Convergence of Nonlinear Observers on $\mathbb{R}^n$ With a Riemannian Metric (Part II)

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Abstract—In "Convergence of Nonlinear Observers on $\mathbb{R}^n$ with a Riemannian Metric (Part I)," we established that a convergent observer with an infinite gain margin can be designed for a given nonlinear system when a Riemannian metric showing that the system is differentially detectable (i.e., the Lie derivative of the Riemannian metric along the system vector field is negative in the space tangent to the output function level sets) and the level sets of the output function are geodesically convex is available. In this paper, we propose techniques for designing a Riemannian metric satisfying the first property in the case where the system is strongly infinitesimally observable (i.e., each time-varying linear system resulting from the linearization along a solution to the system satisfies a uniform observability property) or where it is strongly differentially observable (i.e., the mapping state to output derivatives is an injective immersion) or where it is Lagrangian. Also, we give results that are complementary to those in our previous paper. In particular, we provide a locally convergent observer and make a link to the existence of a reduced order observer. Examples illustrating the results are presented.

Index Terms—Estimation theory, Kalman filters, observers, Riccati equations.

I. INTRODUCTION

WE consider a nonlinear system of the form[1]

$$\dot{x} = f(x), \quad y = h(x) \tag{1}$$

with x in $\mathbb{R}^n$ being the system's state and y in $\mathbb{R}^p$ the measured system's output. We are interested in the design of a function F such that the set

$$\mathcal{A} := \{(x, \hat{x}) \in \mathbb{R}^n \times \mathbb{R}^n : x = \hat{x}\} \tag{2}$$

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[1]If the system is time varying (perhaps due to known exogenous inputs), i.e., $\dot{x} = f(x, t)$, $y = h(x, t)$ most of the results of [1] as well as those here can be extended readily by simply replacing x by $x_e = [x^\top \; t]^\top$, leading to the time-invariant system with dynamics $\dot{x}_e = [f(x, t)^\top 1]^\top =: f_e(x_e)$, $y_e = [h(x, t)^\top t]^\top =: h_e(x_e)$. The drawback of this simplifying viewpoint is that, when time dependence is induced by exogenous inputs, for each input we obtain a different time-varying system. And, maybe even more handicapping, we need to know the time-variations for the design.

is asymptotically stable for the system

$$\dot{x} = f(x), \quad \dot{\hat{x}} = F(\hat{x}, h(x)). \tag{3}$$

A solution to this problem that was proposed in [1] is restated in Theorem 2.3, which is in Section II. It relies on the formalism of Riemannian geometry and gives conditions under which a constructive procedure exists for getting an appropriate function F. This solution requires the satisfaction of mainly two conditions. The first condition is about the geodesic convexity of the level sets of the output function (see point 9 in Appendix A). This condition is not addressed here. Instead, we focus our attention on the second condition, which is a differential detectability property,[2] made precise in Definition 2.1 below. With the terminology used in the study of contracting flows in Riemannian spaces, this property means that f is strictly geodesically monotonic tangentially to the output function level sets. Forthcoming examples related to the so-called harmonic oscillator with unknown frequency will illustrate these notions and provide metrics certifying both weak and strong differential detectability.

In Section II, we establish results complementing those in [1]. In Section II-A, we establish that the differential detectability property only is already sufficient to obtain a locally convergent observer. In Section II-B, we show that this property implies also the existence of a locally convergent reduced order observer, in this way, extending the result established in [2, Corollary 3.1] for the particular case where the metric is Euclidean. The conclusion we draw from Section II is that the design of a locally convergent observer can be reduced to the design a metric exhibiting the differential stability property. Sections III–V are dedicated to such designs in three different contexts.

In Section III, under a uniform observability property of the family of time-varying linear systems resulting from the linearization along solutions to the system, a symmetric covariant 2-tensor giving the strong differential detectability property is shown to exist as a solution to a Riccati equation which, for linear systems, would be an algebraic Riccati equation. Proposition 3.2 establishes this fact. The resulting metric leads to an observer that resembles the Extended Kalman Filter; see, e.g., [3]. In Section III, Proposition 3.5 shows that the metric can instead be taken in the form of an exponentially weighted observability Grammian, leading to an observer design method that is in the spirit of the one proposed in [4].

[2]This expression was suggested to us by Vincent Andrieu.

In Section IV, for systems that are strongly differentially observable [5, Ch. 2.4], we propose an expression for the tensor that is based on the fact that, after writing the system dynamics in an observer form, a high gain observer can be used. This result leads to an observer which has some similarity with the observer for linear systems obtained using Ackerman's formula.

Finally, in Section V, we show how a Riemannian metric can be constructed for Euler-Lagrange systems whose Lagrangian is quadratic in the generalized velocities. This result extends the result in [6].

The design methods proposed in Section III do not necessarily lead to explicit expressions for the metric. Instead, they give numerical procedures to compute it, only involving the solution of ordinary differential equations over a grid of initial conditions. On the other hand, the designs in Sections IV and V involve computations that can be done symbolically. All of these various designs are coordinate independent and do not require to have the system written in some specific form.

To ease the reading, we give a glossary in Appendix A definitions of the main objects we employ from differential geometry.

II. FULL AND REDUCED OBSERVERS UNDER STRONG DIFFERENTIAL DETECTABILITY

In this section, we study what can be obtained when the system satisfies the differential detectability property defined as follows (see items 2, 9, and 11 in Appendix A).

Definition 2.1: The nonlinear system (1) is *strongly differentially detectable* (respectively, *weakly differentially detectable*) on a closed, weakly geodesically convex set $\mathcal{C} \subset \mathbb{R}^n$ with nonempty interior if there exists a symmetric covariant 2-tensor P on $\mathbb{R}^n$ satisfying

$$v^\top \mathcal{L}_f P(x)v < 0 \quad \text{(respectively} \leq 0)$$
$$\forall (x,v) \in \mathcal{C} \times \mathbb{S}^{n-1} : dh(x)v = 0. \quad (4)$$

We illustrate this property with an example

Example 2.2: Consider a harmonic oscillator with unknown frequency. Its dynamics are

$$\dot{x} = f(x) := \begin{pmatrix} x_2 \\ -x_3 x_1 \\ 0 \end{pmatrix}, \quad y = h(x) := x_1 \quad (5)$$

with $(x_1, x_2, x_3) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}_{>0}$. As a candidate to check the differential detectability we pick, in the above coordinates

$$P(x) = \begin{pmatrix} 1 + 2\ell k^2 + 4\ell^2 x_1^2 & -2\ell k & 2\ell x_1 \\ -2\ell k & 2\ell & 0 \\ 2\ell x_1 & 0 & 1 \end{pmatrix} \quad (6)$$

where k and ℓ are strictly positive real numbers. The expression of its Lie derivative $\mathcal{L}_f P$ in these coordinates is

$$\begin{pmatrix} 4\ell k x_3 + 8\ell^2 x_1 x_2 & \star & \star \\ 1 + 2\ell k^2 + 4\ell^2 x_1^2 - 2\ell x_3 & -4\ell k & \star \\ 2\ell k x_1 + 2\ell x_2 & 0 & 0 \end{pmatrix}$$

where the various $\star$ should be replaced by their symmetric values. Then, since we have $(\partial h/\partial x)(x)v = v_1$, where $v = (v_1, v_2, v_3)$, the evaluation of the Lie derivative of P for a vector v in the kernel of dh gives

$$(v_2 \quad v_3) \begin{pmatrix} -4\ell k & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_2 \\ v_3 \end{pmatrix} = -4\ell k v_2^2. \quad (7)$$

This allows us to conclude that the harmonic oscillator with unknown frequency is weakly differentially detectable. Actually, as we shall see later when we use a different metric, it is strongly differentially detectable. △

With this property of differential detectability at hand, we study in the next two subsections what it implies in terms of existence of converging full and then reduced order observers.

A. Local Asymptotic Stabilization of the Set $\mathcal{A}$

In [1, Theorem 3.3 and Lemma 3.6] we have established the following result (see also [7]).

Theorem 2.3: Assume there exist a Riemannian metric P and a closed subset $\mathcal{C}$ of $\mathbb{R}^n$, with nonempty interior, such that

A1: $\mathcal{C}$ is weakly geodesically convex.
A2: There exist a continuous function $\rho : \mathbb{R}^n \to [0, +\infty)$ and a strictly positive real number q such that

$$\mathcal{L}_f P(x) \leq \rho(x) dh(x) \otimes dh(x) - qP(x) \quad \forall x \in \mathcal{C}. \quad (8)$$

A3: There exists a C^2 function $\mathbb{R}^p \times \mathbb{R}^p \ni (y_a, y_b) \mapsto \delta(y_a, y_b) \in [0, +\infty)$ satisfying

$$\delta(h(x), h(x)) = 0, \quad \left.\frac{\partial^2 \delta}{\partial y_a^2}(y_a, y_b)\right|_{y_a = y_b = h(x)} > 0$$

for all $x \in \mathcal{C}$, and, such that, for any pair (x_a, x_b) in $\mathcal{C} \times \mathcal{C}$ satisfying $h(x_a) \neq h(x_b)$ and, for any minimizing geodesic γ^* between $x_a = \gamma^*(s_a)$ and $x_b = \gamma^*(s_b)$ satisfying $\gamma^*(s) \in \mathcal{C}$ for all s in $[s_a, s_b], s_a \leq s_b$, we have

$$\frac{d}{ds}\delta(h(\gamma^*(s)), h(\gamma^*(s_a))) > 0 \quad \forall s \in (s_a, s_b].$$

Then, for any positive real number E there exists a continuous function $k_E : \mathbb{R}^n \to \mathbb{R}$ such that, with the observer given by (see item 4 in Appendix A)

$$F(\hat{x}, y) = f(\hat{x}) - k_E(\hat{x}) \text{grad}_P h(\hat{x}) \frac{\partial \delta}{\partial y_a}(h(\hat{x}), y)^\top \quad (9)$$

the following holds[3]:

$$\mathfrak{D}^+ d(\hat{x}, x) \leq -\frac{q}{4}d(\hat{x}, x)$$

for all $(x, \hat{x}) \in \{(x, \hat{x}) : d(\hat{x}, x) < E\} \bigcap (\text{int}(\mathcal{C}) \times \text{int}(\mathcal{C}))$.

