Asymptotic tracking of a reference trajectory by output-feedback for a class of non linear systems

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ABSTRACT

We consider the problem of approximately tracking a reference trajectory by means of output feedback for a class of nonlinear systems with some non-globally Lipschitz nonlinearities. We solve this problem combining dynamic scaling, homogeneity in the bi-limit and new small gain arguments.

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1. Introduction

The problem of controlling a system in such a way that its state follows a reference trajectory has been widely studied in control theory and it is still an active field of research (see [1–3]). When only the output is available for measurement, it can be formulated as follows. Given:

1. a system $\dot{\eta} = f(\eta, u)$ with output $y = h(\eta)$,
2. a bounded state and input reference trajectory (η_r, u_r), available at each time but not ahead of time

design an output feedback $\dot{w} = \theta(w, y, \eta_r), u = \varphi(w, y, \eta_r)$ which ensures global convergence or, asymptotic closeness (in a sense to be specified) of η to η_r.

This problem is challenging since, as shown in [4], controllability and observability are not sufficient to guarantee the existence of a solution, as is the case for linear systems. Hence, some restrictions have to be imposed on the nonlinear function f.

Given a state feedback controller, the key step to design an output feedback controller, is the synthesis of an appropriate

observer (see [5] for more details). The observer problem is solved in [6–8], for instance by requiring the nonlinear function f to be linear in the unmeasured variable, or in [3] where f is required to be globally Lipschitz with a Lipschitz constant depending on the output.

Recently, the problem of practical tracking has been solved in [2] for a class of systems not globally Lipschitz in the unmeasured state components, under the assumption the reference trajectory is bounded and this bound is known. The aim of this paper is to extend this result to achieve asymptotic tracking while not requiring knowledge of an upper-bound for the reference trajectory.

To illustrate the key ideas of the design and to relate our contribution to existing results we first consider an illustrative example[1]:

$$\begin{cases} \dot{z} = -z + x_2^{1+d}, \\ \dot{x}_1 = x_2, \\ \dot{x}_2 = u + x_2^{1+d} + z, \\ y = x_1, \end{cases} \tag{1}$$

where y in $\mathbb{R}$ is the available measurement, u in $\mathbb{R}$ is the control input and d is a real number in $[0, 1)$. Given a bounded time

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[1] For any real numbers $w \neq 0$ and r, w^r denotes $\text{sign}(w) |w|^r$.

function $t \mapsto (x_{r1}(t), x_{r2}(t), u_r(t))$, together with a solution of

$$\dot{z}_r = -z_r + x_{2,r}^{1+d}$$

we define a state and input reference trajectory $((z_r, x_{r,1}, x_{r,2}), u_r)$, which is an approximate solution of (1), i.e. it solves

$$\begin{cases} \dot{z}_r = -z_r + x_{2,r}^{1+d}, \\ \dot{x}_{r,1} = x_{r,2} + \delta_{r,1}, \\ \dot{x}_{r,2} = u_r + x_{r,2}^{1+d} + z_r + \delta_{r,2}, \end{cases} \tag{2}$$

where $(\delta_{r,1}, \delta_{r,2})$ quantifies the approximation error. The problem is to find an output feedback such that the state (z, x_1, x_2) of (1) approaches (in a sense to be specified) $(z_r, x_{r,1}, x_{r,2})$ despite the presence of $(\delta_{r,1}, \delta_{r,2})$.

Note that system (1) is neither linear nor globally Lipschitz with respect to its unmeasured state components due to the presence of the term x_2^{1+d}. Hence, none of the tracking results developed in [6–8,3] can be used. Moreover functions $z_r, x_{r,1}, x_{r,2}, u_r, \delta_{r,1}$ and $\delta_{r,2}$ are not required to be bounded in norm by some known quantities, and this impedes the use of the technique proposed in [2].

In this paper, the tracking problem is recast into the problem of finding $\tilde{u} = u - u_r$, depending on $(z_r, x_{r,1}, x_{r,2})$ and $\tilde{x}_1 = x_1 - x_{r,1}$ such that the solutions of the error systems

$$\begin{cases} \dot{\tilde{z}} = -\tilde{z} + (x_{r,2} + \tilde{x}_2)^{1+d} - x_{r,2}^{1+d}, \\ \dot{\tilde{x}}_1 = \tilde{x}_2 - \delta_{r,1}, \\ \dot{\tilde{x}}_2 = \tilde{u} + (x_{r,2} + \tilde{x}_2)^{1+d} - x_{r,2}^{1+d} + \tilde{z} - \delta_{r,2}, \end{cases}$$

asymptotically converges to a ball centered at the origin, with radius depending only on the asymptotic behavior of $\delta_{r,1}$ and $\delta_{r,2}$.

To solve this problem we follow a domination approach based on homogeneity. This leads to regarding the term $(x_{r,2} + \tilde{x}_2)^{1+d} - x_{r,2}^{1+d} + \tilde{z} - \delta_{r,2}$ in the definition of $\dot{\tilde{x}}_2$ as a perturbation which can be upper bounded as:

$$|(x_{r,2} + \tilde{x}_2)^{1+d} - x_{r,2}^{1+d} + \tilde{z} - \delta_{r,2}|$$
$$\leq (1 + d)|x_{r,2}|^d |\tilde{x}_2| + |\tilde{x}_2|^{1+d} + |\tilde{z}| + |\delta_{r,2}|. \tag{3}$$

This bound is composed of four terms each of which motivates some particular features of the proposed design.

1. The term $(1 + d)|x_{r,2}|^d |\tilde{x}_2|$ is a known time function which multiplies a linear function of the tracking error. To deal with this kind of term we follow an idea introduced in [9] and design a high-gain output feedback with a dynamic scaling with gain updated from the reference signal $x_{r,2}$.
2. The second term, namely $|\tilde{x}_2|^{1+d}$, is a power of the norm of the tracking error $|\tilde{x}_2|$. To deal with this term, we use the homogeneous in the bi-limit output feedback design tool we have introduced in [10] (see Appendix A for the definition of homogeneity in the bi-limit).
3. The term $|\tilde{z}|$ depends on the state of the appended dynamics. We deal with this one by imposing a minimum phase assumption and invoking a small gain argument. Note that the gain obtained is time-varying (it depends on $x_{r,2}$), and this requires, in the design, to rely on a time-varying small-gain argument.
4. Finally, the term $|\delta_{r,2}|$ coming from the approximation error of the reference is a perturbation which does not necessarily vanish at the origin. This implies that exact tracking cannot be obtained. Nevertheless, the use of high gain allows one to reduce the effect of this disturbance.

In conclusion, the solution to this tracking problem is based on a domination approach and combines high-gain with dynamic scaling and homogeneity in the bi-limit.

In Section 2 the main result of the paper is stated, commented and compared with existing results related to this topic. Section 3 is devoted to the proof of the main result. More precisely, in Section 3.1 we introduce a homogeneous in the bi-limit output feedback design for a chain of integrators compatible with the use of dynamic scaling. With this tool in hand, we propose an output feedback and adjust some of its parameters by studying the closed-loop system in Section 3.2. A brief summary of homogeneity in the bi-limit theory is given in Appendix A, while some technical results are proved in Appendix B.

2. Main result of the paper

2.1. Problem statement and assumptions

Consider a system whose dynamics are described by:

$$\begin{cases} \dot{z} = F(z, x), \\ \dot{x}_1 = x_2 + f_1(z, x), \\ \dot{x}_2 = x_3 + f_2(z, x), \qquad y = x_1, \\ \vdots \\ \dot{x}_n = u + f_n(z, x), \end{cases} \tag{4}$$

where $x = (x_1, \ldots, x_n)$ is in $\mathbb{R}^n$, y is the output in $\mathbb{R}$, u is the input in $\mathbb{R}$ and z in $\mathbb{R}^{n_z}$ is the state of some appended dynamics.[2]

Let $t \mapsto (x_r(t), u_r(t)) \in \mathbb{R}^n \times \mathbb{R}$ be a bounded function to which corresponds a bounded[3] solution $t \mapsto z_r(t) \in \mathbb{R}^{n_z}$ of the appended state dynamics

$$\dot{z}_r = F(z_r, x_r). \tag{5}$$

We consider $t \mapsto (x_r(t), z_r(t), u_r(t)) \in \mathbb{R}^n \times \mathbb{R}$ as a state and input reference trajectory. It is a solution of (4) up to an approximation error $t \mapsto \delta_r(t) = (\delta_{r,1}(t), \ldots, \delta_{r,n}(t)) \in \mathbb{R}^n$, defined as:

$$\begin{cases} \delta_{r,1} = x_{r,2} + f_1(z_r, x_r) - \dot{x}_{r,1}, \\ \delta_{r,2} = x_{r,3} + f_2(z_r, x_r) - \dot{x}_{r,2}, \\ \vdots \\ \delta_{r,n-1} = x_{r,n} + f_{n-1}(z_r, x_r) - \dot{x}_{r,n-1}, \\ \delta_{r,n} = u_r + f_n(z_r, x_r) - \dot{x}_{r,n}, \end{cases} \tag{6}$$

we wish to design an output feedback controller for system (4) to ensure the convergence (or closeness, see inequality (15)) of the solutions (x, z) toward this state reference trajectory (x_r, z_r).

In the design, the z part can be "neglected" provided the appended dynamics with f_i as output and x as input are incremental ISS (see [11]). Specifically, we make the following assumptions.[4]

Assumption 1 (*Minimum-Phase*). There exist non-negative C^1 functions Z_i and C^0 functions γ_i, such that,

(1.1) for each $c \geq 0$, the set $\{\tilde{z} : \exists z_r : |z_r| + \sum_{i=1}^n Z_i(z_r, \tilde{z}) \leq c\}$ is compact.
(1.2) $\frac{\partial Z_i}{\partial z_r}(z_r, \tilde{z})F(z_r, x_r) + \frac{\partial Z_i}{\partial \tilde{z}}(z_r, \tilde{z})\left[F(z_r + \tilde{z}, x_r + \tilde{x}) - F(z_r, x_r)\right]$
$\leq -Z_i(z_r, \tilde{z}) + \gamma_i(z_r, x_r, \tilde{x})$.

As shown in [4] in the context of global asymptotic stabilization of the origin, the tracking problem under consideration might be unsolvable. Consequently, we need to impose some restrictions on the functions f_i. In this regard, we make the following assumption.

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[2] Depending on the structure of the functions F and f_i's these dynamics are usually referred as "inverse dynamics". For instance, this is the case if $F(z, x) = F(z, x_1)$.

[3] With Assumption 1, boundedness of z_r is implied by the boundedness of x_r.

[4] See Section 2.3.2 for some discussions on Assumptions 1–3.

