Integral Action in Output Feedback for Multi-Input Multi-Output Nonlinear Systems

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***Abstract*—We address a particular problem of output regulation for multi-input multi-output nonlinear systems. Specifically, we are interested in making the stability of an equilibrium point and the regulation to zero of an output robust to (small) unmodelled discrepancies between design model and actual system in particular those introducing an offset. We propose a novel procedure which is intended to be relevant to realistic scenarios, as illustrated by a (non academic) example.**

***Index Terms*—Forwarding, high-gain observer, integral action, nonlinear control, non-minimum phase systems, observability, output feedback, robust regulation, semi-global stabilization, uncertain dynamic system.**

I. Introduction

FOR a controlled dynamical system, it is of prime importance in real world applications to be able to design an output feedback control law which achieves asymptotic regulation of a given output while keeping the solutions in some prescribed set, in presence of (constant) uncertainties. We refer to this as the problem of robust output regulation by output feedback.

The problem has been completely solved in the linear framework by Francis and Wonham in the 1970s (see [12]). Important efforts have been done in order to extend this result to the nonlinear case (see, for instance [9], [21]) and many different solutions have been proposed (see among others [10], [14], [23], [35], [4, Ch. 7.2], [1], [18], [25], [38]). Nevertheless, we are still far from having a complete solution to the problem of output regulation in the nonlinear multi-input-multi-output framework similar to what we have in the linear case. Indeed most of the works require a good knowledge of the effects of the disturbances on the system, or they rely on "structural properties" as, for example, *normal forms*, *minimum phase assumption*, *matched uncertainties* or *relative degree uniform in the disturbances*. In particular, for single-input single-output minimum-phase nonlinear systems which possess a well defined relative degree preserved under the effect of disturbances, a complete solution has been given in [25], further improved to the output feedback case in [37]. Under the same assumptions, this approach has been successfully extended in [38] to square multi-input multi-output systems for which the notion of relative degree indices and observability indices coincides. Furthermore, with the technique of the *auxiliary system* introduced in [20], the minimum-phase assumption has been removed in [29] allowing the *zero-dynamics* to be unstable. However, as far as we know, a general solution is still unknown when these structural properties do not hold.

The approach to nonlinear output regulation followed in this paper is motivated by the linear context developed in its full generality in the milestone paper [12] that we find useful to briefly recall here. Consider the linear system

$$\begin{array}{rcl} \dot{x} &=& A_0 x + B_0 u, \\ y &=& C_0 x, \end{array} \qquad y = \begin{pmatrix} y_r \\ y_e \end{pmatrix} = \begin{pmatrix} C_{0,r} \\ C_{0,e} \end{pmatrix} x$$

where the state x is in $\mathbb{R}^n$, the control u is in $\mathbb{R}^m$ and the measured output y is in $\mathbb{R}^p$. The output y is decomposed as $y = (y_r, y_e)$ where y_r, in $\mathbb{R}^r$, with $r \leq m$, is the output to be regulated to zero (without loss of generality). When the system above is supposed to be only an approximation of a process given by

$$\begin{aligned} \dot{x} &= Ax + Bu + Pw, \\ y &= Cx + Qw, \\ y &= \begin{pmatrix} y_r \\ y_e \end{pmatrix} = \begin{pmatrix} C_r \\ C_e \end{pmatrix} x + \begin{pmatrix} Q_r \\ Q_e \end{pmatrix} w \end{aligned}$$

where w is an unknown constant signal to be either rejected or tracked, the *well posed regulator problem with internal stability* (addressed by Wonham for linear systems as shown for instance in [46, Ch. 8]) is that of finding an output feedback law based on the model such that, for all triplets $\{A, B, C\}$ close enough to $\{A_0, B_0, C_0\}$, and for all matrices pairs $\{P, Q\}$, the regulation-stabilization problem is solved, *i.e.* the system admits a stable equilibrium point on which the output to be regulated is equal to zero. According to [9, Proposition 1.6], this problem is solvable if and only if the following three conditions are satisfied:

(A1) the pair (A_0, C_0) is detectable;
(A2) the pair (A_0, B_0) is stabilizable;
(A3) the matrix $\begin{pmatrix} A_0 & B_0 \\ C_{0,r} & 0 \end{pmatrix}$ is right invertible.

Precisely, under the 3 conditions above, it is always possible to design an output feedback law of the form

$$\begin{array}{rcl} \dot{z} &=& y_r \\ \dot{x}_c &=& F x_c + L y \\ u &=& K x_c + M z + N y \end{array}$$

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which solves the regulation problem provided F, L, K, M, and N are chosen such that the following matrix:

$$\begin{pmatrix} A+BNC & BK & BM \\ LC & F & 0 \\ C_r & 0 & 0 \end{pmatrix}$$

is Hurwitz for all triplets $\{A, B, C\}$ close enough to $\{A_0, B_0, C_0\}$, and for all matrices pairs $\{P, Q\}$, Note that in this linear framework no *structural properties* are needed.

Merging the tools available in literature, we try to recover the same result as in the linear case, asking for possibly minimal assumptions but at the same time paying particular attention to proposing a design truly manageable in applications. For example, minimality implies not to ask for any specific structural properties whereas applicability forbids nonlinear changes of coordinates when no expression is known for their inverse. Our answer to the problem uses "bricks" which can be found in other publications (as [40], [34], [5]) that we merge together. But for making this merging process efficient we have to address some (new) specific problems.

As in the linear framework, we extend the system with an integral action. Then, as in [34], we rely on forwarding to design a stabilizing state feedback for the extended system. Next, for transforming this state feedback into an output feedback, it is sufficient to apply the techniques which have been proposed for asymptotic stabilization by output feedback. A lot of effort has been devoted to this question and many results have accumulated (see for instance the survey [2]). In particular the transformation is done by replacing the actual state by a state estimate provided by a tunable observer (*i.e.*, an observer whose dynamics can be made arbitrarily fast). Stability of the overall closed-loop system is established via the common separation principle [40], [7], and output regulation follows from the integral action embedded in the control law.

The tunable observer we propose is, as in [5] (previously inspired by [11] and [30]), a high-gain observer written in the original coordinates and appropriate for our multi-input multi-output (possibly non-square) case. We propose a new set of sufficient conditions which guarantees the existence of such an observer. In contrast with what we have found in the literature (see for instance [8], [17], [15]), our conditions can be verified in the original coordinates thus not requiring the explicit knowledge of the inverse of nonlinear change of coordinates (which may be very hard to find). Also, looking for minimal assumptions, we do not ask for global observability or global uniformity with respect to the inputs. The latter impacts the state feedback design and we show how to address this point (in [34] only a global solution is proposed).

Finally, we show that the proposed solution guarantees robust regulation. Robustness is here with respect to unmodelled effects, not in the system state dimension, but in the approximations of the functions which define its dynamics and measurements. This has been done already in [34] but for the state feedback case and with an assumption on the closed-loop system. Here, we show that if the open-loop model is close enough (in a C^1 sense) to the process, then output regulation is achieved by our output feedback design. However, as opposed to the linear case, where the result is global with respect to the magnitude of the disturbances, an unfortunate consequence of being in our less restrictive context is that we need the perturbations to be small enough.

In this work, for the sake of simplicity, we restrict our attention to systems affine in the input. The extension to the non affine case is possible by considering the system controls as state and their derivatives as virtual controls. See [5] for example.

The paper is organized as follows. Section II is devoted to show the main assumptions and results of this work. In Section III and IV, we present respectively the state feedback design and the observer design. The proofs of the main propositions are given in Section V. Finally, in Section VI, we illustrate the proposed design with a non-academic example inspired from a concrete case study in aeronautics (the regulation of the flight path angle of a simplified longitudinal model of a plane). Two technical lemmas about total stability results are given in the Appendix.

I. Notations

For a set S, $\mathring{S}$ denotes its interior, ∂S denotes its boundary, and $d(x, S)$ denote the distance function of a point x to the set S. When S is a subset of $\mathcal{A} \times \mathcal{B}$ whose points are denoted (a, b), $(S)_a$ denotes the set $\{a \in \mathcal{A} : \exists b \in \mathcal{B} : (a, b) \in S\}$. For a function h and a vector field f, L_f denotes the Lie derivative of h along f, given coordinates x, $L_f h(x) = \frac{\partial h}{\partial x}(x) f(x)$. To any strictly positive real number v, we associate a "saturation" function sat_v defined as a C^1 function bounded by v and satisfying

$$\mathrm{sat}_v(s) = s \qquad \text{if} \quad |s| \leq \frac{v}{1+\varsigma} \tag{1}$$

where ς is a (small) strictly positive real number.

II. Robust Regulation by Output Feedback

A. Problem Statement and Assumptions

For a process, we have available the following dynamical model

$$\dot{x} = f(x) + g(x)u, \qquad y = h(x) = (h_r(x), h_e(x)) \tag{2}$$

where the state x is in $\mathbb{R}^n$, the control u is in $\mathbb{R}^m$, the measured output y is in $\mathbb{R}^p$ the functions $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^{nm}$ and $h : \mathbb{R}^n \to \mathbb{R}^p$ are smooth enough and f and h are zero at the origin. We investigate the problem of regulating at zero the part y_r of the output y decomposed as $y = (y_r, y_e)$ with $y_r \in \mathbb{R}^r$ and $r \leq m$ and this while stabilizing an equilibrium for x. Being aware that the triplet (f, g, h) gives only an approximation of the dynamics of the process, we would like the above regulation-stabilization property to hold not only for this particular triplet but also for any other one in a neighborhood.

The real process is described by equations of the form

$$\dot{x} = \xi(x, u), \qquad y = \zeta(x, u) \tag{3}$$

where the functions $\xi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $\zeta : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ are assumed continuously differentiable (C^1). These functions are unknown but we assume that they are close enough to $f + gu$ and h, respectively, in the sense that the discrepancies

$$|\xi(x, u) - f(x) - g(x)u| \ + \ |\zeta(x, u) - h(x)|$$

and

$$\left|\begin{pmatrix} \frac{\partial \xi}{\partial x}(x,u) - \frac{\partial f}{\partial x}(x) - \frac{\partial g}{\partial x}(x)u & \frac{\partial \xi}{\partial u}(x,u) - g(x) \\ \frac{\partial \zeta}{\partial x}(x,u) - \frac{\partial h}{\partial x}(x) & \frac{\partial \zeta}{\partial u}(x,u) \end{pmatrix}\right|$$

are small enough as made precise later on.

Mimicking the 3 necessary and sufficient conditions for the linear case given in the introduction, we consider the following (sufficient) assumptions that we discuss after their formal statement.

Assumption 1: There exist an open set $\mathcal{O}$ of $\mathbb{R}^n$ containing the origin and an open star-shaped subset $\mathcal{U}$ of $\mathbb{R}^m$, with the origin as star-center, such that, for any strictly positive real number $\bar{u}$ and for any compact subset $\mathfrak{C}$ of $\mathcal{O}$, there exist an integer d, a compact subset $\widehat{\mathfrak{C}}$ of $\mathcal{O}$, a real number $\overline{U}$ and a class-$\mathcal{K}^\infty$ function $\underline{\alpha}$ such that, for each each integer κ, we can find C^1 functions $\vartheta_\kappa : \mathbb{R}^m \times \mathbb{R}^p \times \mathcal{O} \to \mathcal{O}$, $U_\kappa : \mathcal{O} \times \mathcal{O} \to \mathbb{R}_{\geq 0}$, a continuous function $L_{\vartheta_\kappa} : \mathcal{O} \to \mathbb{R}_{\geq 0}$ and a strictly positive real number σ_κ, such that:

1) for any function $t \to u(t)$ with values in $\mathcal{U}(\bar{u})$ defined as

$$\mathcal{U}(\bar{u}) = \{u \in \mathcal{U} \,:\, |u| \leq \bar{u}\} \tag{4}$$

and any bounded function $t \to y(t)$, the set $\widehat{\mathfrak{C}}$ is forward invariant by the flow generated by the following observer

$$\dot{\hat{x}}_\kappa = \vartheta_\kappa(y, \hat{x}_\kappa, u); \tag{5}$$

2) $\forall (x,\hat{x}) \in \mathcal{O} \times \mathcal{O}, \quad U_\kappa(x,\hat{x}) = 0 \quad \Longleftrightarrow \quad x = \hat{x};$

3) $\sigma_\kappa^{-d}\,\underline{\alpha}(|x-\hat{x}|) \leq U_\kappa(x,\hat{x}) \leq \sigma_\kappa^d\,\overline{U} \quad \forall x \in \mathfrak{C}, \forall \hat{x} \in \widehat{\mathfrak{C}};$ (6)

4) $\lim_{\kappa\to\infty} \sigma_\kappa = +\infty$ (7)

5)
$$\frac{\partial U_\kappa}{\partial x}(x,\hat{x}_\kappa)[f(x)+g(x)u] + \frac{\partial U_\kappa}{\partial \hat{x}_\kappa}(x,\hat{x}_\kappa)\vartheta_\kappa(h(x),\hat{x}_\kappa,u) \leq -\sigma_\kappa\, U_\kappa(x,\hat{x}_\kappa) \quad \forall\, u \in \mathcal{U}(\bar{u}), \quad \forall\,(x,\hat{x}_\kappa) \in \mathfrak{C} \times \widehat{\mathfrak{C}}\,. \tag{8}$$

6. For all $(y_a, y_b, \hat{x}_\kappa, u)$ in $\mathbb{R}^{2p} \times \mathcal{O} \times \mathcal{U}(\bar{u})$,

$$|\vartheta_\kappa(y_a,\hat{x}_\kappa,u) - \vartheta_\kappa(y_b,\hat{x}_\kappa,u)| \;\leq\; L_{\vartheta_\kappa}(\hat{x}_\kappa)\,|y_a - y_b| \tag{9}$$

Assumption 2: There exist an open subset $\mathcal{S}$ of $\mathbb{R}^n$ and a continuous function $\beta : \mathcal{S} \to \mathcal{U}$ which is zero at the origin and such that the origin of (2) with $u = \beta(x)$, is an asymptotically and locally exponentially stable equilibrium point with $\mathcal{S}$ as domain of attraction.

