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HOMOGENEOUS APPROXIMATION, RECURSIVE OBSERVER DESIGN, AND OUTPUT FEEDBACK*

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Abstract. We introduce two new tools that can be useful in nonlinear observer and output feedback design. The first one is a simple extension of the notion of homogeneous approximation to make it valid both at the origin and at infinity (homogeneity in the bi-limit). Exploiting this extension, we give several results concerning stability and robustness for a homogeneous in the bi-limit vector field. The second tool is a new recursive observer design procedure for a chain of integrator. Combining these two tools, we propose a new global asymptotic stabilization result by output feedback for feedback and feedforward systems.

Key words. homogeneous approximation, output feedback and observer

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1. Introduction.

The problems of designing globally convergent observers and globally asymptotically stabilizing output feedback control laws for nonlinear systems have been addressed by many authors following different routes. Many of these approaches exploit domination ideas and robustness of stability and/or convergence. In view of possibly clarifying and developing further these techniques we introduce two new tools. The first one is a simple extension of the technique of homogeneous approximation to make it valid both at the origin and at infinity. The second tool is a new recursive observer design procedure for a chain of integrator. Combining these two tools, we propose a new global asymptotic stabilization result by output feedback for feedback and feedforward systems.

To place our contribution in perspective, we consider the following system, for which we want to design a global asymptotic stabilizing output feedback:

$$\dot{x}_1 \;=\; x_2, \qquad \dot{x}_2 \;=\; u + \delta_2(x_1, x_2), \qquad y \;=\; x_1 \;, \tag{1.1}$$

where (see notation (1.4))

$$\delta_2(x_1, x_2) \;=\; c_0\, x_2^{q} + c_\infty\, x_2^{p}, \qquad (c_0, c_\infty) \;\in\; \mathbb{R}^2, \qquad p > q > 0 \;. \tag{1.2}$$

In the domination's approach, the nonlinear function δ_2 is not treated per se in the design but considered as a perturbation. In this framework the output feedback controller is designed on the linear system

$$\dot{x}_1 \;=\; x_2, \qquad \dot{x}_2 \;=\; u, \qquad y \;=\; x_1, \tag{1.3}$$

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and will be suitable for the nonlinear system (1.1), provided the global asymptotic stability obtained for the origin of the closed-loop system is robust to the nonlinear disturbance δ_2. For instance, the design given in [13, 27] provides a linear output feedback controller which is suitable for the nonlinear system (1.1) when $q = 1$ and $c_\infty = 0$. This result has been extended recently in [26] employing a homogeneous output feedback controller which allows us to deal with $p \geq 1$ and $c_0 = 0$.

Homogeneity in the bi-limit and the novel recursive observer design proposed in this paper allow us to deal with the case in which $c_0 \neq 0$ and $c_\infty \neq 0$. In this case, the function δ_2 is such that

1. when $|x_2|$ is small and $q = 1$, $\delta_2(x_2)$ can be approximated by $c_0\, x_2$ and the nonlinearity can be approximated by a linear function;
2. when $|x_2|$ is large, $\delta_2(x_2)$ can be approximated by $c_\infty\, x_2^p$, and hence we have a polynomial growth which can be handled by a weighted homogeneous controller as in [26].

To deal with both linear and polynomial terms we introduce a generalization of weighted homogeneity which highlights the fact that a function becomes homogeneous as the state tends to the origin or to infinity but with different weights and degrees.

The paper is organized as follows. Section 2 is devoted to general properties related to homogeneity. After giving the definition of homogeneous approximation we introduce homogeneous in the bi-limit functions and vector fields (section 2.1) and list some of their properties (section 2.2). Various results concerning stability and robustness for homogeneous in the bi-limit vector fields are given in section 2.3. In section 3 we introduce a novel recursive observer design method for a chain of integrator. Section 4 is devoted to the homogeneous in the bi-limit state feedback. Finally, in section 5, using the previous tools, we establish new results on stabilization by output feedback.

Notation.