Assumption 2 (*Nonlinear Bound*). : There exist positive real numbers q_1 and q_2, a real number $\eth_\infty$ in $\left[0, \frac{1}{n-1}\right)$, a positive real number c_∞, a non-negative continuous function Ω, and non-negative functions μ_i such that:

(2.1) the functions μ_i are zero at zero, C^1 on $(0 + \infty)$ and continuous at 0, the functions $\mu_i^{q_1}$ are convex and:

$$s\mu_i'(s) \le q_2\mu_i(s) \quad \forall s > 0; \tag{7}$$

(2.2) for all i in $\{1, \ldots, n\}$ and all $(z_r, \tilde{z}, x_r, \tilde{x})$ in $\mathbb{R}^{2n_z+2n}$,

$$|f_i(z_r + \tilde{z}, x_r + \tilde{x}) - f_i(z_r, x_r)| \le \Omega(z_r, x_r) \sum_{j=1}^i |\tilde{x}_j|$$

$$+ c_\infty \sum_{j=1}^i |\tilde{x}_j|^{\frac{1-\eth_\infty(n-i-1)}{1-\eth_\infty(n-j)}} + \mu_i(Z_i(z_r, \tilde{z})).$$

Finally, for the appended dynamics, we impose a bound on the gain between its input x and its outputs f_i.

Assumption 3 (*Bound on the Gain*). There exists a strictly positive real number υ such that, for all i in $\{1, \ldots, n\}$ and all $(z_r, x_r, \tilde{x})$ in $\mathbb{R}^{n_z+2n}$, we have,

$$\mu_i((1 + \upsilon)\gamma_i(\tilde{x}, z_r, x_r)) \le \Omega(z_r, x_r) \sum_{j=1}^i |\tilde{x}_j|$$

$$+ c_\infty \sum_{j=1}^i |\tilde{x}_j|^{\frac{1-\eth_\infty(n-i-1)}{1-\eth_\infty(n-j)}}$$

where the functions γ_i, μ_i, Ω and the positive real number c_∞ are as in Assumptions 1 and 2.

2.2. Main result

For systems satisfying Assumptions 1–3, the high-gain output feedback we propose to solve the tracking problem is expressed as:

$$u = u_r + L^{n+b}\phi(\mathfrak{L}^{-1}\hat{x}),$$

$$\dot{\hat{x}} = \mathcal{S}\hat{x} + BL^{n+b}\phi(\mathfrak{L}^{-1}\hat{x}) + L\mathfrak{L}K\left(\frac{\hat{x}_1 - (y - x_{r,1})}{L^b}\right), \tag{8}$$

where

$$\mathfrak{L} = \text{diag}(L^b, \ldots, L^{n+b-1}), \tag{9}$$

$\mathcal{S}$ denotes the left shift matrix of order n, i.e.

$$\mathcal{S}\hat{x} = (\hat{x}_2, \ldots, \hat{x}_{n-1}, 0)^T,$$

and b is a positive real number chosen to satisfy,[5] for $1 \le j \le i \le n$,

$$\frac{1 - \eth_\infty(n-i-1)}{1 - \eth_\infty(n-j)} < \frac{i+b}{j-1+b} < \frac{i}{j-1}, \tag{10}$$

for all $1 \le j \le i \le n$ and with $\eth_\infty$ as given in Assumption 2. Similarly to [10], the functions K and ϕ are designed by following

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[5] This choice is always possible since, for $1 \le j \le i \le n$, we have:

$$\frac{i+b}{j-1+b} < \frac{i}{j-1} \quad \forall b > 0,$$

and

$$1 \le \frac{1 - \eth_\infty(n-i-1)}{1 - \eth_\infty(n-j)} < \frac{i}{j-1} \quad \forall \eth_\infty \in \left[0, \frac{1}{n-1}\right)$$

the procedure described in Section 3.1.3. Unlike [10], the high-gain parameter L is updated on line as:

$$\dot{L} = -a_1 L + L \max\left\{0, a_1(a_2 + 1 - L^\epsilon) + a_3\overline{\Omega}(z_r, x_r)\right\}, \tag{11}$$

where:

$$\overline{\Omega}(z_r, x_r) = \max\left\{\frac{\Omega(z_r, x_r)}{c_\infty}, 1\right\}^{\frac{d_U + \eth_\infty}{1 - \eth_\infty(n-2)}},$$

and where d_U, a_1, a_2, and a_3 are positive real numbers to be defined, with a_1 sufficiently small and d_U, a_2 and a_3 sufficiently large, and ϵ is selected to satisfy, for $1 \le j \le i \le n$,

$$0 < \epsilon < i + b - (b+j-1)\frac{1 - \eth_\infty(n-i-1)}{1 - \eth_\infty(n-j)}. \tag{12}$$

The update law (11) is a modification of the one introduced in [9, (24)] and in [1, (3.12)] or [12, (134)]. Its right-hand side depends only on the state reference trajectory (x_r, z_r). Since the reference trajectory is bounded by assumption, L is upper bounded along any closed-loop solution. Moreover, if $L(0) > a_2^{\frac{1}{\epsilon}}$, $L(t)$ remains larger than $a_2^{\frac{1}{\epsilon}}$ along any closed-loop solution. In particular, $L(t) > 1$ if we select $a_2 \ge 1$. Finally the presence of the term $-a_1 L$ allows one to recover the main property of [9, (24)], i.e. L "follows" its driving term. Specifically, as established in Appendix B, we have:

$$\limsup_{t \to +\infty} L(t) \le \left[a_2 + \frac{a_3}{a_1}\limsup_{t \to +\infty}\{\overline{\Omega}(z_r(t), x_r(t))\}\right]^{\frac{1}{\epsilon}}, \tag{13}$$

$$\liminf_{t \to +\infty} L(t) \ge \left[a_2 + \frac{a_3}{a_1}\liminf_{t \to +\infty}\{\overline{\Omega}(z_r(t), x_r(t))\}\right]^{\frac{1}{\epsilon}}. \tag{14}$$

We are now ready to state the main result of the paper (proved in Section 3).

Theorem 1. *Under Assumptions 1–3, given any strictly positive real numbers b satisfying (10) and a sufficiently large real number d_U, there exist a positive real number c_r and functions K and ϕ such that, for all sufficiently small strictly positive real number a_1 and sufficiently large real numbers a_2 and a_3, the following holds.*

For any bounded state and input reference trajectory $t \mapsto (x_r(t), z_r(t), u_r(t))$, with bounded approximation error $t \mapsto \delta_r(t)$ given by (6), the solutions of system (4) with the output feedback (8), (11) are bounded in positive time and satisfy

$$\limsup_{t \to +\infty} |\mathfrak{L}^{-1}(x(t) - x_r(t))| \le c_r \limsup_{t \to +\infty}\left\{\sum_{i=1}^{n-1}\frac{|\delta_{r,i}(t)|}{L(t)^{i+b}}\right.$$

$$\left. + \left(\frac{|\delta_{r,i}(t)|}{L(t)^{i+b}}\right)^{\frac{1}{1-\eth_\infty(n-i-1)}} + 2\frac{|\delta_{r,n}(t)|}{L(t)^{n+b}}\right\} \tag{15}$$

with $\mathfrak{L}$ defined in (9).

2.3. Some remarks

2.3.1. About the result

If the reference trajectory is an exact solution of (4) (i.e. if $\delta_r(t) = 0$) or if $\delta_r(t)$ converges to zero as t goes to infinity, the output feedback given in Theorem 1 ensures that $(x(t), z(t))$ converges to $(x_r(t), z_r(t))$.

In general, (15) implies

$$\limsup_{t \to +\infty} |y(t) - x_{r,1}(t)| \le c_r \frac{\limsup\limits_{t \to +\infty} L(t)^b}{\liminf\limits_{t \to +\infty} L(t)^{\frac{1+b}{1+\eth_\infty}}}$$

$$\times \limsup_{t \to +\infty}\left\{\sum_{i=1}^{n-1}|\delta_{r,i}(t)| + |\delta_{r,i}(t)|^{\frac{1}{r_{\infty,i}+\eth_\infty}} + 2|\delta_{r,n}(t)|\right\}.$$

Therefore, if the reference trajectory is known in advance, for all $\kappa > 0$ we can select the parameter a_2 strictly larger than

$$\limsup_{t \to +\infty} \left\{ \max \left\{ \frac{a_3}{a_1} \overline{\Omega}(z_r(t), x_r(t)), c_r^\epsilon \right. \right.$$
$$\left. \left. \times \left(\frac{\sum_{i=1}^{n-1} |\delta_{r,i}(t)| + |\delta_{r,i}(t)|^{\frac{1}{r_{\infty,i}+\vartheta_\infty}} + 2|\delta_{r,n}(t)|}{\kappa} \right)^\epsilon 2^b \right\} \right\}.$$

Then, by (13) and (14),

$$\limsup_{t \to +\infty} |y(t) - x_{r,1}(t)| \le \kappa.$$

Hence, as in[6] [2], we recover a practical result in its usual formulation.

On the other hand, we do not need to know in advance the whole reference trajectory or even a bound on it to design the controller and to get asymptotically closeness of the closed loop system solutions toward the reference one. This differs from the results proposed in [2,8].

2.3.2. About the assumptions

Note that for system (1),

- Assumption 1 follows from inequality (3) taking

$$Z_1(z_r, \tilde{z}) = Z_2(z_r, \tilde{z}) = |\tilde{z}|^2,$$

and

$$\gamma_1(\tilde{x}, z_r, x_r) = \gamma_2(\tilde{x}, z_r, x_r) = \left| (1+d)|x_{r,2}|^d |\tilde{x}_2| + |\tilde{x}_2|^{1+d} \right|^2.$$

- Assumption 2 is satisfied setting $\vartheta_\infty = d, c_\infty = \sqrt{2}$ and picking

$$\Omega(z_r, x_r) = \sqrt{2}(1+d)|x_{r,2}|^d, \qquad \mu_1(s) = \mu_2(s) = \sqrt{s},$$
$$q_1 = 2, \qquad q_2 = \frac{1}{2}.$$

- Finally, Assumption 3 is satisfied with $\upsilon = 1$.

Consequently, system (1) belongs to the class of systems satisfying Assumptions 1–3 and Theorem 1 applies.

When compared with what can be found in the textbooks [7,6] or in [8] for instance, our approach allows us to deal with dynamics which, in appropriate coordinates, may have some polynomial growth in the unmeasured state components (as expressed by Assumption 2).

Assumption 1 is more general than the incremental property introduced in [11] (see also [13]) since the gain γ_i in Assumption 1.2 depends also on (z_r, x_r). Nevertheless it retains its main property on the behavior of the solutions. Specifically it implies that, for all bounded time functions $t \mapsto (\mathfrak{x}(t), \mathfrak{x}_r(t))$, if a corresponding solution $t \mapsto \mathfrak{z}_r(t)$ of the appended dynamics of system (4), i.e. solution of:

$$\dot{\mathfrak{z}}_r = F(\mathfrak{z}_r, \mathfrak{x}_r),$$

is bounded, then all solutions $t \mapsto \mathfrak{z}(t)$ of:

$$\dot{\mathfrak{z}} = F(\mathfrak{z}, \mathfrak{x})$$

are bounded as well. If furthermore, for all i, $\gamma_i(\mathfrak{z}_r(t), \mathfrak{x}_r(t), \mathfrak{x}(t) - \mathfrak{x}_r(t))$ goes to zero when t goes to infinity, then $|\mathfrak{z}(t) - \mathfrak{z}_r(t)|$ goes to zero.