Assumption 3: The matrix

$$\begin{pmatrix} \frac{\partial f}{\partial x}(0) & g(0) \\ \frac{\partial h_r}{\partial x}(0) & 0 \end{pmatrix} \tag{10}$$

is right invertible.

Assumption 1 aims to be a counter-part of the detectability condition (A1). But we have to face problems specific to this nonlinear framework.

1) In our construction, we shall rely on the so called separation principle. For nonlinear systems (see [40] for example), it asks for an observer with a *tunability* property, *i.e.*, an observer the speed of convergence of which can be made arbitrary fast (see [27]). This property is provided here by the family of observers (5) satisfying (6)–(8).
2) Observability may depend on the input. This explains why we impose the control to belong to the set $\mathcal{U}$.
3) The tuning of observers for non linear systems may depend on the local Lipschitz constant of the non linearities. This explains why the family of observers depends on the bound $\bar{u}$ of the input.

On the other hand, to reduce the restrictiveness, Assumption 1 is imposed only for system states belonging to an open subset $\mathcal{O}$ of $\mathbb{R}^n$. In Section IV we shall see how the family of observers in this assumption can be designed by following standard high-gain techniques.

Assumption 2 is the counter-part of the stabilizability condition (A2) and claims the existence of a state feedback law which asymptotically stabilizes the system (2). Actually it assumes that a preliminary design step can be done. For it any tool—Lyapunov design, feedback (partial) linearization, passivity, use of structure of uncertainties in combination with gain assignment techniques, etc.—can be exploited. However, because Assumption 1 imposes the control to be in $\mathcal{U}$, we propagate this restriction here, asking the stabilizing control β to take values in that set. On the other hand, we can cope with having an arbitrary domain of attraction $\mathcal{S}$, without asking it to be the full space or any arbitrarily large compact set.

Finally, Assumption 3 corresponds to the non-resonance condition (A3) and states that the first order approximation at the origin of the system (2) does not have any zero at 0.

B. Adding an Integral Action

To solve the problem of regulating y_r to 0 we follow the very classical idea of adding an integral action, namely we consider the extended system

$$\dot{x} = f(x) + g(x)u, \quad \dot{z} = k(x, h_r(x)) \tag{11}$$

where $k : \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^r$ is a C^1 function satisfying,[1] for all x in $\mathbb{R}^n$ and all (y_r^a, y_r^b) in $\mathbb{R}^r \times \mathbb{R}^r$

$$k(x, y_r) = 0 \qquad \Leftrightarrow \qquad y_r = 0\,, \tag{12}$$

$$|k(x,y_r^a) - k(x,y_r^a)| \;\leq\; L_k(x)\,|y_r^a - y_r^b| \tag{13}$$

where $L_k : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a continuous function. Of course the function k can be simply h_r. But, in its choice, we can take advantage of the properties of the physical system under consideration and it can simplify the feedback design or its implementation. An example is given in Section VI. Further details about the design of k will be given in Section III.

As shown in the forthcoming section, the z-dynamics, representing the so-called *internal model unit* in the Francis-Wonham terminology, assures the output y_r to be regulated to zero in presence of uncertainties if the trajectories of the closed-loop system are bounded.

[1]When $L_g L_f^i h_r(x) = 0$, for i in $\{0,\ldots,\rho\}$, (12) can be relaxed in $\left\{ k(x,h_r(x)) = 0,\, L_f h_r(x) = \ldots = L_f^{\rho-1} h_r(x) = 0 \right\} \;\Rightarrow\; h_r(x) = 0$. See [38] for example.

C. Main Results

Assumptions 1 to 3 are sufficient to guarantee the existence of an output feedback law solving the regulation-stabilization problem for the model (2).

Proposition 1: Suppose Assumptions 1, 2 and 3 hold. There exists an open subset $\mathcal{SO}$ of $(\mathcal{S} \cap \mathcal{O}) \times \mathbb{R}^r$ such that, for any of its compact set $\mathcal{C}_{xz}$, there exist an integer $\underline{\kappa}$, a compact subset $\mathcal{C}_{\hat{x}}$ of $\mathcal{O}$, a real number μ, a C^1 function[2] $k : \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^r$ satisfying (12), (13), and a C^1 $\psi_{sat} : \mathbb{R}^n \times \mathbb{R}^r \to \mathcal{U}(\mu)$, such that the origin of the model (2), in closed-loop with the dynamic output feedback

$$\dot{z} = k(\hat{x}, h_r(x)), \quad \dot{\hat{x}} = \vartheta_\kappa(y, \hat{x}, u) \ , \quad u = \psi_{sat}(\hat{x}, z) \quad (14)$$

with $\kappa \geq \underline{\kappa}$, is asymptotically stable and locally exponentially stable with a domain of attraction $\mathcal{A}$ containing the set $\mathring{\mathcal{C}}_{\hat{x}} \times \mathcal{C}_{xz}$.

Proof: This proof follows the same lines as in [5], inspired by [19, Ch. 12.3]. We omit it to save space. It can be found in [6]. ■

In the case where $\mathcal{S}$ and $\mathcal{O}$ are the full space $\mathbb{R}^n$, this result would be a semi-global regulation-stability result. It claims the existence of a dynamic output feedback which asymptotically stabilizes the origin of the model (2). Such a result is not new per se. It is in line with many results related to the separation principle as those in [40], [7] or [19, Ch. 12.3]. However, as written in the introduction, we do not state only "existence" but instead we propose an explicit and workable design. For example, we refer the reader to Section III for the definition[3] of the set $\mathcal{SO}$, the real number μ and the functions k.

In the following propositions, under the Assumptions 1, 2 and 3 and knowing the result of Proposition 1 holds, we study the process (3) in closed-loop with the control law (14) designed for the model (2).

Proposition 2: Let $\overline{\mathfrak{C}}$ be an arbitrary compact subset of the domain of attraction $\mathcal{A}$, given by Proposition 1, which admits the equilibrium as an interior point and is forward invariant for the closed-loop system (2), (14). For any open neighborhood $\mathcal{N}_{\partial\overline{\mathfrak{C}}}$ of the boundary set $\partial\overline{\mathfrak{C}}$, contained in $\mathcal{A}$, there exists a strictly positive real number δ such that, for any pair (ξ, ζ) of C^1 functions which satisfies

$$\begin{aligned} &|\xi(x,u) - [f(x) + g(x)u]| + |\zeta(x,u) - h(x)| \leq \delta \\ &\forall\, (x,u) \in \left(\mathcal{N}_{\partial\overline{\mathfrak{C}}}\right)_x \times \mathcal{U}(\mu) \end{aligned} \quad (15)$$

the closed-loop system (3), (14) has equilibria and at any such point the output y_r is zero.

Proof: See Section V-A. ■

If the domain of attraction were the full space, this result would follow from [39, Section 12]. It says that, when the evaluation, on a "spherical shell"-like set, of the model and process functions are close enough, equilibria where output regulation occurs do exist. If this closeness is everywhere in the domain of attraction, then we have even a solution to the well-posed regulator problem with internal stability.

Proposition 3: For any compact sets $\underline{\mathfrak{C}}$ and $\overline{\mathfrak{C}}$, the latter being forward invariant for the closed-loop system (2), (14), which satisfy

$$\{0\} \subsetneq \underline{\mathfrak{C}} \subsetneq \overline{\mathfrak{C}} \subsetneq \mathcal{A}$$

and for any open neighborhood $\mathcal{N}_{\overline{\mathfrak{C}}}$ of $\overline{\mathfrak{C}}$, contained in $\mathcal{A}$, there exists a strictly positive real number δ such that, to any pair (ξ, ζ) of C^1 functions which satisfies

$$\begin{aligned} &|\xi(x,u) - [f(x) + g(x)u]| + |\zeta(x,u) - h(x)| \leq \delta \\ &\forall\, (x,u) \in \overline{\mathfrak{C}}_x \times \mathcal{U}(\mu) \end{aligned} \quad (16)$$

and

$$\begin{aligned} &\left| \begin{pmatrix} \frac{\partial \xi}{\partial x}(x,u) & \frac{\partial \xi}{\partial u}(x,u) \\ \frac{\partial \zeta}{\partial x}(x,u) & \frac{\partial \zeta}{\partial u}(x,u) \end{pmatrix} - \begin{pmatrix} \frac{\partial f}{\partial x}(x) + \frac{\partial g}{\partial x}(x)u & g(x) \\ \frac{\partial h}{\partial x}(x) & 0 \end{pmatrix} \right| \\ &\leq\ \delta \qquad \forall (x,u) \in \underline{\mathfrak{C}}_x \times \mathcal{U}(\mu) \end{aligned} \quad (17)$$

we can associate a point $\mathcal{x}_e = (x_e, z_e, \hat{x}_e)$ which is an exponentially stable equilibrium point of (3), (14) whose basin of attraction $\mathcal{B}$ contains $\overline{\mathfrak{C}}$. Moreover, any solution $(X(\mathcal{x},t), Z(\mathcal{x},t), \hat{X}(\mathcal{x},t))$ of (3), (14) with initial condition $\mathcal{x}$ in $\mathcal{B}$ satisfies

$$\lim_{t \to +\infty} \zeta_r \left(X(\mathcal{x},t), \psi_{sat}\left(\hat{X}(\mathcal{x},t), Z(\mathcal{x},t)\right)\right) = 0 \, .$$

Proof: See Section V-B. ■

This statement is in the same spirit of those claiming that under the action of (small) perturbations, asymptotic stability is transformed into semiglobal practical stability. However the result stated in Proposition 3 is more general since it claims the existence of a single equilibrium for which the regulated output is zero and it does not require any specific structure of the unmodelled effects.

III. State Feedback Design

A. Design of the State Feedback via Forwarding

In this section we consider the extended system (11) with k any C^1 satisfying (12), (13). Thanks to Assumption 2, we are left with modifying the given state feedback β to obtain a state feedback stabilizing asymptotically the origin for the extended system (11). It is worth noticing that system (11) possesses the so-called feedforward form: this particular structure has been extensively studied in the 90's, in particular by means of the forwarding techniques based on saturations as in [42], [24], or on Lyapunov design with coordinate change as in [31] or coupling term as in [22]. We recall briefly these techniques. They differ on the available knowledge they require. Specifically, Assumption 2 has two consequences:

1. With the converse Lyapunov theorem of [28], we know there exists a C^1 function $V : \mathcal{S} \to \mathbb{R}_{\geq 0}$ which is positive definite and proper on $\mathcal{S}$ and such that the function $x \mapsto \frac{\partial V}{\partial x}(x)\Big(f(x) + g(x)\beta(x)\Big)$ is negative definite on $\mathcal{S}$ and upperbounded by a negative definite quadratic form of x in a neighborhood of the origin.
2. Since the origin of the system (2) in closed-loop with $\beta(x)$ is locally exponentially stable, there exists (see [31,

[2] See the modification given later in (30).

[3] See respectively (25) and (46) for $\mathcal{SO}$, (28) for μ, (23) and (30) for k and (29) for ψ_{sat}.