- $\mathbb{R}_+$ denotes the set $[0, +\infty)$.
- For any nonnegative real number r the function $w \mapsto w^r$ is defined as

$$w^r \;=\; \operatorname{sign}(w)\,|w|^r \qquad \forall\, w \,\in\, \mathbb{R}\,. \tag{1.4}$$

According to this definition,

$$\frac{dw^r}{dw} = r|w|^{r-1}\,,\; w^2 = w|w|\,,\; (w_1 > w_2 \text{ and } r > 0) \Rightarrow w_1^r > w_2^r\,. \tag{1.5}$$

- The function $\mathfrak{H} : \mathbb{R}_+^2 \to \mathbb{R}_+$ is defined as

$$\mathfrak{H}(a, b) \;=\; \frac{a}{1+a}\,[1\,+\,b]\,. \tag{1.6}$$

- Given $r = (r_1, \ldots, r_n)^T$ in $\mathbb{R}_+^n$ and λ in $\mathbb{R}_+$, $\lambda^r \diamond x = (\lambda^{r_1}\, x_1,\, \ldots,\, \lambda^{r_n}\, x_n)^T$ is the dilation of a vector x in $\mathbb{R}^n$ with weight r. Note that

$$\lambda_1^r \diamond (\lambda_2^r \diamond x) \;=\; (\lambda_1\, \lambda_2)^r \diamond x\,.$$

- Given $r = (r_1, \ldots, r_n)^T$ in $(\mathbb{R}_+ \setminus \{0\})^n$, $|x|_r = |x_1|^{\frac{1}{r_1}} + \cdots + |x_n|^{\frac{1}{r_n}}$ is the homogeneous norm with weight r and degree 1. Note that

$$|\lambda^r \diamond x|_r \;=\; \lambda\,|x|_r, \qquad \left|\left(\frac{1}{|x|_r}\right)^r \diamond x\right|_r \;=\; 1\,.$$

- Given r in $(\mathbb{R}_+ \setminus \{0\})^n$, $S_r = \{x \in \mathbb{R}^n \mid |x|_r = 1\}$ is the unity homogeneous sphere. Note that each x in $\mathbb{R}^n$ can be decomposed in polar coordinates; i.e., there exist λ in $\mathbb{R}_+$ and θ in S_r satisfying

$$x = \lambda^r \diamond \theta \quad \text{with} \left\{ \begin{array}{rcl} \lambda &=& |x|_r, \\ \theta &=& \left(\frac{1}{|x|_r}\right)^r \diamond x\,. \end{array} \right. \tag{1.7}$$

2. Homogeneous approximation.

2.1. Definitions. The use of homogeneous approximations has a long history in the study of stability of an equilibrium. It can be traced back to the Lyapunov first order approximation theorem and has been pursued by many authors; see, for example, Massera [16], Hahn [8], Hermes [9], and Rosier [29]. Similarly, this technique has been used to investigate the behavior of the solutions of dynamical systems at infinity; see, for instance, Lefschetz in [14, Chapter IX.5] and Orsi, Praly, and Mareels in [20]. In this section, we recall the definitions of homogeneous approximation at the origin and at infinity and restate and/or complete some related results.

DEFINITION 2.1 (homogeneity in the 0-limit).

- *A function $\phi : \mathbb{R}^n \to \mathbb{R}$ is said to be homogeneous in the* 0*-limit with associated triple (r_0, d_0, ϕ_0), where r_0 in $(\mathbb{R}_+ \setminus \{0\})^n$ is the weight, d_0 in $\mathbb{R}_+$ the degree, and $\phi_0 : \mathbb{R}^n \to \mathbb{R}$ the approximating function, if ϕ is continuous, ϕ_0 is continuous and not identically zero, and, for each compact set C in $\mathbb{R}^n \setminus \{0\}$ and each $\varepsilon > 0$, there exists λ_0 such that*

$$\max_{x \in C} \left| \frac{\phi(\lambda^{r_0} \diamond x)}{\lambda^{d_0}} - \phi_0(x) \right| \leq \varepsilon \qquad \forall \quad \lambda \in (0, \lambda_0]\,.$$

