Automatica 95 (2018) 548–551

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Technical communique

On dynamic regressor extension and mixing parameter estimators: Two Luenberger observers interpretations ☆

Romeo Ortega [a,*], Laurent Praly [b], Stanislav Aranovskiy [c,d], Bowen Yi [e], Weidong Zhang [e]

[a] Laboratoire des Signaux et Systèmes, CNRS, CentraleSupélec, Plateau du Moulon, 91192 Gif-sur-Yvette, France
[b] MINES ParisTech, PSL Research University, Centre for Automatic Control and Systems Theory, 75272 Paris, France
[c] CentaleSupélec –IETR, Avenue de la Boulaie, 35576 Cesson-Sévigné, France
[d] Department of Control Systems and Informatics, ITMO University, 197101 Saint Petersburg, Russia
[e] Department of Automation, Shanghai Jiao Tong University, 200240 Shanghai, China

ARTICLE INFO

Article history:
Received 28 March 2017
Received in revised form 29 March 2017
Accepted 5 June 2017
Available online 25 June 2018

Keywords:
Parameter estimation
Observer design theory
Adaptive systems
Nonlinear systems

ABSTRACT

Dynamic regressor extension and mixing is a new technique for parameter estimation with guaranteed performance improvement – with respect to classical gradient or least-squares estimators – that has proven instrumental in the solution of several open problems in system identification and adaptive control. In this brief note we give two interpretations of this parameter estimator in terms of the recent extensions, to the cases of nonlinear systems and observation of linear functionals for time-varying systems, of the classical Luenberger's state observers.

1. Introduction

A new procedure to design *parameter estimators* for linear and nonlinear regressions, called dynamic regressor extension and mixing (DREM), was recently proposed in Aranovskiy, Bobtsov, Ortega, and Pyrkin (2017). For linear regressions DREM estimators clearly outperform classical gradient or least-squares estimators and its convergence is established without the usual, restrictive requirement of regressor persistency of excitation (PE) (Ioannou & Sun, 1996; Sastry & Bodson, 1989). Instead of PE a non-square integrability condition on the determinant of a designer-dependent extended regressor matrix is imposed. As discussed in the paper, a key feature of DREM is that it ensures the monotonicity of the individual estimation errors. The technique has been successfully applied in a variety of identification and adaptive control problems (Aranovskiy, Bobtsov, Ortega, & Pyrkin, 2016; Gerasimov, Ortega, & Nikiforov, 2018; Pyrkin, Mancilla, Ortega, Bobtsov, & Aranovskiy, 2017).

Pursuing the work reported in Praly (2016b), in this note we prove that the DREM parameter estimator for linear regressions can be derived following the classical Luenberger *state observer* design procedure (Luenberger, 1964) for linear time-invariant (LTI) systems and its recent extension to linear time-varying (LTV) systems (Rotella & Zambettakis, 2013; Shafai & Carroll, 1986; Trumpf, 2007). See Shoshitaishvili (1992) for some early developments of this theory, Andrieu and Praly (2006), Kazantzis and Kravaris (1998) for its extension to nonlinear systems, Astolfi, Karagiannis, and Ortega (2008) for a related observer design based on ideas of immersion and invariance and the recent paper (Afri, Andrieu, Bako, & Dufour, 2017) for an elegant solution of the far more challenging problem of simultaneous state *and* parameter estimation of LTI systems using these observers.

2. Dynamic regressor extension and mixing parameter estimators

In this section we briefly review the application of DREM for linear regressions—referring the interested reader to Aranovskiy et al. (2017) for additional details.

Consider the basic problem of on-line estimation of the constant parameters of the q-dimensional linear regression

$$y(t) = \phi^\top(t)\theta, \tag{1}$$

☆ This paper is partly supported by the Ministry of Education and Science of Russian Federation (14.Z50.31.0031, goszadanie no. 8.8885.2017/8.9), NSFC (61473183, U1509211). The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor A. Pedro Aguiar under the direction of Editor André L. Tits.