The presence of the $\delta_{r,i}$ offers great flexibility for checking our Assumptions. Specifically, it follows from our proof that Assumptions 1 and 2 need not to hold for all (z_r, x_r). It is sufficient they are satisfied with $F(z_r, x_r)$ and $\delta_{r,i}$ given by (6). For example, in the case with no appended dynamics, by letting:

$$\delta_{r,1} = \dot{x}_{r,1} - f_1(x_r), \qquad x_{r,i} = 0, \qquad \delta_{r,i} = f_i(x_r) \quad \forall i \ge 2,$$

Assumption 2 becomes:

$$|f_i(\tilde{x})| \le \Omega(x_r) \sum_{j=1}^i |\tilde{x}_j| + c_\infty \sum_{j=1}^i |\tilde{x}_j|^{\frac{1-\vartheta_\infty(n-i-1)}{1-\vartheta_\infty(n-j)}}.$$

Then, with f_i being locally Lipschitz, around the origin, we recover the result on practical tracking in [2] where standard homogeneity and domination is also used.

In the same way, by letting

$$F(z_r, x_r) = 0, \qquad \delta_{r,i} = f_i(0, x_r) - f_i(z_r, x_r),$$

Assumption 1.2 reduces to a simple ISS property as assumed for example in [8].

Finally, the presence of the function Ω in Assumptions 2 and 3 is an important feature. It is necessary to account for the fact that the local Lipschitz constant depends on around where the solution is and therefore around where (z_r, x_r) is. But, not knowing in advance the reference trajectory, we have no a priori upper bound for this term. This term makes the analysis much more involved and requires ad hoc small gain arguments.

2.3.3. About the reference trajectory

The standard output tracking problem consists of designing a feedback which guarantees boundedness of the closed loop solutions while ensuring convergence of the output to a desired reference signal y_r. To solve this problem, we follow the classical two degrees of freedom design technique. Namely we separate the problem into two subproblems:

- trajectory generation,
- feedback compensation.

Feedback compensation is the object we present in full detail in this paper. The purpose of trajectory generation is to synthesize a state and input reference trajectory (x_r, z_r, u_r) for the system given the desired reference signal y_r. Here we do not require this trajectory to be feasible since a "dynamical mismatch" quantified by δ_r is allowed. In addition we do not require this state and input reference trajectory, and therefore the reference signal to be available ahead of time.

The problem of state and input reference trajectory generation is rendered more difficult by the presence of appended dynamics. When the system we consider is right invertible (see [14]), one way to solve this problem (among many others) is to decompose it in two sub-problems:

1. generation of the time derivatives of the desired reference signal y_r,
2. transformation of these derivatives into a state and input reference trajectory.

The problem of obtaining the time derivative of the desired reference signal is an observer problem. It can be solved once we have a model of the system which generates the reference signal, the so-called exo-system in regulation theory [15]. For example, we could pick as approximate exo-system $y_r^{(m)}(t) = 0$, i.e. $y_r(t)$ is approximated as a polynomial of degree $m-1$ in t (the model used in [2] is $\dot{y}_r = 0$).

Then, to obtain the appropriate signals (x_r, u_r) and z_r, from y_r and its time derivatives $y_r^{(i)}$, we follow a standard inversion

procedure (see [14]). Specifically, if we can partition x in (x_a, x_b) so that (4) takes the specific form[7]

$$\begin{cases} \dot{z} = F(z, x_a), \\ \dot{x}_{a1} = x_{a2} + f_1(x_{a1}), \\ \vdots \\ \dot{x}_{an_a-1} = x_{an_a} + f_{n_a-1}(x_{a1}, \ldots, x_{an_a-1}), \\ \dot{x}_{an_a} = x_{b1} + f_{n_a}(z, x_a), \qquad\qquad y = x_{a1}, \\ \dot{x}_{b1} = x_{b2} + f_{n_a+1}(z, x_a, x_{b1}), \\ \vdots \\ \dot{x}_{bn_b} = u + f_{n_a+n_b}(z, x_a, x_{b1}, \ldots, x_{bn_b}), \end{cases} \qquad (16)$$

then by formal differentiation we obtain functions ϕ_{ai} so that $x_{r,ai} = \phi_{ai}(y_r, \ldots, y_r^{(i-1)})$, and by on-line integration of

$$\dot{z}_r = F(z_r, x_{r,a}),$$

we obtain z_r from $x_{r,a}$. Finally, again by formal differentiation, we obtain functions ϕ_{bi} so that

$$x_{r,bi} = \phi_{bi}(z_r, y_r \ldots, y_r^{(n_a+i-1)}), \qquad u_r = \phi_{bn}(z_r, y_r \ldots, y_r^{(n_a+n_b)}).$$

If, for the z subsystem there exist a particular initial condition $z(0)$ and a bounded time function $t \mapsto x_a(t)$ (considered here as input) such that the corresponding solution $t \mapsto z(t)$ is bounded, then, by the minimum phase Assumption 1.1.2, the above procedure yields a bounded state and input reference trajectory provided that the function $t \mapsto (y_r(t), \ldots, y_r^{(n_a+n_b)}(t))$ is bounded. Note that we rely on a minimum phase assumption and this is not surprising in view of the key role this assumption plays, as discussed in [16].

For system (1) it is possible to compute an exact state and input trajectory from $y_r, \dot{y}_r$ and $\ddot{y}_r$ selecting $x_a = (x_1, x_2)$ with $x_b = \emptyset$. Indeed, we obtain:

$$x_{r,2} = \dot{y}_r, \qquad u_r = \ddot{y}_r - (\dot{y}_r)^{1+d} - z_r, \qquad \dot{z}_r = -z_r + (\dot{y}_r)^{1+d}.$$

Unfortunately, in general, the estimation process yielding the time function $\dot{y}_r(t)$ and $\ddot{y}_r(t)$ from the knowledge of $y_r(t)$ is not exact, and consequently generates an approximation for this reference trajectory (i.e. non-zero δ_1 and δ_2).

The procedure above is only one of many possible solutions. Others may exploit optimal control or flatness [17].

3. Proof of Theorem 1

The proof of inequality (15) is divided into two parts. In the first part we introduce the functions K and ϕ appearing in the output feedback given in (8). In the second part we show how a proper selection of the parameters d_U, a_1, a_2 and a_3 in the high-gain update law (11) yields the result.

3.1. Construction of K and ϕ

3.1.1. Posing the problem as an output feedback stabilization problem

The first step is to pose the tracking problem as a problem of stabilization by output feedback. Setting

$$\tilde{u} = u - u_r, \quad \tilde{x} = x - x_r, \qquad \tilde{z} = z - z_r,$$

we obtain:

$$\begin{cases} \dot{\tilde{z}} = F(z_r + \tilde{z}, x_r + \tilde{x}) - F(z_r, x_r), \\ \dot{\tilde{x}} = \mathcal{S}\tilde{x} + B\tilde{u} + f(z_r + \tilde{z}, x_r + \tilde{x}) - f(z_r, x_r) - \delta_r, \end{cases} \qquad (17)$$

where $B = (0, \ldots, 1)^T$ and $\delta_r = (\delta_{r,1}, \ldots, \delta_{r,n})$.

The objective is to find $\tilde{u}$ depending on the output $\tilde{x}_1 = y - x_{r,1}$ and on the state reference trajectory (z_r, x_r) such that (15) holds.

3.1.2. A high-gain domination approach

The $\tilde{x}$ dynamics of the system (17) have the structure of a chain of integrators disturbed by nonlinear terms depending on the tracking error $(\tilde{x}, \tilde{z})$, the state reference trajectory (x_r, z_r) and the approximation error δ_r. This motivates us to use the domination approach introduced in [18] (see also [19]). In this context, the nonlinear functions (the f_i's) are not used in the design but considered as perturbations and the output feedback is designed on a dominating model which in this case is the chain of integrators. To ensure robustness to these nonlinearities, we employ high-gain techniques. This leads us to design the controller with the scaled coordinates:

$$\tilde{\mathcal{X}}_i = L^{b+1-i}\tilde{x}_i \qquad (18)$$

where L is the updated high-gain and b is a positive real number satisfying (10). Compared to the scaled coordinates used in the high-gain approach in [18], we add an extra high-gain parameter b which has been introduced in [9] and which allows gain adaptation.

3.1.3. Homogeneous in the bi-limit output feedback for a chain of integrators

Following the domination approach, we focus on the dominant part of system (17) in the scaled coordinate (18), i.e. a chain of integrators, with state $\mathcal{X} = (\mathcal{X}_1, \ldots, \mathcal{X}_n)$ in $\mathbb{R}^n$ described by:

$$\dot{\mathcal{X}} = \mathcal{S}\mathcal{X} + Bu, \qquad y = \mathcal{X}_1. \qquad (19)$$

To design the output feedback controller for system (19) we use the tools developed within the framework of homogeneity in the bi-limit, introduced in [10]. (See Appendix A for a brief summary.)

Selecting $\mathfrak{d}_0 = 0$ and with $\mathfrak{d}_\infty$ given by Assumption 2, homogeneity in the bi-limit is obtained for system (19) with the weights $r_0 = (r_{0,1}, \ldots, r_{0,n})$ and $r_\infty = (r_{\infty,1}, \ldots, r_{\infty,n})$ as[8]:

$$r_{0,i} = 1, \qquad r_{\infty,i} = 1 - \mathfrak{d}_\infty(n-i), \qquad i \in \{1, \ldots n\}. \qquad (20)$$

In [10] we have proposed an output feedback for system (19) given by:

$$u = \phi(\hat{\mathcal{X}}), \qquad \dot{\hat{\mathcal{X}}} = \mathcal{S}\hat{\mathcal{X}} + B\phi(\hat{\mathcal{X}}) + K(\hat{\mathcal{X}}_1 - \mathcal{X}_1), \qquad (21)$$

where $\hat{\mathcal{X}} = (\hat{\mathcal{X}}_1, \ldots, \hat{\mathcal{X}}_n)$ is in $\mathbb{R}^n$, K is a homogeneous in the bi-limit vector field with weights r_0 and r_∞, and degrees 0 and $\mathfrak{d}_\infty$ and ϕ is homogeneous in the bi-limit function with weights r_0 and r_∞, and degrees 1 and $1 + \mathfrak{d}_\infty$. Setting:

$$E = (e_1 \ldots, e_n)^T = \hat{\mathcal{X}} - \mathcal{X}$$

the chain of integrators (19) with the controller (21) can be described by:

$$\begin{cases} \dot{\hat{\mathcal{X}}} = \mathcal{S}\hat{\mathcal{X}} + B\phi(\hat{\mathcal{X}}) + K(e_1) \\ \dot{E} = \mathcal{S}E + K(e_1). \end{cases} \qquad (22)$$

In [10], the design of K and ϕ is performed recursively by following an observer/controller approach in such a way that there exists homogeneous in the bi-limit Lyapunov function U of degree d_U satisfying, for some real number c_1,

$$\frac{\partial U}{\partial E}(\hat{\mathcal{X}}, E)(\mathcal{S}E + K(e_1)) + \frac{\partial U}{\partial \mathcal{X}}(\hat{\mathcal{X}}, E)(\mathcal{S}\hat{\mathcal{X}} + B\phi(\hat{\mathcal{X}}) + K(e_1))$$

$$\leq -c_1\left(U(\hat{\mathcal{X}}, E) + U(\hat{\mathcal{X}}, E)^{\frac{d_U+\mathfrak{d}_\infty}{d_U}}\right). \qquad (23)$$

---

[7] x_b is not an argument of F and z is not an argument of $f_1, \ldots, f_{n_a-1}$.