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[3]$\mathfrak{D}^+ d(\hat{x}, x)$ is the upper right Dini derivative along the solution, i.e., with $(\hat{X}((\hat{x}, x), t), X(x, t))$ denoting a solution of (3)

$$\mathfrak{D}^+ d(\hat{x}, x) = \limsup_{t \searrow 0} \frac{d(\hat{X}((\hat{x}, x), t), X(x, t)) - d(\hat{x}, x)}{t}.$$

Theorem 2.3 establishes that, when assumptions A1–A3 hold, for every given positive number E, an observer with vector field as in (9) renders the set $\mathcal{A}$ in (2) asymptotically stable with a domain of attraction containing the set $\{(x, \hat{x}) : d(\hat{x}, x) < E\} \bigcap (\text{int}(\mathcal{C}) \times \text{int}(\mathcal{C}))$.

Condition A2 is a stronger version of what we have called differential detectability in the introduction. We come back to it extensively below.

Condition A3 is a restrictive way of saying that the output level sets are geodesically convex. Fortunately, even without assumption A3, inspired by [6, Theorem 1], we can design an observer making the set (2) asymptotically stable. As opposed to Theorem 2.3, its domain of attraction cannot be made arbitrarily large.

Proposition 2.4: Assume there exist a Riemannian metric P and a closed subset $\mathcal{C}$ of $\mathbb{R}^n$, with nonempty interior, such that:

A1': $\mathcal{C}$ is weakly geodesically convex and there exist coordinates denoted x and positive numbers $\underline{p}$ and $\bar{h}_1$ such that, for each x in $\mathcal{C}$, we have

$$\underline{p} \le |P(x)|, \quad |\text{Hess}_P h(x)| \le \bar{h}_1 \qquad (10)$$

where $\text{Hess}_P h$ is the p-uplet of the Hessian of the components h_i of h; see item 5 in Appendix A.

A2': There exist a positive real number $\bar{\rho}$ and a strictly positive real number $\underline{q}$ such that

$$\mathcal{L}_f P(x) \le \bar{\rho} dh(x) \otimes dh(x) - \underline{q} P(x) \quad \forall x \in \mathcal{C}. \quad (11)$$

A3': There exists a C^2 function $\mathbb{R}^p \times \mathbb{R}^p \ni (y_a, y_b) \mapsto \delta(y_a, y_b) \in [0, +\infty)$ and positive real numbers $\bar{\delta}_1$ and $\underline{\delta}_2$ satisfying

$$\delta(h(x), h(x)) = 0, \quad \left.\frac{\partial^2 \delta}{\partial y_a^2}(y_a, y_b)\right|_{y_a = y_b = h(x)} > \underline{\delta}_2 I \quad (12)$$

for all $x \in \mathcal{C}$

$$\left|\frac{\partial \delta}{\partial y_a}(h(x_a), h(x_b))\right| \le \bar{\delta}_1 d(x_a, x_b) \quad (13)$$

for all $(x_a, x_b) \in \mathcal{C} \times \mathcal{C}$.

Then, with the observer given by

$$F(\hat{x}, y) = f(\hat{x}) - k \, \text{grad}_P h(\hat{x}) \frac{\partial \delta}{\partial y_a}(h(\hat{x}), y)^\top \quad (14)$$

the following holds:

$$\mathfrak{D}^+ d(\hat{x}, x) \le -\underline{r} d(\hat{x}, x) \quad (15)$$

for all $(x, \hat{x}) \in \{(x, \hat{x}) : d(\hat{x}, x) \le (\varepsilon/k)\} \bigcap (\mathcal{C} \times \mathcal{C})$ when we have

$$k \ge \frac{\bar{\rho}}{2\underline{\delta}_2}, \quad \underline{q} > \underline{r}, \quad \varepsilon := \frac{(\underline{q} - \underline{r})\underline{p}}{2\bar{h}_1 \bar{\delta}_1}. \quad (16)$$

Remark 2.5: We make the following observations:

1) A key difference with respect to the result in Theorem 2.3 is that, in the latter, the domain of attraction gets larger

with the increase of the observer gain, while the domain of attraction guaranteed by the result in Proposition 2.4 decreases when k increases.

2) When there exists a positive real number $\bar{h}_2$ satisfying

$$\left|\frac{\partial h}{\partial x}(x)\right| \le \bar{h}_2 \qquad \forall x \in \mathcal{C}$$

a function δ satisfying A3' is simply

$$\delta(y_a, y_b) = |y_a - y_b|^2.$$

Indeed, let $\gamma^* : [s_a, s_b] \to \mathbb{R}^n$ be a minimizing geodesic between x_a and x_b that stays in $\mathcal{C}$. We have

$$\left|\frac{\partial \delta}{\partial y_a}(h(x_a), h(x_b))\right|$$

$$= 2|h(x_a) - h(x_b)|$$

$$= 2\left|\int_{s_a}^{s_b} \frac{\partial h}{\partial x}(\gamma^*(r)) \frac{d\gamma^*}{ds}(r) dr\right|$$

$$= 2\int_{s_a}^{s_b} \sqrt{\frac{\partial h}{\partial x}(\gamma^*(r)) P(\gamma^*(r))^{-1} \frac{\partial h}{\partial x}(\gamma^*(r))^\top}$$

$$\times \sqrt{\frac{d\gamma^*}{ds}(r)^\top P(\gamma^*(r)) \frac{d\gamma^*}{ds}(r)} dr$$

$$\le \frac{2\bar{h}_2}{\sqrt{\underline{p}}} d(x_a, x_b).$$

$\square$

Proof: It is sufficient to show that the vector field $\hat{x} \mapsto F(\hat{x}, y)$ is geodesically strictly monotonic with respect to P (uniformly in y), at least when $\hat{x}$ and x are sufficiently close. See [1, Lemma 2.2] and the discussion before it. With the coordinates given by assumption A1' and item 5 in Appendix A, we have

$$\mathcal{L}_F P(\hat{x}, y) = \mathcal{L}_f P(\hat{x}) - k\mathcal{L}_{\text{grad}_P h} P(\hat{x}, y) \otimes \frac{\partial \delta}{\partial y_a}(h(\hat{x}), y)^\top$$

$$- 2k\frac{\partial h}{\partial x}(\hat{x})^\top \frac{\partial^2 \delta}{\partial y_a^2}(h(\hat{x}), y) \frac{\partial h}{\partial x}(\hat{x})$$

$$= \mathcal{L}_f P(\hat{x}) - 2k \, \text{Hess}_P h(\hat{x}) \otimes \frac{\partial \delta}{\partial y_a}(h(\hat{x}), y)^\top$$

$$- 2k\frac{\partial h}{\partial x}(\hat{x})^\top \frac{\partial^2 \delta}{\partial y_a^2}(h(\hat{x}), y) \frac{\partial h}{\partial x}(\hat{x}).$$

Here, the notation $\text{Hess}_P h \otimes v$, with v a vector in $\mathbb{R}^p$ stands for $\sum_{i=1}^p \text{Hess}_P h_i v_i$, where each $\text{Hess}_P h_i v_i$ is a covariant

2-tensor. So, with (10)–(13) and (16), we obtain successively

$$\mathcal{L}_F P(\hat{x}, y) \leq \mathcal{L}_f P(\hat{x}) + 2k\bar{h}_1\bar{\delta}_1 d(\hat{x}, x) - 2k\underline{\delta}_2 \frac{\partial h}{\partial x}(\hat{x})^\top \frac{\partial h}{\partial x}(\hat{x})$$

$$\leq -\underline{q} P(\hat{x}) + k\frac{2\bar{h}_1\bar{\delta}_1}{\underline{p}} d(\hat{x}, x) P(\hat{x})$$

$$- (2k\underline{\delta}_2 - \bar{\rho}) \frac{\partial h}{\partial x}(\hat{x})^\top \frac{\partial h}{\partial x}(\hat{x})$$

$$\leq -\underline{r} P(\hat{x})$$

for all $(x, \hat{x}) \in \{(x, \hat{x}) : d(\hat{x}, x) \leq (\varepsilon/k)\} \bigcap (\mathcal{C} \times \mathcal{C})$. Since $\mathcal{C}$ is weakly geodesically convex, (15) follows by integration along a minimizing geodesic. ∎

The proofs of Theorem 2.3 and Proposition 2.4 differ mainly on the way the term $\mathrm{Hess}_P h(\hat{x}) \otimes (\partial\delta/\partial y_a)(h(\hat{x}), y)^\top$ is handled. With Assumption A3, related to the geodesic convexity of the output level sets, it can be shown to be harmless because of its sign. Instead, with Assumption A3' only, we go with upper bounds and show it is harmless at least when $\hat{x}$ and x are sufficiently close. Hence, a local convergence result in the latter case and a regional one in the former are obtained.

B. A Link Between the Existence of P and a Reduced Order Observer

In [2, Corollary 3.1] it is established that, if, in some coordinates, the expression of the metric P is constant and that of h is linear, then there exists a reduced order observer. In this section, we establish a similar result without imposing the metric to be Euclidean. The interest of a reduced order observer is that there is no correction term to design. This task is replaced by that of finding appropriate coordinates. In our context, the existence of such coordinates is guaranteed by the following result from [8].

Theorem 2.6 ([8, p. 57, Section 19]): Let P be a complete Riemannian metric on $\mathbb{R}^n$. Assume $p = 1$ and h has rank 1 at x_0 in $\mathbb{R}^n$. Then, there exists a neighborhood $\mathcal{N}_{x_0}$ of x_0 on which there exists coordinates

$$x = (y, \chi)$$

such that, for each x in $\mathcal{N}_{x_0}$, the expression of h and P in these coordinates can be decomposed as

$$y = h((y, \chi)) \qquad (17)$$

$$P((y, \chi)) = \begin{pmatrix} P_{yy}(y, \chi) & 0 \\ 0 & P_{\chi\chi}(y, \chi) \end{pmatrix} \qquad (18)$$

with $P_{yy}(y, \chi)$ in $\mathbb{R}^{p \times p}$ and $P_{\chi\chi}(y, \chi)$ in $\mathbb{R}^{(n-p) \times (n-p)}$.

