x_1 - a\ln(x_1+a)\le s} \frac{2}{a}\left(x_1-c+(1-b)a\right)^2$.

and the vector K is chosen such that there exist positive definite matrices P and Q satisfying:

$$\begin{aligned} &P\,(A-KC) \;+\; (A-KC)^\top P \;\leq\; -2\,Q\ ,\\ &2\left|\tau(x)^\top PB\,\varphi_m(\tau(x))\right| \leq \tau(x)^\top Q\,\tau(x) \quad \forall x \in \mathfrak{C}\ . \end{aligned}$$

Since $\varphi_m(0) = 0$, by invoking the high gain observer paradigm (see [9] for example), we are guaranteed that P and Q exist when K is in the form:

$$K = \begin{pmatrix} \ell k_1 \\ \vdots \\ \ell^m k_m \end{pmatrix}\ ,$$

where the gain ℓ is to be chosen large enough depending on the Lipschitz constant:

$$Lip \;=\; \sup_{x\in\mathfrak{C}} \left|\frac{\partial\varphi_m}{\partial z}(\tau(x))\right|\ .$$

Then, since we have:

$$\dot{\overbrace{\tau(x)}} \;=\; (A-KC)\,\tau(x) \;+\; B\,\varphi_m(\tau(x)) \;+\; K\,h(x)\ ,$$

we obtain, for all x in $\mathfrak{C}$,

$$\begin{aligned} \dot{\overbrace{\tau(x)^\top P\,\tau(x)}} &\leq -\tau(x)^\top Q\,\tau(x) \;+\; 2\,\tau(x)^\top PKh(x)\ ,\\ &\leq -\left(1-\tfrac{\rho k^2 p^2}{\lambda_{\min}(Q)}\right)\tau(x)^\top Q\,\tau(x) \;+\; \tfrac{1}{\rho}|h(x)|^2, \end{aligned}$$

where k and p are norms of K and P, and ρ is a strictly positive real number. With this, Lemma 1 gives us again a strict Lyapunov function on $\mathfrak{C}$. A simpler expression is:

$$V_s(x) \;=\; V(x) + \mu_a\,\tau(x)^\top P\,\tau(x)$$

where μ_a is a real number to be chosen. We get:

$$\dot{\overbrace{V_s(x)}} \;\leq\; -\left[1-\frac{\mu_a}{\rho}\right]|h(x)|^2 \;-\; \mu_a\left(1-\frac{\rho k^2 p^2}{\lambda_{\min}(Q)}\right)\tau(x)^\top Q\,\tau(x)\ .$$

Since τ is injective on $\mathcal{O}$ and zero at the origin, the right hand side is negative definite when μ_a and ρ satisfy:

$$\mu_a \;<\; \rho \;<\; \frac{\lambda_{\min}(Q)}{k^2 p^2}\ .$$

Remark 1. Our construction uses the function τ which comes directly from the successive Lie derivatives of a square root of $-L_f V$. This is related, though simpler, to what is done in [10] for a Jurdjevic–Quinn control design with the introduction of an auxiliary scalar field and used also in [1, Sections 4.4 and 5.2].

Our construction gives also a tool simpler than the one proposed in [1, Section 5.4] although with some resemblance. But, of course, here, we have the stronger assumption of strong differential observability.

Example 2. Consider the system:

$$\dot x_1 \;=\; \sigma(1-x_2)x_1\ , \quad \dot x_2 \;=\; \left(1-\frac{x_2}{x_1}\right)x_2\ ,$$

with (x_1, x_2) in:

$$\mathcal{O} \;=\; (\mathbb{R}_{>0})^2$$

and σ a constant strictly positive real number. This is the system [11, (6)] with:

$$\begin{aligned} &\mathcal{M}_1 = \mathcal{M}_2 = 1\ ,\ D^* = \frac{1}{\sigma}\ ,\ u = y - x\ ,\\ &x_1 = \exp(x)\ ,\ x_2 = \exp(y)\ . \end{aligned}$$

In [11] the authors exhibit a strict Lyapunov function allowing them to prove that $(1,1)$ is an asymptotically stable equilibrium with $\mathcal{O}$ as domain of attraction. By following the approach described above we can get another expression of a strict Lyapunov function valid, at least, on any compact subset of $\mathcal{O}$.

Our starting point is to observe that:

$$(x_1-1)\,x_2\,dx_1 \;-\; \sigma(1-x_2)x_1^2\,dx_2 \;=\; 0$$

is an exact differential equation with:

$$V(x) \;=\; \ln(x_1) \;+\; \frac{1}{x_1} \;-\; \sigma\ln(x_2) \;+\; \sigma\,x_2$$

as potential. This function is defined, C^1 and proper on $\mathcal{O}$ with a unique stationary point (global minimum) at $(1,1)$. Moreover it satisfies:

$$\dot{\overbrace{V(x)}} \;=\; -\sigma\,\frac{1}{x_1}\,(1-x_2)^2\ . \tag{14}$$

Hence V is a weak global Lyapunov function on $\mathcal{O}$ showing the stability of the point $(1,1)$. Let the compact set:

$$\mathfrak{C}_v \;=\; \{x\in\mathcal{O} : V(x) \leq v\}$$

where v is any strictly positive real number. Let also the output function $h : \mathcal{O} \to \mathbb{R}$ be defined from (14) as:

$$h(x_1, x_2) \;=\; x_2 - 1\ .$$

There exists a real number α satisfying:

$$-\dot{\overbrace{V(x)}} \;\geq\; \alpha\,h(x)^2 \qquad \forall x\in\mathfrak{C}_v\ .$$