* Corresponding author.
E-mail addresses: ortega@lss.supelec.fr (R. Ortega), Laurent.Praly@minesparistech.fr (L. Praly), stanislav.aranovskiy@centralesupelec.fr (S. Aranovskiy), yibowen@ymail.com (B. Yi), wdzhang@sjtu.edu.cn (W. Zhang).

https://doi.org/10.1016/j.automatica.2018.06.011

where[1] $y \in \mathbb{R}$ and $\phi \in \mathbb{R}^q$ are known, bounded functions of time and $\theta \in \mathbb{R}^q$ is the vector of unknown parameters. The standard gradient estimator

$$\dot{\hat{\theta}} = \Gamma\phi(y - \phi^\top\hat{\theta}),$$

with a positive definite adaptation gain $\Gamma \in \mathbb{R}^{q\times q}$ yields the error equation

$$\dot{\tilde{\theta}} = -\Gamma\phi\phi^\top\tilde{\theta}, \tag{2}$$

where $\tilde{\theta} := \hat{\theta} - \theta$ are the parameter estimation errors. Taking the derivative of the function $|\tilde{\theta}|^2$ it is easy to show that

$$|\tilde{\theta}(0)| \geq |\tilde{\theta}(t)|, \ \forall\, t \geq 0. \tag{3}$$

That is, the *norm* of the parameter error is non-increasing.

It is also well-known (Ioannou & Sun, 1996; Sastry & Bodson, 1989) that the zero equilibrium of the LTV system (2) is globally *exponentially* stable if and only if the regressor vector ϕ is PE, that is, if

$$\int_t^{t+T} \phi(s)\phi^\top(s)ds \geq \delta I_q, \tag{4}$$

for some $T, \delta > 0$ and for all $t \geq 0$, which will be denoted as $\phi \in$ PE. In many practical circumstances $\phi \notin$ PE. To overcome this problem recent studies (Barabanov & Ortega, 2016; Praly, 2016a) have derived strictly weaker conditions for global (but non-uniform) *asymptotic* stability of (2) –hence for convergence of the parameter errors to zero – which is sufficient in many applications. Unfortunately, these new conditions are non-robust and remain hard to verify.

To overcome the limitation imposed by the PE condition and improve the transient performance of the estimator the DREM procedure, introduced in Aranovskiy et al. (2017), generates q new, one-dimensional, regression models to *independently* estimate each of the parameters.

The first step in DREM is to introduce a *linear, single-input q-output, $\mathcal{L}_\infty$-stable* operator $\mathcal{H} : \mathcal{L}_\infty \to \mathcal{L}_\infty^q$, and define the vector $Y \in \mathbb{R}^q$ and the matrix $\Phi \in \mathbb{R}^{q\times q}$ as

$$\begin{aligned} Y &:= \mathcal{H}y \\ \Phi &:= \mathcal{H}\phi^\top. \end{aligned} \tag{5}$$

Clearly, because of linearity and $\mathcal{L}_\infty$ stability, these signals satisfy

$$Y = \Phi\theta + \epsilon_t, \tag{6}$$

with ϵ_t a vector of exponentially decaying terms neglected, without loss of generality, in the sequel—see Remark 3 in Aranovskiy et al. (2017) where the effect of these terms is rigorously analyzed.

The elements of the operator $\mathcal{H}$ may be simple, exponentially stable *LTI filters* of the form[2]

$$\mathcal{H}_i(p) = \frac{\alpha_i}{p + \beta_i}, \ i \in \bar{q} := \{1, 2, \ldots, q\}$$

with $p := \frac{d}{dt}$ and $\alpha_i \neq 0$, $\beta_i > 0$; in this case ϵ_t accounts for the effect of the initial conditions of the filters. Another option of interest are *delay operators*, that is $[\mathcal{H}_i(\cdot)](t) := (\cdot)(t - d_i)$, where $d_i \in \mathbb{R}_+$. See Section 4 for the case of general LTV operators.

[1] When clear from the context, in the sequel the arguments of the functions are omitted.

[2] In the sequel the clarification $i \in \bar{q}$ is omitted for brevity.