Proof: See [8, p. 57, Section 19]. A sketch of another proof is as follows. Note first that, the Constant Rank Theorem implies the existence of a neighborhood of x_0 on which coordinates $(y, \bar{\chi})$ are defined and satisfy $h(x) = h((y, \bar{\chi})) = y$. Let the expression of the metric in the $(y, \bar{\chi})$-coordinates be

$$\overline{P}((y, \bar{\chi})) = \begin{pmatrix} \overline{P}_{yy}(y, \bar{\chi}) & \overline{P}_{y\bar{\chi}}(y, \bar{\chi}) \\ \overline{P}_{\bar{\chi}y}(y, \bar{\chi}) & \overline{P}_{\bar{\chi}\bar{\chi}}(y, \bar{\chi}) \end{pmatrix}$$

and let $\varphi(y, \bar{\chi})$ denote the solution, evaluated at time $h(x_0)$, of the time-varying system $(d\mathfrak{x}/d\mathfrak{y}) = -\overline{P}_{\bar{\chi}\bar{\chi}}(\mathfrak{y}, \mathfrak{x})^{-1}\overline{P}_{\bar{\chi}y}(\mathfrak{y}, \mathfrak{x})$ issued from $\mathfrak{x} = \bar{\chi}$ at time $\mathfrak{y} = y$. The proof can be completed by showing that the function φ defined this way on a neighborhood of x_0 satisfies all the required properties for $(y, \chi) = (y, \varphi(y, \bar{\chi}))$ to be the appropriate coordinates in the neighborhood of x_0. ∎

Example 2.7: Consider the matrix P in (6) with $y = x_1, \bar{\chi} = (x_2, x_3)$. We have

$$\overline{P}_{\bar{\chi}y}(y, \bar{\chi}) = \begin{pmatrix} -2\ell k \\ 2\ell y \end{pmatrix}, \quad \overline{P}_{\bar{\chi}\bar{\chi}}(y, \bar{\chi}) = \begin{pmatrix} 2\ell & 0 \\ 0 & 1 \end{pmatrix}.$$

This leads to the system

$$\frac{d\mathfrak{x}}{d\mathfrak{y}} = \mathfrak{f}(\mathfrak{y}, \mathfrak{x}) = -\overline{P}_{\bar{\chi}\bar{\chi}}(\mathfrak{y}, \mathfrak{x})^{-1}\overline{P}_{\bar{\chi}y}(\mathfrak{y}, \mathfrak{x}) = \begin{pmatrix} k \\ -2\ell\mathfrak{y} \end{pmatrix}$$

the solutions of which, at time $\mathfrak{y}$, going through $\mathfrak{x}_0$ at time $\mathfrak{y}_0$, are

$$\mathfrak{X}(\mathfrak{x}_0, \mathfrak{y}_0; \mathfrak{y}) = \mathfrak{x}_0 + \begin{pmatrix} k[\mathfrak{y} - \mathfrak{y}_0] \\ -\ell[\mathfrak{y}^2 - \mathfrak{y}_0^2] \end{pmatrix}.$$

So in particular, we get

$$\varphi((y, \bar{\chi})) = \mathfrak{X}((x_2, x_3), y; 0) = \begin{pmatrix} x_2 - ky \\ x_3 + \ell y^2 \end{pmatrix}.$$

From the proof above, it follows that the coordinates (y, χ) satisfying (18) in Theorem 2.6 are:

$$(y, \chi_1, \chi_2) = \varphi(x) = \varphi((y, \bar{\chi})) = (x_1, x_2 - kx_1, x_3 + \ell x_1^2). \qquad (19)$$

They are defined on the open set

$$\Omega = \mathcal{N}_{x_0} = \varphi(\mathbb{R}^2 \times \mathbb{R}_{>0}) \qquad (20)$$

and they give

$$P_{yy}((y, \chi)) = 1, \quad P_{\chi\chi}((y, \chi)) \begin{pmatrix} 2\ell & 0 \\ 0 & 1 \end{pmatrix}.$$

△

Let us express the differential detectability and the observer (9) in the special coordinates given by Theorem 2.6. The dynamics of (1) in the coordinates (y, χ) are

$$\dot{y} = f_y(y, \chi), \quad \dot{\chi} = f_\chi(y, \chi).$$

We notice that, by decomposing a tangent vector as $v = \begin{pmatrix} v_y \\ v_\chi \end{pmatrix}$, and since $(\partial h/\partial y)(x_0) \neq 0$, we find that (17) gives, for every $x = (y, \chi)$ in $\mathcal{N}_{x_0}$

$$\frac{\partial h}{\partial x}(x)v = 0 \iff \frac{\partial h}{\partial y}(y, \chi)v_y = 0 \iff v_y = 0.$$

It follows that, with expression (18) and in (y, χ) coordinates, condition A2 in (8) is as follows:

$$
2v_\chi^\top P_{\chi\chi}(y,\chi)\frac{\partial f_\chi}{\partial \chi}(\chi)v_\chi + \frac{\partial}{\partial y}\left(v_\chi^\top P_{\chi\chi}(y,\chi)v_\chi\right)f_y(y,\chi)
$$

$$
+ \frac{\partial}{\partial \chi}\left(v_\chi^\top P_{\chi\chi}(y,\chi)v_\chi\right)f_\chi(y,\chi) \leq -qv_\chi^\top P_{\chi\chi}(y,\chi)v_\chi \quad (21)
$$

for all (y, χ, v_χ) such that $(y, \chi) \in \mathcal{N}_{x_0}, v_\chi \in \mathbb{S}^{n-2}$. Also our observer (9) takes the form

$$
\dot{\hat{y}} = f_y(\hat{y}, \hat{\chi}) - k_E((\hat{y}, \hat{\chi}))\frac{1}{P_{yy}((\hat{y}, \hat{\chi}))}\frac{\partial \delta}{\partial y_a}(\hat{y}, y)
$$

$$
\dot{\hat{\chi}} = f_\chi(\hat{y}, \hat{\chi}).
$$

The remarkable fact here is that there is no "correction term" in the dynamics of $\hat{\chi}$. Hence, we may expect that, if P is a complete Riemannian metric for which there exist coordinates defined on some open set Ω satisfying (17), (18), and (21) (with Ω replacing $\mathcal{N}_{x_0}$), then the system

$$
\dot{\hat{\chi}} = f_\chi(y, \hat{\chi}) \quad (22)
$$

(with y instead of $\hat{y}$!) could be an appropriate reduced order observer in charge of estimating the unmeasured components χ. To show that this is indeed the case, we equip $\mathbb{R}^{n-p}$, in which this reduced order observer lives, with the y dependent Riemannian metric $\chi \mapsto P_{\chi\chi}(y, \chi)$. For each fixed y, we define the distance

$$
d_\chi(\chi_a, \chi_b; y) = \min_{\gamma_\chi}\int_{s_a}^{s_b}\sqrt{\frac{d\gamma_\chi}{ds}(s)^\top P_{\chi\chi}(y, \gamma_\chi(s))\frac{d\gamma_\chi}{ds}(s)}ds \quad (23)
$$

where γ_χ is any piecewise C^1 path satisfying $\gamma_\chi(s_a) = \chi_a$, $\gamma_\chi(s_b) = \chi_b$. With this, we have the following result for the reduced order observer (22).

Proposition 2.8: Let $P_{\chi\chi}$ be a y-dependent Riemannian metric on $\mathbb{R}^{n-p}$ and $\mathcal{C}$ be a closed subset of $\mathbb{R}^n$, with nonempty interior, satisfying:

A1": $\mathcal{C}$ is weakly $P_{\chi\chi}$-geodesically convex in the following sense: if (χ_a, χ_b, y) is such that (y, χ_a) and (y, χ_b) are in $\mathcal{C}$, then there exists a minimizing geodesic $[s_a, s_b] \ni s \mapsto \gamma_\chi^*(s)$ in the sense of (23) such that $(y, \gamma_\chi^*(s))$ is in $\mathcal{C}$ for all s in $[s_a, s_b]$. Also, there exist coordinates denoted χ and positive numbers $\underline{p}, \overline{p}_{y1}, \bar{f}_{y1}$, such that, for each (y, χ) in $\mathcal{C}$, we have

$$
\underline{p}I_{n-p} \leq P_{\chi\chi}(y, \chi), \quad \left|\frac{\partial P_{\chi\chi}}{\partial y}(y, \chi)\right| \leq \overline{p}_{y1}
$$

$$
\left|\frac{\partial f_y}{\partial \chi}(y, \chi)\right| \leq \bar{f}_{y1}.
$$

A2": There exists a strictly positive real number q such that (21) holds on $\mathcal{C} \times \mathbb{S}^{n-p-1}$.

Then, along the solutions to the system

$$
\dot{y} = f_y(y, \chi), \quad \dot{\chi} = f_\chi(y, \chi), \quad \dot{\hat{\chi}} = f_\chi(y, \hat{\chi})
$$

the following holds:

$$
\mathfrak{D}^+ d_\chi(\hat{\chi}, \chi; y) \leq -\underline{r}d_\chi(\hat{\chi}, \chi; y)
$$

for all $(\chi, \hat{\chi}, y)$ such that $(y, \chi), (y, \hat{\chi}) \in \mathcal{C}$ and

$$
d_\chi(\hat{\chi}, \chi) \leq \frac{(q - 2\underline{r})\underline{p}\sqrt{\underline{p}}}{\bar{p}_{y1}\bar{f}_{y1}}. \quad (24)
$$

The rationale is that, if the system is strongly differentially detectable (see Definition 2.1), then there exists a reduced order observer that is exponentially convergent as long as (y, χ) and $(y, \hat{\chi})$ are in $\mathcal{C}$ and the coordinates $x = (y, \chi)$ exist, which, when $p = 1$, we know is the case on a neighborhood of any point where h has rank 1.

Proof: Let $(\chi, \hat{\chi}, y)$ be such that (y, χ) and $(y, \hat{\chi})$ are in $\mathcal{C}$. From our assumption, there exists a minimizing geodesic $[s, \hat{s}] \ni s' \mapsto \gamma_\chi^*(s')$ such that $(y, \gamma_\chi^*(s'))$ is in $\mathcal{C}$ for all s' in $[s, \hat{s}]$. By following the same steps as in [9, Proof of Theorem 2] and with [1, (36)], we can show that we have

$$
\mathfrak{D}^+ d_\chi(\hat{\chi}, \chi; y)
$$

$$
\leq \int_s^{\hat{s}}\frac{\frac{d\gamma_\chi^*}{ds}(r)^\top\left[\mathcal{L}_{f_\chi}P_{\chi\chi}(y, \gamma_\chi^*(r)) + \frac{\partial P_{\chi\chi}}{\partial \chi}(y, \gamma_\chi^*(r))\dot{y}\right]\frac{d\gamma_\chi^*}{ds}(r)}{2\sqrt{\frac{d\gamma_\chi^*}{ds}(r)^\top P_{\chi\chi}(y, \gamma_\chi^*(r))\frac{d\gamma_\chi^*}{ds}(r)}}dr
$$

where $\dot{y} = f_y(y, \chi)$. So our result holds if the term between brackets is upper bounded by $-2\underline{r}P(y, \gamma_\chi^*(r))$. Note that, in the coordinates given by A1", (21) can be rewritten as

$$
\mathcal{L}_{f_\chi}P_{\chi\chi}(y, \gamma_\chi^*) + \frac{\partial P_{\chi\chi}}{\partial y}(y, \gamma_\chi^*)\dot{y} \leq -qP_{\chi\chi}(y, \gamma_\chi^*)
$$

$$
+ \frac{\partial P_{\chi\chi}}{\partial y}(y, \gamma_\chi^*)\left[f_y(y, \chi) - f_y(y, \gamma_\chi^*)\right] \quad (25)
$$

for all (χ, γ_χ^*, y) such that (y, χ) and (y, γ_χ^*) are in $\mathcal{C}$. But we have also

$$
\left|\frac{\partial P}{\partial y}(y, \gamma_\chi^*(r))\left[f_y(y, \chi) - f_y(y, \gamma_\chi^*(r))\right]\right|
$$

$$
\leq \bar{p}_{y1}\bar{f}_{y1}\frac{d_\chi(\hat{\chi}, \chi; y)}{\sqrt{\underline{p}}}\frac{P_{\chi\chi}(y, \gamma_\chi^*(r))}{\underline{p}}.
$$

Hence, the result holds when (24) holds. ∎

In this proof, we see that the restriction (24) disappears and q can be zero, if $\bar{p}_{y1}$ is zero, i.e., if $P_{\chi\chi}$ does not depend on y. This is indeed the case when the level sets of the output function are totally geodesic as shown in [1]. Hence, we have the following result.