- *A vector field $f = \sum_{i=1}^n f_i \frac{\partial}{\partial x_i}$ is said to be homogeneous in the* 0*-limit with associated triple $(r_0, \mathfrak{d}_0, f_0)$, where r_0 in $(\mathbb{R}_+ \setminus \{0\})^n$ is the weight, $\mathfrak{d}_0$ in $\mathbb{R}$ is the degree, and $f_0 = \sum_{i=1}^n f_{0,i} \frac{\partial}{\partial x_i}$ is the approximating vector field, if, for each i in $\{1, \dots, n\}$, $\mathfrak{d}_0 + r_{0,i} \geq 0$ and the function f_i is homogeneous in the* 0*-limit with associated triple $(r_0, \mathfrak{d}_0 + r_{0,i}, f_{0,i})$.*

This notion of local approximation of a function or of a vector field can be found in [9, 29, 2, 10].

Example 2.2. The function $\delta_2 : \mathbb{R} \to \mathbb{R}$ introduced in the illustrative system (1.1) is homogeneous in the 0-limit with associated triple $(r_0,\, d_0,\, \delta_{2,0}) = (1, q, c_0\, x_2^q)$. Furthermore, if $q < 2$, then the vector field $f(x_1, x_2) = (x_2, \delta_2(x_2))$ is homogeneous in the 0-limit with associated triple

$$(r_0,\, \mathfrak{d}_0,\, f_0) = \Big((2-q, 1),\, q-1,\, (x_2, c_0\, x_2^q)\Big)\,. \tag{2.1}$$

DEFINITION 2.3 (homogeneity in the ∞-limit).

- *A function $\phi : \mathbb{R}^n \to \mathbb{R}$ is said to be homogeneous in the ∞-limit with associated triple $(r_\infty, d_\infty, \phi_\infty)$, where r_∞ in $(\mathbb{R}_+ \setminus \{0\})^n$ is the weight, d_∞ in*

$\mathbb{R}_+$ is the degree, and $\phi_\infty : \mathbb{R}^n \to \mathbb{R}$ is the approximating function, if ϕ is continuous, ϕ_∞ is continuous and not identically zero, and, for each compact set C in $\mathbb{R}^n \setminus \{0\}$ and each $\varepsilon > 0$, there exists λ_∞ such that

$$\max_{x \in C} \left| \frac{\phi(\lambda^{r_\infty} \diamond x)}{\lambda^{d_\infty}} - \phi_\infty(x) \right| \leq \varepsilon \qquad \forall \lambda \geq \lambda_\infty .$$

- *A vector field $f = \sum_{i=1}^n f_i \frac{\partial}{\partial x_i}$ is said to be homogeneous in the ∞-limit with associated triple $(r_\infty, \mathfrak{d}_\infty, f_\infty)$, where r_∞ in $(\mathbb{R}_+ \setminus \{0\})^n$ is the weight, $\mathfrak{d}_\infty$ in $\mathbb{R}$ is the degree, and $f_\infty = \sum_{i=1}^n f_{\infty,i} \frac{\partial}{\partial x_i}$ is the approximating vector field, if, for each i in $\{1, \ldots, n\}$, $\mathfrak{d}_\infty + r_{\infty,i} \geq 0$ and the function f_i is homogeneous in the ∞-limit with associated triple $(r_\infty, \mathfrak{d}_\infty + r_{\infty,i}, f_{\infty,i})$.*

Example 2.4. The function $\delta_2 : \mathbb{R} \to \mathbb{R}$ given in the illustrative system (1.1) is homogeneous in the ∞-limit with associated triple $(r_\infty, d_\infty, \delta_{2,\infty}) = (1, p, c_\infty x_2^p)$. Furthermore, when $p < 2$, the vector field $f(x_1, x_2) = (x_2, \delta_2(x_2))$ is homogeneous in the ∞-limit with associated triple

$$(r_\infty, \mathfrak{d}_\infty, f_\infty) = \Big((2-p, 1), p-1, (x_2, c_\infty x_2^p)\Big) . \tag{2.2}$$