Lemma IV.2]) a C^1 function $H : \mathcal{S} \to \mathbb{R}^r$ satisfying

$$\frac{\partial H}{\partial x}(x)\left(f(x)+g(x)\beta(x)\right)=k(x,h_r(x))\,,\ H(0)=0. \tag{18}$$

Depending on whether or not we know the function V and/or the function H or only its first-order approximation at the origin leads to different designs.

a) *Forwarding with V and H known*

When both V and H are known, a stabilizer ψ for the system (11), is

$$\psi(x,z) = \beta(x) - J(x,[L_gV(x)-(z-H(x))^\top L_gH(x)]^\top)$$

with H defined by (18), and with $J : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ any continuous function satisfying, for all $x \in \mathbb{R}^n$,

$$v^\top J(x,v) > 0 \quad \forall\, v \neq 0\,, \quad \det\left(\frac{\partial J}{\partial v}(x,0)\right) \neq 0. \tag{19}$$

Following [31], this can be established under Assumptions 2 and 3 with the function $V_e : \mathcal{S}\times\mathbb{R}^r \to \mathbb{R}_{\geq 0}$ defined as

$$V_e(x,z) = V(x) + \frac{1}{2}(z-H(x))^\top(z-H(x)). \tag{20}$$

Remark 1: If V is known from the design of β, it may not be proper on $\mathcal{S}$. To make it proper we first define $v_{\mathcal{S}}$ as

$$v_{\mathcal{S}} = \inf_{x\notin\mathcal{S}} V(x)$$

and we replace $V(x)$ by $\frac{V(x)}{v_{\mathcal{S}}-V(x)}$. See [41]. Unfortunately, with this modification, the domain of definition of this new function V may be a strict subset of $\mathcal{S}$. In the following, we still call $\mathcal{S}$ this domain on which V is proper.

b) *Forwarding with V unknown but H known*

When V is unknown, but H is known, there exists a function $\gamma : \mathcal{S} \to \mathbb{R}_{\geq 0}$ with strictly positive values such that a state feedback for the system (11) is

$$\psi(x,z) = \beta(x) + \gamma(x) L_gH(x)^\top J(x, z-H(x)) \tag{21}$$

with H defined by (18), and $J : \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^r$ bounded and satisfying (19). This can be established with the Lyapunov function (20).

c) *Forwarding with V unknown and H approximated*

Instead of solving the partial differential equation (18) for H, and using (21), we pick

$$\psi(x,z) = \beta(x) + \gamma(x) g(x)^\top {H_0}^\top J(x, z-H_0x)$$

where H_0 is obtained as

$$H_0 = \frac{\partial k}{\partial y_r}(0,0)\frac{\partial h_r}{\partial x}(0)\left[\frac{\partial f}{\partial x}(0)+g(0)\frac{\partial \beta}{\partial x}(0)\right]^{-1}. \tag{22}$$

The corresponding Lyapunov function is

$$V_e(x,z) = d(V(x)) + \sqrt{1+\tfrac{1}{2}(z-H_0x)^\top(z-H_0x)} - 1$$

where $d : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a C^1 function with strictly positive derivative, to be chosen large enough (see [31]). In the case where the system

$$\dot{x} = f(x) + g(x)(\beta(x)+v)$$

with v as input is input to state stable with restriction, *i.e.*, provided $|v|$ is bounded by some given strictly positive real number Δ, then following [42], the state feedback can be chosen as

$$\psi(x,z) = \beta(x) + \epsilon J\left(x, \frac{g(0)^\top H_0^\top(z-H_0x)}{\epsilon}\right)$$

with $J : \mathbb{R}^x \times \mathbb{R}^m \to \mathbb{R}^m$ bounded and satisfying (19) and ϵ is a small enough strictly positive real number.

Whatever design route a), b) or c) we follow, we obtain the following lemma.

Lemma 1: Under Assumptions 2 and 3, the function V_e is positive definite and proper on $\mathcal{S}\times\mathbb{R}^r$. Its derivative along the extended system (11) in closed-loop with $u=\psi(x,z)$ is negative definite on $\mathcal{S}\times\mathbb{R}^r$ and upperbounded by a negative definite quadratic form of (x,z) in a neighborhood of the origin. Consequently, for the corresponding closed-loop system, the origin is asymptotically stable with $\mathcal{S}\times\mathbb{R}^r$ as domain of attraction[4] and locally exponentially stable.

Proof: Since V is positive definite and proper on $\mathcal{S}$, V_e is positive definite and proper on $\mathcal{S}\times\mathbb{R}^r$. Also the derivative of V_e along the solutions of the closed oop system is negative definite in $(x,\psi(x,z))$ and upperbounded by a negative definite quadratic form of $(x,\psi(x,z))$ in a neighborhood of the origin (see [31], [42] for example). With this, to complete the proof, it is sufficient to show the existence of a real number $c>0$ such that

$$|z| \leq c\,|\psi(0,z)|\,.$$

Since we have

$$\psi(0,z) = J(0, L_gH(0)z)\ \text{, respectively} = J(0, H_0g(0)z)$$

where the function J satisfies (19), the above inequality holds if $L_gH(0)$, respectively, $H_0g(0)$, is right invertible. Note that smoothness of k and (12) implies

$$\frac{\partial k}{\partial x}(x,0) = 0 \qquad \forall\, x\in\mathbb{R}^n\,.$$

As a consequence by differentiating (18) which holds at least in a neighborhood of the origin, using (22), and since f and β are zero at the origin, we have

$$\frac{\partial H}{\partial x}(0) = H_0\,.$$

Assume the matrix $H_0g(0)$ is not right invertible, *i.e.* the exists a vector v in $\mathbb{R}^r$ such that

$$v^\top H_0 g(0) = 0\,.$$

[4]Recall Remark 1.

Then we have

$$\begin{pmatrix} v^\top H_0 & -v^\top \frac{\partial k}{\partial y_r}(0,0) \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial x}(0) & g(0) \\ \frac{\partial h_r}{\partial x}(0) & 0 \end{pmatrix} = 0$$

which contradicts Assumption 3.

Remark 2:

1) Because the set $\mathcal{U}$ in Assumption 1 is star-shaped, while satisfying (19), the function J can always be chosen such that the function ψ above defined takes values in $\mathcal{U}$.
2) A drawback of the integral action is the possible wind-up. To prevent this phenomenon, in all the above, $\dot{z}$ can be modified in

$$\dot{z} = k(x, y_r) + c\,[\mathrm{sat}_{\bar{z}}(z + H(x)) - (z + H(x))] \tag{23}$$

with $H(x)$ replaced by $H_0 x$ when needed and where the saturation function is defined in (1), $c > 0$ is a constant real number and $\bar{z} > 0$ should be chosen large enough to allow the z-dynamics to converge to the right equilibrium point. This modification does not change anything to the asymptotic stability which can be established with the same Lyapunov functions.

B. Adding Saturations for Output Feedback Design

If we were to design a state feedback, we could stop here. But the output feedback we design is based on the previous state feedback and augmented with an observer. Since during the transient the estimated state may differ consistently form the real state, we need a mechanism to prevent any bad closed-loop effects during these periods. As proposed in [26], we use saturation.

First we define the set $\mathcal{SO}$ where we would like the state to be confined. For this, let $\mathcal{S}$ be given by Assumption 2, maybe modified as explained in Remark 1 above. Similarly, let $\mathcal{O}$ be given by Assumption 1 (maybe modified later as in (44)). Let also the function V_e, positive definite and proper on $\mathcal{S} \times \mathbb{R}^r$, be given by the above design of the state feedback or a converse Lyapunov theorem [28] satisfying

$$\begin{aligned} &\dot{V}_e(x,z) \\ &= \frac{\partial V_e}{\partial x}(x,z)[f(x) + g(x)\psi(x,z)] + \frac{\partial V_e}{\partial z}(x,z)k(x, h_r(x)) \\ &= -W_e(x,z) \end{aligned} \tag{24}$$

where the function W_e defined here is positive definite on $\mathcal{S} \times \mathbb{R}^r$. Then, if $\mathcal{S}$ is not a subset of $\mathcal{O}$, we let v_∞ be the real number defined as

$$\mathrm{v}_\infty = \inf_{(x,z) \in (\mathcal{S}\times\mathbb{R}^r)\setminus(\mathcal{O}\times\mathbb{R}^r)} V_e(x,z) \, .$$

If not, formally let v_∞ be infinity. We define the open set[5]

$$\mathcal{SO} = \{(x,z) \in \mathcal{S} \times \mathbb{R}^r \ : \ V_e(x,z) < \mathrm{v}_\infty\} \, . \tag{25}$$

This set in non empty since it contains the origin.

In the same way, to each real number v in $[0, \mathrm{v}_\infty)$ we associate the set

$$\Omega_\mathrm{v} = \{(x,z) \in \mathcal{S} \times \mathbb{R}^r \ : \ V_e(x,z) \le \mathrm{v}\} \, . \tag{26}$$

[5]See the further modification (46).

It is a compact subset of $\mathcal{SO}$. Also, from Lemma 1, it is forward invariant for the extended system (11) in closed-loop with $u = \psi(x,z)$. On the other hand, for any $\mathcal{C}_{xz}$, compact subset of $\mathcal{SO}$, we can find real numbers $\mathrm{v}_1 < \mathrm{v}_2$ satisfying

$$\mathcal{C}_{xz} \subsetneq \Omega_{\mathrm{v}_1} \subsetneq \Omega_{\mathrm{v}_2} \subsetneq \mathcal{SO} \, . \tag{27}$$

Then, with μ the real number defined as

$$\mu = (1 + \varsigma) \max_{(x,z) \in \Omega_{\mathrm{v}_2}} |\psi(x,z)| \tag{28}$$

with ς a small number as in (1), we consider the subset $\mathcal{U}(\mu) \subset \mathcal{U}$ (see (4)). As $\mathcal{U}$ in Assumption 1, it is star-shaped with the origin as a star-center. Let then the function $\psi_{sat} : \mathbb{R}^n \times \mathbb{R}^r \to \mathcal{U}(\mu)$ be

$$\psi_{sat}(x,z) = \mathrm{sat}_\mu(\psi(x,z)) \, . \tag{29}$$

It is bounded and Lipschitz and, as ψ, it is C^1 on a neighborhood of the origin. Similarly, we modify the function k (defined in (12)) by saturating its argument x. Namely, we replace

$$k(x, h(x)) \qquad \text{by} \qquad k(\mathrm{sat}_{\bar{x}}(x), h_r(x)) \tag{30}$$

with $\bar{x} = (1+\varsigma)\max_{(x,z)\in\Omega_{\mathrm{v}_2}} |x|$.

IV. Observer Design

In this section we focus on the design of *tunable* observers of the form (5) satisfying Assumption 1, and we refer in particular to high-gain observers. A lot of attention has been devoted to this type of observers and many results are available at least for the single output case. See for example the survey [27] and the references therein. Here, we are interested in some specific aspects as

(a) the possibility of writing the dynamics of the observer in the original coordinates;
(b) the multi-output case; as far as we know at the time we write this text, the study of tunable observers in the multi-output case is far from being conclusive. Only some sufficient conditions are known (see, for instance [27], [43], [17], [18], [15], [7]);
(c) the fact that observability holds only on $\mathcal{O}$, a (possibly) strict subset of the full space $\mathbb{R}^n$.

To introduce them, we find useful to start with a very brief reminder on single output high gain observers.

A. Reminder on High Gain Observers in the Single Output Case

It is known (see [15, Theorem 3.4.1] for example) that, for a single-input single-output system of the form

$$\dot{x} = f(x) + g(x)u, \quad y = h(x), \quad x \in \mathbb{R}^n, \ u, y \in \mathbb{R} \tag{31}$$

which is observable uniformly with respect to the input and is differentially observable of order n_o, there exists an injective immersion $\Phi : \mathbb{R}^n \to \mathbb{R}^{n_o}$, obtained as

$$\eta = \Phi(x) = \begin{pmatrix} h(x) & L_f h(x) & \cdots & L_f^{n_o - 1} h(x) \end{pmatrix}^\top \tag{32}$$

which puts the system (31) into the so called observability (triangular) normal form

$$\dot{\eta} = A_{n_o}\eta + B_{n_o} b(\eta) + D_{n_o}(\eta)u, \qquad y = C_{n_o}\eta \tag{33}$$

where

$$A_{n_o} = \begin{pmatrix} 0_{n_o-1\times 1} & I_{n_o-1\times n_o-1} \\ 0 & 0_{1\times n_o-1} \end{pmatrix},\ B_{n_o} = \begin{pmatrix} 0_{n_o-1\times 1} \\ 1 \end{pmatrix},$$

$$C_{n_o} = (1 \quad 0_{1\times n_o-1}),$$

$$D_{n_o}(\eta) = (d_1(\eta_1), \ldots, d_i(\eta_1,\ldots,\eta_i), \ldots, d_{n_o}(\eta))^\top \qquad (34)$$

and where $b(\cdot)$, $d_i(\cdot)$ are locally Lipschitz function. An observer for the system (31) is

$$\begin{aligned}\dot{\hat\eta} &= A_{n_o}\hat\eta + B_{n_o} b(\hat\eta) + D_{n_o}(\hat\eta)u \\ &\quad + K_{n_o} L_{n_o}(\ell)(y - C_{n_o}\hat\eta), \\ \hat x &= \Phi^{\ell-inv}(\hat\eta),\end{aligned} \qquad (35)$$

where K_{n_o} is such that $(A_{n_o} - K_{n_o} C_{n_o})$ is Hurwitz, $L_{n_o}(\ell) = \mathrm{diag}(\ell, \ldots, \ell^{n_o})$ and $\Phi^{\ell-inv}$ is any locally Lipschitz left inverse function of Φ satisfying

$$\Phi^{\ell-\mathrm{inv}}(\Phi(x)) = x \qquad \forall x \in \mathbb{R}^n \,.$$

In the η-coordinates, it is a standard high gain observer the dynamics of which can be made arbitrary fast by increasing the high-gain parameter ℓ (see for instance [8]).