According to (13), we define:

$$\tau(x) \;=\; \begin{pmatrix} h(x) \\ L_f h(x) \end{pmatrix} \;=\; \begin{pmatrix} x_2 - 1 \\ \left(1-\frac{x_2}{x_1}\right)x_2 \end{pmatrix}\ .$$

The corresponding function is a diffeomorphism on $\mathcal{O}$. Then, by letting $z = \tau(x)$, we obtain:

$$\dot z_1 \;=\; z_2 \quad , \qquad \dot z_2 \;=\; \varphi_2(x)\ ,$$

where:

$$\varphi_2(x) \;=\; \left(1-2\frac{x_2}{x_1}\right)\left(1-\frac{x_2}{x_1}\right)x_2 + \sigma\frac{x_2^2}{x_1}(1-x_2)\ .$$

It follows from what has been done above for the general case that, for each strictly positive real number v , for all μ_a sufficiently small and ℓ sufficiently large[3], the function:

$$\left[\ln(x_1) + \frac{1}{x_1} - \sigma \ln(x_2) + \sigma x_2\right] + \mu_a \left[[x_2 - 1]^2 - \frac{1}{\ell} [x_2 - 1] \left[\left(1 - \frac{x_2}{x_1}\right) x_2 \right] + \frac{1}{\ell^2} \left[\left(1 - \frac{x_2}{x_1}\right) x_2 \right]^2 \right]$$

is a strict Lyapunov function on $\mathfrak{C}_v$. Hence, we are left with tuning the two real numbers ℓ and μ_a to get a strict Lyapunov function, whereas in [11], it is a function which has to be tuned. However in the latter the obtained Lyapunov function is strict on $\mathcal{O}$ and not only on the compact subset $\mathfrak{C}_v$ as in our case.

2.3. Solution with a nonlinear Luenberger observer

Again to simplify the presentation, we assume (12), instead of (6), and we choose a compact subset $\mathfrak{C}$ of $\mathcal{O}$ which is forward invariant for (1). Following [8, 6, 12], a nonlinear Luenberger observer takes the form:

$$\dot{z} = F z + G h(x) ,$$

where z is in $\mathbb{R}^m$, and F is a Hurwitz matrix. Consider the partial differential equation:

$$L_f \tau^*(x) = F \tau(x) + G h(x) . \tag{15}$$

According to [8, Theorem 2], a solution exists on $\mathfrak{C}$ provided the real part of the eigen values of F are negative enough. It is C^1 and injective if:

- the system has no (backward) indistinguishable pair of states,
- the dimension of z is large enough ($m \geq 2(n+1)$)
- and the eigen values of F are outside a zero Lebesgue measure set. See [8, Theorem 3].

Let P be a positive definite symmetric matrix satisfying:

$$P F + F^\top P = -Q < 0 .$$

We get:

$$\begin{aligned} &\overbrace{V(x) + \mu_a \tau(x)^\top P \tau(x)}^{\cdot} \\ &\quad \leq -|h(x)|^2 + 2\mu_a \tau(x)^\top P L_f \tau(x) , \\ &\quad \leq -|h(x)|^2 - \mu_a \tau(x)^\top Q \tau(x) + 2\mu_a \tau(x)^\top P G h(x) , \\ &\quad \leq -\tfrac{1}{2} |h(x)|^2 - \mu_a \left[1 - \tfrac{4\mu_a p^2 g^2}{\lambda_{\min}(Q)}\right] \tau(x)^\top Q \tau(x) . \end{aligned}$$

Hence, by choosing μ_a in the open interval $\left(0, \frac{\lambda_{\min}(Q)}{4p^2 g^2}\right)$, the following is a strict Lyapunov function:

$$V_s(x) = V(x) + \mu_a \tau(x)^\top P \tau(x) .$$

[3] The observer gains are $k_1 = \ell$ and $k_2 = \ell^2$ where ℓ is to be tuned depending on the Lipschitz constant on $\mathfrak{C}_v$ of φ_2 with respect to z.

The interest of this approach is that distinguishability only is sufficient instead of strong differential observability for the high gain approach. But the drawback is the problem of finding an expression for a solution to the partial differential equation (15). Fortunately approximations are allowed as claimed in [8, Theorem 5].

Example 3. Consider the system:

$$\dot{x}_1 = x_2 , \quad \dot{x}_2 = -x_1 - x_2 \mathrm{exp}\left(-\frac{1}{x_2^2}\right) , \tag{16}$$

already studied in [13, Example 1] and [14, Example 3.4]. We have:

$$\overbrace{V(x)}^{\cdot} = \overbrace{x_1^2 + x_2^2}^{\cdot} = -2 x_2^2 \mathrm{exp}\left(-\frac{1}{x_2^2}\right) .$$

We define the output function h as:

$$y = h(x) = x_2 \mathrm{exp}\left(-\frac{1}{2x_2^2}\right) .$$

With such a function the system (16) is observable but not differentially observable. So we cannot proceed with a high gain observer. Unfortunately, we do not know either how to go with a nonlinear Luenberger observer since we have no expression for a solution to the partial differential equation corresponding to (15).