Premultiplying (6) by the *adjunct matrix* of Φ, denoted adj$\{\Phi\}$, we get q *scalar* regressors of the form

$$\mathcal{Y}_i(t) = \Delta(t)\theta_i, \tag{7}$$

where we defined the scalar function $\Delta \in \mathbb{R}$

$$\Delta := \det\{\Phi\}, \tag{8}$$

and the vector $\mathcal{Y} \in \mathbb{R}^q$

$$\mathcal{Y} := \text{adj}\{\Phi\}Y. \tag{9}$$

The estimation of the parameters θ_i from the scalar regression form (7) can be easily carried out via

$$\dot{\hat{\theta}}_i = \gamma_i\Delta(\mathcal{Y}_i - \Delta\hat{\theta}_i), \tag{10}$$

with adaptation gains $\gamma_i > 0$. From (7) it is clear that the latter equations are equivalent to

$$\dot{\tilde{\theta}}_i = -\gamma_i\Delta^2\tilde{\theta}_i. \tag{11}$$

A first important advantage of DREM is that the *individual* parameter errors satisfy

$$|\tilde{\theta}_i(0)| \geq |\tilde{\theta}_i(t)|, \ \forall\, t \geq 0, \tag{12}$$

that is *strictly stronger* than the monotonicity property (3). Moreover, solving the simple scalar differential equation (11) we conclude that

$$\lim_{t\to\infty} \tilde{\theta}_i(t) = 0 \quad \Longleftrightarrow \quad \Delta(t) \notin \mathcal{L}_2, \tag{13}$$

that is, parameter convergence is established without the restrictive PE assumption. In Aranovskiy et al. (2017) the relationship between the condition $\Delta \notin \mathcal{L}_2$ and $\phi \in$ PE is thoroughly discussed—see also Section 5.

3. DREM as a gradient descent of a Kazantzis–Kravaris–Luenberger observer

In this section we derive DREM following the Luenberger's observer design procedure proposed in Luenberger (1964)—see also the extension to nonlinear systems, originally proposed in Kazantzis and Kravaris (1998) and refined in Andrieu and Praly (2006). First, we select ℓ negative real (or complex with negative real part) numbers λ_j and consider the system

$$\begin{aligned} \dot{z}_j &= \lambda_j(z_j - y) \\ \dot{T}_j &= \lambda_j(T_j - \phi), \ j \in \bar{\ell} := \{1, \ldots, \ell\}. \end{aligned} \tag{14}$$

Defining

$$\begin{aligned} T &:= \begin{bmatrix} T_1 & \cdots & T_\ell \end{bmatrix} \\ z &:= \text{col}(z_1, \ldots, z_\ell) \\ \Lambda &:= \text{diag}\{\lambda_1, \ldots, \lambda_\ell\} \end{aligned}$$

we have that

$$\frac{d}{dt}(z - T^\top\theta) = -\Lambda(z - T^\top\theta),$$

from which it is clear that $z(t) \to T^\top(t)\theta$. It follows that, if the matrix $T^\top$ is left-invertible, we can generate an asymptotically convergent estimate of θ as

$$\hat{\theta} = (T^\top)_{\mathrm{L}}z,$$

where $(T^\top)_{\mathrm{L}} \in \mathbb{R}^{q\times\ell}$ is the left-inverse of $T^\top$.

It should be underscored that the construction above is *exactly* the one proposed in Luenberger (1964) for LTI systems adapted to the particular problem when the "state" θ verifies $\dot{\theta} = 0$ and the output matrix is time varying, that is, $\phi^\top(t)$.

The crucial issues of left-invertibility of $T^\top$ and the actual computation of $(T^\top)_L$ – first discussed in Kazantzis and Kravaris (1998) – are topics of current research extensively studied in the observers literature. The key question here is the extension to non-stationary systems of the sufficient conditions reported for the stationary case in Andrieu and Praly (2006). See also Section 5 for some further discussion.