Definition 2.5 (homogeneity in the bi-limit). *A function $\phi : \mathbb{R}^n \to \mathbb{R}$ (or a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$) is said to be homogeneous in the bi-limit if it is homogeneous in the 0-limit and homogeneous in the ∞-limit.*

Remark 2.6. If a function ϕ (resp., a vector field f) is homogeneous in the bi-limit, then the approximating function ϕ_0 or ϕ_∞ (resp., the approximating vector field f_0 or f_∞) is homogeneous in the standard sense[1] (with the same weight and degree).

Example 2.7. As a consequence of Examples 2.2 and 2.4, the vector field $f(x_1, x_2) = (x_2, \delta_2(x_2))$ is homogeneous in the bi-limit with the associated triples given in (2.1) and (2.2) as long as $0 < q < p < 2$.

Example 2.8. The function $x \mapsto |x|_{r_0}^{d_0} + |x|_{r_\infty}^{d_\infty}$, where (d_0, d_∞) are in $\mathbb{R}_+^2$ and (r_0, r_∞) are in $(\mathbb{R}_+ \setminus \{0\})^{2n}$, is homogeneous in the bi-limit with associated triples $\big(r_0, d_0, |x|_{r_0}^{d_0}\big)$ and $\big(r_\infty, d_\infty, |x|_{r_\infty}^{d_\infty}\big)$, provided that

$$\frac{d_\infty}{r_{\infty,i}} > \frac{d_0}{r_{0,i}} \quad \forall i \in \{1, \ldots, n\} . \tag{2.3}$$

Example 2.9. We recall (1.6) and consider two homogeneous and positive definite functions $\phi_0 : \mathbb{R}^n \to \mathbb{R}_+$ and $\phi_\infty : \mathbb{R}^n \to \mathbb{R}_+$ with weights (r_0, r_∞) in $(\mathbb{R}_+ \setminus \{0\})^{2n}$ and degrees (d_0, d_∞) in $(\mathbb{R}_+ \setminus \{0\})^2$. The function $x \mapsto \mathfrak{H}(\phi_0(x), \phi_\infty(x))$ is positive definite and homogeneous in the bi-limit with associated triples (r_0, d_0, ϕ_0) and $(r_\infty, d_\infty, \phi_\infty)$. This way of constructing a homogeneous in the bi-limit function from two positive definite homogenous functions is extensively used in this paper.

2.2. Properties of homogeneous approximations. To begin, we note that the weight and degree of a homogeneous in the 0- (resp., ∞-) limit function are

[1] This is proved by noting that, for all x in $\mathbb{R}^n$ and all μ in $\mathbb{R}_+ \setminus \{0\}$,

$$\frac{\phi_0(\mu^{r_0} \diamond x)}{\mu^{d_0}} = \frac{1}{\mu^{d_0}} \lim_{\lambda \to 0} \frac{\phi\left(\lambda^{r_0} \diamond (\mu^{r_0} \diamond x)\right)}{\lambda^{d_0}} = \lim_{\lambda \to 0} \frac{\phi\left((\lambda\mu)^{r_0} \diamond x\right)}{(\lambda\mu)^{d_0}} = \phi_0(x) ,$$

and similarly for the homogeneous in the ∞-limit function.

not uniquely defined. Indeed, if ϕ is homogeneous in the 0- (resp., ∞-) limit with associated triple (r_0, d_0, ϕ_0) (resp., $(r_\infty, d_\infty, \phi_\infty)$), then it is also homogeneous in the 0- (resp., ∞-) limit with associated triple $(k\, r_0, k\, d_0, \phi_0)$ (resp., $(k\, r_\infty, k\, d_\infty, \phi_\infty)$) for all $k > 0$. (Simply change λ into λ^k.)