But, instead of the output y we can use as well a function of y, as long as we do not lose observability. Here we note that the function:

$$x_2 \mapsto y = x_2 \mathrm{exp}\left(-\frac{1}{2x_2^2}\right)$$

is strictly increasing and therefore invertible. Let h^{-1} denote its inverse. So, instead of y, we use x_2 as an output and actually even a function g of x_2. In this case a Luenberger observer is for instance:

$$\dot{z} = -z + g(x_2) .$$

The partial differential equation corresponding to (15) is:

$$\frac{\partial \tau}{\partial x_1}(x_1, x_2)\, x_2 - \frac{\partial \tau}{\partial x_2}(x_1, x_2) \left[x_1 + x_2 \mathrm{exp}\left(-\frac{1}{x_2^2}\right)\right] = -\tau(x_1, x_2) + g(x_2) .$$

We find that by selecting g as:

$$g(x_2) = 2x_2 - x_2 \mathrm{exp}\left(-\frac{1}{x_2^2}\right) = \frac{2h^{-1}(y)^2 - y^2}{h^{-1}(y)} ,$$

an expression of a solution τ is simply:

$$\tau^*(x_1, x_2) = x_1 + x_2 .$$

Since the function $(x_1, x_2) \mapsto (h(x), \tau(x)) = (x_1, x_1 + x_2)$ is injective, the above observer is appropriate. It gives us the auxiliary function:

$$V_a(x) = [x_1 + x_2]^2 .$$

It satisfies:

$$\overbrace{V_a(x)}^{\cdot} = -2\left[[x_1+x_2]^2 - [x_1+x_2]g(x_2)\right] , \\ \leq -V_a(x) + \gamma(|y|) ,$$

where γ is a class $\mathcal{K}^\infty$ function satisfying:

$$\left|\frac{2h^{-1}(y)^2 - y^2}{h^{-1}(y)}\right|^2 \leq 4h^{-1}(|y|)^2 = \gamma(|y|) \leq 4V(x) .$$

Hence we can get a strict Lyapunov function from Lemma 1 in appendix with:

$$\alpha(s) = 2s^2 , \ \alpha_a(v) = \frac{v}{\omega} , \ \gamma^{-1}(s) = h\left(\frac{\sqrt{s}}{2}\right) , \ \kappa(v) = 4v .$$

3. Extensions via examples

The procedure described in Section 2 is only under its very elementary form. Many variations are possible. The following examples illustrate some degrees of freedom.

Example 4 ([15, Theorem 2.3]). Consider the linear time-varying system:

$$\dot{x}_1 = A x_1 + B\psi(t)^\top x_2 \quad , \quad \dot{x}_2 = -\psi(t) B^\top x_1, \tag{17}$$

with A an $n_1 \times n_1$ matrix, B a vector in $\mathbb{R}^{n_1}$, x_1 in $\mathbb{R}^{n_1}$, x_2 in $\mathbb{R}^{n_2}$. The uniform complete observability of this system is studied in a complementary section of this report. In [15, Theorem 2.3 ii)], it is established that the origin is uniformly asymptotically stable if the function $\psi : \mathbb{R} \to \mathbb{R}^{n_2}$ is bounded, locally integrable and there exist strictly positive real numbers θ, β and T satisfying:

$$A + A^\top \leq -\beta^2 I$$

and[4]:

$$\int_t^{t+T} \left(\int_t^s \psi(r)dr\right)\left(\int_t^s \psi(r)dr\right)^\top ds \geq \theta^2 I . \tag{18}$$

This result can be re-established as follows. The Lyapunov function:

$$V(x) = |x_1|^2 + |x_2|^2$$

satisfies:

$$\overbrace{V(x)}^{\cdot} \leq -\beta^2 |x_1|^2 .$$

So the origin is uniformly stable. Let:

$$h(x) = \beta x_1 .$$

To make sure there exists a convergent observer, we check observability. The trick here is to consider the system:

$$\dot{x}_1 = B\psi(t)^\top x_2 + Au \ , \ \dot{x}_2 = -\psi(t)B^\top u \ , \ y = x_1 , \tag{19}$$

with $u = x_1$ as (known) input. In [15, pages 46-48], it is proved that, under (18), this system (19) is uniformly completely observable. So, according to [16, Lemma 5 and Theorem 2], there exist strictly positive real numbers $\underline{p}$ and $\overline{p}$ and a solution $t \mapsto P(t)$ to the following Riccati equation:

$$\dot{P}(t) \\ = 2\mathsf{Sym}\left(\begin{pmatrix}0 & B\psi(t)^\top \\ 0 & 0\end{pmatrix} P(t)\right) - P(t)\begin{pmatrix}I & 0\\ 0 & 0\end{pmatrix}P(t) + I$$

satisfying:

$$\underline{p} \leq P(t) \leq \overline{p} \qquad \forall t \geq 0 .$$

With this, we have a Kalman filter for the system (19). There is no need to write it. It is sufficient to know that the error system can be analyzed with the quadratic form given by $P(t)^{-1}$. This leads to:

$$V_s(x,t) = |x_1|^2 + |x_2|^2 + \mu_a \begin{pmatrix}x_1^\top & x_2^\top\end{pmatrix} P(t)^{-1}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

as a candidate strict Lyapunov function. It satisfies:

$$\overbrace{V_s(x,t)}^{\cdot} \\ \leq -[\beta^2 - \mu_a]|x_1|^2 - \mu_a\left|P(t)^{-1}\begin{pmatrix}x_1\\ x_2\end{pmatrix}\right|^2 \\ +2\mu_a \begin{pmatrix}x_1^\top & x_2^\top\end{pmatrix} P(t)^{-1}\begin{pmatrix}A\\ -\psi(t)B^\top\end{pmatrix} x_1 , \\ \leq -\Big[\beta^2 - \mu_a - 2\mu_a[a^2 + \overline{\psi}^2 b^2]\Big]|x_1|^2 - \frac{\mu_a}{2}\left|P(t)^{-1}\begin{pmatrix}x_1\\ x_2\end{pmatrix}\right|^2 ,$$

where a is a norm of A, b a norm for B and $\overline{\psi}$ a bound for $|\psi|$. So, for μ_a strictly smaller than $\frac{\beta^2}{1+2[a^2+\overline{\psi}^2 b^2]}$ V_s is a strict Lyapunov function.