Instead of "solving" for $\hat\theta$ the equation $z = T^\top\hat\theta$ an alternative route to generate the parameter estimate is to apply a *gradient descent* method to solve the optimization problem

$$\min_{\hat\theta(t)}\{[z(t) - T^\top(t)\hat\theta(t)]^\top Q(t)[z(t) - T^\top(t)\hat\theta(t)]\} \tag{15}$$

where the matrix $Q : \mathbb{R}_+ \to \mathbb{R}^{q\times q}$ is positive definite. This yields the on-line estimator

$$\dot{\hat\theta} = \Gamma T Q(z - T^\top\hat\theta). \tag{16}$$

We are in position to present the first result of the note.

Proposition 1. *Consider the linear regression (1). The DREM estimator (5), (8), (9), (10)* exactly *coincides with the Luenberger observer (14) with the gradient descent (16) selecting*

$$\begin{aligned} \ell &= q \\ \mathcal{H}_i(p) &= \frac{-\lambda_i}{p-\lambda_i} \\ Q(t) &= adj\{T(t)\}adj\{T^\top(t)\} \\ \Gamma &= diag\{\gamma_1, \ldots, \gamma_q\}. \end{aligned} \tag{17}$$

Proof. The proof follows directly noting that, for the choices of the DREM parameters (17), we have that $\Phi = T^\top$, and the fact that

$$T\mathrm{adj}\{T\} = \Delta I_q.$$

4. DREM as a gradient descent of a functional observer for LTV Systems

The problem of functional observers for LTV systems is formulated as follows. Given a classical LTV system

$$\begin{aligned} \dot x &= A(t)x + B(t)u \\ y &= C(t)x, \end{aligned} \tag{18}$$

where $x \in \mathbb{R}^q$, $u \in \mathbb{R}^p$, $y \in \mathbb{R}^m$ and a linear functional

$$v = M(t)x \tag{19}$$

with $v \in \mathbb{R}^\ell$ we want to design an observer for the signal v. In Trumpf (2007) the following result is established.

Proposition 2. *Define a completely observable ℓ-dimensional system*

$$\begin{aligned} \dot z &= F(t)z + G(t)u + H(t)y \\ w &= P(t)z, \end{aligned} \tag{20}$$

with all the solutions of $\dot x = F(t)x$ converging to zero. The system (20) is a global asymptotic observer[3] *of the linear functional (19) for the system (18)* if and only *if there exists a continuously differentiable $n \times q$ matrix $T(t)$ solution of the equations*

$$\begin{aligned} G(t) &= T(t)B(t) \\ \dot T(t) &= F(t)T(t) - T(t)A(t) + H(t)C(t) \\ M(t) &= P(t)T(t). \end{aligned} \tag{21}$$

[3] That is, for all $x(0) \in \mathbb{R}^q$, $z(0) \in \mathbb{R}^\ell$ and all continuous, bounded inputs u we have $\lim_{t\to\infty}(v(t) - w(t)) = 0$.

To apply this result for the estimation of the parameters of the linear regression (1) we make the assignments

$$\begin{aligned} p &= m = 1 \\ x &= \theta, u = 0 \\ A(t) &= 0,\ B(t) = 0,\ C(t) = \phi^\top(t). \end{aligned}$$

The observer (20) and the conditions (21) become then

$$\begin{aligned} \dot z &= F(t)z + H(t)y, \\ w &= P(t)z \end{aligned} \tag{22}$$

and

$$\begin{aligned} \dot T(t) &= F(t)T(t) + H(t)\phi^\top(t) \\ M(t) &= P(t)T(t), \end{aligned} \tag{23}$$

respectively. Now, we select the dimension of the observer equal to the dimension of the systems state θ, that is $\ell = q$, and choose

$$M(t) = \det\{T(t)\}I_q. \tag{24}$$

The linear functional to be observed (19) takes a decoupled form

$$v_i = \det\{T(t)\}\theta_i.$$

Moreover, the conditions for existence of the functional observer (23) are satisfied with the choice

$$P(t) = \mathrm{adj}\{T(t)\}.$$

Invoking Proposition 2 we have the following result.