Proposition 2.9: Under conditions A1" and A2" in Proposition 2.8 with q possibly zero, if $P_{\chi\chi}$ does not depend on y, we have

$$
\mathfrak{D}^+ d_\chi(\hat{\chi}, \chi) \leq -qd(\hat{\chi}, \chi) \quad (26)
$$

for all $(\chi, \hat{\chi}, y)$ such that (y, χ) and $(y, \hat{\chi})$ are in $\mathcal{C}$.

Again, the rationale is that if, the system is strongly (respectively weakly) differentially detectable and the output function level sets are totally geodesic, then there exists a reduced order observer which makes the zero error set $\{(y, \chi, \hat{\chi}) : \chi = \hat{\chi}\}$ exponentially stable (respectively stable) as long as (y, χ) and $(y, \hat{\chi})$ are in $\mathcal{C}$ and the coordinates $x = (y, \chi)$ exist.

Example 2.10: Consider the harmonic oscillator with unknown frequency (5). Its dynamics expressed in the coordinates (y, χ_1, χ_2) we have obtained in (19) are

$$\dot{y} = \chi_1 + ky$$
$$\dot{\chi}_1 = -y(\chi_2 - \ell y^2) - k(\chi_1 + ky)$$
$$\dot{\chi}_2 = 2\ell y(\chi_1 + ky). \tag{27}$$

In Example 2.2, we have shown this system is weakly differentially detectable with a metric the expression of which in the (y, χ_1, χ_2) coordinates is

$$P((y, \chi_1, \chi_2)) = \left[\left[\frac{\partial \varphi}{\partial x}(x) \right]^{-1} \right]^\top P(x) \left[\frac{\partial \varphi}{\partial x}(x) \right]^{-1}$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2\ell & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{28}$$

As already observed in Example 2.7, the decomposition given in (18) of Theorem 2.6 with even the $P_{\chi\chi}$ block independent of y. So the assumptions of Proposition 2.9 are satisfied with $\mathcal{C} = \mathbb{R}^3$, but with $q = 0$ and the zero error set (with Ω given in (20))

$$\mathcal{Z} = \left\{ (y, \chi_1, \chi_2, \hat{\chi}_1, \hat{\chi}_2) \in \Omega \times \mathbb{R}^2 : \chi_2 = \hat{\chi}_2 \right\}$$

is globally stable. To check that we have actually global stability, we note that the Lie derivative of the $P_{\chi\chi}$ block of P in (28) along the vector field given by (27) satisfies for all y

$$2 \, \mathsf{Sym} \left(\begin{pmatrix} 2\ell & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -k & -y \\ 2\ell y & 0 \end{pmatrix} \right) = \begin{pmatrix} -4\ell k & 0 \\ 0 & 0 \end{pmatrix}$$

where for a matrix A, $\mathsf{Sym}(A) = (A + A^\top)/2$. This establishes that the vector field f_χ defined as

$$f_\chi(y, \chi) = \begin{pmatrix} -y(\chi_2 - \ell y^2) - k(\chi_1 + ky) \\ 2\ell y(\chi_1 + ky) \end{pmatrix}$$

is weakly geodesically monotonic uniformly in y. This implies that the flow it generates is a weak contraction. The solutions of the harmonic oscillator being bounded, the same holds for the solutions of

$$\dot{\hat{\chi}} = f_\chi(y, \hat{\chi}). \tag{29}$$

Then, according to [10, Theorem 2], the set[4]

$$\mathcal{Z} \setminus \left(\varphi \left(\{(0,0)\} \times \mathbb{R}_{>0} \right) \times \mathbb{R}^2 \right)$$

with φ defined in (19), is globally asymptotically stable for the interconnected system (5), (29). △

---

[4]This means that the initial condition for (x_1, x_2) is not the origin.

III. DESIGN OF RIEMANNIAN METRIC P FOR LINEARLY RECONSTRUCTIBLE SYSTEMS

We have seen in [1, Theorem 2.9] (see also [11, Proposition 3.2]) that differential detectability implies that each linear (time varying) system given by the first order approximation of (1) (assumed to be forward complete) along any of its solution is uniformly detectable. In [11, Proposition 3.2], it is also shown that, if this uniform linear detectability is strengthened into a uniform reconstructibility property (or, say, uniform infinitesimal observability [5, Section I.2.1]), then a Riemannian metric exhibiting differential detectability does exist. In this section, we recover this last property through the solution of a Riccati equation and propose a numerical method to compute the metric P.[5]

To do all this, we assume the existence of a backward invariant open set Ω for the system (1). This implies that, for each x in Ω, there exists a strictly positive real number σ_x, possibly infinite, such that the corresponding solution to (1), $t \mapsto X(x,t)$, is defined with values in Ω over $(-\infty, \sigma_x)$. For each such x, the linearization of f and h evaluated along $t \mapsto X(x,t)$ gives the functions $t \mapsto A_x(t) = (\partial f/\partial x)(X(x,t))$ and $t \mapsto C_x(t) = (\partial h/\partial x)(X(x,t))$, which are defined on $(-\infty, \sigma_x)$. To these functions, we associate the following family of linear time-varying systems with state ξ in $\mathbb{R}^n$ and output η in $\mathbb{R}^p$:

$$\dot{\xi} = A_x(t)\xi, \quad \eta = C_x(t)\xi \tag{30}$$

which is parameterized by the initial condition x of the chosen solution $t \mapsto X(x,t)$. Below, Φ_x denotes the state transition matrix for (30). It satisfies

$$\frac{\partial \Phi_x}{\partial s}(t, s) = A_x(t)\Phi_x(t, s), \quad \Phi_x(s, s) = I.$$

Definition 3.1 (Reconstructibility): The family of systems (30) is said to be reconstructible on a set Ω if there exist strictly positive real numbers τ and ε, such that we have

$$\int_{-\tau}^{0} \Phi_x(t, 0)^\top C_x(t)^\top C_x(t) \Phi_x(t, 0) dt \geq \varepsilon I \quad \forall x \in \Omega. \tag{31}$$

Proposition 3.2: Let Q be a symmetric contravariant 2-tensor. Assume there exist:

i) an open set $\Omega \subset \mathbb{R}^n$ that is backward invariant for (1) and on which the family of systems (30) is reconstructible;
ii) coordinates for x such that the derivatives of f and h are bounded on Ω and we have

$$0 < \underline{q}I \leq Q(x) \leq \overline{q}I \quad \forall x \in \Omega. \tag{32}$$

Then, there exists a symmetric covariant 2-tensor P defined on Ω, which admits a Lie derivative $\mathcal{L}_f P$, satisfying

$$\mathcal{L}_f P(x) = dh(x) \otimes dh(x) - P(x)Q(x)P(x) \quad \forall x \in \Omega \tag{33}$$

---

[5]Some of the material in this section is in [12], which we reproduce here for the sake of completeness.

and there exist strictly positive real numbers $\underline{p}$ and $\overline{p}$ such that, in the coordinates given above, we have

$$0 < \underline{p}I \le P(x) \le \overline{p}I \quad \forall\, x \in \Omega. \tag{34}$$

Proof: The proof of Proposition 3.2 can be found in [12]. It relies on a fixed point argument, the core of which is the fact the flow generated by the differential Riccati equation is a contraction. This fact, first established for the discrete time case in [13], is proved in [14] for the continuous-time case. ■

Remark 3.3: In his introduction of Riccati differential equations for matrices in [15] and [16], Radon has shown that such equations can be solved via two coupled linear differential equations. (See also [17].) In our framework, this leads to obtain a solution to (33) by solving in (α, β) the coupled system

$$\sum_{i=1}^{n} \frac{\partial \alpha}{\partial x_i}(x) f_i(x) = -\frac{\partial f}{\partial x}(x)^\top \alpha(x) + \frac{\partial h}{\partial x}(x)\frac{\partial h}{\partial x}(x)^\top \beta(x)$$

$$\sum_{i=1}^{n} \frac{\partial \beta}{\partial x_i}(x) f_i(x) = Q(x)\alpha(x) + \frac{\partial f}{\partial x}(x)\beta(x) \tag{35}$$

with β invertible and then picking $P(x) = \alpha(x)\beta(x)^{-1}$. □

Remark 3.4: Our observer in (3) with right-hand side given by (9) or (14) resembles the Extended Kalman filter for a particular choice of δ. In fact, when the metric is obtained by solving (33), the observer we obtain from (9) (or (14)) with $\delta(y_a, y_b) = |y_a - y_b|^2$ resembles an Extended Kalman Filter (see [3], for instance) since, in some coordinates, our observer is

$$\dot{\hat{x}} = f(\hat{x}) - 2k_E(\hat{x})P(\hat{x})^{-1}\frac{\partial h}{\partial x}(\hat{x})^\top (h(\hat{x}) - y) \tag{36}$$

$$\sum_{i=1}^{n} \frac{\partial P}{\partial x_i}(\hat{x}) f_i(\hat{x}) = -P(\hat{x})\frac{\partial f}{\partial x}(\hat{x}) - \frac{\partial f}{\partial x}(\hat{x})^\top P(\hat{x})$$
$$+ \frac{\partial h}{\partial x}(\hat{x})^\top \frac{\partial h}{\partial x}(\hat{x}) - P(\hat{x})Q(\hat{x})P(\hat{x}) \tag{37}$$

while the corresponding extended Kalman filter would be

$$\dot{\hat{x}} = f(\hat{x}) - P^{-1}\frac{\partial h}{\partial x}(\hat{x})^\top (h(\hat{x}) - y) \tag{38}$$

$$\dot{P} = -P\frac{\partial f}{\partial x}(\hat{x}) - \frac{\partial f}{\partial x}(\hat{x})^\top P + \frac{\partial h}{\partial x}(\hat{x})^\top \frac{\partial h}{\partial x}(\hat{x}) - PQ(\hat{x})P. \tag{39}$$

The expressions for $\dot{\hat{x}}$ in (38) and (36) are the same except for the presence of k_E in (36). On the other hand, (37) and (39) are significantly different. The former is a partial differential equation which can be solved off-line as an algebraic Riccati equation. If the assumptions in Proposition 3.2 are satisfied, (37) has a solution, guaranteed to be bounded and positive definite on Ω. Nevertheless, assumption A3 of Theorem 2.3 may not hold but then according to Proposition 2.4, we have a locally convergent observer.