The same steps as the ones followed here for a linear adaptive system can be followed for a nonlinear one, as the system studied in [17].

Example 5 (Forwarding). In this example we construct a strict Lyapunov function for a system the feedback of which is designed by forwarding. We remain at the simplest level and refer to [18] for more elaborate techniques and to [19] for even more details.

With an appropriate choice of the coordinates[5] for x_2, the system is:

$$\dot{x}_1 = a(x_1) + b(x_1)u , \quad \dot{x}_2 = A x_2 + c(x_1)u . \tag{20}$$

We assume:

A1: we know a C^1 positive definite and proper function V_1 and a continuous positive definite function $\bar{\alpha}$ satisfying:

$$L_a V(x_1) \leq -\bar{\alpha}(|x_1|) - |L_b V_1(x_1)|^2 .$$

A2: $$A + A^\top \leq 0 . \tag{21}$$

[4] If the derivative $\dot{\psi}$ exists almost everywhere and satisfies $\int_t^{t+T} |\dot{\psi} s)|^2 ds \leq \Psi$ then $\int_t^{t+T} \psi(s)\psi(s)^\top ds \geq \bar{\theta}^2 I$ implies (18).

[5] See [18, Section IV].

A3: The pair $(A, c(0)^\top)$ is detectable.

With letting:

$$V(x) \;=\; V_1(x_1) \;+\; x_2^\top x_2 \;,$$

we obtain:

$$\overbrace{V(x)}^{\cdot} \;\leq\; -\bar{\alpha}(|x_1|) \;-\; |L_b V_1(x_1)|^2 \\ \qquad + \; \left(L_b V_1(x_1) + 2 x_2^\top c(x_1)\right) u \;.$$

This motivates us for selecting:

$$u \;=\; -c(x_1)^\top x_2 \;. \tag{22}$$

Indeed this gives:

$$\overbrace{V(x)}^{\cdot} \;\leq\; -\bar{\alpha}(|x_1|) \;-\; \frac{7}{4}\,\left|x_2^\top c(x_1)\right|^2 \;.$$

With Assumption A3, we can conclude that the origin is asymptotically stable. But V is not a strict Lyapunov function. To go on following our procedure we pick the function h as:

$$h(x) \;=\; \left(\begin{array}{c} x_1 \\ c(x_1)^\top x_2 \end{array}\right) \;.$$

We note that inequality (21) gives:

$$\left[A - c(0)c(0)^\top\right] \;+\; \left[A - c(0)c(0)^\top\right]^\top \;\leq\; -2c(0)c(0)^\top \;.$$

It follows from Assumption A3 that the matrix $A - c(0)c(0)^\top$ is strictly Hurwitz. So there exists a positive definite matrix P satisfying:

$$P\left[A - c(0)c(0)^\top\right] + \left[A - c(0)c(0)^\top\right]^\top P = -I \;. \tag{23}$$

With this, the observer we consider is:

$$\dot{z} \;=\; A\,z + c(x_1)u + c(0)\left[c(x_1)^\top x_2 - c(0)^\top z\right] \;,$$

with u known, given by (22). The problem here, created by the second component of h, is the argument x_1 instead of 0 in the term between brackets. If we ignore it, a solution to the partial differential equation corresponding to (15) is simply:

$$\tau(x) \;=\; x_2 \;.$$

The injectivity requirement is trivially satisfied. And, in view of (23), the function $\mathcal{V}$ is to be:

$$\mathcal{V}(x,z) \;=\; [z - x_2]^\top P [z - x_2] \;.$$

It corresponds the auxiliary function:

$$V_a(x) \;=\; \mathcal{V}(0, \tau(x)) \;=\; x_2^\top P x_2$$

which satisfies:

$$\begin{aligned} &\overbrace{V_a(x)}^{\cdot} \\ &= 2x_2^\top P\left[A - c(x_1)c(x_1)^\top\right] x_2 \;, \\ &= 2x_2^\top P\left[A - c(0)c(0)^\top\right] x_2 \\ &\qquad -2x_2^\top P c(x_1)c(x_1)^\top x_2 + 2x_2^\top P c(0)c(0)^\top x_2 \;, \\ &= -|x_2|^2 + 2x_2^\top P\left[c(0)c(0)^\top - c(x_1)c(x_1)^\top\right] x_2 \;. \end{aligned}$$

We are in the case discussed in Remark 2 in appendix where:

$$\gamma(x) \;=\; 2x_2^\top P\left[c(0)c(0)^\top - c(x_1)c(x_1)^\top\right] x_2$$

does not depend only on $h(x)$. Fortunately, the continuity of c implies the existence of a strictly positive real number δ such that we have the property:

$$|x_1| \;\geq\; \delta \quad \forall x_1 : \; \frac{1}{\omega} \leq 2\,\mathsf{Sym}\left(P\left[c(0)c(0)^\top - c(x_1)c(x_1)^\top\right]\right) .$$

On another hand, the function V being proper, there exists a class $\mathcal{K}$ function κ which satisfies, for all $x : |x_1| \;\geq\; \delta$,

$$2\,\mathsf{Sym}\left(P\left[c(0)c(0)^\top - c(x_1)c(x_1)^\top\right]\right) \leq \kappa(V(x))\bar{\alpha}(|x_1|)\,P \,.$$

This implies (see (26)):

$$\frac{1}{1 + x_2^\top P x_2}\,\gamma(x) \;\leq\; \kappa(V(x))\,\alpha(|x_1|) \qquad \forall x : |x_1| \;\geq\; \delta \;.$$

From all this and Remark 2, it follows that:

$$V_s(x) \;=\; \omega \int_0^{V(x)} \kappa(s) ds \;+\; \int_0^{x_2^\top P x_2} \frac{1}{1+v} dv$$

is a strict Lyapunov function.