B. On the Possibility of Writing the Dynamic of the Observer in the Original Coordinates

As already observed in [30], a main issue in implementing the observer (35) is about the function $\Phi^{\ell-\mathrm{inv}}$ for which we have typically no analytical expression, meaning that we have to solve on-line a minimization problem as

$$\hat x = \mathrm{argmin}_x |\eta(x) - \hat\eta| \,.$$

Luckily, as noticed in [11] and also proposed in [30], this difficulty can be rounded when Φ is a diffeomorphism. Indeed in this case η is simply another set of coordinates for x and the observer (35) can be simply rewritten in the original x coordinates as

$$\dot{\hat x} = f(\hat x) + g(\hat x)u + \left(\frac{\partial\Phi}{\partial x}(\hat x)\right)^{-1} K_n L_n(\ell)(y - h(\hat x)) \,.$$

As a consequence there is no need to find the inverse mapping of the function Φ but, (infinitely) more simply, only to invert the matrix $\frac{\partial\Phi}{\partial x}(\hat x)$. But for Φ to be a diffeomorphism, we need n_o to be equal to n, i.e., to have the (full order) observer to have the smallest possible dimension.

C. High Gain Observer in the Multi-Output Case

As shown in [43], in the multi-input multi-output case (2), a typical expression for Φ is

$$\Phi(x) = \Big(\Phi_1(x) \quad \cdots \quad \Phi_p(x)\Big)^\top,$$

$$\Phi_i(x) = \Big(h_i(x) \quad L_f h_i(x) \quad \cdots \quad L_f^{p_i} h_i(x)\Big)^\top \qquad (36)$$

where h_i is the i-th component of h and p_i are integers called the observability indexes and $\sum_{i=1}^p p_i \geq n$. The dynamics of system (2) expressed in these coordinates is

$$\dot\eta = A\eta + B\bar b(\eta) + D(\eta)u, \qquad y = C\eta \qquad (37)$$

where

$$\begin{aligned} A &= \mathrm{blckdiag}\left(A_{p_1}, \ldots, A_{p_p}\right), \\ B &= \mathrm{blckcol}\left(B_{p_1}, \ldots, B_{p_p}\right), \\ C &= \mathrm{blckrow}\left(C_{p_1}, \ldots, C_{p_p}\right), \\ \bar b(\eta) &= \mathrm{col}\big(b_1(\eta), \ldots, b_p(\eta)\big), \\ D(\eta) &= \mathrm{blckcol}\left(D_{p_1}(\eta), \ldots, D_{p_p}(\eta)\right) \end{aligned}$$

where $\bar b(\eta)$ and $D(\eta)$ are locally Lipschitz functions. However, even when the system is observable uniformly in the input, the functions $\bar b$ and D may not have the triangular structure we need for the design of a high-gain observer. Conditions under which we do get triangular dependence for $\bar b(\eta)$ and $D(\eta)$ have been studied for instance in [8] and [17]. Following the idea of writing the observer in the original coordinates and imposing Φ to be a diffeomorphism, an alternative condition under which we have an appropriate structure is given by the following (technical) assumption, for which we do not need to know the inverse of Φ.

Assumption 4: There exist

i) an open set $\mathcal{O} \subset \mathbb{R}^n$ containing the origin and a star-shaped set $\mathcal{U}$ with the origin as star-center,

ii) a C^1 function $\Phi : \mathcal{O} \to \mathbb{R}^n$,

iii) sequences of matrices $L_\ell \in \mathbb{R}^{n\times n}$, $M_\ell \in \mathbb{R}^{n\times n}$ and $N_\ell \in \mathbb{R}^{p\times p}$, a matrix $C \in \mathbb{R}^{p\times n}$, with L_ℓ and M_ℓ invertible,

iv) matrix functions $u \in \mathcal{U} \mapsto K(u) \in \mathbb{R}^{n\times p}$ and $u \in \mathcal{U} \mapsto A(u) \in \mathbb{R}^{n\times n}$,

v) and, for any positive real number $\bar u$, there exist a positive definite symmetric matrix $P \in \mathbb{R}^{n\times n}$ and strictly positive real numbers ν and d,

such that

O1) the function Φ is a diffeomorphism on the set $\mathcal{O}$ and $\Phi(0) = 0$,

O2) $C\,\Phi(x) = h(x)$,

O3) the matrices $A(u), K(u), P, C$ satisfy, for any $u \in \mathcal{U}(\bar u)$,

$$\begin{aligned} P(A(u) - K(u)C) + (A(u) - K(u)C)^\top P &\leq -2\nu P \,, \\ A(u)\, L_\ell = L_\ell\, M_\ell\, A(u) \,, \quad N_\ell\, C\, L_\ell &= C \end{aligned}$$

O4) the matrix M_ℓ is such that $M_\ell P^{-1}$ is symmetric and satisfies

$$\lim_{\ell\to+\infty} \lambda_{\min}(M_\ell P^{-1}) = +\infty$$

O5)
$$\begin{aligned} \lambda_{\max}\left(L_\ell M_\ell P^{-1} L_\ell^\top\right) &\leq \lambda_{\min}(M_\ell P^{-1})^d, \\ 1 &\leq \lambda_{\min}\left(L_\ell M_\ell P^{-1} L_\ell^\top\right) \lambda_{\min}(M_\ell P^{-1})^d \,. \end{aligned}$$

Moreover, for any compact set $\mathfrak{C}$ and $\widehat{\mathfrak{C}}$ satisfying

$$\mathfrak{C} \subset \widehat{\mathfrak{C}} \subset \mathcal{O}$$

there exists a sequence of positive real numbers c_ℓ such that

O6) $\lim_{\ell\to+\infty} c_\ell = 0$,

O7) the function $B : \mathbb{R}^{n\times m} \to \mathbb{R}^n$ defined as

$$B(\Phi(x), u) = L_f \Phi(x) + L_g \Phi(x) u - A(u)\, \Phi(x) \tag{38}$$

satisfies, for all $x_a \in \mathfrak{C}$, $x_b \in \widehat{\mathfrak{C}}$ and $u \in \mathcal{U}(\bar{u})$,

$$\left|P^{\frac{1}{2}} M_\ell^{-1} L_\ell^{-1} \left[B(\Phi(x_a), u) - B(\Phi(x_b), u)\right]\right| \leq c_\ell \left|P^{\frac{1}{2}} L_\ell^{-1} \left[\Phi(x_a) - \Phi(x_b)\right]\right| . \tag{39}$$

Remark 3:

1) As shown in the next Lemma, the existence of a high-gain observer for the system (2) is guaranteed if Assumption 4 holds. In particular the properties O1, O2, O3, O6, and O7 guarantee the existence of a converging observer in the original coordinates whereas properties O4 and O5 assure its tunability property.
2) We remark that these conditions can be checked without need of finding formally the inverse mapping Φ^{-1}. In particular, given a system and a candidate diffeomorphism Φ (property O1), one can immediately check properties O2 (linear dependence of the diffeomorphism on the output) Then, if this properties holds, one can fix the degrees of freedom $K(u)$, M_ℓ, N_ℓ, L_ℓ, P which properly defines the high-gain observer as shown later in Lemma 2 (see (40)) and check also the Lipschitz condition (39) in 07. Finally, property O3 guarantees the convergence of the observer (see proof of Lemma 2).
3) The conditions of Assumption 4 are satisfied in the single-output case considered in Section IV-A when $n_o = n$, by choosing Φ as in (32), and picking

$$L_\ell = \mathrm{diag}(1, \ell, \ldots, \ell^{n-1}) , \quad M_\ell = I_n \ell , \quad N_\ell = 1$$
$$A(u) = A_n, \quad B(\Phi(x), u) = B_n b(\Phi(x)) + D_n(\Phi(x))u$$

$C = C_n$ and $c_\ell = 1/\ell$, where the triplet A_n, B_n, C_n and the functions $b(\cdot)$, $D(\cdot)$ are given in (34). In this case, we set $L_n(\ell) = L_\ell M_\ell N_\ell$ and $K(u) = K_n$ in the observer (35).
4) In this assumption, A is allowed to be input-dependent to allow a broader class of nonlinear systems. For instance it can be verified that the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = u, \qquad y = -x_1 + x_2 + x_2^2$$

cannot be transformed in the form (33) but it satisfies Assumption 4.
5) In some cases, the nonlinear terms (38) can be disregarded in the high gain observer design (usually also called dirty derivative observer). This is possible, for example, when the notions of observability indexes and relative degree indexes coincide (see [38] among others). In this case, these nonlinear terms act through their bound and not their Lipschitzness. Unfortunately a very specific structure is required because otherwise the gain between these nonlinear terms and some estimation error is increasing with the observer gain. Here, we intend to consider a broader class of systems and thus we do need to have these terms present in the observer.

Lemma 2: Under Assumption 4, for any compact set $\mathfrak{C}$ and $\widehat{\mathfrak{C}}$ satisfying $\mathfrak{C} \subset \widehat{\mathfrak{C}} \subset \mathcal{O}$, the family of systems

$$\dot{\hat{x}}_\ell = f(\hat{x}_\ell) + g(\hat{x}_\ell)u + \left(\frac{\partial \Phi}{\partial x}(\hat{x}_\ell)\right)^{-1} L_\ell M_\ell K(u) N_\ell \big[y - h(\hat{x}_\ell)\big] \tag{40}$$

indexed by ℓ in $\mathbb{R}_{>0}$ satisfies points 2 to 6 of Assumption 1.

Proof: We let

$$\eta = \Phi(x), \quad \hat{\eta}_\ell = \Phi(\hat{x}_\ell), \quad \tilde{\eta}_\ell = \eta - \hat{\eta}_\ell . \tag{41}$$

With (38) and (40), systems (2) and (40) are transformed in

$$\dot{\eta} = A(u)\eta + B(\eta, u)$$
$$\dot{\hat{\eta}}_\ell = A(u)\hat{\eta}_\ell + B(\hat{\eta}_\ell, u) + L_\ell M_\ell K(u) N_\ell C(\eta - \hat{\eta}_\ell)$$

With Assumption 4 and the notations (41), we define the Lyapunov Function

$$U_\ell(x, \hat{x}) = \frac{1}{2}(\eta - \hat{\eta}_\ell)^\top [L_\ell M_\ell P^{-1} L_\ell^\top]^{-1} (\eta - \hat{\eta}_\ell) .$$

As Φ, it is defined on $\mathcal{O} \times \mathcal{O}$ and it takes values in $\mathbb{R}_{\geq 0}$. Also, because the matrix $L_\ell M_\ell P^{-1} L_\ell^\top$ is positive definite, we have

$$\forall (x, \hat{x}_\ell) \in \mathcal{O} \times \mathcal{O}, \quad U_\ell(x, \hat{x}_\ell) = 0 \iff x = \hat{x}_\ell .$$

So point 2 of Assumption 1 holds. Also, we get

$$\dot{U}_\ell(x, \hat{x}) = (\eta - \hat{\eta}_\ell)^\top \big[L_\ell^{-\top} P M_\ell^{-1} L_\ell^{-1}\big] \times \left[(A(u) - L_\ell M_\ell K(u) N_\ell C)(\eta - \hat{\eta}_\ell) + B(\eta, u) - B(\hat{\eta}_\ell, u)\right] \tag{42}$$

which, with using O3 and (39), gives, for all $(x, \hat{x})$ in $\mathfrak{C} \times \widehat{\mathfrak{C}}$ and for all $u \in \mathcal{U}(\bar{u})$,

$$\dot{U}_\ell \leq -\nu |P^{\frac{1}{2}} L_\ell^{-1} \tilde{\eta}_\ell|^2 + c_\ell |P^{\frac{1}{2}} L_\ell^{-1} \tilde{\eta}_\ell|^2 .$$

Then, with O6, there exists a $\underline{\ell}$ such that, for any $\ell \geq \underline{\ell}$

$$\dot{U}_\ell(x, \hat{x}) \leq -\frac{\nu}{2} \tilde{\eta}_\ell^\top L_\ell^{-\top} P L_\ell^{-1} \tilde{\eta}_\ell \qquad \forall (x, \hat{x}) \in \mathfrak{C} \times \widehat{\mathfrak{C}} . \tag{43}$$