The differential Riccati (39) of the extended Kalman filter is an ordinary differential equation with P being part of the observer state. The corresponding observer is also known to be locally convergent but under the extra assumption that P

is bounded and positive definite. See [18] for instance. Unfortunately, even when the assumptions in Proposition 3.2 are satisfied, we have no guarantee that P has such properties except may be if $\hat{x}$ remains close enough to x (which is what is to be proved). □

The quadratic term $P(x)Q(x)P(x)$ in the "algebraic Riccati equation" (33), can be replaced by $\lambda P(x)$. Specifically we have the following reformulation of [11, Proposition 3.2].

Proposition 3.5: Under the conditions of Proposition 3.2, there exists $\underline{\lambda} > 0$ such that, for each $\lambda > \underline{\lambda}$, there exists a symmetric covariant 2-tensor P defined on Ω that admits a Lie derivative $\mathcal{L}_f P$ satisfying

$$\mathcal{L}_f P(x) = dh(x) \otimes dh(x) - \lambda P(x) \quad \forall\, x \in \Omega \tag{40}$$

and there exist strictly positive real numbers $\underline{p}$ and $\overline{p}$ such that the expression of P in the coordinates given by the assumption satisfies (34).

Proof: See [12]. ■

Remark 3.6: When the metric is given by (40), the observer we obtain from (9) with $\delta(y_a, y_b) = |y_a - y_b|^2$ resembles the Kleinman's observer, dual of the Kleinman's controller proposed in [4]. Indeed, in some coordinates, our observer is

$$\dot{\hat{x}} = f(\hat{x}) - 2k_E(\hat{x})P(\hat{x})^{-1}\frac{\partial h}{\partial x}(\hat{x})^\top (h(\hat{x}) - y)$$

$$P(x) = \lim_{T \to \infty} \int_{-T}^{0} \exp(\lambda t)\Phi_x(t,0)^\top C_x(t)^\top C_x(t)\Phi_x(t,0)dt$$

the latter being a solution to (40). Correspondingly, Kleinman's observer would be

$$\dot{\hat{x}} = f(\hat{x}) - W(\hat{x})^{-1}\frac{\partial h}{\partial x}(\hat{x})^\top (h(\hat{x}) - y)$$

$$W(x) = \int_{-T}^{0} \Phi_x(t,0)^\top C_x(t)^\top C_x(t)\Phi_x(t,0)dt$$

with T positive. □

Example 3.7: For the harmonic oscillator with unknown frequency (5), it can be checked that the following expression of P is a solution to (40):

$$P(x) = \begin{pmatrix} \frac{\lambda^2 + 2x_3}{\lambda(\lambda^2 + 4x_3)}, & \star, & \star \\ -\frac{1}{(\lambda^2 + 4x_3)}, & \frac{2}{\lambda(\lambda^2 + 4x_3)}, & \star \\ \frac{-\lambda^3 x_1 + (\lambda^2 - 4x_3)x_2}{\lambda^2(\lambda^2 + 4x_3)^2}, & \frac{(3\lambda^2 + 4x_3)x_1 - 4\lambda x_2}{\lambda^2(\lambda^2 + 4x_3)^2}, & a \end{pmatrix} \tag{41}$$

where the various $\star$ should be replaced by their symmetric values and

$$a = \frac{6\lambda^4 + 12\lambda^2 x_3 + 16x_3^2}{\lambda^3(\lambda^2 + 4x_3)^3}x_1^2 - \frac{4(5\lambda^2 + 4x_3)}{\lambda^2(\lambda^2 + 4x_3)^3}x_1 x_2$$
$$+ \frac{4(5\lambda^2 + 4x_3)}{\lambda^3(\lambda^2 + 4x_3)^3}x_2^2.$$

△

One way to prove Proposition 3.2, respectively, Proposition 3.5, is to show that the system

$$\dot{x} = f(x)$$

$$\dot{\pi} = F(x, \pi) = -\pi \frac{\partial f}{\partial x}(x) - \frac{\partial f}{\partial x}(x)^\top \pi + \frac{\partial h}{\partial x}(x)^\top \frac{\partial h}{\partial x}(x) - \pi Q(x)\pi$$

respectively

$$\dot{x} = f(x)$$

$$\dot{\pi} = F(x, \pi) = -\pi \frac{\partial f}{\partial x}(x) - \frac{\partial f}{\partial x}(x)^\top \pi + \frac{\partial h}{\partial x}(x)^\top \frac{\partial h}{\partial x}(x) - \lambda \pi$$

admits an invariant manifold of the form $\{(x, \pi) : \pi = P(x)\}$. These facts suggest the following method to approximate P.

Given x in Ω at which P is to be evaluated, pick $T > 0$ large enough, and perform the following steps[6]:

Step 1) Compute the solution $[-T, 0] \ni t \mapsto X(x, t)$ to (1) backward in time from the initial condition x at time $t = 0$, up to a negative time $t = -T$.

Step 2) Using the function $[-T, 0] \ni t \mapsto X(x, t)$ obtained in Step 1, compute the solution $[-T, 0] \ni t \mapsto \Pi(t)$ with initial condition $\pi(-T) = \underline{p}I_n$, to

$$\dot{\pi} = -\pi \frac{\partial f}{\partial x}\left(X(x, t)\right) - \frac{\partial f}{\partial x}\left(X(x, t)\right)^\top \pi$$

$$+ \frac{\partial h}{\partial x}\left(X(x, t)\right)^\top \frac{\partial h}{\partial x}\left(X(x, t)\right) - \pi Q\left(X(x, t)\right)\pi$$

respectively to

$$\dot{\pi} = -\pi \frac{\partial f}{\partial x}\left(X(x, t)\right) - \frac{\partial f}{\partial x}\left(X(x, t)\right)^\top \pi$$

$$+ \frac{\partial h}{\partial x}\left(X(x, t)\right)^\top \frac{\partial h}{\partial x}\left(X(x, t)\right) - \lambda \pi$$

with λ large enough.

Step 3) Define the value of P at x as the value $\Pi(0)$.

By griding the state space of x and approximating P at each such x, the method suggested above can be considered as a design tool, at least for low dimensional systems. Note that the computations in Step 1 and Step 2 only require the use of a scheme for integration of ordinary differential equations. In the following example, we employ this method to approximate the metric P for the harmonic oscillator after a convenient reparameterization allowing a reduction of the number of points needed in a grid for a given desired precision.

Example 3.8: The second version of the proposed algorithm applied to the harmonic oscillator in (5) leads to an approximation of the analytic expression of the metric P given in Example 3.7. To this end, we exploit the fact that $\sqrt{x_3}$ and t

have the same dimension and, similarly, $x_1/\sqrt{x_3}$, and x_2/x_3 have the same dimension. To exploit this property, we let

$$r = \sqrt{x_3 x_1^2 + x_2^2}, \qquad \cos(\theta) = \frac{\sqrt{x_3} x_1}{r}$$

$$\sin(\theta) = \frac{x_2}{r}, \qquad\qquad \omega = \frac{\lambda}{\sqrt{x_3}}.$$

Then, it can be checked that the metric P can be factorized as

$$P(x_1, x_2, x_3, \lambda) = M(x_3)^{-1} P\left(\cos(\theta), \sin(\theta), 1, \omega\right) M(x_3)^{-1}$$

where $M(x_3) = \mathrm{diag}(x_3^{1/4}, \sqrt{x_3} x_3^{1/4}, (x_3 \sqrt{x_3} x_3^{1/4}/r))$. This shows that it is sufficient to know the function $(\theta, \omega) \ni (\mathbb{S}^1 \times \mathbb{R}_{>0}) \mapsto P(\cos(\theta), \sin(\theta), 1, \omega)$ and the value of x_3 to know the function P everywhere on $(\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}_{>0}^2$. Further using the fact that

$$\frac{\partial h}{\partial x}(x_1, x_2, x_3) = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$$

the gain of the proposed observer reduces to

$$P(x_1, x_2, x_3, \lambda)^{-1} \frac{\partial h}{\partial x}(x_1, x_2, x_3)^\top$$

$$= \begin{pmatrix} \sqrt{x_3} & 0 & 0 \\ 0 & x_3 & 0 \\ 0 & 0 & \frac{x_3^2}{r} \end{pmatrix} P\left(\cos(\theta), \sin(\theta), 1, \omega\right)^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

This shows that it is sufficient to know the function

$$(\theta, \omega) \ni (\mathbb{S}^1 \times \mathbb{R}_{>0}) \mapsto \begin{pmatrix} P_{11}^{-1}\left(\cos(\theta), \sin(\theta), 1, \omega\right) \\ P_{12}^{-1}\left(\cos(\theta), \sin(\theta), 1, \omega\right) \\ P_{13}^{-1}\left(\cos(\theta), \sin(\theta), 1, \omega\right) \end{pmatrix}$$

to know the observer gain everywhere on $(\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}_{>0}^2$. Hence, it is sufficient to grid the circle $\mathbb{S}^1$ with m_θ points and the strictly positive real numbers with m_ω points, and therefore to store only $3 * m_\theta * m_\omega$ values in which the above function is interpolated.

We note that for the computation of P using the algorithm above, since a closed-form expression of the solutions to (5) is available, Step 1 of the algorithm is not needed. To compute an approximation of P, we define a grid of the (θ, ω)-region $[-\pi, \pi] \times [4, 7]$ with $m_\theta * m_\omega$ points with $m_\theta = 360$ and $m_\omega = 100$. The value of T used in the simulations is chosen as a function of ω, namely, $T(\omega)$, so as to guarantee a desired absolute error for the approximation of P for the given point (θ, ω) from the grid.