This function has some similarities with the one given in [20, Proposition 3.2] and obtained from the stabilizability of the pair $(A, c(0))$.

4. Conclusion

On one hand, auxiliary functions, as proposed by [3], are shown in [1] to be efficient for strictifying a given Lyapunov function when combined with the output-to-state stability formalism advocated in [2].

On another hand zero-state detectability is often used complementary to LaSalle invariance principle to establish asymptotic stability.

The combination of these two facts leads to the procedure we have proposed for the strictification of a given Lyapunov function. Specifically, an observer is constructed based on the zero-state detectability. This gives us a Lyapunov function describing the convergence of the associated error system. This function is then used to construct an auxiliary function allowing us to propose a candidate strict Lyapunov function.

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Appendix

An existence result for μ and μ_a

From all the many published results concerning cascaded systems, small gain theorems, strictification of Lyapunov functions, ... we have extracted what follows without any novelty but formatted for our specific framework.

Lemma 1. *Let $\mathcal{O}$ be an open neighborhood of the origin in $\mathbb{R}^n$, $V : \mathcal{O} \to \mathbb{R}_{\geq 0}$ be a C^1 positive definite function, $h : \mathcal{O} \to \mathbb{R}^p$ be a C^0 function which is zero at the origin, $V_a : \mathcal{O} \to \mathbb{R}_{\geq 0}$ be a C^1 function, γ be a class $\mathcal{K}$ function, $\alpha_a : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a C^0 positive definite function and ω be a real number strictly larger than 1, such that the function $x \mapsto \alpha_a(V_a(x)) + |h(x)|$ is positive definite on $\mathcal{O}$ and we have, for all x in $\mathcal{O}$,*

$$\begin{aligned} L_f V(x) &\leq -\alpha(|h(x|)) , \\ L_f V_a(x) &\leq -\omega\, \alpha_a(V_a(x)) + \gamma(|h(x)|) . \end{aligned} \quad (24)$$

Then, for any compact subset[6] *$\mathfrak{C}$ in $\mathcal{O}$, there exist a class $\mathcal{K}^\infty$ function μ and a class $\mathcal{K}$ function μ_a such that the following is is a strict Lyapunov function on $\mathfrak{C}$:*

$$V_s(x) = \mu(V(x)) + \mu_a(V_a(x)) .$$

Proof. Because V is positive definite on $\mathcal{O}$, γ is a class $\mathcal{K}^\infty$ function and h is continuous and zero at the origin, there exists a class $\mathcal{K}$ function κ satisfying:

$$\gamma(|h(x)|) \leq \kappa(V(x)) \qquad \forall x \in \mathfrak{C} .$$

Let:

$$\mu(v) = \omega \int_0^v \kappa(s) ds , \quad \mu_a(v) = \int_0^v \kappa_a(s) ds . \quad (25)$$

with the notation:

$$\kappa_a(s) = \alpha \circ \gamma^{-1} \circ \alpha_a(s) .$$

The function μ is of class $\mathcal{K}^\infty$ and the function μ_a is of class $\mathcal{K}$. They give:

$$\begin{aligned} L_f V_s(x) \leq -\omega\, \kappa(V(x))\, \alpha(|h(x)|) \\ - \kappa_a(V_a(x))\, \omega\, \alpha_a(V_a(x)) \\ + \kappa_a(V_a(x))\, \gamma(|h(x)|) . \end{aligned}$$

If we have $\alpha_a(V_a(x)) \geq \gamma(|h(x)|)$, this yields :

$$\begin{aligned} L_f V_s(x) \leq -\omega\, \kappa(V(x))\, \alpha(|h(x)|) \\ - [\omega - 1]\, \kappa_a(V_a(x))\, \alpha_a(V_a(x)) \qquad \forall x \in \mathfrak{C} . \end{aligned}$$

If instead we have $\gamma(|h(x)|) > \alpha_a(V_a(x))$, this yields :

$$\begin{aligned} L_f V_s(x) &\leq -[\omega - 1]\, \kappa(V(x))\, \alpha(|h(x)|) \\ &\quad - \omega\, \kappa_a(V_a(x))\, \alpha_a(V_a(x)) \\ &\quad - \kappa(V(x)) \left[\alpha(|h(x)|) - \alpha \circ \gamma^{-1} \circ \alpha_a(V_a(x))\right] , \\ &\leq -[\omega - 1]\, \alpha(|h(x)|)\, \kappa(V(x)) \\ &\quad - \omega\, \kappa_a(V_a(x))\, \alpha_a(V_a(x)) \qquad \forall x \in \mathfrak{C} . \end{aligned}$$

For the right hand side to be zero we must have $V(x)$ or $(\alpha_a(V_a(x)), h(x))$ zero. But the functions V and $x \mapsto \alpha_a(V_a(x)) + |h(x)|$ being positive definite, this implies $L_f V_s$ is negative definite and therefore that V_s is a strict Lyapunov function on $\mathfrak{C}$.

[6] We can have $\mathfrak{C}$ non compact and equal to $\mathcal{O}$ if V is proper on $\mathcal{O}$.