Since we have $P \geq \lambda_{\min}(P) \lambda_{\min}(M_\ell P^{-1})\, P M_\ell^{-1}$, we obtain, for all $(x, \hat{x})$ in $\mathfrak{C} \times \widehat{\mathfrak{C}}$,

$$\dot{U}_\ell(x, \hat{x}) \leq -\frac{\nu\, \lambda_{\min}(P) \lambda_{\min}(M_\ell P^{-1})}{2}\, U_\ell(x, \hat{x}) .$$

So, with O4, points 4 and 5 of Assumption 1 hold when we choose the integer κ as the integer part of the ratio $\ell/\underline{\ell}$ and with

$$\sigma_\kappa = \frac{\nu\, \lambda_{\min}(P) \lambda_{\min}(M_\ell P^{-1})}{2} .$$

Next, we have

$$U_\ell(x, \hat{x}) \lambda_{\min}(L_\ell M_\ell P^{-1} L_\ell^\top) = \frac{\tilde{\eta}_\ell^\top\, (L_\ell M_\ell P^{-1} L_\ell^\top)^{-1}\, \tilde{\eta}_\ell}{\lambda_{\max}((L_\ell M_\ell P^{-1} L_\ell^\top)^{-1})}$$
$$\leq |\eta - \hat{\eta}_\ell|^2$$
$$\times\ |\eta - \hat{\eta}_\ell|^2 \leq \frac{\tilde{\eta}_\ell^\top\, (L_\ell M_\ell P^{-1} L_\ell^\top)^{-1}\, \tilde{\eta}_\ell}{\lambda_{\min}((L_\ell M_\ell P^{-1} L_\ell^\top)^{-1})}$$
$$\leq U_\ell(x, \hat{x})\, \lambda_{\max}(L_\ell M_\ell P^{-1} L_\ell^\top).$$

So, with O5, we get

$$U_\ell(x,\hat{x})\lambda_{\min}(M_\ell P^{-1})^{-d} \leq |\Phi(x)-\Phi(\hat{x}_\ell)|^2 \leq U_\ell(x,\hat{x})\lambda_{\min}(M_\ell P^{-1})^{d}.$$

Since Φ is a diffeomorphism defined on $\mathcal{O}$, for any compact subsets $\mathfrak{C}$ and $\widehat{\mathfrak{C}}$ of $\mathcal{O}$, there exist real numbers $\overline{\Phi}$ and $L_{\Phi^{-1}}$, independent of ℓ, such that, for all x in $\mathfrak{C}$ and $\hat{x}_\ell$ in $\widehat{\mathfrak{C}}$, we have

$$|x-\hat{x}_\ell| = |\Phi^{-1}(\Phi(x)) - \Phi^{-1}(\Phi(\hat{x}_\ell))| \leq L_{\Phi^{-1}}|\Phi(x)-\Phi(\hat{x}_\ell)| \leq \overline{\Phi}\,.$$

This gives

$$|x-\hat{x}_\ell|^2 \frac{1}{L_{\Phi^{-1}}^2}\lambda_{\min}(M_\ell P^{-1})^{-d} \leq U_\ell(x,\hat{x}) \leq \overline{\Phi}^2\lambda_{\min}(M_\ell P^{-1})^{d}.$$

So, with O4, point 3 of Assumption 1 holds.

Finally, point 6 of Assumption 1 holds too. Indeed, by definition of the set $\mathcal{U}(\bar{u})$, the matrices $K(u), M_\ell, N_\ell, L_\ell$ and the diffeomorphism Φ, there exists a positive definite function $L_{\vartheta_\ell}(\hat{x}_\ell)$ such that

$$\left|\left(\frac{\partial\Phi}{\partial x}(\hat{x}_\ell)\right)^{-1} L_\ell M_\ell K(u) N_\ell\right| \leq L_{\vartheta_\ell}(\hat{x}_\ell)$$

for any $\ell > 0$, $u \in \mathcal{U}(\bar{u})$ and $\hat{x}_\ell \in \widehat{\mathfrak{C}}$. ∎

D. Taking Care of Observability Restricted to $\mathcal{O}$ by an Observer Modification

In the above (2), we are missing point 1 of Assumption 1, namely $\widehat{\mathfrak{C}}$ may not be forward invariant. The problem is that the observer (47) does not guarantee that $\hat{x}_\ell$ remains in $\mathcal{O}$ and therefore that $\frac{\partial\Phi}{\partial x}(\hat{x}_\ell)$ is invertible. To overcome this problem, as in [30], we modify this observer, here not by projection, but by considering a dummy measured output (extending the results in [5]). To make our point clear, we introduce the following assumption.

Assumption 5: Given the set $\mathcal{O}$ and the diffeomorphism Φ of Assumption 4, for any compact subset $\mathfrak{C}$ of $\mathcal{O}$, we know of a C^1 function $h_2 : \mathcal{O} \to \mathbb{R}_{\geq 0}$ such that:

H1. the set $\{x \in \mathbb{R}^n \ : \ h_2(x) < 1\}$ is a subset of $\mathcal{O}$;

H2. the function $x \mapsto \frac{h_2(x)}{\left|\frac{\partial h_2}{\partial x}(x)\right|}$ is continuous on $\mathcal{O}$;

H3. for any real number s in $[0,1]$, and any x_1 and x_2 in $\mathcal{O}$ satisfying

$$h_2(x_1) \leq s \quad , \qquad h_2(x_2) \leq s$$

we have $h_2(x) \leq s$ for all x which satisfies for some λ in $[0,1]$

$$\Phi(x) = \lambda\Phi(x_1) + (1-\lambda)\Phi(x_2)\,.$$

This means nothing but the fact that, for any s in $[0,1]$, the image by Φ of the set $\{x \in \mathbb{R}^n : h_2(x) \leq s\}$ is convex;

H4. the set $\mathcal{O}_{mod}$ defined as

$$\mathcal{O}_{mod} = \{x \in \mathbb{R}^n \ : \ h_2(x) \leq 0\} \tag{44}$$

contains $\mathfrak{C}$ and has a non empty interior which contains the origin;

H5. the set $\widehat{\mathfrak{C}} = \left\{x \in \mathbb{R}^n \ : \ |h_2(x)| \leq \frac{1}{2}\right\}$ is compact.

Remark 4:

1) There is a systematic way to define this function h_2 when, given the compact set $\mathfrak{C}$, we know a positive definite symmetric matrix Q and a real number R satisfying

$$\Phi(\mathfrak{C}) \subset \{\eta \in \mathbb{R}^n : \eta^\top Q\eta \leq R\} \subset \Phi(\mathcal{O})\,.$$

Indeed, in this case we let ϱ be the number defined as

$$\varrho = \sup_{R:\{\eta:\eta^\top Q\eta\leq R\}\subset\Phi(\mathcal{O})} R\,.$$

Since $\mathcal{O}$ is a neighborhood of the origin, ϱ is strictly positive. Then we select a real number ϵ in $(0,1)$ and let

$$h_2(x) = \max\left\{\frac{\Phi(x)^\top Q\Phi(x)}{\varrho(1+\epsilon)} - \epsilon\,,\, 0\right\}^2\,. \tag{45}$$

With this choice and since Φ is a diffeomorphism, we can check that Properties H1 to H5 are satisfied.

2) We may dislike the convexity property mentioned in H3 of Assumption 5. Unfortunately, it is in some sense necessary. Indeed, our objective with the modification E is to preserve the high-gain paradigm. This means in particular that we choose to keep an Euclidean distance in the image by Φ as a Lyapunov function for studying the error dynamics. Also we need an infinite gain margin, as defined in Definition 2.8 in [36], since the correction term must dominate all the other ones in the expression of $\dot{\hat{x}}$ when h_2 becomes too large. Then, as proved in Lemma 2.7 [36], with such constraints, the convexity assumption is necessary. This implies that, if we want to remove the convexity assumption, we have to find another class of observers.

We are interested in the function h_2 because it satisfies the property

$$h_2(x) = 0 \qquad \forall x \in \mathcal{O}_{mod}\,.$$

This leads us to introduce a dummy measured output

$$y_2 = h_2(x)\,.$$

Indeed y_2 is zero when x is in $\mathcal{O}_{mod}$. But $\mathcal{O}_{mod}$ being a strict subset of $\mathcal{O}$, we have here a stronger constraint. To deal with this restriction, we need to "reduce" the set $\mathcal{SO}$ by modifying its definition given in (25) into

$$\begin{aligned} \mathrm{v}_\infty &= \inf_{(x,z)\in(\mathcal{S}\times\mathbb{R}^r)\setminus(\mathcal{O}_{mod}\times\mathbb{R}^r)} V(x,z) \\ \mathcal{SO} &= \left\{(x,z)\in(\mathcal{S}\times\mathbb{R}^r) \ : \ V(x,z) < \mathrm{v}_\infty\right\}. \end{aligned} \tag{46}$$

With Assumption 5, point 1 of Assumption 1 can be established via a modification of the observer.

Lemma 3: Under Assumption 5, let $\Phi : \mathcal{O} \to \mathbb{R}^n$ be a diffeomorphism, $\bar{u}$ be a positive real number, $t \to u(t)$ be a continuous function with values in $U(\bar{u})$ and $t \to y(t)$ be a continuous bounded function. The set $\widehat{\mathfrak{C}}$ given in H5 is forward invariant for any system in the family, indexed by ℓ in $\mathbb{R}_{>0}$

$$\dot{\hat{x}}_\ell = f(\hat{x}_\ell) + g(\hat{x}_\ell)u + \left(\frac{\partial \Phi}{\partial x}(\hat{x}_\ell)\right)^{-1} L_\ell M_\ell K(u) N_\ell \big[y - h(\hat{x}_\ell)\big] + E(\hat{x}_\ell, u, y) \tag{47}$$

where the term E is defined as

$$E(\hat{x}_\ell, u, y) = -\tau_\ell\,(\hat{x}_\ell, u, y) \times \left(\frac{\partial \Phi}{\partial x}(\hat{x}_\ell)\right)^{-1} L_\ell M_\ell P^{-1} L_\ell^\top \left(\frac{\partial \Phi}{\partial x}(\hat{x}_\ell)\right)^{-1\top} \frac{\partial h_2}{\partial x}(\hat{x}_\ell)^\top h_2(\hat{x}_\ell) \tag{48}$$

where τ_ℓ is a C^1 function to be chosen large.[6] If all conditions of Assumption 4 hold and the model state x remains in $\mathcal{O}_{mod}$, then all the points of Assumption 1 are satisfied.

Proof: First we observe that

$$\frac{\partial h_2}{\partial x}(\hat{x}_\ell)\dot{\hat{x}}_\ell = R(\hat{x}_\ell, u, y) - \tau_\ell(\hat{x}_\ell)\left|(M_\ell P^{-1})^{\frac{1}{2}} L_\ell^\top \left(\frac{\partial \Phi}{\partial x}(\hat{x}_\ell)\right)^{-1\top} \frac{\partial h_2}{\partial x}(\hat{x}_\ell)^\top\right|^2 h_2(\hat{x}_\ell)$$

where we have let

$$R(\hat{x}_\ell, u, y) = \frac{\partial h_2}{\partial x}(\hat{x}_\ell) \times \left[f(\hat{x}_\ell) + g(\hat{x}_\ell)u + \left(\frac{\partial \Phi}{\partial x}(\hat{x}_\ell)\right)^{-1} L_\ell M_\ell K(u) N_\ell [y - h(\hat{x}_\ell)]\right].$$

This motivates us for choosing τ_ℓ satisfying

$$\tau_\ell(\hat{x}_\ell, u, y) \geq \frac{8 h_2(\hat{x}_\ell)^2\, R(\hat{x}_\ell, u, y)}{\left|(M_\ell P^{-1})^{\frac{1}{2}} L_\ell^\top \left(\frac{\partial \Phi}{\partial x}(\hat{x}_\ell)\right)^{-1\top} \frac{\partial h_2}{\partial x}(\hat{x}_\ell)^\top\right|^2} \tag{49}$$

which can be computed on-line.

Thanks to H2, the function $x \mapsto \tau_\ell(x)$ defined this way is continuous on $\mathcal{O}$. So we can use τ_ℓ as long as $\hat{x}_\ell$ is in $\mathcal{O}$. It implies that $\overbrace{h_2(\hat{x}_\ell)}^{\cdot}$ is non positive when $h_2(\hat{x}_\ell)$ is strictly larger than $\frac{1}{2}$. This implies that, for each s in $[\frac{1}{2}, 1]$ the set $\{(\hat{x}_\ell) : h_2(\hat{x}_\ell) \leq s\}$ is forward invariant and so is the compact set $\widehat{\mathfrak{C}}$ in particular. This says that point 1 of Assumption 1 hold. On the other hand, the modification E augments $\dot{U}_\ell$ in (42) with

$$-\tau_\ell(\hat{x}_\ell)\,[\Phi(\hat{x}_\ell) - \Phi(x)]^\top \left(\frac{\partial \Phi}{\partial x}(\hat{x}_\ell)\right)^{-1\top} \frac{\partial h_2}{\partial x}(\hat{x}_\ell)^\top h_2(\hat{x}_\ell)\,.$$

But, when $h_2(x)$ is zero which is the case when the model state x remains in $\mathcal{O}_{mod}$ and when $h_2(\hat{x}_\ell)$ is in $[0, 1]$, the convexity property of h_2 in H3 gives

$$0 \;\leq\; [\Phi(\hat{x}_\ell) - \Phi(x)]^\top \left(\frac{\partial \Phi}{\partial x}(\hat{x}_\ell)\right)^{-1\top} \frac{\partial h_2}{\partial x}(\hat{x}_\ell)^\top h_2(\hat{x}_\ell)\,.$$

[6] see (49).