Fig. 1 shows state estimates $\hat{x}$ using the observer in (14) for a periodic solution to (5). These solutions start from the same initial condition and are such that the state estimates asymptotically converge to the periodic solution. The solid blue/darkest solution corresponds to the estimate obtained using in (14) the analytic expression of P in (41) with parameter $\lambda = 8$, which is a large enough value to satisfy the desired precision. The other solutions in Fig. 1 correspond to estimates obtained with different computed values of P using our algorithm. The dash dot blue/gray solution is obtained when observer gain is discretized over the chosen grid and provided to the observer using nearest point interpolation. The dashed red/dark solution is obtained

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[6]In the case where the system is time varying and its time variations are dealt with as explained in footnote 1, these steps do require the knowledge of the time functions. This imposes a difficulty when, for instance, the time functions are induced by inputs provided by a feedback law.
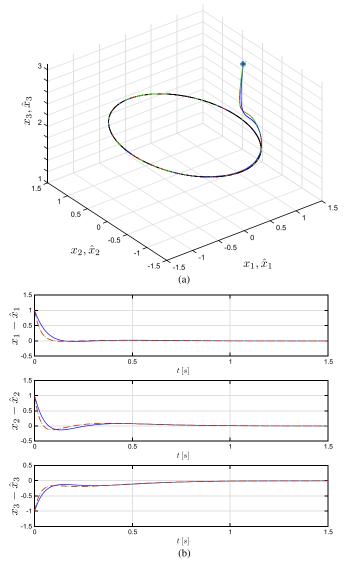
Fig. 1. Solutions to the observer converging to the estimate obtained with exact gain with $\lambda = 8$ (solid blue/darkest), with exact gain discretized over a grid (dash dot blue/gray), and with computed and interpolated gain (dashed red/dark). (a) Solutions. (b) Estimation errors.

when the observer gain is computed (over the same grid) using the algorithm proposed above. For each simulation, the error trajectories converge to zero. Note that the error between the dash dot blue/gray solution and the dashed red/dark solution is quite small. As the figures suggest, the estimates obtained with the approximated gains are close to the one obtained with its analytical expression. Additional numerical analysis confirms that the error between the solutions gets smaller as the number of points and the quality of the interpolation are increased. △

IV. DESIGN OF RIEMANNIAN METRIC P FOR STRONGLY DIFFERENTIALLY OBSERVABLE SYSTEMS

According to [5, Def. 4.2 of Ch. 2], the nonlinear system (1) is strongly differentially observable of order n_o on an open set Ω if, for the positive integer n_o, the function $\mathcal{H}_{n_o} : \Omega \to \mathbb{R}^{m \times n_o}$ defined as

$$\mathcal{H}_{n_o}(x) = \left(h(x), L_f h(x), \dots, L_f^{n_o-1} h(x) \right)^\top \quad (42)$$

is an injective immersion, i.e., an injective map whose differential is injective at each point x in Ω.

Example 4.1: For the system (5) in Example 3.7, successive derivatives of y lead to

$$\mathcal{H}_3(x) = (x_1, x_2, -x_3 x_1)^\top$$

$$\mathcal{H}_4(x) = (x_1, x_2, -x_3 x_1, -x_3 x_2)^\top.$$

The map $\mathcal{H}_3$ is an injective immersion on $\Omega_3 = (\mathbb{R} \setminus \{0\}) \times \mathbb{R} \times \mathbb{R}_{>0}$ which is not an invariant set. Instead $\mathcal{H}_4$ is an injective immersion on $\Omega_4 = (\mathbb{R}^2 \times \mathbb{R}_{>0}) \setminus (\{(0,0)\} \times \mathbb{R}_+)$ which is an invariant set. Hence, the system in Example 3.7 is strongly differentially observable of order 4 on the invariant set Ω_4. △

The property that $\mathcal{H}_{n_o}$ is an injective immersion implies that the family of systems (30) is reconstructible (on Ω). According to Section III, this property further implies that differential detectability holds with a metric obtained as a solution of (33) or of (40). But we can take advantage of the strong observability property to give another more explicit expression for the metric. Precisely, we assume the following properties.

B: There are coordinates for x in Ω such that:

- $\mathcal{H}_{n_o}$ is Lipschitz and a uniform immersion, i.e., assume the existence of strictly positive real numbers $\underline{h}$ and $\overline{h}$ such that we have

$$\underline{h}I \le \frac{\partial \mathcal{H}_{n_o}}{\partial x}(x)^\top \frac{\partial \mathcal{H}_{n_o}}{\partial x}(x) \le \overline{h}I \quad \forall x \in \Omega. \quad (43)$$

- There exists a strictly positive real number ν such that, in the given coordinates for x, we have the following Lipschitz-like condition[7]

$$\left| \frac{\partial L_f^{n_o} h}{\partial x}(x) \right| \le \frac{1}{\nu} \left| \frac{\partial \mathcal{H}_{n_o}}{\partial x}(x) \right|. \quad (44)$$

To exploit these properties, we note first that we have

$$L_f \mathcal{H}_{n_o}(x) = \frac{\partial \mathcal{H}_{n_o}}{\partial x}(x) f(x) = \mathcal{A}\mathcal{H}_{n_o}(x) + \mathcal{B}L_f^{n_o} h(x)$$

$$y = h(x) = \mathcal{C}\mathcal{H}_{n_o}(x) \quad (45)$$

where $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are given by

$$\mathcal{A} = \begin{pmatrix} 0 & I_m & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & I_m \\ 0 & \dots & \dots & \dots & 0 \end{pmatrix}, \qquad \mathcal{B} = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ I_m \end{pmatrix}$$

$$\mathcal{C} = (I_m \quad 0 \quad \dots \quad \dots \quad 0).$$

Then, among the many results known about high gain observers, we have the following property.

---

[7] We say that (44) is a Lipschitz-like condition since, the function $\mathcal{H}_{n_o}$, being injective, has a left inverse $\mathcal{H}_{n_o}^{\mathrm{li}}$ satisfying $\mathcal{H}_{n_o}^{\mathrm{li}}(\mathcal{H}_{n_o}(x)) = x$. Consequently, we have $L_f^{n_o} h = L_f^{n_o}(h \circ \mathcal{H}_{n_o}^{\mathrm{li}} \circ \mathcal{H}_{n_o})$. It follows that, if the function $\xi \mapsto L_f^{n_o}(h \circ \mathcal{H}_{n_o}^{\mathrm{li}})(\xi)$ is Lipschitz, then (44) holds.

Lemma 4.2: Given ν satisfying (44), there exist an $(m \times n_o) \times (m \times n_o)$ symmetric positive definite matrix $\mathcal{P}_\nu$, a $(m \times n_o) \times m$ column vector $\mathcal{K}_\nu$, and a strictly positive real number q satisfying

$$\mathcal{P}_\nu(\mathcal{A} - \mathcal{K}_\nu \mathcal{C}) + (\mathcal{A} - \mathcal{K}_\nu \mathcal{C})^\top \mathcal{P}_\nu + 2qI_{m \times n_o} + \frac{1}{q\nu^2}\mathcal{P}_\nu \mathcal{B}\mathcal{B}^\top \mathcal{P}_\nu \leq 0. \tag{46}$$

With Lemma 4.2, we pick P as the metric induced by the immersion $\mathcal{H}_{n_o}$. (See [19, Ex. 2, Ch. II].) Namely, in the coordinates x given by assumption B so that (43) and (44) hold, we express P on Ω as

$$P(x) = \frac{\partial \mathcal{H}_{n_o}}{\partial x}(x)^\top \mathcal{P}_\nu \frac{\partial \mathcal{H}_{n_o}}{\partial x}(x). \tag{47}$$

Remark 4.3: The above design of P relies strongly on the high gain observer technique. Nevertheless, the observer we obtain differs from a usual high gain observer, at least when n_o is strictly larger than n, i.e., $\mathcal{H}_{n_o}$ is an injective immersion and not a diffeomorphism. Indeed, the state $\hat{x}$ of our observer lives in $\mathbb{R}^n$, whereas the state of a usual high gain observer would live in $\mathbb{R}^{n_o}$, not diffeomorphic to $\mathbb{R}^n$, and a left inverse of $\mathcal{H}_{n_o}$ would be needed to extract $\hat{x}$ from this state.

Proposition 4.4: Suppose that, with $\mathcal{H}_{n_o}$ defined in (42), Assumption B holds and let $\mathcal{P}_\nu$ be any symmetric positive definite matrix satisfying (46). Then, (47) defines a positive definite symmetric covariant 2-tensor which satisfies the differential detectability property (4) on Ω.

Here, similar to Ackerman's formula for linear systems, where the observer gain uses the inverse of the observability matrix, the gain of our observer, namely, $P(\hat{x})^{-1}(\partial h/\partial x)(\hat{x})^\top$, resulting from expressing the metric as in (47) is obtained by writing the system in an observable form. This form can be obtained using $(\partial \mathcal{H}_{n_o}/\partial x)(x)$ as the observability matrix, the inverse of which also appears in the gain of our observer.

Proof: We proceed by establishing the needed properties for P.

- *P is a symmetric covariant 2-tensor*: Let $\widetilde{x}$ be other coordinates related to x by $\widetilde{x} = \varphi(x)$ with φ being a diffeomorphism. Let also $\widetilde{h}$, $\widetilde{P}$, and $\widetilde{\mathcal{H}}_{n_o}$ denote the expression of h, P, and $\mathcal{H}_{n_o}$ in the coordinates $\widetilde{x}$, respectively. They satisfy

$$\widetilde{h}(\widetilde{x}) = h(x), \qquad \widetilde{f}(\widetilde{x}) = \frac{\partial \varphi}{\partial x}(x)f(x)$$

$$\frac{\partial h}{\partial x}(x) = \frac{\partial \widetilde{h}}{\partial \widetilde{x}}(\widetilde{x})\frac{\partial \varphi}{\partial x}(x), \qquad \mathcal{H}_{n_o}(x) = \widetilde{\mathcal{H}}_{n_o}(\widetilde{x})\frac{\partial \varphi}{\partial x}(x)$$

$$P(x) = \frac{\partial \varphi}{\partial x}(x)^\top \widetilde{P}(\widetilde{x})\frac{\partial \varphi}{\partial x}(x)$$

the latter showing that P satisfies the rule a linear operator should obey under a change of coordinates to be a symmetric covariant 2-tensor.