[8] Note that $r_{\infty,i}$ increases with i and is in $(0, 1]$.

To combine this tool with dynamic scaling we need to establish a specific property on the Lyapunov function U. This property is homogeneous in the bi-limit version of the one given in [9, equation (16)] or in [3, Lemma A1] and also used in the context of observer design in [20]. Namely, given the diagonal matrix

$$D = \text{diag}(b, 1 + b, \ldots, n - 1 + b),$$

where b is a positive real number satisfying (10), the function ϕ and the vector field K are selected such that the Lyapunov function U satisfies (23), and also:

$$\frac{\partial U}{\partial E}(\hat{\mathfrak{x}}, E)DE + \frac{\partial U}{\partial \hat{\mathfrak{x}}}(\hat{\mathfrak{x}}, E)D\hat{\mathfrak{x}} \geq c_2 U(\hat{\mathfrak{x}}, E), \qquad (24)$$

for some positive real number c_2.

Such a property can be obtained by modifying the recursive procedure given in [10] as claimed in the following statement the proof of which is omitted but can be found in [21].

Theorem 2. *Let d_U be a positive real number satisfying $d_U \geq 2 \max_{1 \leq j \leq n} r_{0,j} + \mathfrak{d}_\infty$. There exists homogeneous in the bi-limit function $\phi : \mathbb{R}^n \to \mathbb{R}_+$ with associated triples $(r_0, 1, \phi_0)$ and $(r_\infty, 1 + \mathfrak{d}_\infty, \phi_\infty)$, homogeneous in the bi-limit vector field $K : \mathbb{R} \to \mathbb{R}^n$, with associated triples $(r_0, \mathfrak{d}_0, K_0)$ and $(r_\infty, \mathfrak{d}_\infty, K_\infty)$ and a positive definite, proper and C^1 function $U : \mathbb{R}^{2n} \to \mathbb{R}_+$, homogeneous in the bi-limit with associated triples (r_0, d_U, U_0) and $(r_\infty, d_U, U_\infty)$, such that the following holds.*

1. *The homogeneous approximating functions U_0 and U_∞ are positive definite and proper and for all j in $\{1, \ldots, n\}$, the functions $\frac{\partial U}{\partial e_j}$ and $\frac{\partial U}{\partial \hat{\mathfrak{x}}_j}$ are homogeneous in the bi-limit with approximating functions $\left(\frac{\partial U_0}{\partial e_j}, \frac{\partial U_\infty}{\partial e_j} \right)$ and $\left(\frac{\partial U_0}{\partial \hat{\mathfrak{x}}_j}, \frac{\partial U_\infty}{\partial \hat{\mathfrak{x}}_j} \right)$ respectively.*
2. *There exist two positive real numbers c_1 and c_2 such that (23) and (24) are satisfied.*

We choose d_U satisfying also $\frac{d_U - 1}{1 + \mathfrak{d}_\infty} > q_1$, with q_1 given in Assumption 2. In this way the functions μ_i^q are convex for all $q \geq \frac{d_U - 1}{1 + \mathfrak{d}_\infty}$. In what follows the parameters d_U and b, the vector field K and the function ϕ are assumed fixed, it remains to select the parameters a_1, a_2 and a_3 appearing in the high-gain updated law (11).

3.2. Properties of the closed loop system

3.2.1. ISS property with respect to the $\delta_{r,i}$'s

The system (4) with the controller (8), (11), introduced in Section 2.2, can be fully described by the high-gain update law (11), the $\tilde{z}$ dynamics in (17) and the equations[9]

$$\begin{cases} \dot{E} = L(\mathcal{S}E + K(e_1)) - \frac{\dot{L}}{L}DE - \Delta_f(L) - \Delta_\delta(L), \\ \dot{\hat{\mathfrak{x}}} = L(\mathcal{S}\hat{\mathfrak{x}} + B\phi(\hat{\mathfrak{x}}) + K(e_1)) - \frac{\dot{L}}{L}D\hat{\mathfrak{x}} \end{cases} \qquad (25)$$

where $E = (e_1, \ldots, e_n)^T$ and $\hat{\mathfrak{x}} = (\hat{\mathfrak{x}}_1, \ldots, \hat{\mathfrak{x}}_n)^T$ are defined as:

$$\hat{\mathfrak{x}} = \mathcal{L}^{-1}\hat{x}, \qquad E = \mathcal{L}^{-1}(\hat{x} - \tilde{x}) \qquad (26)$$

and

$$\Delta_f(L) = \mathcal{L}^{-1}(f(z_r + \tilde{z}, x_r + \tilde{x}) - f(z_r, x_r)),$$
$$\Delta_\delta(L) = \mathcal{L}^{-1}(\delta_{r,1}, \ldots, \delta_{r,n})$$

are regarded as perturbations.

By inequality (23), the function U obtained from Theorem 2 satisfies, along the solutions of system (25),

$$\dot{\overline{U(\hat{\mathfrak{x}}, E)}} \leq -c_1 L\theta \left(U(\hat{\mathfrak{x}}, E) \right) + T_{DS} + T_{Dist}, \qquad (27)$$

where θ is the function defined on $\mathbb{R}_+$ as:

$$\theta(s) = s + s^{\frac{\mathfrak{d}_\infty + d_U}{d_U}}, \qquad (28)$$

and

$$T_{DS} = -\frac{\dot{L}}{L} \left(\frac{\partial U}{\partial \hat{\mathfrak{x}}}(\hat{\mathfrak{x}}, E)D\hat{\mathfrak{x}} + \frac{\partial U}{\partial E}(\hat{\mathfrak{x}}, E)DE \right),$$

$$T_{Dist} = -\frac{\partial U}{\partial E}(\hat{\mathfrak{x}}, E)[\Delta_f(L) + \Delta_\delta(L)].$$

The above discussion and Assumption 2 yield the following result, the proof of which is given in Appendix B.2.

Lemma 1. *There exist two positive real numbers c_3 and c_4 such that, for all sufficiently small strictly positive real numbers a_1 and sufficiently large real numbers a_2 and a_3, inequality (27) becomes:*

$$\dot{\overline{U(\hat{\mathfrak{x}}, E)}} \leq -4c_3 L\theta \left(U(\hat{\mathfrak{x}}, E) \right) + c_4 L^{1-\epsilon} \sum_{i=1}^n \zeta_i(Y_i)$$

$$+ Lc_4 \sum_{i=1}^n \zeta_i \left(\frac{\delta_{r,i}}{L^{i+b}} \right), \qquad (29)$$

where

$$Y_i = L^{\epsilon - (i+b)} \mu_i(Z_i(z_r, \tilde{z})) \qquad (30)$$

and the ζ_i are C^1, convex, strictly increasing and homogeneous in the bi-limit functions with weights 1 and $r_{\infty,i} + \mathfrak{d}_\infty$ and degrees d_U and $d_U + \mathfrak{d}_\infty$, defined as[10]:

$$\zeta_i(s) = \int_0^s \max\{|\sigma|^{d_U-1}, |\sigma|^{\frac{d_U - r_{\infty,i}}{r_{\infty,i} + \mathfrak{d}_\infty}}\} d\sigma, \quad \forall i \in \{1 \ldots, n-1\},$$
$$\zeta_n(s) = \int_0^s \min\{|\sigma|^{d_U-1}, |\sigma|^{\frac{d_U-1}{1+\mathfrak{d}_\infty}}\} d\sigma. \qquad (31)$$

If the functions Z_i's were not present (via Y_i), inequality (29) would give readily an ISS property between $\frac{\delta_{r,i}}{L^{i+b}}$ and $(\hat{\mathfrak{x}}, E)$. To prove this claim note that the function

$$s \in \mathbb{R}^n \mapsto \left(\sum_{i=1}^{n-1} |s_i| + |s_i|^{\frac{1}{r_{\infty,i}+\mathfrak{d}_\infty}} + \frac{|s_n|}{1 + |s_n|} \left(1 + |s_n|^{\frac{1}{1+\mathfrak{d}_\infty}} \right) \right)^{d_U}$$

is positive definite, homogeneous in the bi-limit with weights 1 and $r_{\infty,i} + \mathfrak{d}_\infty$ and degrees d_U and d_U and its homogeneous approximating functions (i.e. $\sum_{i=1}^n |s_i|^{d_U}$ and $\sum_{i=1}^n |s_i|^{\frac{d_U}{r_{\infty,i}+\mathfrak{d}_\infty}}$) are positive definite. Furthermore, since the function $s \in \mathbb{R}^n \mapsto \theta^{-1}\left(\frac{c_4}{c_3} \sum_{i=1}^n \zeta_i(s_i) \right)$ is homogeneous in the bi-limit with the same weights and degree[11] by Claim A.3 in Appendix A, there exists a

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[9] Note that $\dot{\overline{\mathcal{L}^{-1}}} = -L^{-1}\dot{L}D\mathcal{L}^{-1}$.

[10] Recall that $d_U - 1 \geq \frac{d_U - r_{\infty,i}}{r_{\infty,i}}$ for all i in $\{1, \ldots, n-1\}$ and $d_U - 1 \geq \frac{d_U-1}{1+\mathfrak{d}_\infty}$.