We conclude that, when all conditions of Assumption 4 hold, (43) holds even with the modification E. Hence, from the proof of Lemma 2, points 2 to 5 of Assumption 1 hold. Finally, with (48) and (49), the function defined by the right hand side of (47) satisfies also the point 6 of the Assumption 1. ■

Remark 5: An important feature is that, thanks to the additional term E, no other modification (as saturation) is needed. This modification, in fact, guarantees that the estimate state $\hat{x}_\ell$ remains in a compact subset of $\mathcal{O}$ which depends on the choice of the parameters.

V. Proofs of Propositions

A. Proof of Proposition 2

We denote

$$\mathcal{X} = (x, z, \hat{x})\,,$$

$$\varphi_m(\mathcal{X}) = \begin{pmatrix} f(x) + g(x)\psi_{sat}(\hat{x}, z) \\ k(\hat{x}, h_r(x)) \\ \vartheta_\kappa(h(x), \hat{x}, \psi_{sat}(\hat{x}, z)) \end{pmatrix},$$

$$\varphi_p(\mathcal{X}) = \begin{pmatrix} \xi(x, \psi_{sat}(\hat{x}, z) \\ k(\hat{x}, \zeta_r(x, \psi_{sat}(\hat{x}, z))) \\ \vartheta_\kappa(\zeta(x, \psi_{sat}(\hat{x}, z)), \hat{x}, \psi_{sat}(\hat{x}, z)) \end{pmatrix}. \tag{50}$$

A first elementary remark is that, if $\mathcal{X}_e = (x_e, z_e, \hat{x}_e)$ is an equilibrium point of φ_p, then we have in particular

$$0 = \dot{z}|_{\mathcal{X} = \mathcal{X}_e} = k(\hat{x}_e, h_r(x_e))\,.$$

With (12) this implies $h_r(x_e)$ is zero.

To prove the existence of $\mathcal{X}_e$, we use Lemma 4 given in the Appendix. In particular, from points 1 and 6 of Assumption 1, we know that, even when the observer in (14) is fed with $y = \zeta(x, u)$ and not with $h(x)$, it admits a forward invariant compact subset $\widehat{\mathfrak{C}}$ of $\mathcal{O}$. So with

$$L = \sup_{\mathcal{X} \in \widehat{\mathfrak{C}}} \{L_{\vartheta_\kappa}(\hat{x}), L_k(\hat{x})\}$$

with L_{ϑ_κ} given by (9) and $L_k(\hat{x})$ given by (13), we have, for all $(x, z, \hat{x}, u)$ in $\mathbb{R}^n \times \mathbb{R}^r \times \widehat{\mathfrak{C}} \times \mathcal{U}$,

$$|\varphi_p(\mathcal{X}) - \varphi_m(\mathcal{X})| \leq |\xi(x, u) - [f(x) + g(x)u]| \;+\; 2L\,|\zeta(x, u) - h(x)|\,.$$

Hence (56) holds when (15) is satisfied with

$$\delta = \frac{1}{1 + 2L} \frac{\inf_{\mathcal{X} \in C} \mathcal{V}(\mathcal{X})}{2 \sup_{\mathcal{X} \in C} \left|\frac{\partial \mathcal{V}}{\partial \mathcal{X}}(\mathcal{X})\right|}\,.$$

B. Proof of Proposition 3

Firstly, note that smoothness of k and (12) implies the existence[7] of a continuous function $\pi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfying $\pi(0) = 0$ and

$$|y_r| \;\leq\; [1 + |x| + |y_r|^2]\, \pi(|k(x, y_r)|) \quad \forall (x, y_r) \in \mathbb{R}^n \times \mathbb{R}^p .$$

As a consequence, in view of the Lemma 5 given in the Appendix, Proposition 3 holds if (16) and (17) imply (57) and (58). In the Proof of Proposition 2 we have seen that (16) implies (57). So we are left with proving that (16) and (17) imply (58). By using again the notations (50) and by dropping the arguments we see that

$$\left|\frac{\partial \varphi_p}{\partial x}(x) - \frac{\partial \varphi_m}{\partial x}(x)\right| \leq |\Delta_{k\vartheta}(\Delta_p - \Delta_m)\Delta_u| + |(\Delta_2 + \Delta_{2\vartheta}\Delta_u)|\,|\Delta_y|$$

where

$$\Delta_{k\vartheta} = \begin{pmatrix} I & 0 & 0 \\ 0 & \frac{\partial k}{\partial y_r} & \frac{\partial \vartheta_\kappa}{\partial y} \end{pmatrix}^\top , \quad \Delta_u = \begin{pmatrix} I & 0 & 0 \\ 0 & \frac{\partial \psi_{sat}}{\partial z} & \frac{\partial \psi_{sat}}{\partial \hat{x}} \end{pmatrix},$$

$$\Delta_p = \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial u} \\ \frac{\partial \zeta}{\partial x} & \frac{\partial \zeta}{\partial u} \end{pmatrix}, \qquad \Delta_m = \begin{pmatrix} \frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}\psi_{sat} & g \\ \frac{\partial h}{\partial x} & 0 \end{pmatrix},$$

$$\Delta_2 = \begin{pmatrix} 0 & 0 & 0 \\ \frac{\partial^2 k}{\partial y_r^2}\frac{\partial h_r}{\partial x} & 0 & \frac{\partial^2 k}{\partial \hat{x}^2} \\ \frac{\partial^2 \vartheta_\kappa}{\partial y^2}\frac{\partial h}{\partial x} & 0 & \frac{\partial^2 \vartheta_\kappa}{\partial \hat{x}^2} \end{pmatrix}, \quad \Delta_{2\vartheta} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & \frac{\partial^2 \vartheta_\kappa}{\partial u^2} \end{pmatrix},$$

$$\Delta_y = \zeta(x, \psi_{sat}(\hat{x}, z)) - h(x).$$

Recall that by construction the functions ψ_{sat}, k and ϑ are C^1. Hence, by letting (where the arguments are dropped for the sake of compactness)

$$L_{k\vartheta} = \sup_{(x,z,\hat{x})\in \underline{\mathfrak{C}}} \left\{ \left|\frac{\partial k}{\partial y_r}\right|, \left|\frac{\partial \vartheta_\kappa}{\partial y}\right| \right\},$$

$$L_h = \sup_{x\in(\underline{\mathfrak{C}})_x} \left\{ \left|\frac{\partial h}{\partial x}\right| \right\},$$

$$L_u = \sup_{(z,\hat{x})\in(\underline{\mathfrak{C}})_{z,\hat{x}}} \left\{ \left|\frac{\partial \psi_{sat}}{\partial z}\right|, \left|\frac{\partial \psi_{sat}}{\partial \hat{x}}\right| \right\},$$

$$L_{2k} = \sup_{(x,z,\hat{x})\in \underline{\mathfrak{C}}} \left\{ \left|\frac{\partial^2 k}{\partial y_r^2}\right|, \left|\frac{\partial^2 k}{\partial \hat{x}^2}\right| \right\},$$

$$L_{2\vartheta} = \sup_{(x,z,\hat{x})\in \underline{\mathfrak{C}}} \left\{ \left|\frac{\partial^2 \vartheta_\kappa}{\partial y_r^2}\right|, \left|\frac{\partial^2 \vartheta_\kappa}{\partial u^2}\right|, \left|\frac{\partial^2 \vartheta_\kappa}{\partial \hat{x}^2}\right| \right\}$$

and $L_2 = \max\{L_{2k}, L_{2\vartheta}\}$, we have, for all $(x, z, \hat{x})$ in $\underline{\mathfrak{C}}$

$$\left|\frac{\partial \varphi_p}{\partial x}(x) - \frac{\partial \varphi_m}{\partial x}(x)\right| \leq 4L_u L_{k\vartheta}\,|\Delta_p - \Delta_m| + 2L_2(1 + L_u + L_h)\,|\Delta_y|\,.$$

The proof can be completed by using (16) and (17) in place of Δ_y and $(\Delta_p - \Delta_m)$ and by properly defining δ.

[7]The function π is a smoothened version of

$$s \to \sup_{(x,y_r):|k(x,y_r)|\leq s} \frac{|y_r|}{1 + |x| + |y_r|^2}$$

VI. Illustration of the Proposed Design via the Longitudinal Model of a Plane

As an illustration we consider a non academic but still very simplified model of the longitudinal dynamics of a fixed-wing vehicle flying at high speed, given (see [33], [34]) by

$$\begin{aligned} \dot{v} &= e - g\sin(\gamma) \\ \dot{\gamma} &= \pounds\, v \sin(\theta - \gamma) - \frac{g\cos(\gamma)}{v} \\ \dot{\theta} &= q \end{aligned} \tag{51}$$

where v is the modulus of the speed, γ is the path angle, θ is the pitch angle, q is the pitch rate, g is the standard gravitational acceleration and $\pounds$ is an aerodynamic lift coefficient. This model makes sense for v strictly positive only.

The problem is to regulate γ at 0, with v remaining close to a prescribed cruise speed v_0, using the pitch rate q and the thrust e as controls, and with γ and θ as only measurements. So here, by using the notation introduced in Section II

$$x = (\theta, \gamma, v)\,, \quad u = (e, q)\,, \quad y = (\theta, \gamma)\,, \quad y_r = \gamma\,.$$

A. Choice of the Function k in the Integral Action

We select

$$k(x, h(x)) = v\sin(\gamma)\,.$$

The motivation is that, then the integrator state z has the same dynamics as the altitude of the vehicle (not taken into account in this illustration).

B. State Feedback Design

To design the state feedback ψ and the associated Lyapunov function V_e, we start by noting that the so called phugoid mode is conservative (see for instance [3, Section VII.4]). Precisely, we have that the following function remains constant along the solutions when $e = 0$ and $\sin(\theta - \gamma) = \frac{g}{\pounds v_0^2}$

$$\mathcal{I}(v, \gamma) = \frac{v^3}{3v_0^3} - \frac{v}{v_0}\cos(\gamma)$$

This can be checked by looking the time derivative of $\mathcal{I}$. Also, the open sublevel set of $\mathcal{I}$

$$\mathcal{S} = \left\{(v,\gamma) \,:\, \frac{v^3}{3v_0^3} \;-\; \frac{v}{v_0}\cos(\gamma) < 0\right\}$$

is the largest sublevel set not containing a point of the type $(0,\gamma)$. Namely it is the largest sublevel set of $\mathcal{I}$ where the model (51) is well defined. Moreover in this set the γ-component of any point is in $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. Also $\mathcal{I}$ is positive definite in $v - v_0$ and γ on $\mathcal{S}$. We conclude that $\mathcal{I} + \frac{2}{3}$, restricted to $\mathcal{S}$ is a candidate for playing the role of a Lyapunov function. Also forwarding with the functions V and H known is possible since when $e = 0$, the function

$$z \;-\; H(v) = z + \frac{v^2}{2g}$$

remains constant along the solutions of the following (z, v)-subsystem

$$\begin{aligned}\dot{z} &= \; v\,\sin(\gamma)\\ \dot{v} &= \; e - g\,\sin(\gamma)\,.\end{aligned}$$

Finally, we can complete the design of a state feedback by applying backstepping from the fact that θ given as

$$\theta = \gamma \;+\; \arcsin\left(\frac{g}{\pounds v_0^2}\right)$$

is stabilizing for the (z, v, γ)-subsystem. All this leads to the following (weak)[8] Control Lyapunov function

$$\begin{aligned}V(z,v,\gamma) = \frac{v^3}{3v_0^3} \;+\; \frac{2}{3} \;-\; \frac{v}{v_0}\;\cos(\gamma)\frac{k_1}{4}\left(\frac{2gz+v^2-v_0^2}{v_0^2}\right)^2\\ +\;\frac{k_2}{2}\left[\theta-\gamma-\arcsin\left(\frac{g}{\pounds v_0^2}\right)\right]^2\end{aligned}$$

with $k_1 > 0$ and $k_2 > 0$ arbitrary numbers, and the following feedback law

$$\begin{aligned}e &= -\text{sat}_{k_e}\left(k_3\left[\frac{v^2}{v_0^2}-\cos(\gamma)+k_1\frac{2gz+v^2-v_0^2}{v_0^2}\frac{v}{v_0}\right]\right),\\ q &= -\left(\frac{\pounds v_0^2\sin(\theta-\gamma)-g}{\theta-\gamma-\arcsin\left(\frac{g}{\pounds v_0^2}\right)}\frac{v^2}{k_2 v_0^3}\sin(\gamma)+\frac{g\cos(\gamma)}{v}\right.\\ &\qquad \left.-\pounds v\sin(\theta-\gamma)+k_4\left[\theta-\gamma-\arcsin\left(\frac{g}{\pounds v_0^2}\right)\right]\right),\end{aligned}$$

with $k_3 > 0$ and $k_4 > 0$ arbitrary real numbers and $k_e > 0$ and $k_q > 0$ arbitrary saturation levels. With LaSalle invariance principle it is possible to prove that $(v, \gamma, \theta) = \left(v_0, 0, \arcsin\left(\frac{g}{\pounds v_0^2}\right)\right)$ is the only asymptotically stable equilibrium point of the system (51). It is worth noticing that the proposed Lyapunov function V does not give enough degrees of freedom to improve performance and increase the domain of attraction. More appropriate designs are possible by choosing different Lyapunov functions (see [33]). Finally, according to Section III-B, for its use in the output feedback, the state feedback law q above has to be modified by adding a saturation (see in particular the function ψ_{sat} in (29)).