Remark 2. *The proof above relies only on the property:*

$$\kappa_a(V_a(x))\, \gamma(|h(x)|) \leq \kappa(V(x))\, \alpha(|h(x)|) . \quad (26)$$
$$\forall x \in \mathfrak{C} : \alpha_a(V_a(x)) \leq \gamma(x) .$$

In the case where, to satisfy (24), we must have γ depending on the whole x and not only on $h(x)$, it is sufficient that, with κ a continuous function satisfying:

$$\kappa(v) \geq \sup_{x:\, V(x) \leq v} \gamma(x) ,$$

there exists a continuous positive definite function κ_a satisfying:

$$\kappa_a(v) \leq \inf_{x \in \mathfrak{C}: V_a(x) = v,\, \alpha_a(v) \leq \gamma(x)} \frac{\kappa(V(x))\alpha(|h(x)|)}{\gamma(x)} .$$

Indeed (26) follows readily and we can pick (25) again.

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Complements

Uniform complete observability of (17)

We reproduce here what can be found in [15, pages 46-48].

For the system:

$$\dot{x}_1 = B\psi(t)^\top x_2 \ , \ \dot{x}_2 = 0 \ , \ y = x_1 \ , \tag{27}$$

the fundamental matrix associated with the linear part is:

$$\Phi(s,t) \ = \ \begin{pmatrix} I & B\mathfrak{T}(s,t)^\top \\ 0 & I \end{pmatrix}$$

with the notation:

$$\mathfrak{T}(s,t) \ = \ \left(\int_t^s \psi(r)dr\right) \ .$$

The corresponding observability Grammian is:

$$\begin{aligned} &\Gamma(t,t+T) \\ =& \int_t^{t+T} \Phi(s,t)^\top \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \Phi(s,t)ds \ , \\ =& \begin{pmatrix} T & B\int_t^{t+T} \mathfrak{T}(s,t)^\top ds \\ \int_t^{t+T} \mathfrak{T}(s,t)dsB^\top & B^\top B\int_t^{t+T} \mathfrak{T}(s,t)\mathfrak{T}(s,t)^\top ds \end{pmatrix} \ . \end{aligned}$$

It gives:

$$\gamma_x(t,t+T) \ = \ x^\top\Gamma(t,t+T)x \ = \ \int_t^{t+T} |x_1+B\mathfrak{T}(s,t)^\top x_2|^2 ds$$

where:

$$\begin{aligned} &\big||x_1|^2 - |x_1 + B\mathfrak{T}(s,t)^\top x_2|^2\big| \\ &\quad= \ 2\left|\int_t^s x_2^\top \varphi(r)B^T[x_1 + B\mathfrak{T}(r,t)^\top x_2]dr\right| \\ &\quad\leq \ 2|B|\sqrt{\int_t^s |x_2|^2|\varphi(r)|^2dr}\sqrt{\int_t^s |x_1 + B\mathfrak{T}(r,t)^\top x_2|^2dr} \\ &\quad\leq \ 2|B||x_2|\sqrt{\int_t^s |\varphi(r)|^2dr}\sqrt{\gamma_x(t,s)} \end{aligned}$$

This yields:

$$\begin{aligned} &T\,|x_1|^2 \\ &\leq \ \gamma_x(t,t+T) \ + \ 2|B||x_2|\int_t^{t+T}\sqrt{\int_t^s |\varphi(r)|^2dr}\sqrt{\gamma_x(t,s)}ds \\ &\leq \ \gamma_x(t,t+T) \ + \ 2|B||x_2|T\sqrt{\int_t^{t+T} |\varphi(r)|^2dr}\sqrt{\gamma_x(t,t+T)} \end{aligned}$$

On another hand, we have:

$$\begin{aligned} &\int_t^{t+T} |B\mathfrak{T}(r,t)^\top x_2|^2dr \\ &\qquad= \ \int_t^{t+T} |-x_1 + [x_1 B\mathfrak{T}(r,t)^\top x_2]|^2dr \\ &\qquad\leq \ 2T\,|x_1|^2 \ + \ 2\gamma_x(t,t+T) \end{aligned}$$

With (18), we have finally obtained, for all x,

$$\begin{aligned} \theta^2|B|^2|x_2|^2 \ \leq& \ 4\sqrt{x^\top\Gamma(t,t+T)x} \quad \times \\ \times& \left[\sqrt{x^\top\Gamma(t,t+T)x} \ + \ |B||x_2|T\sqrt{\int_t^{t+T} |\varphi(r)|^2dr}\right] \end{aligned}$$

The function φ being bounded, this implies uniform complete observability of the system (27).

Note that, with the same kind of arguments as above, we can get:

$$\begin{aligned} &\int_t^{t+T} |\mathfrak{T}(r,t)^\top x_2|^2dr \\ &\quad\leq \ 2\int_t^{t+T}\left|\int_t^s x_2^\top\varphi(r)\mathfrak{T}(r,t)^\top x_2dr\right|ds \\ &\quad\leq \ 2T\sqrt{\int_t^{t+T} |\varphi(r)^\top x_2|^2dr}\sqrt{\int_t^{t+T} |\mathfrak{T}(r,t)^\top x_2|^2dr} \\ &\quad\leq \ 4T^2\int_t^{t+T} |\varphi(r)^\top x_2|^2dr \end{aligned}$$

Hence (18) implies the same of type of bound for φ as for $\mathfrak{T}$.