- *P is positive definite*: Using (47) and the positive definiteness of $\mathcal{P}_\nu$, we have

$$0 < \lambda_{\min}(\mathcal{P}_\nu)\underline{h}I \leq P(x) \leq \lambda_{\max}(\mathcal{P}_\nu)\overline{h}I \quad \forall x \in \Omega.$$

- *P satisfies (4)*: With (45) and (51), we obtain

$$L_f P(x) = \frac{\partial \mathcal{H}_{n_o}}{\partial x}(x)^\top (\mathcal{P}_\nu \mathcal{A} + \mathcal{A}^\top \mathcal{P}_\nu)\frac{\partial \mathcal{H}_{n_o}}{\partial x}(x)$$

$$+ \frac{\partial \mathcal{H}_{n_o}}{\partial x}(x)^\top \mathcal{P}_\nu \mathcal{B}\frac{\partial L_f^{n_o}h}{\partial x}(x) + \frac{\partial L_f^{n_o}h}{\partial x}(x)^\top \mathcal{B}^\top \mathcal{P}_\nu \frac{\partial \mathcal{H}_{n_o}}{\partial x}(x)$$

from where it follows that:

$$\mathcal{L}_f P(x) \leq \frac{\partial \mathcal{H}_{n_o}}{\partial x}(x)^\top \left(\mathcal{P}_\nu \mathcal{K}_\nu \mathcal{C} + \mathcal{C}^\top \mathcal{K}_\nu^\top \mathcal{P}_\nu - 2qI \right.$$

$$\left. - \frac{1}{q\nu^2}\mathcal{P}_\nu \mathcal{B}\mathcal{B}^\top \mathcal{P}_\nu \right) \frac{\partial \mathcal{H}_{n_o}}{\partial x}(x)$$

$$+ \frac{\partial \mathcal{H}_{n_o}}{\partial x}(x)^\top \mathcal{P}_\nu \mathcal{B}\frac{\partial L_f^{n_o}h}{\partial x}(x)$$

$$+ \frac{\partial L_f^{n_o}h}{\partial x}(x)^\top \mathcal{B}^\top \mathcal{P}_\nu \frac{\partial \mathcal{H}_{n_o}}{\partial x}(x)$$

$$\leq \frac{\partial \mathcal{H}_{n_o}}{\partial x}(x)^\top \mathcal{P}_\nu \mathcal{K}_\nu \frac{\partial h}{\partial x}(x) + \frac{\partial h}{\partial x}(x)^\top \mathcal{K}_\nu^\top \mathcal{P}_\nu \frac{\partial \mathcal{H}_{n_o}}{\partial x}(x)$$

$$- q\left(2\frac{\partial \mathcal{H}_{n_o}}{\partial x}(x)^\top \frac{\partial \mathcal{H}_{n_o}}{\partial x}(x) \right.$$

$$\left. - \nu^2 \frac{\partial L_f^{n_o}h}{\partial x}(x)^\top \frac{\partial L_f^{n_o}h}{\partial x}(x) \right).$$

Then, using (44), we get

$$v^\top \mathcal{L}_f P(x)v \leq -qv^\top \frac{\partial \mathcal{H}_{n_o}}{\partial x}(x)^\top \frac{\partial \mathcal{H}_{n_o}}{\partial x}(x)v$$

$$\leq -\frac{q\underline{h}}{\lambda_{\max}(\mathcal{P}_\nu)\overline{h}}v^\top P(x)v$$

for all (x, v) such that $(\partial h/\partial x)(x)v = 0$, which is (4) in the given coordinates). $\blacksquare$

Example 4.5: With the above, we see that a Riemannian metric, appropriate for the design of an observer for the harmonic oscillator with unknown frequency in Example 3.7, can be parameterized on $(\mathbb{R}^2 \times \mathbb{R}_{>0}) \setminus (\{(0,0)\} \times \mathbb{R}_{>0})$ as

$$P(x) = \begin{pmatrix} 1 & 0 & -x_3 & 0 \\ 0 & 1 & 0 & -x_3 \\ 0 & 0 & -x_1 & -x_2 \end{pmatrix} \mathcal{P} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x_3 & 0 & -x_1 \\ 0 & -x_3 & -x_2 \end{pmatrix}$$

where $\mathcal{P}$ remains to be designed as a positive definite symmetric 4×4 matrix. $\triangle$

V. DESIGN OF RIEMANNIAN METRIC P FOR LAGRANGIAN SYSTEMS

In this section, we show that, besides differentially observable systems studied above Lagrangian systems make another family for which we can easily get an expression for a Riemannian metric that satisfies the differential detectability property introduced in Definition 2.1, at least with symbolic computations and with no need to solve any equation. To show this, we follow the ideas in the seminal contribution [6] and employ the metric used in [19] and [20].

Let $\mathcal{Q}$ be an $\overline{n}$-dimensional configuration manifold equipped with a Riemannian metric g. Once we have a chart for $\mathcal{Q}$ with coordinates q_k, with $k \in \{1, 2, \ldots, \overline{n}\}$, we have also coordinates (q_k, v_l) with $(k, l) \in \{1, 2, \ldots, \overline{n}\}^2$ for its tangent bundle with q being the generalized position and v the generalized velocity. Assume we have a Lagrangian $\mathfrak{L} : \mathcal{TQ} \to \mathbb{R}$ of the form $\mathfrak{L}(q, v) = (1/2)v^\top g(q)v - U(q)$, where the scalar function U is the potential energy. The corresponding Euler-Lagrange equations written via any chart are

$$\dot{q}_k = v_k, \quad \dot{v}_l = -\mathfrak{C}^l_{\mathfrak{ab}} v_\mathfrak{a} v_\mathfrak{b} + S_l(q, t) \tag{49}$$

where $k, l, \mathfrak{a}, \mathfrak{b} \in \{1, 2, \ldots, \overline{n}\}$; S is a source term, a known time-varying vector field on $\mathbb{R}^{\overline{n}}$; $\mathfrak{a}$, $\mathfrak{b}$ are dummy indices used for summation in Einstein notation[8]; and $\mathfrak{C}^l_{\mathfrak{ab}}$ are the Christoffel symbols associated with the metric g, namely

$$\mathfrak{C}^l_{\mathfrak{ab}}(q) = \frac{1}{2}\left(g(q)^{-1}\right)_{lm}\left(\frac{\partial g_{m\mathfrak{a}}}{\partial x_\mathfrak{b}}(q) + \frac{\partial g_{m\mathfrak{b}}}{\partial x_\mathfrak{a}}(q) - \frac{\partial g_{\mathfrak{ab}}}{\partial x_m}(q)\right).$$

We consider the measurement y is q, namely $y = h(q, v) = q$.

The metric we propose below is for the tangent bundle $\mathcal{TQ}$. There are many ways of defining a Riemannian metric for the tangent bundle of a Riemannian manifold [21]. We follow the same route as the one proposed in [6] to study the local convergence of an observer by considering the following modification of the Sasaki metric (see [20, (3.5)] or [19, page 55]):

$$P(q, v) = \begin{pmatrix} P_{qq}(q, v) & P_{qv}(q, v) \\ P_{vq}(q, v) & P_{vv}(q, v) \end{pmatrix}$$

where the entries of the $\overline{n} \times \overline{n}$-dimensional blocks P_{qq}, P_{qv}, P_{vq}, and P_{vv} are, respectively, P_{ij}, $P_{i\beta}$, $P_{\alpha j}$, and $P_{\alpha\beta}$, defined as

$$\begin{aligned} P_{ij}(q, v) &= ag_{ij}(q) - c\left(g_{i\mathfrak{b}}(q)\mathfrak{C}^\mathfrak{b}_{\mathfrak{a}j}(q)v_\mathfrak{a} + g_{\mathfrak{a}j}(q)\mathfrak{C}^\mathfrak{a}_{\mathfrak{b}i}(q)v_\mathfrak{b}\right) \\ &\quad + bg_{\mathfrak{cd}}(q)\mathfrak{C}^\mathfrak{c}_{\mathfrak{a}i}(q)\mathfrak{C}^\mathfrak{d}_{\mathfrak{b}j}(q)v_\mathfrak{a}v_\mathfrak{b} \\ P_{i\beta}(q, v) &= -cg_{i\beta}(q) + bg_{\beta\mathfrak{b}}(q)\mathfrak{C}^\mathfrak{b}_{\mathfrak{a}i}(q)v_\mathfrak{a} \\ P_{\alpha j}(q, v) &= -cg_{\alpha j}(q) + bg_{\alpha\mathfrak{a}}(q)\mathfrak{C}^\mathfrak{a}_{\mathfrak{b}j}(q)v_\mathfrak{b} \\ P_{\alpha\beta}(q, v) &= bg_{\alpha\beta}(q) \end{aligned}$$

where a, b, and c are strictly positive real numbers satisfying $c^2 < ab$, $g_{\mathfrak{ab}}$ are the entries of the metric g; and, here and below, roman indices i, j, and k are used to index the components of q, Greek indices α, β, and γ to index the components of v, and $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, and $\mathfrak{d}$ are dummy roman or Greek indices.

We obtain

$$\begin{aligned} (\eta^\top \ \omega^\top)P\begin{pmatrix}\eta \\ \omega\end{pmatrix} &= \eta_i P_{ij}\eta_j + \eta_i P_{i\beta}\omega_\beta + \omega_\alpha P_{\alpha j}\eta_j + \omega_\alpha P_{\alpha\beta}\omega_\beta \\ &= a\eta_i g_{ij}\eta_j + b(\omega_\alpha + \mathfrak{C}^\alpha_{\mathfrak{a}i}v_\mathfrak{a}\eta_i)g_{\alpha\beta}\left(\omega_\beta + \mathfrak{C}^\beta_{\mathfrak{b}j}v_\mathfrak{b}\eta_j\right) \\ &\quad - 2c\eta_i g_{i\beta}\left(\omega_\beta + \mathfrak{C}^\beta_{\mathfrak{a}j}v_\mathfrak{a}\eta_j\right). \end{aligned}$$

Since g is positive definite and $c^2 < ab$, we see that P takes positive definite values.

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[8] $\sum_m a_m b_{mk}$ is denoted $a_m b_{mk}$ where the fact that the index m is used twice means that we should sum in m.