[8] Its derivative along the solutions may be only non positive.

C. Design of the High-Gain Observer

To obtain an observer we check that the conditions of Assumptions 4 are satisfied. Let γ_{dot} be defined as the following function

$$\gamma_{dot}(\theta,\gamma,v) = \pounds v\sin(\theta-\gamma) - \frac{g\cos(\gamma)}{v}.$$

Then let $(\eta_1, \eta_2, \eta_3) = \Phi((\theta, \gamma, v)) = (\theta, \gamma, \gamma_{dot}(\theta, \gamma, v))$. It is defined on the set

$$\mathcal{O} = \left(-\frac{\pi}{2};\frac{\pi}{2}\right)\times\left(-\frac{\pi}{2};\frac{\pi}{2}\right)\times(0;+\infty)$$

and (θ, γ, v) can be recovered from its values (η_1, η_2, η_3) in the following subset[9] of $\Phi(\mathcal{O})$

$$\begin{aligned}\Xi = \Big\{\eta\in\mathbb{R}^3 : \eta_1 \in \left(-\frac{\pi}{2};\frac{\pi}{2}\right),\; \eta_2\in\left(-\frac{\pi}{2};\frac{\pi}{2}\right),\\ \eta_3 < -2\sqrt{g\pounds|\eta_1-\eta_2|}\quad\text{if}\quad(\eta_1-\eta_2)\le 0\Big\}\,.\end{aligned}$$

Note also that $\partial\Phi/\partial x$ is always non-singular on the set $\mathcal{O}$ because $\partial\gamma_{dot}/\partial v$ cannot be equal to 0 when $\eta \in \Xi$. Hence, the function Φ is a diffeomorphism satisfying Assumption O1. Then, with C defined as $C = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}$, Assumption O2 also holds.

Now let $A, B(\Phi(x), u), L_\ell, M_\ell$ and N_ℓ be defined as

$$A = \begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix},\; B(\cdot) = \begin{pmatrix} u_1\\ 0\\ \frac{\partial\gamma_{dot}}{\partial\theta}u_1+\frac{\partial\gamma_{dot}}{\partial v}u_2\\ +\frac{\partial\gamma_{dot}}{\partial\gamma}\gamma_{dot}-\frac{\partial\gamma_{dot}}{\partial v}g\sin(\gamma)\end{pmatrix}$$

$L_\ell = \text{diag}(1, 1, \ell)$, $M_\ell = \text{diag}(\ell, \ell, \ell)$, $N_\ell = \text{diag}(1, 1)$. Also, given any strictly positive number ν, let P be a symmetric positive definite matrix defined as

$$P = \begin{pmatrix}* & * & *\\ * & * & p_{23}\\ * & p_{23} & p_{33}\end{pmatrix}$$

where $2p_{23} \le -\nu p_{33}$. Then there exists a real number ρ such that we have $PA + A^\top P - \rho\, C^\top C \;\le\; -\nu P$. This implies the existence of a real number $\underline{\nu_k} > 0$ such that, for any $\nu_k \ge \underline{\nu_k}$, with $K = \nu_k P^{-1} C^\top$ assumptions O3 to O7 are satisfied.

D. Design of the Correction Term

Following Section IV, the function $h_2(x)$ can be defined as

$$h_2(x) = h_2^1(x) + h_2^2(x) + h_2^3(x) + h_2^4(x)$$

[9] We use $|\eta_1 - \eta_2|$ to upper bound $\cos\eta_2\sin(\eta_1 - \eta_2)$.

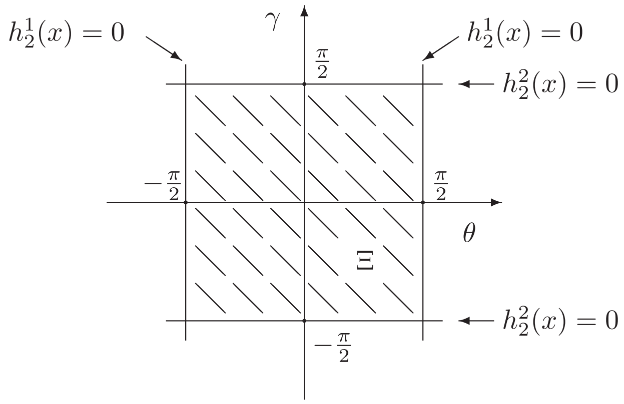

Fig. 1. Design of the functions h_2^1, h_2^2.

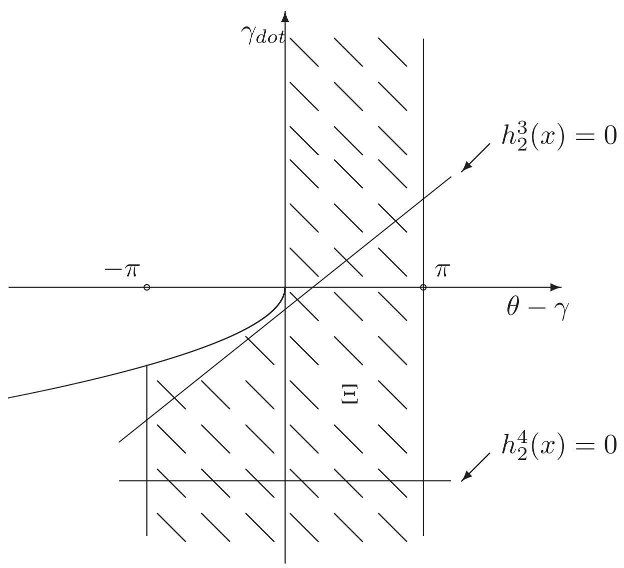

Fig. 2. Design of the functions h_2^3, h_2^4.

with

$$h_2^1(x) = \max\left\{\frac{4\theta^2}{\pi^2} - \varepsilon_1; 0\right\}^2, \quad h_2^2(x) = \max\left\{\frac{4\gamma^2}{\pi^2} - \varepsilon_2; 0\right\}^2,$$

$$h_2^3(x) = \max\left\{\varepsilon_3 (\theta - \gamma) - \gamma_{dot} - \varepsilon_4; 0\right\}^2,$$

$$h_2^4(x) = \max\left\{\frac{\gamma_{dot}}{\gamma_{dot\ \max}} - \varepsilon_5; 0\right\}^2$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5$ and $\gamma_{dot\ \max}$ are constants to be properly chosen. The functions $h_{2,1}$ and $h_{2,2}$ take care respectively of θ and γ to stay in the set Ξ as showed in Fig. 1, whereas functions $h_{2,3}$ and $h_{2,4}$ take care of $f(\theta, \gamma, v)$ as in Fig. 2.

The correction term E is defined as in Lemma 3. Finally the functions U_κ and σ_κ can be defined as in the proof of Lemma 2.

VII. Conclusions

Robust asymptotic output regulation by output feedback has been investigated. Our design technique follows the very usual approach of stabilizing the origin of the model augmented with integrators of the output errors. To do so, we assume we have already a stabilizing state feedback for the model but not asking for any specific structure nor for normal form nor for minimum phase. For the augmented model we redesign the state feedback by applying forwarding. The output feedback is obtained by introducing a high-gain observer expressed in the original coordinates. The output regulation is shown to be robust to any small enough (in a C^1 sense) unstructured discrepancy between model and process in open loop. In establishing our main propositions we obtained new results, which may have their own interest. They concern high-gain observers for multi-output systems (Lemma 2) and persistence of equilibria under small perturbations (Lemma 4).

The design we propose is illustrated by the regulation of the flight path angle for a simplified longitudinal model of a plane.

Appendix

We study here how the stability properties of a given model described by

$$\dot{x} = \varphi_m(x) \tag{52}$$

are propagated to a process described by

$$\dot{x} = \varphi_p(x) \tag{53}$$

when they are close enough. Lemma 4 concerns persistence of an equilibrium under small perturbations, whereas Lemma 5 combines total stability and hyperbolicity and is a variation of Theorem 6 in [34].

Lemma 4: Let a C^1 function $\varphi_m : \mathbb{R}^n \to \mathbb{R}^n$ be given such that the origin is an asymptotically stable equilibrium point of (52), with $\mathcal{A}$ as domain of attraction. Let $\overline{\mathfrak{C}}$ be an arbitrary compact subset of $\mathcal{A}$ which admits the equilibrium as an interior point and is forward invariant for the system (52). For any open neighborhood $\mathcal{N}_{\partial\overline{\mathfrak{C}}}$ of the boundary set $\partial\overline{\mathfrak{C}}$, contained in $\mathcal{A}$, there exists a strictly positive real number δ such that, for any C^1 function φ_p satisfying

$$|\varphi_m(x) - \varphi_p(x)| \ \le\ \delta \quad \forall\, x \in \mathcal{N}_{\partial\overline{\mathfrak{C}}}$$

the system (53) has an equilibrium in the interior of $\overline{\mathfrak{C}}$.

Proof: To prove the existence of an equilibrium, we use [16, Theorem 8.2] which says that a forward invariant set which is homeomorphic to the closed unit ball of $\mathbb{R}^n$ contains an equilibrium. As a consequence of asymptotic stability, we know the existence of a forward invariant set by using a converse Lyapunov theorem. It may not be homeomorphic to the closed unit ball. Therefore, our first task is to show the existence of such set satisfying the required properties.

The equilibrium of (52) being asymptotically attractive and interior to $\overline{\mathfrak{C}}$ which is forward invariant, $\overline{\mathfrak{C}}$ is attractive. It is also stable due to the continuity of solutions with respect to initial conditions uniformly on compact time subsets of the domain of definition. So it is asymptotically stable with the same domain of attraction $\mathcal{A}$ as the equilibrium. It follows from [45, Theorem 3.2] that there exist C^∞ functions $V : \mathcal{A} \to \mathbb{R}_{\ge 0}$ and $V_{\overline{\mathfrak{C}}} : \mathcal{A} \to \mathbb{R}_{\ge 0}$ which are proper on $\mathcal{A}$ and a class $\mathcal{K}_\infty$

function α satisfying

$$\alpha(|\mathcal{X}|) \leq V(\mathcal{X}), \qquad V(0) = 0,$$

$$\alpha(d(\mathcal{X}, \overline{\mathfrak{C}})) \leq V_{\overline{\mathfrak{C}}}(\mathcal{X}), \quad V_{\overline{\mathfrak{C}}}(\mathcal{X}) = 0 \quad \forall\, \mathcal{X} \in \overline{\mathfrak{C}},$$

$$\frac{\partial V}{\partial \mathcal{X}}(\mathcal{X})\, \varphi_m(\mathcal{X}) \leq -V(\mathcal{X}) \quad \forall\, \mathcal{X} \in \mathcal{A},$$

$$\frac{\partial V_{\overline{\mathfrak{C}}}}{\partial \mathcal{X}}(\mathcal{X})\, \varphi_m(\mathcal{X}) \leq -V_{\overline{\mathfrak{C}}}(\mathcal{X}) \quad \forall\, \mathcal{X} \in \mathcal{A}\, .$$