5.1. A strict global Lyapunov for the system in Example 2

After a time change $dt \mapsto x_1dt$ and a coordinate change $x_1 \mapsto \chi_1 = \frac{1}{x_1}$, the dynamics are:

$$\dot{\chi}_1 = -\sigma(1-x_2) \ , \quad \dot{x}_2 = \left(\frac{1}{\chi_1} - x_2\right)x_2 \ .$$

Assuming x_2 is measured, a reduced order observer is:

$$\dot{z} = -\sigma(1-x_2) - x_2^2 + \frac{x_2}{z+1-x_2} \ .$$

It gives an estimate of $\chi_1 + x_2$ and provides the auxiliary function:

$$V_a(x) \ = \ \frac{1}{2}\left(\chi_1 + x_2 - 2\right)^2 \ = \ \frac{1}{2}\left(\frac{1}{x_1} + x_2 - 2\right)^2 \ .$$

We have:

$$\overbrace{V_a(x)}^{\cdot}$$

$$= (\chi_1 + x_2 - 2)\left(-\sigma(1-x_2) - x_2^2 + \frac{x_2}{\chi_1}\right) ,$$
$$= (\chi_1 + x_2 - 2) \times$$
$$\times \left(-\sigma(1-x_2) + \frac{x_2}{\chi_1}\left[(1-x_2)^2 - x_2(\chi_1 + x_2 - 2)\right]\right) ,$$
$$= -\frac{x_2^2}{\chi_1}(\chi_1 + x_2 - 2)^2$$
$$+ (\chi_1 + x_2 - 2)(1-x_2)\left(-\sigma + \frac{x_2}{\chi_1}(1-x_2)\right)$$
$$\leq -\frac{x_2^2}{2\chi_1}(\chi_1 + x_2 - 2)^2 + \frac{\chi_1}{2x_2^2}\left(-\sigma + \frac{x_2}{\chi_1}(1-x_2)\right)^2 (1-x_2)^2 ,$$
$$\leq -\frac{x_2^2}{2x_1}(1 + x_1(x_2 - 2))^2$$
$$+ \frac{1}{2x_1 x_2^2}(-\sigma + x_1 x_2(1-x_2))^2 (1-x_2)^2 .$$

Going back to the initial time, we have:

$$\dot{\overbrace{V_a(x)}} \leq -\frac{x_2^2}{2x_1^2}(1 + x_1(x_2 - 2))^2 + \frac{(-\sigma + x_1 x_2(1-x_2))^2}{2\sigma x_1 x_2^2}\frac{\sigma}{x_1}(1-x_2)^2 .$$

It follows that a strict Lyapunov function is given by:

$$V_s(x) = \mu(V(x)) + V_a(x) ,$$

with μ satisfying:

$$\mu'(V(x)) \geq \frac{(-\sigma + x_1 x_2(1-x_2))^2}{2\sigma x_1 x_2^2} .$$

Such a function exists since V is proper on $(\mathbb{R}_{>}0)^2$.

This construction is to be compared with what is in the first part of the proof of Theorem 2 in [11].

Generalization of Example 4

In [17], the authors consider the following more general form of the system in Example 4. In that paper, the authors give an expression of a strict Lyapunov function for the system:

$$\begin{aligned} \dot{x}_1 &= F(t,x_1) + \Phi(t,x_1)x_2 , \\ \dot{x}_2 &= -\Phi(t,x_1)^\top \frac{\partial V_1}{\partial x_1}(t,x_1)^\top , \end{aligned} \tag{28}$$

where the function $x_1 \mapsto V_1(t,x_1)$ is positive definite, radially unbounded, both uniformly in t, and satisfies:

$$\frac{\partial V_1}{\partial t}(t,x_1) + \frac{\partial V_1}{\partial x_1}(t,x_1)F(t,x_1) \leq -\alpha(|x_1|) .$$

To find an expression of a strict Lyapunov function for this system we start with picking:

$$V(t,x) = V_1(t,x_1) + \frac{1}{2}|x_2|^2 , \quad W(t,x) = -\alpha(|x_1|) .$$

Then, as a preliminary task for designing an observer, we rewrite the system dynamics as:

$$\begin{aligned} \dot{x}_1 &= \Phi_o(t)x_2 + u_1 + \delta_1 , \\ \dot{x}_2 &= u_2 , \\ y &= x_1 , \end{aligned}$$

where u_1 and u_2 are known inputs, y is the measured output and δ_1 is a disturbance defined as:

$$u_1 = F(t,x_1) , \quad u_2 = -\Phi(t,x_1)\frac{\partial V_1}{\partial x_1}(t,x_1)^\top ,$$
$$\delta_1 = K(t,x_1)x_2 ,$$

with the notation:

$$K(t,x_1) = \Phi_o(t) - \Phi(t,x_1) \quad , \qquad \Phi_o(t) = \Phi(t,0) .$$

Because of smoothness, uniform in t, and the fact that $F(t,0)$ and $\frac{\partial V_1}{\partial x_1}(t,0)$ are zero, there exists a class $\mathcal{K}^\infty$ function, linearly bounded on a neighborhood of zero, satisfying :

$$|F(t,x_1)| + |K(t,x_1)| + \left|\Phi(t,x_1)^\top \frac{\partial V_1}{\partial x_1}(t,x_1)^\top\right| \leq \rho(|x_1|) .$$

With ignoring the disturbance for the time being, we can go with a Kalman filter as an observer. This yields the auxiliary function:

$$V_a(t,x) = x^\top P(t)^{-1} x$$

with P the solution of the Riccati equation coming with the Kalman filter. Because of the persistence of excitation and the boundedness of Φ_o, there exist strictly positive real numbers $\underline{p}$ and $\overline{p}$ such that:

$$\underline{p} \leq P(t) \leq \overline{p} \qquad \forall t .$$

Note that we have:

$$\left| x^\top P(t)^{-1} \begin{pmatrix} F(t,x_1) + K(t,x_1)x_2 \\ -\Phi(t,x_1)^\top \frac{\partial V_1}{\partial x_1}(t,x_1)^\top \end{pmatrix} \right|$$
$$\begin{aligned} &\leq [|x_1| + |x_2|]\frac{\rho(|x_1|)}{\underline{p}}[2 + |x_2|] , \\ &\leq \frac{1}{\underline{p}}\Big[\rho(|x_1|)|x_1| + \frac{\rho(|x_1|)^2}{\varepsilon} + \varepsilon|x_2|^2 \\ &\qquad + \rho(|x_1|)|x_1|\frac{1+|x_2|^2}{2} + \rho(|x_1|)|x_2|^2\Big] , \\ &\leq \varpi_1(|x_1|) + [\varpi_2(|x_1|) + \varepsilon]|x_2|^2 , \end{aligned}$$