[11] $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ defined in (28) is a bijective, homogeneous in the bi-limit function, and satisfies all assumptions of [10, Proposition 2.11]. This implies homogeneity in the bi-limit of its inverse map $\theta^{-1} : \mathbb{R}_+ \to \mathbb{R}_+$ with approximating homogeneous functions s and $s^{\frac{d_U}{\mathfrak{d}_\infty + d_U}}$. Then [10, Proposition 2.10] implies that the function $\theta^{-1}\left(\frac{c_4}{c_3} \sum_{i=1}^n \zeta_i(s_i) \right)$ is homogeneous in the bi-limit with degrees d_U and d_U.

positive real number c_5 such that:

$$\theta^{-1}\left(\frac{c_4}{c_3}\sum_{i=1}^{n}\zeta_i(s_i)\right)$$

$$\leq c_5^{d_U}\left(\sum_{i=1}^{n-1}|s_i|+|s_i|^{\overline{r_{\infty,i}+\mathfrak{d}\infty}}+\frac{|s_n|}{1+|s_n|}\left(1+|s_n|^{\frac{1}{1+\mathfrak{d}\infty}}\right)\right)^{d_U},$$

$$\leq c_5^{d_U}\left(\sum_{i=1}^{n-1}|s_i|+|s_i|^{\overline{r_{\infty,i}+\mathfrak{d}\infty}}+2|s_n|\right)^{d_U}. \qquad (32)$$

Hence, without Y_i in the inequality (29), we would have (see [22]):

$$\limsup_{t\to+\infty} U(\hat{\mathfrak{x}},E(t)) \leq c_5^{d_U}\limsup_{t\to+\infty}\left[\sum_{i=1}^{n-1}\left(\frac{|\delta_{r,i}(t)|}{L(t)^{i+b}}\right)\right.$$

$$\left.+\left(\frac{|\delta_{r,i}(t)|}{L(t)^{i+b}}\right)^{\frac{1}{r_{\infty,i}+\mathfrak{d}\infty}}+2\frac{|\delta_{r,n}(t)|}{L(t)^{n+b}}\right]^{d_U}. \quad (33)$$

In addition, since the function U is positive definite, homogeneous in the bi-limit with degrees d_U and d_U and its homogeneous approximating functions are positive definite, and since $r_{\infty,i}$ is smaller or equal to 1, Claim A.3 in Appendix A implies the existence of a positive real number c_{13} such that:

$$U(\mathfrak{X},E) \geq c_{13}\left|\hat{\mathfrak{x}}-E\right|^{d_U} = c_{13}\left|\mathfrak{L}^{-1}\tilde{x}\right|^{d_U}, \qquad (34)$$

which implies that (15) holds with $c_r = \frac{c_5}{c_{13}^{d_U}}$.

3.2.2. Small-gain arguments

To establish an inequality like (33) in the presence of Z_i (or Y_i), we need a more advanced argument relying on a small gain theorem.

First of all, note that, Assumptions 1 and 3 yield the following result, the proof of which is in Appendix B.3.

Lemma 2. *There exist two positive real numbers c_6 and c_7 such that, for all sufficiently small strictly positive real numbers a_1 and sufficiently large real numbers a_2 and a_3, we have, along the trajectories of the closed-loop system,*

$$\overline{\dot{\zeta_i(Y_i)}} \leq -c_6\zeta_i(Y_i)+c_7\overline{\varOmega}(z_r,x_r)\theta\left(U(\hat{\mathfrak{x}},E)\right), \quad \forall i\{1,\ldots,n\}. \quad (35)$$

Lemma 2 quantifies the ISS gain between $(\hat{\mathfrak{x}},E)$ and Y_i, which, unfortunately, depends on (x_r,z_r) (through the function $\overline{\varOmega}$). Nevertheless, due to the special structure of the high-gain update law (11), an inequality similar to (33) can still be obtained.

Suppose that a_1 is sufficiently small and that a_2 and a_3 are sufficiently large such that Lemmas 1 and 2 apply. Then, inequality (35) yields, for all $0 \leq s \leq t$,

$$\zeta_i(Y_i(t)) \leq \exp(c_6(s-t))\zeta_i(Y_i(s))$$

$$+\int_s^t \exp(c_6(r-t))c_7\overline{\varOmega}(r)\theta(U(r))dr,$$

where we have used the compact notation:

$$U(t) = U(\hat{\mathfrak{x}}(t),E(t)), \qquad Y_i(t) = L(t)^{\epsilon-(i+b)}\mu_i(Z_i(z_r(t),\tilde{z}(t))),$$
$$\overline{\varOmega}(t) = \overline{\varOmega}(z_r(t),x_r(t)).$$

The inequality above, together with definition (30) and inequality (29) divided by $L^{1-\epsilon}$, yields:

$$\frac{\dot{\overline{U(t)}}}{L(t)^{1-\epsilon}} \leq -4c_3L(t)^\epsilon\theta(U(t))+L(t)^\epsilon c_4\sum_{i=1}^{n}\zeta_i\left(\frac{\delta_{r,i}(t)}{L(t)^{i+b}}\right)$$

$$+c_4\exp(c_6(s-t))\sum_{i=1}^{n}\zeta_i\left(L(s)^{\epsilon-(i+b)}(\mu_i(Z_i(s)))\right)$$

$$+nc_4c_7\int_s^t\exp(c_6(r-t))\left[\overline{\varOmega}(r)\theta(U(r))\right]dr, \qquad (36)$$

where :

$$Z_i(t) = Z_i(z_r(t),\tilde{z}(t)).$$

This yields:

$$\frac{\dot{\overline{U(t)}}}{L(t)^{1-\epsilon}} \leq -4c_3L(t)^\epsilon\theta(U(t))+L(t)^\epsilon c_4\sum_{i=1}^{n}\zeta_i\left(\frac{\delta_{r,i}(t)}{L(t)^{i+b}}\right)$$

$$+c_4\exp(c_6(s-t))\sum_{i=1}^{n}\zeta_i\left(L(s)^{\epsilon-(i+b)}(\mu_i(Z_i(s)))\right)$$

$$+nc_4c_7\sup_{[s,t]}\left\{\exp\left(\frac{c_6}{2}(r-t)\right)\theta(U(r))\right\}$$

$$\times\int_s^t\exp\left(\frac{c_6}{2}(r-t)\right)\overline{\varOmega}(r)dr. \qquad (37)$$

Now, assume for the time being that L satisfies[12]:

$$L(t)^\epsilon \geq \frac{1}{c_3}\left[nc_4c_7\int_s^t\exp\left(\frac{c_6}{2}(r-t)\right)\overline{\varOmega}(r)dr+1\right] \quad \forall t \geq 0. \qquad (38)$$

In this case (37) becomes, for all $s \leq t$,

$$\frac{\dot{\overline{U(t)}}}{L(t)^{1-\epsilon}} \leq -4c_3L(t)^\epsilon\theta(U(t))+L(t)^\epsilon c_4\sum_{i=1}^{n}\zeta_i\left(\frac{\delta_{r,i}(t)}{L(t)^{i+b}}\right)$$

$$+c_4\exp(c_6(s-t))\sum_{i=1}^{n}\zeta_i\left(L(s)^{\epsilon-(i+b)}(\mu_i(Z_i(s)))\right)$$

$$+[c_3L(t)^\epsilon-1]\exp\left(-\frac{c_6}{2}t\right)\sup_{[s,t]}\left\{\exp\left(\frac{c_6}{2}r\right)\theta(U(r))\right\},$$

$$\leq -[4c_3L(t)^\epsilon+1]\theta(U(t))+L(t)^\epsilon c_4\sum_{i=1}^{n}\zeta_i\left(\frac{\delta_{r,i}(t)}{L(t)^{i+b}}\right)$$

$$+c_4\exp(c_6(s-t))\sum_{i=1}^{n}\zeta_i\left(L(s)^{\epsilon-(i+b)}(\mu_i(Z_i(s)))\right)$$

$$+c_3L(t)^\epsilon\exp\left(-\frac{c_6}{2}t\right)\sup_{[s,t]}\left\{\exp\left(\frac{c_6}{2}r\right)\theta(U(r))\right\}. \qquad (39)$$

Since the functions δ and L are upper bounded on $\mathbb{R}_+$ and $L(0) > a_2^{\frac{1}{\epsilon}} \geq 1$, there exist ∂_m and L_m that satisfy:

$$\partial_m \geq \sum_{i=1}^{n}\zeta_i\left(\frac{\delta_{r,i}(t)}{L(t)^{i+b}}\right), \qquad L_m \geq L(t) \geq a_2^{\frac{1}{\epsilon}} \geq 1, \quad \forall t \geq 0. \quad (40)$$

Then, given the initial conditions $L(0)$ and $Z_i(0)$, we can find a sufficiently large positive real number U_m, strictly larger than $U(0)$, and satisfying, for all i,

$$\theta(U_m) \geq c_4\sum_{i=1}^{n}\zeta_i\left(L(0)^{\epsilon-(i+b)}(\mu_i(Z_i(0)))\right)+c_4L_m^\epsilon\partial_m. \qquad (41)$$

---

[12] This property will be established at the end of the proof.

Suppose there exists a time t_m in the positive time domain of existence of the solution such that $U(t_m) = U_m$ and $U(r) < U_m$ for all r in $[0, t_m)$. This implies

$$\sup_{[0,t_m]} \left\{ \exp\left(\frac{c_6}{2}r\right) \theta(U(r)) \right\} = \exp\left(\frac{c_6}{2}t_m\right) \theta(U(t_m)).$$

On the other hand, by (41), setting $s = 0$ and $t = t_m$ in (39) yields

$$\frac{\dot{\widehat{U(t_m)}}}{L(t_m)^{1-\epsilon}} \leq -3c_3 L(t_m)^\epsilon \theta(U(t_m)) < 0 .$$

This contradicts the definition of t_m. Consequently, U_m upper bounds the function $t \mapsto U(t)$ in the positive time domain of the existence of the solution. Since $t \mapsto L(t)$ is bounded, the same holds for $t \mapsto (\hat{x}(t), \tilde{x}(t))$ and therefore also for $t \mapsto \gamma_i(z_r(t), x_r(t), \tilde{x}(t))$. Then, by integration, we obtain from Assumption 1.2 that, for all i, the function $t \mapsto Z_i(z_r(t), \tilde{z}(t))$ is bounded on the positive time domain of existence of the solution. Hence, by Assumption 1.1, the same holds for $t \mapsto \tilde{z}(t)$. This implies that the positive time domain of existence is $[0, +\infty)$ and that the closed loop solution is bounded on this interval.