It is straightforward to show that if ϕ and ζ are two functions homogeneous in the 0- (resp., ∞-) limit, with weights $r_{\phi,0}$ and $r_{\zeta,0}$ (resp., $r_{\phi,\infty}$ and $r_{\zeta,\infty}$), degrees $d_{\phi,0}$ and $d_{\zeta,0}$ (resp., $d_{\phi,\infty}$ and $d_{\zeta,\infty}$), and approximating functions ϕ_0 and ζ_0 (resp., ϕ_∞ and ζ_∞), then the following hold:

P1: If there exists k in $\mathbb{R}_+$ such that $k\, r_{\phi,0} = r_{\zeta,0}$ (resp., $k\, r_{\phi,\infty} = r_{\zeta,\infty}$), then the function $x \mapsto \phi(x)\, \zeta(x)$ is homogeneous in the 0- (resp., ∞-) limit with weight $r_{\zeta,0}$, degree $k\, d_{\phi,0} + d_{\zeta,0}$ (resp., $r_{\zeta,\infty}$, $k\, d_{\phi,\infty} + d_{\zeta,\infty}$) and approximating function $x \mapsto \phi_0(x)\, \zeta_0(x)$ (resp., $x \mapsto \phi_\infty(x)\, \zeta_\infty(x)$).

P2: If, for each j in $\{1, \ldots, n\}$, $\frac{d_{\phi,0}}{r_{\phi,0,j}} < \frac{d_{\zeta,0}}{r_{\zeta,0,j}}$ (resp., $\frac{d_{\phi,\infty}}{r_{\phi,\infty,j}} > \frac{d_{\zeta,\infty}}{r_{\zeta,\infty,j}}$), then the function $x \mapsto \phi(x) + \zeta(x)$ is homogeneous in the 0- (resp., ∞-) limit with degree $d_{\phi,0}$ and weight $r_{\phi,0}$ (resp., $d_{\phi,\infty}$ and $r_{\phi,\infty}$) and approximating function $x \mapsto \phi_0(x)$ (resp., $x \mapsto \phi_\infty(x)$). In this case we say that the function ϕ *dominates* the function ζ in the 0-limit (resp., in the ∞-limit).

P3: If the function $\phi_0 + \zeta_0$ (resp., $\phi_\infty + \zeta_\infty$) is not identically zero and, for each j in $\{1, \ldots, n\}$, $\frac{d_{\phi,0}}{r_{\phi,0,j}} = \frac{d_{\zeta,0}}{r_{\zeta,0,j}}$ (resp., $\frac{d_{\phi,\infty}}{r_{\phi,\infty,j}} = \frac{d_{\zeta,\infty}}{r_{\zeta,\infty,j}}$), then the function $x \mapsto \phi(x) + \zeta(x)$ is homogeneous in the 0- (resp., ∞-) limit with degree $d_{\phi,0}$ and weight $r_{\phi,0}$ (resp., $d_{\phi,\infty}$ and $r_{\phi,\infty}$) and approximating function $x \mapsto \phi_0(x) + \zeta_0(x)$ (resp., $x \mapsto \phi_\infty(x) + \zeta_\infty(x)$).

Some properties of the composition or inverse of functions are given in the following two propositions, the proofs of which are given in Appendices A and B.

PROPOSITION 2.10 (composition function). *If $\phi : \mathbb{R}^n \to \mathbb{R}$ and $\zeta : \mathbb{R} \to \mathbb{R}$ are homogeneous in the 0- (resp., ∞-) limit functions, with weights $r_{\phi,0}$ and $r_{\zeta,0}$ (resp., $r_{\phi,\infty}$ and $r_{\zeta,\infty}$), degrees $d_{\phi,0} > 0$ and $d_{\zeta,0} \geq 0$ (resp., $d_{\phi,\infty} > 0$ and $d_{\zeta,\infty} \geq 0$), and approximating functions ϕ_0 and ζ_0 (resp., ϕ_∞ and ζ_∞), then $\zeta \circ \phi$ is homogeneous in the 0- (resp., ∞-) limit with weight $r_{\phi,0}$ (resp., $r_{\phi,\infty}$), degree $\frac{d_{\zeta,0}\, d_{\phi,0}}{r_{\zeta,0}}$ (resp., $\frac{d_{\zeta,\infty}\, d_{\phi,\infty}}{r_{\zeta,\infty}}$), and approximating function $\zeta_0 \circ \phi_0$ (resp., $\zeta_\infty \circ \phi_\infty$).*