Since $\overline{\mathfrak{C}}$ is compact and $\mathcal{N}_{\partial \overline{\mathfrak{C}}}$ is a neighborhood of its boundary, there exists a strictly positive real number $\overline{d}$ such that the set $\{x \in \mathcal{A}\colon d(x, \overline{\mathfrak{C}}) \in (0, \overline{d}]\}$ is a subset of $\mathcal{N}_{\partial \overline{\mathfrak{C}}}$. Then, with the notations

$$v_{\overline{\mathfrak{C}}} = \sup_{\mathcal{X} \in \mathcal{A}\colon d(\mathcal{X}, \overline{\mathfrak{C}}) \leq \overline{d}} V(x) \quad , \quad \varpi = \frac{\alpha(\overline{d})}{2 v_{\overline{\mathfrak{C}}}}$$

and since α is of class $\mathcal{K}_\infty$, we obtain the implications

$$\begin{aligned} V_{\overline{\mathfrak{C}}}(x) + \varpi V(x) = \alpha(\overline{d}) &\Rightarrow \alpha(d(x, \overline{\mathfrak{C}})) \leq V_{\overline{\mathfrak{C}}}(x) \leq \alpha(\overline{d})\, , \\ &\Rightarrow d(x, \overline{\mathfrak{C}}) \leq \overline{d}\, , \\ &\Rightarrow V(x) \leq v_{\overline{\mathfrak{C}}}. \end{aligned}$$

With our definition of ϖ, this yields also

$$\begin{aligned} \alpha(\overline{d}) - \varpi\, V(x) = V_{\overline{\mathfrak{C}}}(x) &\Rightarrow 0 \; < \; \frac{\alpha(\overline{d})}{2} \leq V_{\overline{\mathfrak{C}}}(x)\, , \\ &\Rightarrow 0 < d(x, \overline{\mathfrak{C}}) \leq \overline{d}\, , \\ &\Rightarrow x \in \mathcal{N}_{\partial \overline{\mathfrak{C}}} \setminus \overline{\mathfrak{C}}\, . \end{aligned} \tag{54}$$

On the other hand, with the compact notation

$$\mathcal{V}(x) = V_{\overline{\mathfrak{C}}}(x) + \varpi V(x)$$

we have

$$\frac{\partial \mathcal{V}}{\partial x}(x)\, \varphi_m(x) \; < \; -\mathcal{V}(x) \quad \forall x \in \mathcal{A}\, . \tag{55}$$

All this implies that $\mathcal{V}$ is a Lyapunov Function for (52) on $\mathcal{A}$ in the sense of [44, Page 324] and that the sublevel set $\{x \in \mathcal{A} : \mathcal{V}(x) \leq \alpha(\overline{d})\}$ is contained in $\mathcal{N}_{\partial \overline{\mathfrak{C}}} \cup \overline{\mathfrak{C}}$. It follows from [44, Corollary 2.3][10] that the level set $\{x \in \mathcal{A} : \mathcal{V}(x) = \alpha(\overline{d})\}$ is homeomorphic to the unit sphere. But, with the fact that the origin is asymptotically stable and the arguments used in the proof of [44, Theorem 1.2], this implies that the sublevel set $\{x \in \mathcal{A} : \mathcal{V}(x) \leq \alpha(\overline{d})\}$ is homeomorphic to the closed unit ball. Then, since the set $C = \{x \in \mathcal{N}_{\partial \overline{\mathfrak{C}}} : d(x, \overline{\mathfrak{C}}) \in [0, \overline{d}]\}$ is a compact subset of $\mathcal{N}_{\partial \overline{\mathfrak{C}}} \subset \mathcal{A}$, the real number

$$G = \sup_{\mathcal{X} \in C} \left| \frac{\partial \mathcal{V}}{\partial x}(x) \right|$$

is well defined and strictly positive. We get, for all x in C

$$\begin{aligned} \frac{\partial \mathcal{V}}{\partial x}(x)\varphi_p(x) &= \frac{\partial \mathcal{V}}{\partial x}(x)\varphi_m(x) + \frac{\partial \mathcal{V}}{\partial x}(x)[\varphi_p(x) - \varphi_m(x)]\, , \\ &\leq -\mathcal{V}(x) + G \sup_{\mathcal{X} \in C} |\varphi_p(x) - \varphi_m(x)|\, . \end{aligned}$$

So, if φ_p satisfies

$$|\varphi_p(x) - \varphi_m(x)| \leq \frac{\inf_{\mathcal{X} \in C} \mathcal{V}(x)}{2G} \quad \forall x \in \mathcal{N}_{\partial \overline{\mathfrak{C}}} \tag{56}$$

we have, for all x in $\{x \in \mathcal{A} :\, \mathcal{V}(x) = \alpha(\overline{d})\}$

$$\frac{\partial \mathcal{V}}{\partial x}(x)\varphi_p(x) \leq -\frac{1}{2}\mathcal{V}(x)\, .$$

This implies the compact sublevel set $\{x : \mathcal{V}(x) \leq \alpha(\overline{d})\}$ is homeomorphic to the closed unit ball and forward invariant for the system (53). With [16, Theorem 8.2], we conclude that this sublevel set contains an equilibrium of this system. ■

Lemma 5: Let a C^1 function $\varphi_m : \mathbb{R}^n \to \mathbb{R}^n$ be given such that the origin is an exponentially stable equilibrium point of (52) with $\mathcal{A}$ as domain of attraction. For any compact sets $\underline{\mathfrak{C}}$ and $\overline{\mathfrak{C}}$, the latter being forward invariant for the above system, which satisfy

$$\{0\} \subsetneq \underline{\mathfrak{C}} \subsetneq \overline{\mathfrak{C}} \subsetneq \mathcal{A}$$

there exists a strictly positive real number δ such that, for any C^1 function $\varphi_p : \mathbb{R}^n \to \mathbb{R}^n$ satisfying

$$|\varphi_p(x) - \varphi_m(x)| \leq \delta, \quad \forall x \in \overline{\mathfrak{C}}, \tag{57}$$

$$\left| \frac{\partial \varphi_p}{\partial x}(x) - \frac{\partial \varphi_m}{\partial x}(x) \right| \leq \delta, \quad \forall x \in \underline{\mathfrak{C}} \tag{58}$$

there exists an exponentially stable equilibrium point x_e of (53) with basin of attraction containing the compact set $\overline{\mathfrak{C}}$.

Proof: Let Π be a positive definite symmetric matrix and a a strictly positive real number satisfying

$$\Pi \frac{\partial \varphi_m}{\partial x}(0) + \frac{\partial \varphi_m}{\partial x}(0)^\top \Pi \; \leq -a\Pi, \quad \lambda_{\min}(\Pi) = 1$$

where $\lambda_{\max}$ and $\lambda_{\min}$ respectively stand for max and min eigenvalues. By continuity there exists a strictly positive real number p_0 such that we have, for all x satisfying $x^\top \Pi x \leq p_0$,

$$\begin{aligned} \Pi \frac{\partial \varphi_m}{\partial x}(x) + \frac{\partial \varphi_m}{\partial x}(x)^\top \Pi \; &\leq \; -\frac{a}{2}\Pi \\ x^\top \Pi \varphi_m(x) \; &\leq \; -\frac{a}{4} x^\top \Pi x. \end{aligned}$$

Let $\varphi_p : \mathbb{R}^n \to \mathbb{R}^n$ be any C^1 function satisfying

$$|\varphi_p(x) - \varphi_m(x)| \leq \frac{a}{4}\sqrt{\frac{p_0}{12 \lambda_{\max}(\Pi)}}, \forall x : x^\top \Pi x = \frac{p_0}{6}. \tag{59}$$

[10] Thanks to the contribution of Freedman [13] and Perelman [32] the restriction on the dimension is not needed.

We obtain

$$\begin{aligned} x^\top \Pi \varphi_p(x) &= x^\top \Pi \varphi_m(x) + x^\top \Pi[\varphi_p(x) - \varphi_m(x)], \\ &\le x^\top \Pi \varphi_m(x) + \frac{a}{8} x^\top \Pi x \\ &\quad + \frac{2}{a}[\varphi_p(x) - \varphi_m(x)]^\top \Pi [\varphi_p(x) - \varphi_m(x)] \end{aligned}$$

and therefore $x^\top \Pi \varphi_p(x) \le -\frac{a}{16} x^\top \Pi x$ for all $x : x^\top \Pi x = \frac{p_0}{6}$. In this condition, it follows from [16, Theorem 8.2] that, for each function φ_p satisfying (59), there exits a point x_e satisfying

$$\varphi_p(x_e) = 0, \quad (x_e)^\top \Pi x_e \ \le \ \frac{p_0}{6}. \tag{60}$$

Assume further that φ_p satisfies

$$\left| \frac{\partial \varphi_p}{\partial x}(x) - \frac{\partial \varphi_m}{\partial x}(x) \right| \le \frac{a}{8\lambda_{\max}(\Pi)}, \quad \forall x : x^\top \Pi x \le p_0. \tag{61}$$

In this case, we have, for all x satisfying $x^\top \Pi x \le p_0$

$$\begin{aligned} \Pi \frac{\partial \varphi_p}{\partial x}(x) + \frac{\partial \varphi_p}{\partial x}(x)^\top \Pi &= \left[\frac{\partial \varphi_p}{\partial x}(x) - \frac{\partial \varphi_m}{\partial x}(x) \right]^\top \Pi \\ &+ \Pi \frac{\partial \varphi_m}{\partial x}(x) + \frac{\partial \varphi_m}{\partial x}(x)^\top \Pi + \Pi \left[\frac{\partial \varphi_p}{\partial x}(x) - \frac{\partial \varphi_m}{\partial x} x) \right] \\ &\le -\frac{a}{4}\Pi. \end{aligned}$$

Note also that we have

$$[x_e + s(x - x_e)]^\top \Pi [x_e + s(x - x_e)] \le p_0,$$
$$\forall (x, x_e, s): \ s \in [0,1], \ (x_e)^\top \Pi x_e \le \frac{p_0}{6}, \ x^\top \Pi x \le \frac{p_0}{3}.$$

Then, with

$$\begin{aligned} \varphi_p(x) &= \varphi_p(x) - \varphi_p(x_e) \\ &= \int_0^1 \frac{\partial \varphi_p}{\partial x}(x_e + s(x - x_e)) ds [x - x_e] \end{aligned}$$

and (60), we get, for all x satisfying $x^\top \Pi x \le \frac{p_0}{3}$

$$\begin{aligned} &[x - x_e]^\top \Pi \varphi_p(x) \\ &= \int_0^1 \left([x - x_e]^\top \Pi \frac{\partial \varphi_p}{\partial x}(x_e + s(x - x_e))[x - x_e] \right) ds, \\ &\le -\frac{a}{4}[x - x_e]^\top \Pi [x - x_e]. \end{aligned}$$

Let

$$\delta_1 = \min \left\{ \frac{a}{4}\sqrt{\frac{p_0}{12\lambda_{\max}(\Pi)}}, \frac{a}{8\lambda_{\max}(\Pi)} \right\}$$

and reduce p_0 if necessary to have that x satisfying $(x_e)^\top \Pi x_e \le p_0$ is in $\underline{\mathfrak{C}}$. Then (57) and (58) with $\delta = \delta_1$ implies (59) and therefore (60). We have established that the system (53) has an exponentially stable equilibrium with basin of attraction containing the compact set $\{x \in \mathbb{R}^n : x^\top \Pi x \le \frac{p_0}{3}\}$. Now, with $\overline{d}$ and $\mathcal{V} = V_{\overline{\mathfrak{C}}} + \varpi V$ as defined in the proof of Lemma 4, we let $\underline{v}$ be a strictly positive real number such that we have

$$x^\top \Pi x \ \le \ \frac{p_0}{3} \quad \forall x \in \mathcal{A} : \mathcal{V}(x) \le \underline{v} \tag{62}$$

Let also $C = \{x \in \mathcal{A} : \underline{v} \le \mathcal{V}(x) \ , \ d(x, \overline{\mathfrak{C}}) \in [0, \overline{d}]\}$. It is a compact subset of $\mathcal{N}_{\overline{\mathfrak{C}}} \subset \mathcal{A}$. By mimicking the same steps as in the proof of Lemma 4, we can obtain that, if φ_p satisfies

$$|\varphi_p(x) - \varphi_m(x)| \le \frac{\inf_{x \in C} \mathcal{V}(x)}{2G}, \quad \forall x \in \overline{\mathfrak{C}} \tag{63}$$

we have

$$\frac{\partial \mathcal{V}}{\partial x}(x)\varphi_p(x) \le -\frac{1}{2}\mathcal{V}(x) \quad \forall x \in C.$$

This implies the compact set $\{x \in \mathcal{A} : \mathcal{V}(x) \le \underline{v}\}$ is asymptotically stable for the system (53) with basin of attraction $\mathcal{B}$ containing the compact set $\{x \in \mathcal{A} : \mathcal{V}(x) \le \alpha(\overline{d})\}$ which contains $\overline{\mathfrak{C}}$. Since, with (62), we have

$$\{x \in \mathcal{A} : V(x) \le \underline{v}\} \subset \left\{ x \in \mathbb{R}^n : x^\top \Pi x \le \frac{p_0}{3} \right\}.$$

with (59), (61), and (63) we have established our result with

$$\delta = \min \left\{ \delta_1, \frac{\inf_{x \in C} V(x)}{2 \sup_{x \in C} \left| \frac{\partial V}{\partial x}(\mathcal{X}) \right|} \right\}.$$

■

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