It remains to establish (15). Let $\underline{U}$ be any positive real number satisfying

$$\underline{U} > c_5^{d_U} \limsup_{t \to +\infty} \left(\sum_{i=1}^{n-1} \frac{|\delta_{r,i}(t)|}{L(t)^{i+b}} \right.$$
$$\left. + \left(\frac{|\delta_{r,i}(t)|}{L(t)^{i+b}} \right)^{\frac{1}{r_{\infty,i+\partial\infty}}} + 2\frac{|\delta_{r,n}(t)|}{L(t)^{b+n}} \right)^{d_U}, \tag{42}$$

with c_5 given in (32). By definition of lim sup, there exists t_l such that

$$\underline{U} \geq c_5^{d_U} \left(\sum_{i=1}^{n-1} \left(\frac{|\delta_{r,i}(t)|}{L(t)^{i+b}} \right) + \left(\frac{|\delta_{r,i}(t)|}{L(t)^{i+b}} \right)^{\frac{1}{r_{\infty,i+\partial\infty}}} + 2\frac{|\delta_{r,n}(t)|}{L(t)^{b+n}} \right)^{d_U},$$
$$\forall t \geq t_l. \tag{43}$$

Furthermore, using (32), we obtain:

$$L(t)^\epsilon c_4 \sum_{i=1}^{n} \zeta_i \left(\frac{\delta_{r,i}(t)}{L(t)^{i+b}} \right) \leq c_3 L(t)^\epsilon \theta\left(\underline{U}\right), \quad \forall t \geq t_l. \tag{44}$$

Let $Z_{i,m}$ be a bound for the function $t \mapsto Z_i(t)$ and:

$$t_k = \frac{1}{c_6} \ln \left(\frac{c_4 \sum_{i=1}^{n} \zeta_i(a_2^{1-\frac{i+b}{\epsilon}} \mu_i(Z_{i,m}))}{c_3 a_2 \theta\left(\underline{U}\right)} \right).$$

By (40), we have:

$$c_4 \exp(c_6(s-t)) \sum_{i=1}^{n} \zeta_i \left(L(s)^{\epsilon-(i+b)}(\mu_i(Z_i(s))) \right) \leq c_3 L(t)^\epsilon \theta\left(\underline{U}\right),$$
$$\forall t \geq s + t_k. \tag{45}$$

Define now

$$\mathcal{U}(s) = \sup_{r \geq s} U(r),$$

and assume that we have:

$$\mathcal{U}(s) \geq \underline{U} \quad \forall s \geq t_l. \tag{46}$$

Then, by (44) and (45), (39) with $s \geq t_l$, gives:

$$\frac{\dot{\widehat{U(t)}}}{L(t)^{1-\epsilon}} \leq -[4c_3 L(t)^\epsilon + 1]\theta(U(t)) + 3c_3 L(t)^\epsilon \theta(\mathcal{U}(s))$$
$$\forall t \geq s + t_k.$$

So, for each time $t \geq s+t_k$ for which we have $\theta(U(t)) \geq \frac{3}{4}\theta(\mathcal{U}(s))$, we obtain:

$$\dot{\widehat{U(t)}} < -\theta(U(t)).$$

It follows from the proof of [23, Theorem 1] that we have:

$$\limsup_{t \to +\infty} \theta(U(t)) \leq \frac{3}{4}\theta(\mathcal{U}(s)).$$

But the definition (28) of θ gives:

$$\theta(a) \leq \rho\theta(b) \Rightarrow a \leq \rho^{\frac{d_U}{\partial\infty+d_U}} b \quad \forall a, b \geq 0, \forall \rho \in [0, 1].$$

This yields:

$$\limsup_{t \to +\infty} U(t) \leq \left(\frac{3}{4}\right)^{\frac{d_U}{\partial\infty+d_U}} \mathcal{U}(s)$$

and this for all $s \geq t_l$. By taking the limit for s going to infinity, we get

$$\limsup_{t \to +\infty} U(t) \leq \left(\frac{3}{4}\right)^{\frac{d_U}{\partial\infty+d_U}} \limsup_{t \to +\infty} U(t)$$

and therefore $\limsup_{t \to +\infty} U(t) = 0$, which contradicts (46). We conclude that, for each $\underline{U}$ satisfying (42), there exists s satisfying:

$$\mathcal{U}(s) = \sup_{r \geq s} U(r) < \underline{U}$$

from which (15) follows.

To complete the proof it remains to show that the property (38) is satisfied. By letting

$$M = L^\epsilon,$$

(11) gives:

$$\dot{M} = -\epsilon a_1 M + \epsilon M \max\left\{0, a_1(a_2 + 1 - M) + a_3\overline{\Omega}(z_r, x_r)\right\}.$$

Since $M \geq a_2 \geq 1$, when:

$$a_1(a_2 + 1) + a_3\overline{\Omega}(z_r, x_r) \geq a_1 M,$$

we get:

$$\dot{M} \geq -\epsilon a_1 M + \epsilon \left[a_1(a_2 + 1 - M) + a_3\overline{\Omega}(z_r, x_r)\right],$$
$$\geq -2\epsilon a_1 M + \epsilon \left[a_1(a_2 + 1) + a_3\overline{\Omega}(z_r, x_r)\right]$$

and, when:

$$a_1(a_2 + 1) + a_3\overline{\Omega}(z_r, x_r) < a_1 M,$$

we get:

$$\dot{M} = -2\epsilon a_1 M + \epsilon a_1 M,$$
$$\geq -2\epsilon a_1 M + \epsilon \left[a_1(a_2 + 1) + a_3\overline{\Omega}(z_r, x_r)\right].$$

By integration this yields:

$$M(t) \geq \exp(-2\epsilon a_1 t)\left(M(0) - \frac{a_2 + 1}{2}\right)$$
$$+ \varepsilon a_3 \int_0^t \exp(2\epsilon a_1(r-t))\Omega(z_r(r), x_r(r))dr + \frac{a_2 + 1}{2}.$$

Picking $L(0)^\epsilon = M(0) \geq a_2$, the inequality (38) holds provided a_1 is chosen sufficiently small and a_2 and a_3 sufficiently large.

4. Conclusion

We have solved a tracking problem by output feedback for minimum phase non-linear systems which globally admit a strict normal form. Unlike most existing results, we allow nonlinearities in the model satisfying a polynomial type growth in the unmeasured state components. In particular, the result obtained generalizes the one obtained in [2] since asymptotic tracking may be obtained without knowing an upper-bound on the reference trajectory.

This has been achieved by exploiting the tools of domination, homogeneity in the bi-limit, dynamic scaling and a novel time varying small gain argument.

Appendix A. Homogeneity in the bi-limit

For details on the notion of homogeneity in the bi-limit the reader is referred to [10]. For completeness, we recall the definition and state the main properties used in the paper.

Given a weight $r = (r_1, \ldots, r_n)$ in $(\mathbb{R}_+/\{0\})^n$, we define the dilation of a vector x in $\mathbb{R}^n$ as

$$\lambda^r \diamond x = \left(\lambda^{r_1} x_1, \ldots, \lambda^{r_n} x_n\right)^{\mathrm{T}}.$$

Definition 1 (*Homogeneity in the 0-limit*).

- A continuous function $\phi : \mathbb{R}^n \to \mathbb{R}$ is said to be homogeneous in the 0-limit with the associated triple (r_0, d_0, ϕ_0), where r_0 in $(\mathbb{R}_+/\{0\})^n$ is the weight, [13] d_0 in $\mathbb{R}_+$ the degree and $\phi_0 : \mathbb{R}^n \to \mathbb{R}$ the approximating function, respectively, if ϕ_0 is continuous and not identically zero and, for each compact set C in $\mathbb{R}^n \setminus \{0\}$ and each $\varepsilon > 0$, there exists λ^* such that we have:

$$\max_{x \in C} \left| \frac{\phi(\lambda^{r_0} \diamond x)}{\lambda^{d_0}} - \phi_0(x) \right| \leq \varepsilon \quad \forall \lambda \in (0, \lambda^*].$$

- A vector field $f = \sum_{i=1}^n f_i \frac{\partial}{\partial x_i}$ is said to be homogeneous in the 0-limit with the associated triple $(r_0, \eth_0, f_0)$, where $f_0 = \sum_{i=1}^n f_{0,i} \frac{\partial}{\partial x_i}$, if, for each i in $\{1, \ldots, n\}$, the function f_i is homogeneous in the 0-limit with the associated triple $\left(r_0, \eth_0 + r_{0,i}, f_{0,i}\right)$. [14]

Definition 2 (*Homogeneity in the ∞-limit*).

- A continuous function $\phi : \mathbb{R}^n \to \mathbb{R}$ is said to be homogeneous in the ∞-limit with the associated triple $(r_\infty, d_\infty, \phi_\infty)$ where r_∞ in $(\mathbb{R}_+/\{0\})^n$ is the weight, d_∞ in $\mathbb{R}_+$ the degree and $\phi_\infty : \mathbb{R}^n \to \mathbb{R}$ the approximating function, respectively, if ϕ_∞ is continuous and not identically zero and, for each compact set C in $\mathbb{R}^n \setminus \{0\}$ and each $\varepsilon > 0$, there exists λ^* such that we have:

$$\max_{x \in C} \left| \frac{\phi(\lambda^{r_\infty} \diamond x)}{\lambda^{d_\infty}} - \phi_\infty(x) \right| \leq \varepsilon \quad \forall \lambda \in [\lambda^*, +\infty).$$

- A vector field $f = \sum_{i=1}^n f_i \frac{\partial}{\partial x_i}$ is said to be homogeneous in the ∞-limit with the associated triple $(r_\infty, \eth_\infty, f_\infty)$, with $f_\infty = \sum_{i=1}^n f_{\infty,i} \frac{\partial}{\partial x_i}$, if, for each i in $\{1, \ldots, n\}$, the function f_i is homogeneous in the ∞-limit with the associated triple $(r_\infty, \eth_\infty + r_{\infty,i}, f_{\infty,i})$.

Definition 3 (*Homogeneous in the bi-limit function*). A continuous function $\phi : \mathbb{R}^n \to \mathbb{R}$ (or a vector field f) is said to be homogeneous in the bi-limit if it is homogeneous in the 0-limit and homogeneous in the ∞-limit.

We now recall some properties of homogeneous in the bi-limit functions. Let η and γ be two continuous homogeneous in the bi-limit functions with weights r_0, r_∞, degrees $d_{\eta,0}$, $d_{\eta,\infty}$ and $d_{\gamma,0}$, $d_{\gamma,\infty}$, and continuous approximating functions $\eta_0, \eta_\infty, \gamma_0, \gamma_\infty$.

Claim A.1. *The function* $x \mapsto \eta(x)\gamma(x)$ *is homogeneous in the bi-limit with the associated triples* $(r_0, d_{\eta,0} + d_{\gamma,0}, \eta_0\gamma_0)$ *and* $(r_\infty, d_{\eta,\infty} + d_{\gamma,\infty}, \eta_\infty\gamma_\infty)$.