PROPOSITION 2.11 (inverse function). *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a bijective homogeneous in the 0- (resp., ∞-) limit function with associated triple $\left(1, d_0, \varphi_0\, x^{d_0}\right)$ with $\varphi_0 \neq 0$ and $d_0 > 0$ (resp., $\left(1, d_\infty, \varphi_\infty\, x^{d_\infty}\right)$ with $\varphi_\infty \neq 0$ and $d_\infty > 0$). Then the inverse function $\phi^{-1} : \mathbb{R} \to \mathbb{R}$ is a homogeneous in the 0- (resp., ∞-) limit function with associated triple $(1, \frac{1}{d_0}, (\frac{x}{\varphi_0})^{\frac{1}{d_0}})$ (resp., $(1, \frac{1}{d_\infty}, (\frac{x}{\varphi_\infty})^{\frac{1}{d_\infty}})$).*

Despite the existence of well-known results concerning the derivative of a homogeneous function, it is not possible to say anything, in general, when dealing with homogeneity in the limit. For example, the function

$$\phi(x) = x^3 + x^2 \sin(x^2) + x^3 \sin(1/x) + x^2, \qquad x \in \mathbb{R}\ ,$$

is homogeneous in the bi-limit with associated triples

$$\left(1, 2, x^2\right), \qquad \left(1, 3, x^3\right)\ .$$

However, its derivative is homogeneous in neither the 0-limit nor the ∞-limit. Nevertheless the following result holds, the proof of which is elementary.

PROPOSITION 2.12 (integral function). *If the function $\phi : \mathbb{R}^n \to \mathbb{R}$ is homogeneous in the 0- (resp., ∞-) limit with associated triple (r_0, d_0, ϕ_0) (resp., $(r_\infty, d_\infty, \phi_\infty)$),* *then the function* $\Phi_i(x) = \int_0^{x_i} \phi(x_1, \dots, x_{i-1}, s, x_{i+1}, \dots, x_n)\, ds$ *is homogeneous in the* 0- *(resp.,* ∞*-) limit with associated triple* $(r_0, d_0 + r_{0,i}, \Phi_{i,0})$ *(resp.,* $(r_\infty, d_\infty + r_{\infty,i}, \Phi_{i,\infty})$*), with* $\Phi_{i,0}(x) = \int_0^{x_i} \phi_0(x_1, \dots, x_{i-1}, s, x_{i+1}, \dots, x_n)\, ds$ *(resp.,* $\Phi_{i,\infty}(x) = \int_0^{x_i} \phi_\infty(x_1, \dots, x_{i-1}, s, x_{i+1}, \dots, x_n)\, ds$*).*

By exploiting the definition of homogeneity in the bi-limit, it is possible to establish results which are straightforward extensions of well-known results based on the standard notion of homogeneity. These results are given as corollaries of the following key technical lemma, the proof of which is given in Appendix C.

LEMMA 2.13 (key technical lemma). *Let* $\eta : \mathbb{R}^n \to \mathbb{R}$ *and* $\gamma : \mathbb{R}^n \to \mathbb{R}_+$ *be two functions homogeneous in the bi-limit, with weights* r_0 *and* r_∞*, degrees* d_0 *and* d_∞*, and approximating functions,* η_0*,* η_∞ *and* γ_0*,* γ_∞ *such that the following hold:*

$$\begin{aligned}
\{\ x \in \mathbb{R}^n \setminus \{0\} \ :\ \gamma(x) = 0\ \} &\subseteq \{\ x \in \mathbb{R}^n \ :\ \eta(x) < 0\ \}\ ,\\
\{\ x \in \mathbb{R}^n \setminus \{0\} \ :\ \gamma_0(x) = 0\ \} &\subseteq \{\ x \in \mathbb{R}^n \ :\ \eta_0(x) < 0\ \}\ ,\\
\{\ x \in \mathbb{R}^n \setminus \{0\} \ :\ \gamma_\infty(x) = 0\ \} &\subseteq \{\ x \in \mathbb{R}^n \ :\ \eta_\infty(x) < 0\ \}\ .
\end{aligned}$$