Proposition 3. *Consider the linear regression (1). For any observable pair $(F(t), H(t))$, with all the solutions of $\dot x = F(t)x$ converging to zero, the functional observer*

$$\begin{aligned} \dot z &= F(t)z + H(t)y \\ \dot T(t) &= F(t)T(t) + H(t)\phi^\top \\ w &= \mathrm{adj}\{T(t)\}z \end{aligned} \tag{25}$$

ensures

$$\lim_{t\to\infty}(w_i(t) - \det\{T(t)\}\theta_i) = 0$$

for all $T(0) \in \mathbb{R}^{q\times q}$ and all $z(0) \in \mathbb{R}^q$.

If $\det\{T(t)\} \neq 0$ we can obtain an estimate of θ as

$$\hat\theta = \frac{1}{\det\{T(t)\}}w.$$

To avoid the possibility of a singularity in the calculation above we can proceed as done in Section 3 to apply a *gradient descent* method to solve the optimization problem

$$\min_{\hat\theta_i(t)}\{[w_i(t) - \det\{T(t)\}\hat\theta_i(t)]^2\}. \tag{26}$$

This yields the on-line estimator

$$\dot{\hat\theta}_i = \gamma_i\det\{T(t)\}(w_i - \det\{T(t)\}\hat\theta_i). \tag{27}$$

The derivations above established the following result.

Proposition 4. *The functional observer (25) with the gradient descent (27)* exactly *coincides with the DREM estimator of Section 2 with the LTV operator*

$$\mathcal{H} = [pI_q - F(t)]^{-1}H(t)$$

and the identities

$$Y = z,\ \Phi = T,\ \mathcal{Y} = w,\ \Delta = \det\{T(t)\}.$$

5. Concluding remarks

It has been shown that the DREM parameter estimator can be interpreted as a Luenberger observer with a gradient descent search in two different ways. First, as a "standard" full state observer (14) where the gradient search is done for the quadratic criterion (15) with the weighting matrix

$$Q(t) = \mathrm{adj}\{T(t)\}\mathrm{adj}\{T^\top(t)\}.$$

Second, as a functional observer for LTV systems of the form (25) with the weighting matrix of the state functional to be observed (19) selected as (24) and the standard quadratic criterion to be minimized (26).

For the case when the dimension of the observer state z is equal to the dimension of the unknown vector θ, that is, when $\ell = q$, the problem of left invertibility of the matrix T discussed in Section 3 reduces to the full rank condition of T mentioned in Section 4, and it is hardly verified in practice. On the other hand, in DREM the necessary and sufficient convergence condition (13) of the observers gradient descent search, which is related with the previous full rank assumption, remains an essentially open question.

One interesting possibility that has been considered in the literature (Afri et al., 2017; Andrieu & Praly, 2006) is to take the dimension of z *larger* than the one of θ, that is, $\ell > q$. Propositions 3 and 4 apply *verbatim* selecting

$$Q(t) = T^\top(t)\big(\mathrm{adj}\{T(t)T^\top(t)\}\big)^2 T(t),$$

in the former, and

$$\begin{aligned} w &= \mathrm{adj}\{T^\top(t)T(t)\}T^\top(t)z \\ P(t) &= \mathrm{adj}\{T^\top(t)T(t)\}T^\top(t), \end{aligned}$$

in the latter. Unfortunately, the existing results for observers – *without* the gradient descent step – reported in Andrieu and Praly (2006), Kazantzis and Kravaris (1998), Rotella and Zambettakis (2013) and Trumpf (2007) do not seem applicable for the investigation of the non square-integrability condition (13) for convergence of DREM. Current research is under way along this direction.

It should be pointed out that an important component of DREM is the flexibility in the selection of the operator $\mathcal{H}$. It has been shown in several applications (Aranovskiy et al., 2016; Gerasimov et al., 2018; Pyrkin et al., 2017) that choosing them as LTI filters and/or delays, with different values of the coefficients α_i, β_i and d_i, is essential to guarantee the performance improvement of the estimator. The selection of these coefficients is done invoking considerations of signals bandwidth and frequency content. Although little is known on the effect of time-varying operators on a signals spectrum, choosing them to be *time-varying* –as suggested in Proposition 4 –widens our options and opens up a new interesting area of research. In this respect, the optimality results reported for the Extended Instrumental Variable method (in the stochastic framework) might prove useful also for DREM, see Gilson (2015) for a recent survey of this method.