Claim A.2. *If the degrees satisfy* $d_{\eta,0} \geq d_{\gamma,0}$ *and* $d_{\eta,\infty} \leq d_{\gamma,\infty}$, *and* $\gamma(x) \geq 0$, *and we have the following implications for all non-zero x in $\mathbb{R}^n$:*

$$\gamma(x) = 0 \Rightarrow \eta(x) < 0,$$
$$\gamma_0(x) = 0 \Rightarrow \eta_0(x) < 0,$$
$$\gamma_\infty(x) = 0 \Rightarrow \eta_\infty(x) < 0,$$

then there exists a real number k^* such that, for all $k \geq k^*$, and for all non-zero x in $\mathbb{R}^n$:

$$\eta(x) < k\gamma(x), \qquad \eta_0(x) < k\gamma_0(x), \qquad \eta_\infty(x) < k\gamma_\infty(x).$$

Claim A.3. *If the degrees satisfy* $d_{\eta,0} \geq d_{\gamma,0}$ *and* $d_{\eta,\infty} \leq d_{\gamma,\infty}$ *and the functions γ, γ_0 and γ_∞ are positive definite then there exists a positive real number c satisfying $\eta(x) \leq c\gamma(x)$ for all x in $\mathbb{R}^n$.*

Appendix B. Technical proofs

B.1. Proof of inequalities (13) *and* (14)

For any $t_0 \geq 1$, let

$$\bar{L}_{t_0} = \left(a_2 + \frac{1}{t_0} + \frac{a_3}{a_1} \sup_{s \geq t_0} \overline{\Omega}(s) \right)^{\frac{1}{\epsilon}}.$$

Then from (11)

$$\dot{L}(t) \leq -a_1 L(t) + L(t) \max\left\{ 0, a_1 \left(1 - \frac{1}{t_0} + \bar{L}_{t_0}^\epsilon - L(t)^\epsilon \right) \right\}$$
$$\forall t \geq t_0.$$

It follows that

$$\dot{L}(t) \leq -\frac{a_1}{t_0} L(t) \quad \forall t \geq t_0 \geq 1 : L(t) \geq \bar{L}_{t_0}.$$

From the proof of [23, Theorem 1], this yields:

$$\limsup_{t \to +\infty} L(t) \leq \bar{L}_{t_0}.$$

As a result inequality (13) follows allowing t_0 to go to $+\infty$.

Similarly, let

$$\underline{L}_{t_0} = \left(a_2 - \frac{1}{t_0} + \frac{a_3}{a_1} \inf_{s \geq t_0} \overline{\Omega}(s) \right)^{\frac{1}{\epsilon}}.$$

This gives

$$\overline{\underline{L}_{t_0} - L(t)} \leq -\frac{a_1}{t_0} L(t) \leq -\frac{a_4}{t_0} (\underline{L}_{t_0} - L(t))$$
$$\forall t \geq t_0 : \underline{L}_{t_0} \geq L(t) \quad (\geq a_2^{\frac{1}{\epsilon}}),$$

---

where:

$$a_4 = \frac{a_1 a_2^{\frac{1}{\epsilon}}}{\underline{L}_{t_0} - a_2^{\frac{1}{\epsilon}}}.$$

It follows that

$$\liminf_{t \to +\infty} L(t) \geq \underline{L}_{t_0},$$

from which (14) follows letting t_0 go to $+\infty$.

B.2. Proof of Lemma 1

This proof is composed of two steps. We first give a bound on T_{DS} and then on T_{Dist}.

Bound on the term T_{DS}. Eq. (24), and the expression of $\dot{L}$ in (11), give:

$$T_{DS} \leq -[a_1(a_2 - L^\epsilon) + a_3 \overline{\Omega}(z_r, x_r)]$$
$$\times \left(\frac{\partial U}{\partial \hat{\mathfrak{x}}}(\hat{\mathfrak{x}}, E)D\hat{\mathfrak{x}} + \frac{\partial U}{\partial E}(\hat{\mathfrak{x}}, E)DE \right).$$

From Theorem 2, point (1), for each i in $\{1, \ldots, n\}$ the functions $\frac{\partial U}{\partial e_i}(\hat{\mathfrak{x}}, E)$ and $\frac{\partial U}{\partial \hat{\mathfrak{x}}_i}(\hat{\mathfrak{x}}, E)$ are homogeneous in the bi-limit with the same weights (r_0, r_0) and (r_∞, r_∞) and degrees $d_U - 1$ and $d_U - r_{\infty,i}$. Hence, by Claim A.1 in Appendix A, the function $(\hat{\mathfrak{x}}, E) \mapsto \frac{\partial U}{\partial \hat{\mathfrak{x}}}(\hat{\mathfrak{x}}, E)D\hat{\mathfrak{x}} + \frac{\partial U}{\partial E}(\hat{\mathfrak{x}}, E)DE$ is homogeneous in the bi-limit with weights (r_0, r_0) and (r_∞, r_∞) and degrees d_U and d_U. Hence Claim A.3 yields a positive real number c_{11} such that:

$$\frac{\partial U}{\partial \hat{\mathfrak{x}}}(\hat{\mathfrak{x}}, E)D\hat{\mathfrak{x}} + \frac{\partial U}{\partial E}(\hat{\mathfrak{x}}, E)DE \leq c_{11}U(\hat{\mathfrak{x}}, E).$$

Finally, using (24) once again and since $a_2^{\frac{1}{\epsilon}} \leq L^\epsilon \leq L$, we get

$$T_{DS} \leq -c_2 a_3 \overline{\Omega}(z_r, x_r)U(\hat{\mathfrak{x}}, E) + c_{11}a_1 L U(\hat{\mathfrak{x}}, E). \quad (47)$$

Bound on the term T_{Dist}. By Assumption 2, (9), (26) and (20), we have, for all i,

$$|(\Delta_f(L))_i| = L^{1-i-b}|f_i(z_r + \tilde{z}, x_r + \tilde{x}) - f_i(z_r, x_r)|,$$

$$\leq \Omega(z_r, x_r) \sum_{j=1}^{i} L^{j-i}|\hat{\mathfrak{x}}_j - e_j| + c_\infty L^{1-i-b}$$

$$\times \sum_{j=1}^{i} |L^{b+j-1}(\hat{\mathfrak{x}}_j - e_j)|^{\frac{r_{\infty,i}+\mathfrak{d}_\infty}{r_{\infty,j}}} + L^{1-i-b}\mu_i(Z_i).$$

Inequality (12) implies that, for all $L \geq a_2^{\frac{1}{\epsilon}} \geq 1$,

$$\frac{1}{a_2} \geq L^{-\epsilon} \geq L^{(b+j-1)\frac{r_{\infty,i}+\mathfrak{d}_\infty}{r_{\infty,j}}-i-b}.$$

Consequently, for all $(\hat{\mathfrak{x}}, E)$ in $\mathbb{R}^{2n}$ and $L \geq a_2^{\frac{1}{\epsilon}} \geq 1$,

$$|(\Delta_f(L))_i| \leq \Omega(z_r, x_r) \sum_{j=1}^{i} |\hat{\mathfrak{x}}_j - e_j|$$

$$+ \frac{c_\infty}{a_2} L \sum_{j=1}^{i} |\hat{\mathfrak{x}}_j - e_j|^{\frac{r_{\infty,i}+\mathfrak{d}_\infty}{r_{\infty,j}}} + L^{1-i-b}\mu_i(Z_i).$$

On the other hand, the functions $\left|\frac{\partial U}{\partial e_i}(E)\right| |\hat{\mathfrak{x}}_j - e_j|$ and $\left|\frac{\partial U}{\partial e_i}(E)\right|$
$\left|\hat{\mathfrak{x}}_j - e_j\right|^{\frac{r_{\infty,i}+\mathfrak{d}_\infty}{r_{\infty,j}}}$ are homogeneous in the bi-limit with weights (r_0, r_0) and (r_∞, r_∞) and degrees d_U and $d_U + r_{\infty,j} - r_{\infty,i}$ ($\leq d_U$)

and, respectively, $d_U - 1 + \frac{r_{\infty,i}+\mathfrak{d}_\infty}{r_{\infty,j}}$ ($\geq d_U$) and $d_U + \mathfrak{d}_\infty$. Hence, Claim A.3 yields positive real numbers c_{12} and c_{13} such that:

$$|T_{Dist}| \leq c_{12}\Omega(z_r, x_r)U(\hat{\mathfrak{x}}, E) + \frac{c_\infty c_{13}}{a_2}L\theta\left(U(\hat{\mathfrak{x}}, E)\right)$$

$$+ \sum_{i=1}^{n} \left|\frac{\partial U}{\partial e_i}(\hat{\mathfrak{x}}, E)\right| L^{1-i-b}\mu_i(Z_i) + \sum_{i=1}^{n} \left|\frac{\partial U}{\partial e_i}(\hat{\mathfrak{x}}, E)\right| L^{1-i-b}\delta_{r,i}.$$
(48)

Collecting (27), (47) and (48) with $\overline{\Omega} \geq \frac{\Omega}{c_\infty}$, we obtain

$$\dot{\overline{U(\hat{\mathfrak{x}}, E)}} \leq -\left[c_1 - c_{11}a_1 - \frac{c_\infty c_{13}}{a_2}\right]L\theta\left(U(\hat{\mathfrak{x}}, E)\right)$$

$$- [c_2 a_3 - c_{12}c_\infty]\overline{\Omega}(z_r, x_r)U(\hat{\mathfrak{x}}, E)$$

$$+ \sum_{i=1}^{n} \left|\frac{\partial U}{\partial e_i}(\hat{\mathfrak{x}}, E)\right| L^{1-i-b}\mu_i(Z_i) + \sum_{i=1}^{n} \left|\frac{\partial U}{\partial e_i}(\hat{\mathfrak{x}}, E)\right| L^{1-i-b}\delta_{r,i}.$$

Hence setting $c_3 = \frac{c_1}{10}$, for all sufficiently small a_1 and large a_2 and a_3, we have

$$\dot{\overline{U(\hat{\mathfrak{x}}, E)}} \leq -5c_3 L\theta\left(U(\hat{\mathfrak{x}}, E)\right) + \sum_{i=1}^{n} \left|\frac{\partial U}{\partial e_i}(\hat{\mathfrak{x}}, E)\right| \frac{\mu_i(Z_i)}{L^{i+b-1}}$$

$$+ \sum_{i=1}^{n} \left|\frac{\partial U}{\partial e_i}(\hat{\mathfrak{x}}, E)\right| \frac{\delta_{r,i}}{L^{i+b-1}}.$$
(49)

The function $(\hat{\mathfrak{x}}, E, \overline{\delta}_{r,i}) \mapsto \frac{\partial U}{\partial e_i}(\hat{\mathfrak{x}}, E)\overline{\delta}_{r,i}$ is homogeneous in the bi-limit with degrees d_U and $d_U + \mathfrak{d}_\infty$ and the weights 1 and $r_{\infty,i} + \mathfrak{d}_\infty$ for $\overline{\delta}_{r,i}$. Hence there exists a positive real number c_4 satisfying for any E, i, Z_i and $L \geq 1$,

$$\left|\frac{\partial U}{\partial e_i}(\hat{\mathfrak{x}}, E)\right| \frac{\delta_{r,i}}{L^{i+b}} \leq \frac{c_3}{2n}\theta\left(U(\hat{\mathfrak{x}}, E)\right) + c_4\zeta_i\left(\frac{\delta_{r,i}}{L^{i+b}}\right),$$

$$L^{1-\epsilon} \left|\frac{\partial U}{\partial e_i}(\hat{\mathfrak{x}}, E)\right| \frac{\mu_i(Z_i)}{L^{i+b-\epsilon}} \leq L\frac{c_3}{2n}\theta\left(U(\hat{\mathfrak{x}}, E)\right) + c_4 L^{1-\epsilon}\zeta_i\left(\frac{\mu_i(Z_i)}{L^{i+b-\epsilon}}\right),$$

where ζ_i is the homogeneous in the bi-limit function defined in (31). Hence, inequality (49) becomes (29) and yields the claim of Lemma 1. □

B.3. Proof of Lemma 2

Before entering the proof of Lemma 2, it is useful to list some properties of the functions μ_i and ζ_i. The functions $\mu_i^{q_1}$ are zero at zero and convex, hence by (7), we have:

$$\frac{1}{q_1}\mu_i(s) \leq s\mu_i'(s) \leq q_2\mu_i(s) \quad \forall s > 0. \quad (50)$$

Similarly, by definition (31) of the function ζ_i, we have:

$$\zeta_i(s) \leq s\zeta_i'(s) \leq \frac{d_U + \mathfrak{d}_\infty}{1 - \mathfrak{d}_\infty(n-2)}\zeta_i(s) \quad \forall s \geq 0. \quad (51)$$

By integration, this gives:

$$\zeta_i(rs) \leq r^{\frac{d_U+\mathfrak{d}_\infty}{1-\mathfrak{d}_\infty(n-2)}}\zeta_i(s) \quad \forall s \geq 0, \forall r \geq 1. \quad (52)$$

Also, since d_U satisfies $\frac{d_U-1}{1+\mathfrak{d}_\infty} > q_1$, the functions $s \mapsto \zeta_i(\mu_i(s))$ are convex, C^1 and zero at zero. It follows that we have the equations in Box I:

Proof of Lemma 2. By Box I and Y_i defined in (30), Assumption 1.2 gives the equation in Box II.