Then there exists a real number c^* *such that, for all* $c \geq c^*$ *and for all* x *in* $\mathbb{R}^n \setminus \{0\}$*,*

$$\eta(x) - c\,\gamma(x) < 0\ , \quad \eta_0(x) - c\,\gamma_0(x) < 0\ , \quad \eta_\infty(x) - c\,\gamma_\infty(x) < 0\ . \tag{2.4}$$

Example 2.14. To illustrate the importance of this lemma, consider, for (x_1, x_2) in $\mathbb{R}^2$, the functions

$$\eta(x_1, x_2) = x_1\, x_2 - |x_1|^{\frac{r_1 + r_2}{r_1}}, \qquad \gamma(x_1, x_2) = |x_2|^{\frac{r_1 + r_2}{r_2}}\ ,$$

with $r_1 > 0$ and $r_2 > 0$. They are homogeneous in the standard sense, and therefore in the bi-limit, with the same weight $r = (r_1, r_2)$ and the same degree $d = r_1 + r_2$. Furthermore, the function γ takes positive values, and for all (x_1, x_2) in $\{(x_1, x_2) \in \mathbb{R}^2 \setminus \{0\} \ :\ \gamma(x_1, x_2) = 0\}$ we have

$$\eta(x_1, x_2) \ = \ -|x_1|^{\frac{r_1 + r_2}{r_1}} \ < \ 0\ .$$

Thus Lemma 2.13 yields the existence of a positive real number c^* such that for all $c \geq c*$, we have

$$x_1\, x_2 - |x_1|^{\frac{r_1 + r_2}{r_1}} - c\,|x_2|^{\frac{r_1 + r_2}{r_2}} < 0 \qquad \forall\ (x_1, x_2) \in \mathbb{R}^2 \setminus \{0\}\ . \tag{2.5}$$

This is a generalization of the procedure known as the completion of the squares in which, however, the constant c_1^* is not specified.

COROLLARY 2.15. *Let* $\phi : \mathbb{R}^n \to \mathbb{R}$ *and* $\zeta : \mathbb{R}^n \to \mathbb{R}_+$ *be two homogeneous in the bi-limit functions with the same weights* r_0 *and* r_∞*, degrees* $d_{\phi,0}$*,* $d_{\phi,\infty}$ *and* $d_{\zeta,0}$*,* $d_{\zeta,\infty}$*, and approximating functions* η_0*,* ϕ_∞ *and* ζ_0*,* ζ_∞*. If the degrees satisfy* $d_{\phi,0} \geq d_{\zeta,0}$ *and* $d_{\phi,\infty} \leq d_{\zeta,\infty}$*, and the functions* ζ*,* ζ_0 *and* ζ_∞ *are positive definite, then there exists a positive real number* c *satisfying*

$$\phi(x) \ \leq \ c\,\zeta(x) \quad \forall\, x \ \in \ \mathbb{R}^n\ .$$

Proof. Consider the two functions

$$\eta(x) \ := \ \phi(x) + \zeta(x), \qquad \gamma(x) \ := \ \zeta(x)\ .$$

By property P2 (or P3)[2] in section 2.2, they are homogeneous in the bi-limit with degrees $d_{\zeta,0}$ and $d_{\zeta,\infty}$. The function γ and its homogeneous approximations being positive definite, all assumptions of Lemma 2.13 are satisfied. Therefore there exists a positive real number c such that

$$c\,\gamma(x) \;>\; \eta(x) \;>\; \phi(x) \qquad \forall\, x \in \mathbb{R}^n \setminus \{0\}\ .$$

Finally, by continuity of the functions ϕ and ζ at zero, we can obtain the claim. ☐

2.3. Stability and homogeneous approximation. A very basic property of asymptotic stability is its robustness. This fact was already known to Lyapunov, who proposed his second method, in which (local) asymptotic stability of an equilibrium is established by looking at the first order approximation of the system. The case of local homogeneous approximations of higher degree has been investigated by Massera [16], Hermes [9], Rosier [29], and Kawski [12].