References

Afri, C., Andrieu, V., Bako, L., & Dufour, P. (2017). State and parameter estimation: A nonlinear Luenberger observer approach. *IEEE Transactions on Automatic Control*, *62*(2), 973–980.

Andrieu, V., & Praly, L. (2006). On the existence of a Kazantzis-Kravaris-Luenberger observer. *SIAM Journal on Control and Optimization*, *45*(2), 432–456.

Aranovskiy, S., Bobtsov, A., Ortega, R., & Pyrkin, A. (2016). Improved transients in multiple frequencies estimation via dynamic regressor extension and mixing. In *12th IFAC international workshop on adaptation and learning in control and signal processing*, Eindhoven, The Netherlands, 29/06–01/07.

Aranovskiy, S., Bobtsov, A., Ortega, R., & Pyrkin, A. (2017). Performance enhancement of parameter estimators via dynamic regressor extension and mixing. *IEEE Transactions on Automatic Control*, *62*(2) (See also arXiv preprint arXiv: 1509.02763).

Astolfi, A., Karagiannis, D., & Ortega, R. (2008). *Nonlinear and adaptive control with applications*. Berlin: Springer-Verlag, Communications and Control Engineering.

Barabanov, N. E., & Ortega, R. (2016). On global asymptotic stability of $\dot{x} = -\phi(t)\phi^\top(t)x$ with $\phi(t)$ bounded and not persistently exciting. In *LSS-centralesupelec Int. report*, December 20.

Gerasimov, D., Ortega, R., & Nikiforov, V. (2018). Relaxing the high-frequency gain sign assumption in direct model reference adaptive control. *European Journal of Control* (accepted for publication).

Gilson, M. (2015). What has Instrumental Variable method to offer for system identification? In *8th IFAC international conference on mathematical modelling, MATHMOD 2015*, Vienna, Austria, February. (See also hal-01242758).

Ioannou, P., & Sun, J. (1996). *Robust adaptive control*. NJ: Prentice–Hall.

Kazantzis, N., & Kravaris, C. (1998). Nonlinear observer design using Lyapunov's auxiliary theorem. *Systems & Control Letters*, *34*(5), 241–247.

Luenberger, D. G. (1964). Observers for multivariable systems. *IEEE Transactions on Automatic Control*, *11*(2), 190–197.

Praly, L. (2016a). Convergence of the gradient algorithm for linear regression models in the continuous and discrete-time cases. In *Int. report MINES Paristech, centre automatique et systèmes*, December 26, (revised January 15, 2017). https://hal.archives-ouvertes.fr/hal-01423048.

Praly, L. (2016b). Nonlinear Luenberger observer for estimation of a linear regression model. In *CAS-Mines Int. report*, November 29.

Pyrkin, A., Mancilla-David, F., Ortega, R., Bobtsov, A., & Aranovskiy, S. (2017). Identification of photovoltaic arrays' maximum power extraction point via dynamic regressor extension and mixing. *International Journal on Adaptive Control and Signal Processing*, 1337–1349. http://dx.doi.org/10.1002/acs.2768.

Rotella, F., & Zambettakis, I. (2013). On functional observers for linear time-varying systems. *IEEE Transactions on Automatic Control*, *58*(5), 1354–1361.

Sastry, S., & Bodson, M. (1989). *Adaptive control: Stability, convergence and robustness*. New Jersey: Prentice-Hall.

Shafai, B., & Carroll, R. L. (1986). Minimal-order observer designs for linear time-varying multivariable systems. *IEEE Transactions on Automatic Control*, *AC-31*(8), 757–761.

Shoshitaishvili, A. (1992). On control branching systems with degenerate linearization. In *Proc. IFAC NOLCOS*, Bordeaux, France, June (pp. 495–500) (See also: A. Shoshitaishvili, Singularities for projections of integral manifolds with applications to control and observation problems, in Advances in Soviet Mathematics, Vol. 1: Theory of singularities and its applications, V.I. Arnol'd, Editor, 1991).

Trumpf, J. (2007). Observers for linear time-varying systems. *Linear Algebra and Applications*, *425*, 303–312.