To check that we have the differential detectability property (4), we rewrite (49) in the following compact form:

$$\dot{q} = v, \quad \dot{v} = f_v(q, v, t), \quad y = h(q, v) = q.$$

Since we have

$$\frac{\partial h}{\partial(q, v)}(q, v)^\top \frac{\partial h}{\partial(q, v)}(q, v) = \begin{pmatrix} I_{\overline{n}} & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{2\overline{n} \times 2\overline{n}}$$

inequality (4) is satisfied if we have, for some strictly positive real number $\overline{q}$

$$\begin{aligned} (P_{vq} \quad P_{vv})\begin{pmatrix} I \\ \frac{\partial f_v}{\partial v} \end{pmatrix} &+ \begin{pmatrix} I & \frac{\partial f_v}{\partial v}^\top \end{pmatrix}\begin{pmatrix} P_{qv} \\ P_{vv} \end{pmatrix} \\ &+ \frac{\partial P_{vv}}{\partial q}v + \frac{\partial P_{vv}}{\partial v}f_v \leq -\overline{q}P_{vv}. \end{aligned}$$

With the component-wise expression of f_v in (49), the symmetry of g, and using Kronecker's delta to denote the identity entries, the left-hand side above is nothing but

$$\begin{aligned} &\left[(-cg_{\alpha\mathfrak{c}} + bg_{\alpha\mathfrak{a}}\mathfrak{C}^\mathfrak{a}_{\mathfrak{b}\mathfrak{c}}v_\mathfrak{b})\delta_{\mathfrak{c}\beta} - bg_{\alpha\mathfrak{a}}\left(\mathfrak{C}^\mathfrak{a}_{\mathfrak{b}\beta} + \mathfrak{C}^\mathfrak{a}_{\beta\mathfrak{b}}\right)v_\mathfrak{b}\right] \\ &+ \left[\delta_{\alpha\mathfrak{c}}(-cg_{\mathfrak{c}\beta} + bg_{\beta\mathfrak{a}}\mathfrak{C}^\mathfrak{a}_{\mathfrak{b}\mathfrak{c}}v_\mathfrak{b}) - (\mathfrak{C}^\mathfrak{a}_{\alpha\mathfrak{b}} + \mathfrak{C}^\mathfrak{a}_{\mathfrak{b}\alpha})v_\mathfrak{b}g_{\mathfrak{a}\beta}\right] + b\frac{\partial g_{\alpha\beta}}{\partial q_\mathfrak{b}}v_\mathfrak{b} \\ &= -2cg_{\alpha\beta} - b\left[g_{\alpha\mathfrak{a}}\mathfrak{C}^\mathfrak{a}_{\beta\mathfrak{b}} + g_{\beta\mathfrak{a}}\mathfrak{C}^\mathfrak{a}_{\alpha\mathfrak{b}} - \frac{\partial g_{\alpha\beta}}{\partial q_\mathfrak{b}}\right]v_\mathfrak{b} = -2cg_{\alpha\beta}. \end{aligned}$$

Hence, (4) holds since b and c are strictly positive, and the entries of P_{vv} are $bg_{\alpha\beta}$.

Example 5.1: Consider a system with $\mathfrak{L}(q, v) = (1/2)\exp(-2q)v^2$ for all $q, v \in \mathbb{R}$ as Lagrangian. The associated metric and its Christoffel symbols are $g(q) = \exp(-2q)$, $\mathfrak{C} = -1$. Then, the system dynamics are given by $\dot{q} = v$, $\dot{v} = v^2$. Since the (unique) Christoffel symbol is $\mathfrak{C} = -1$, we get

$$P(q, v) = \exp(-2q)\begin{pmatrix} a + 2cv + bv^2 & -c - bv \\ -c - bv & b \end{pmatrix}.$$

$\triangle$

VI. CONCLUSION

We have established that strong differential detectability is already sufficient for the observer proposed in [1] to guarantee that, at least locally, a Riemannian distance between the estimated state and the system state decreases along solutions. Moreover in such a case, the existence of a full order observer implies the existence of a reduced order one. This extends the result in [2, Corollary 3.1] established for the particular case of an Euclidean metric.

The design of the metric, exhibiting the strong differential detectability property and consequently allowing us to design an observer, is possible when the system is strongly infinitesimally observable (i.e., each time-varying linear system resulting from the linearization along a solution to the system satisfies a uniform observability property). In such a case, one needs the solution of an "algebraic" (actually a partial differential equation) Riccati equation. This leads to an observer which resembles an Extended Kalman Filter.

With the same strong infinitesimal observability property, we can also proceed with a linear equation instead of the quadratic Riccati equation. In this case the metric we obtain is nothing but an exponentially weighted observability Grammian.

The two designs above need the solution of a partial differential equation. But thanks to the method of characteristics, it can be obtained off-line by solving ordinary differential equations on a sufficiently large time interval and over a grid of initial conditions in the system state space.

A simpler design is possible when the system is strongly differentially observable (i.e., the mapping state to output derivatives is an injective immersion). Indeed in this case the metric can be expressed as a linear combination of functions which can be obtained by symbolic computations. It then remains to choose the linear coefficients.

As already shown in [6], another case where the metric can be obtained via symbolic computations is for Euler-Lagrange systems whose Lagrangian is quadratic in the generalized velocities.

Unfortunately, to obtain observers for which convergence holds globally or at least regionally and not only locally, the metric may need to satisfy an extra property. As shown in [1], one such property is geodesic convexity of the level sets of the output function. This condition leads to additional algebraic equations involving the Hessian of the output function.

APPENDIX

A. Notations and Short Glossary of Riemannian Geometry

1) $\mathbb{S}^n$ denotes the n-dimensional unit sphere.

2) Given a function $h : \mathbb{R}^n \to \mathbb{R}^p$, dh denotes its differential form whose expression in coordinates x is $(\partial h_k / \partial x_j)(x)$ for each k in $\{1, \ldots, p\}$ and each j in $\{1, \ldots, n\}$. With $\otimes$, a tensor product, $dh(x) \otimes dh(x)$ is a symmetric covariant 2-tensor whose expression in coordinates x is $\sum_{k=1}^{p} (\partial h_k / \partial x_j)(x)(\partial h_k / \partial x_j)(x)$.

3) A Riemannian metric is a symmetric covariant 2-tensor with positive definite values. The associated Christoffel symbols in coordinates x are

$$\Gamma_{ij}^l = \frac{1}{2} \sum_k (P^{-1})_{kl} \left[\frac{\partial P_{ik}}{\partial x_j} + \frac{\partial P_{jk}}{\partial x_i} - \frac{\partial P_{ij}}{\partial x_k} \right].$$

4) Given a Riemannian metric P and a real valued function h, $\mathrm{grad}_P h$ denotes the (Riemannian) gradient of h. It is its first covariant derivative. Its expression in coordinates x is (see [22, Sections 1.2 and 2])

$$\mathrm{grad}_P h(x) = P(x)^{-1} \frac{\partial h}{\partial x}(x)^\top.$$

5) Given a Riemannian metric P and a real valued function h, $\mathrm{Hess}_P h$ denotes the (Riemannian) Hessian of h. It is its second covariant derivative. Its expression in coordinates x is

$$[\mathrm{Hess}_P h(x)]_{ij} = \frac{\partial^2 h}{\partial x_i \partial x_j}(x) - \sum_l \Gamma_{ij}^l(x) \frac{\partial h}{\partial x_l}(x).$$

It satisfies (see [22, Sections 1.2 and 2])

$$\mathcal{L}_{\mathrm{grad}_P h} P(x) = 2 \, \mathrm{Hess}_P h(x). \tag{50}$$

6) The length of a C^1 path γ between points x_a and x_b is defined as

$$L(\gamma) \Big|_{s_a}^{s_b} = \int_{s_a}^{s_b} \sqrt{\frac{d\gamma}{ds}(s)^\top P(\gamma(s)) \frac{d\gamma}{ds}(s)} \, ds$$

where $\gamma(s_a) = x_a$ and $\gamma(s_b) = x_b$.

7) The Riemannian distance $d(x_a, x_b)$ is the minimum of $L(\gamma)|_{s_a}^{s_b}$ among all possible piecewise C^1 paths γ between x_a and x_b. A minimizer giving the distance is called a minimizing geodesic and is denoted γ^*.

8) A topological space equipped with a Riemannian distance is complete when every geodesic can be maximally extended to $\mathbb{R}$.

9) A subset S of $\mathbb{R}^n$ is said to be weakly geodesically convex if, for any pair of points (x_a, x_b) in $S \times S$, there exists a minimizing geodesic γ^* between $x_a = \gamma^*(s_a)$ and $x_b = \gamma^*(s_b)$ satisfying $\gamma^*(s) \in S$ for all $s \in [s_a, s_b]$. A trivial consequence is that any two points in a weakly geodesically convex can be linked by a minimizing geodesic.

10) Given a C^1 function $h : \mathbb{R}^n \mapsto \mathbb{R}^p$ and a closed subset $\mathcal{C}$ of $\mathbb{R}^n$, the set

$$S = \{x \in \mathbb{R}^n : h(x) = 0\} \cap \mathcal{C}$$

is said to be totally geodesic if, for any pair (x, v) in $S \times \mathbb{R}^n$ such that $(\partial h / \partial x)(x)v = 0$ and $v^\top P(x)v = 1$, any geodesic γ with $\gamma(0) = x$, $(d\gamma/ds)(0) = v$ satisfies $h(\gamma(s)) = 0$ for all $s \in J_\gamma$, where J_γ is the maximal interval containing 0 so that $\gamma(J_\gamma)$ is contained in $\mathcal{C}$.

11) Given a set of coordinates for x, the Lie derivative $\mathcal{L}_f P$ of a symmetric covariant 2-tensor P is, for all v in $\mathbb{R}^n$

$$
\begin{aligned}
v^\top \mathcal{L}_f P(x) v = \lim_{t \to 0} &\left[\frac{\left[\left(I + t \frac{\partial f}{\partial x}(x) \right) v \right]^\top P(X(x,t))}{t} \right. \\
&\left. \times \frac{\left[\left(I + t \frac{\partial f}{\partial x}(x) \right) v \right]}{t} - \frac{v^\top P(x) v}{t} \right] \\
= &\frac{\partial}{\partial x} \left(v^\top P(x) v \right) f(x) + 2 v^\top P(x) \left(\frac{\partial f}{\partial x}(x) v \right)
\end{aligned}
$$

where $t \mapsto X(x,t)$ is the solution to (1). If there exist coordinates in $\mathbb{R}^n$ denoted x and a function $\varphi : \mathbb{R}^n \to \mathbb{R}^p$ such that the expression of P is

$$P(x) = \frac{\partial \varphi}{\partial x}(x)^\top \mathcal{P} \frac{\partial \varphi}{\partial x}(x)$$

where $\mathcal{P}$ is a symmetric matrix, then we have

$$\mathcal{L}_f P(x) = \frac{\partial L_f \varphi}{\partial x}(x)^\top \mathcal{P} \frac{\partial \varphi}{\partial x}(x) + \frac{\partial \varphi}{\partial x}(x)^\top \mathcal{P} \frac{\partial L_f \varphi}{\partial x}(x) \tag{51}$$

where $L_f \varphi$ is the image by φ of the vector field f (in $\mathbb{R}^n$). Indeed, we have

$$v^\top \mathcal{L}_f P(x) v = 2 v^\top \frac{\partial \varphi}{\partial x}(x)^\top \mathcal{P} \frac{\partial L_f \varphi}{\partial x}(x) v.$$

We would like the reader to distinguish the notation $\mathcal{L}_f P$ for the Lie derivative of a symmetric covariant 2-tensor from $L_f \varphi$, which is used for the more usual Lie derivative of a function φ, or equivalently, the vector field induced by a function.

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