Since $\dot{L} + a_1 L$ is non-negative and ϵ satisfies (12), we have:

$$[\epsilon - (i + b)]\frac{\dot{L}}{L}\zeta_i'(Y_i)Y_i \leq [(i + b) - \epsilon]a_1\zeta_i'(Y_i)Y_i.$$

$$\frac{\partial}{\partial r}\zeta_i(r\mu_i(s)) = \zeta_i'(r\mu_i(s))\mu_i(s), \qquad \frac{\partial}{\partial s}\zeta_i(r\mu_i(s)) \quad = r\zeta_i'(r\mu_i(s))\mu_i'(s) \quad \text{if } s > 0,$$
$$= 0 \qquad \text{if } s = 0$$
$$0 \le \frac{\partial}{\partial s}\zeta_i(r\mu_i(s_1)) \le \frac{\partial}{\partial s}\zeta_i(r\mu_i(s_2)) \quad \forall 0 \le s_1 \le s_2, \ \forall r \ge 0.$$

Box I.

$$\dot{\overline{\zeta_i(Y_i)}} \le [\epsilon - (i+b)]\frac{\dot{L}}{L}\zeta_i'(Y_i)Y_i + L^{\epsilon-(i+b)}\zeta_i'(Y_i)\mu_i'(Z_i(z_r, \tilde{z}))\Big[-Z_i(z_r, \tilde{z}) + \gamma_i(\tilde{x}, z_r, x_r)\Big] \quad \text{if } Z_i \ne 0,$$
$$= 0 \qquad \text{if } Z_i = 0.$$

Box II.

To exhibit an upper bound on $L^{\epsilon-(i+b)}\zeta_i'(Y_i)\mu_i'(Z_i(z_r, \tilde{z}))\Big[-Z_i(z_r, \tilde{z}) + \gamma_i(\tilde{x}, z_r, x_r)\Big]$ we distinguish two cases:

(1) $\gamma_i(\tilde{x}, z_r, x_r) \le \dfrac{Z_i(z_r, \tilde{z})}{1+\upsilon}$, (2) $\gamma_i(\tilde{x}, z_r, x_r) > \dfrac{Z_i(z_r, \tilde{z})}{1+\upsilon}$,

with υ given in Assumption 3.

Case (1): (50) yields:

$$L^{\epsilon-(i+b)}\zeta_i'(Y_i)\mu_i'(Z_i(z_r, \tilde{z}))\Big[-Z_i(z_r, \tilde{z}) + \gamma_i(\tilde{x}, z_r, x_r)\Big]$$
$$\le -\frac{\upsilon}{1+\upsilon}L^{\epsilon-(i+b)}\zeta_i'(Y_i)\mu_i'(Z_i(z_r, \tilde{z}))Z_i(z_r, \tilde{z}),$$
$$\le -\frac{\upsilon}{q_1(1+\upsilon)}\zeta_i'(Y_i)L^{\epsilon-(i+b)}\mu_i(Z_i(z_r, \tilde{z})),$$
$$\le -\frac{\upsilon}{q_1(1+\upsilon)}\zeta_i'(Y_i)Y_i.$$

Case (2): By using successively the second equation in Box I, (50) and (51), we get:

$$L^{\epsilon-(i+b)}\zeta_i'(Y_i)\mu_i'(Z_i(z_r, \tilde{z}))\Big[-Z_i(z_r, \tilde{z}) + \gamma_i(\tilde{x}, z_r, x_r)\Big]$$
$$\le L^{\epsilon-(i+b)}\zeta_i'\left(L^{\epsilon-(i+b)}\mu_i\left((1+\upsilon)\gamma_i(\tilde{x}, z_r, x_r)\right)\right)$$
$$\times \mu_i'\left((1+\upsilon)\gamma_i(\tilde{x}, z_r, x_r)\right)\gamma_i(\tilde{x}, z_r, x_r) - \frac{1}{q_1}\zeta_i'(Y_i)Y_i,$$
$$\le \zeta_i'\left(L^{\epsilon-(i+b)}\mu_i\left((1+\upsilon)\gamma_i(\tilde{x}, z_r, x_r)\right)\right)$$
$$\times \frac{q_2}{1+\upsilon}L^{\epsilon-(i+b)}\mu_i\left((1+\upsilon)\gamma_i(\tilde{x}, z_r, x_r)\right) - \frac{1}{q_1}\zeta_i'(Y_i)Y_i,$$
$$\le \frac{d_U + \vartheta_\infty}{1 - \vartheta_\infty(n-2)}\frac{q_2}{1+\upsilon}\zeta_i\left(L^{\epsilon-(i+b)}\mu_i\right.$$
$$\times \left.\left((1+\upsilon)\gamma_i(\tilde{x}, z_r, x_r)\right)\right) - \frac{1}{q_1}\zeta_i'(Y_i)Y_i,$$

where by Assumption 3, (26) and $L \ge 1$, we have:

$$L^{\epsilon-(i+b)}\mu_i\left((1+\upsilon)\gamma_i(\tilde{x}, z_r, x_r)\right)$$
$$\le L^{\epsilon-(i+b)}\sum_{j=1}^{i}\Omega(z_r, x_r)|\tilde{x}_j| + c_\infty|\tilde{x}_j|^{\frac{r_{\infty,i}+\vartheta_\infty}{r_{\infty,j}}}$$
$$\le \max\left\{\frac{\Omega(z_r, x_r)}{c_\infty}, 1\right\}c_\infty\sum_{j=1}^{i}|\hat{x}_j - e_j| + |\hat{x}_j - e_j|^{\frac{r_{\infty,i}+\vartheta_\infty}{r_{\infty,j}}}.$$

Since the function $(s_j) \mapsto \sum_{j=1}^{i}|s_j| + |s_j|^{\frac{r_{\infty,i}+\vartheta_\infty}{r_{\infty,j}}}$ is homogeneous in the bi-limit with weights r_0 and r_∞ and degrees 1 and $r_{\infty,i}+\vartheta_\infty$, the function ζ_i is homogeneous in the bi-limit with weights 1 and $r_{\infty,i}+\vartheta_\infty$ and degrees d_U and $d_U + \vartheta_\infty$, and the function $(\hat{x}, E) \mapsto \theta\left(U(\hat{x}, E)\right)$ is positive definite, proper and homogeneous in the bi-limit with weights (r_0, r_0) and (r_∞, r_∞) and degrees d_U and $d_U + \vartheta_\infty$, there exists a positive real number c_7 such that we have:

$$\frac{d_U + \vartheta_\infty}{1 - \vartheta_\infty(n-2)}\frac{q_2}{1+\upsilon}\zeta_i$$
$$\times \left(c_\infty\sum_{j=1}^{i}|\hat{x}_j - e_j| + |\hat{x}_j - e_j|^{\frac{r_{\infty,i}+\vartheta_\infty}{r_{\infty,j}}}\right) \le c_7\theta\left(U(\hat{x}, E)\right).$$

Hence, with (52), we have obtained:

$$\frac{d_U + \vartheta_\infty}{1 - \vartheta_\infty(n-2)}\frac{q_2}{1+\upsilon}\zeta_i\left(L^{\epsilon-(i+b)}\mu_i((1+\upsilon)\gamma_i(\tilde{x}, z_r, x_r))\right)$$
$$\le c_7\max\left\{\frac{\Omega(z_r, x_r)}{c_\infty}, 1\right\}^{\frac{d_U+\vartheta_\infty}{1-\vartheta_\infty(n-2)}}\theta\left(U(\hat{x}, E)\right).$$

Combining the two cases, we get:

$$L^{\epsilon-(i+b)}\zeta_i'(Y_i)\mu_i'(Z_i(z_r, \tilde{z}))\Big[-Z_i(z_r, \tilde{z}) + \gamma_i(\tilde{x}, z_r, x_r)\Big]$$
$$\le -\frac{\upsilon}{q_1(1+\upsilon)}\zeta_i'(Y_i)Y_i$$
$$+ c_7\max\left\{\frac{\Omega(z_r, x_r)}{c_\infty}, 1\right\}^{\frac{d_U+\vartheta_\infty}{1-\vartheta_\infty(n-2)}}\theta\left(U(\hat{x}, E)\right)$$

which implies for Box II:

$$\dot{\overline{\zeta_i(Y_i)}} \le -\left(\frac{\upsilon}{q_1(1+\upsilon)} - [(i+b)-\epsilon]a_1\right)\zeta_i'(Y_i)Y_i$$
$$+ c_7\max\left\{\frac{\Omega(z_r, x_r)}{c_\infty}, 1\right\}^{\frac{d_U+\vartheta_\infty}{1-\vartheta_\infty(n-2)}}\theta\left(U(\hat{x}, E)\right),$$
$$\le -c_6\zeta_i(Y_i) + c_7\max\left\{\frac{\Omega(z_r, x_r)}{c_\infty}, 1\right\}^{\frac{d_U+\vartheta_\infty}{1-\vartheta_\infty(n-2)}}\theta\left(U(\hat{x}, E)\right),$$

where to get the latter inequality we have used (51) and chosen c_6 satisfying:

$$c_6 < \frac{\upsilon}{q_1(1+\upsilon)}$$

and a_1 satisfying

$$a_1 < \frac{\frac{\upsilon}{q_1(1+\upsilon)} - c_6}{n+b}.$$

This completes the proof of the Lemma. □

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