where ε is a strictly positive real number, strictly smaller than $\frac{1}{3\overline{p}^2}$, and the functions ϖ_1 and ϖ_2 are defined as:

$$\begin{aligned} \varpi_1(|x_1|) &= \frac{\rho(|x_1|)}{\underline{p}}\left[|x_1| + \frac{\rho(|x_1|)}{\varepsilon} + \frac{1}{2}|x_1|\right] , \\ \varpi_2(|x_1|) &= \frac{\rho(|x_1|)}{\underline{p}}\left[\frac{1}{2}|x_1| + 1\right] . \end{aligned}$$

They are 0 at 0. This yields:

$$
\begin{aligned}
&\overbrace{V_a(t,x)}^{\cdot} \\
&= \quad 2x^\top P(t)^{-1} \begin{pmatrix} F(t,x_1)+K(t,x_1)x_2 \\ -\Phi(t,x_1)^\top \frac{\partial V_1}{\partial x_1}(t,x_1)^\top \end{pmatrix} \;+\; |x_1|^2 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - \; \left|P(t)^{-1}x\right|^2 \;, \\
&\leq \quad \varpi_1(|x_1|) \;+\; [\varpi_2(|x_1|)+\varepsilon]\,|x_2|^2 \;+\; |x_1|^2 \\
&\qquad\qquad\qquad - \; \tfrac{1}{3\overline{p}^2}\left[|x_1|^2+|x_2|^2\right] \;-\; \tfrac{2}{3\overline{p}}V_a(t,x) \;, \\
&\leq \quad \gamma(x) \;-\; \frac{2}{3\overline{p}}V_a(t,x) \;.
\end{aligned}
$$

with the notation:

$$
\begin{aligned}
\gamma(x) \;=\; \varpi_1(|x_1|) + \left[1-\frac{1}{3\overline{p}^2}\right]|x_1|^2 \\
+\max\left\{0, \varpi_2(|x_1|)+\varepsilon-\frac{1}{3\overline{p}^2}\right\}\;|x_2|^2 \;.
\end{aligned}
$$

We let:

$$
\xi_1(v) \;=\; \inf_{x\,:\,\underline{p}v\leq|x|^2\leq\overline{p}v\;,\;\gamma(x)\geq\frac{1}{3\overline{p}^2}|x|^2} |x_1| \tag{29}
$$

May be with increasing ϖ_1, this is a continuous function according to Berge Maximum Theorem. Since

$$
\lim_{|x_1|\to 0} \max\left\{0, \varpi_2(|x_1|)+\varepsilon-\frac{1}{3\overline{p}^2}\right\} \;=\; 0 \;,
$$

ξ_1 is positive definite and we have :

$$
|x_1| \geq \xi_1(V_a(t,x)) \qquad \forall (t,x) \;:\; \gamma(x)\geq \frac{1}{3\overline{p}}V_a(t,x) \;.
$$

Let also κ be a continuous function satisfying:

$$
\kappa(V(t,x)) \geq \gamma(x) \qquad \forall (t,x) \;. \tag{30}
$$

We pick:

$$
\mu(v) \;=\; 2\int_0^v \kappa(s)ds \quad , \qquad \mu_a(v) \;=\; \int_0^v \alpha\circ\xi_1(s)ds
$$

With:

$$
V_s(t,x) \;=\; \mu(V(t,x)) \;+\; \mu_a(V_a(t,x)) \;,
$$

we obtain:

$$
\begin{aligned}
\overbrace{V_s(t,x)}^{\cdot} \;=\; -\kappa(V(t,x))\alpha(|x_1|) - \frac{1}{6\overline{p}}\alpha\circ\xi_1(V_a(t,x))V_a(t,x) \\
-\kappa(V(t,x))\alpha(|x_1|) - \alpha\circ\xi_1(V_a(t,x))\left[\tfrac{1}{3\overline{p}}V_a(t,x)-\gamma(x)\right] \;.
\end{aligned}
$$

There is no problem when $\frac{1}{3\overline{p}}V_a(t,x)-\gamma(x)>0$. In the other case, we get:

$$
\begin{aligned}
\overbrace{V_s(t,x)}^{\cdot} &\leq -\kappa(V(t,x))\alpha(|x_1|) - \frac{1}{6\overline{p}}\alpha\circ\xi_1(V_a(t,x))V_a(t,x) \\
&\qquad - \left[\kappa(V(t,x))\alpha(|x_1|) - \alpha\circ\xi_1(V_a(t,x))\gamma(x)\right] \;, \\
&\leq -\kappa(V(t,x))\alpha(|x_1|) - \frac{1}{6\overline{p}}\alpha\circ\xi_1(V_a(t,x))V_a(t,x) \\
&\qquad -\gamma(x)\left[\alpha(|x_1|) - \alpha\circ\xi_1(V_a(t,x))\right] \;, \\
&\leq -\kappa(V(t,x))\alpha(|x_1|) - \frac{1}{6\overline{p}}\alpha\circ\xi_1(V_a(t,x))V_a(t,x) \\
&\qquad \forall x \;:\; \gamma(x) \geq \frac{1}{3\overline{p}}V_a(t,x) \;.
\end{aligned}
$$

So we have an expression of a strict Lyapunov function up to solving the optimization problems (29) and (30).