PROPOSITION 2.16 (see [29]). *Consider a homogeneous in the* 0*-limit vector field* $f\ :\mathbb{R}^n \to \mathbb{R}^n$ *with associated triple* $(r_0, \mathfrak{d}_0, f_0)$. *If the origin of the system*

$$\dot{x} \;=\; f_0(x)$$

is locally asymptotically stable, then the origin of

$$\dot{x} \;=\; f(x)$$

is locally asymptotically stable.

Consequently, a natural strategy to ensure local asymptotic stability of an equilibrium of a system is to design a stabilizing homogeneous control law for the homogeneous approximation in the 0-limit (see [9, 12, 5], for instance).

Example 2.17. Consider the system (1.1), with $q = 1$ and $p > q$, and the linear control law

$$u \;=\; -(c_0+1)\,x_2 \;-\; x_1\ .$$

The closed-loop vector field is homogeneous in the 0-limit with degree $\mathfrak{d}_0 = 0$, weight $(1,1)$ (i.e., we are in the linear case), and associated vector field $f_0(x_1,x_2) = (x_2, -x_1 - x_2)^T$. Selecting the Lyapunov function of degree 2,

$$V_0(x_1,x_2) \;=\; \frac{1}{2}\,|x_1|^2 \;+\; \frac{1}{2}\,|x_2 \,+\, x_1|^2\ ,$$

yields

$$\frac{\partial V_0}{\partial x}(x)\,f_0(x) \;=\; -|x_1|^2 \;-\; |x_2 \,+\, x_1|^2\ .$$

It follows, from Lyapunov's second method, that the control law locally asymptotically stabilizes the equilibrium of the system. Furthermore, local asymptotic stability is preserved in the presence of any perturbation which does not change the approximating homogeneous function, i.e., in the presence of perturbations which are dominated by the linear part (see P2 in section 2.2).

[2]If $\phi_0(x)+\zeta_0(x) = 0$ (resp., $\phi_\infty(x)+\zeta_\infty(x) = 0$), the proof can be completed by replacing ζ with 2ζ.

In the context of homogeneity in the ∞-limit, we have the following result.

PROPOSITION 2.18. *Consider a homogeneous in the ∞-limit vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ with associated triple $(r_\infty, \mathfrak{d}_\infty, f_\infty)$. If the origin of the system*

$$\dot{x} \;=\; f_\infty(x)$$

is globally asymptotically stable, then there exists an invariant compact subset of $\mathbb{R}^n$, denoted $\mathcal{C}_\infty$, which is globally asymptotically stable[3] *for the system*

$$\dot{x} \;=\; f(x) \;.$$

The proof of the proposition is given in Appendix D.

As in the case of homogeneity in the 0-limit, this property can be used to design a feedback, ensuring boundedness of solutions.

Example 2.19. Consider the system (1.1) with $0 < q < p < 2$ and the control law

$$u \;=\; -\frac{1}{2-p}\, x_1^{\frac{p-1}{2-p}} x_2 \,-\, x_1^{\frac{p}{2-p}} \,-\, c_\infty\, x_2^p - \left(x_2 \,+\, x_1^{\frac{1}{2-p}}\right)^p \;. \tag{2.6}$$

This control law is such that the closed-loop vector field is homogeneous in the ∞-limit with degree $\mathfrak{d}_\infty = p-1$, weight $(2-p,1)$, and associated vector field $f_\infty(x_1,x_2) = (x_2, -\frac{1}{2-p}\, x_1^{\frac{p-1}{2-p}} x_2 - x_1^{\frac{p}{2-p}} - (x_2 + x_1^{\frac{1}{2-p}})^p)^T$. For the homogeneous Lyapunov function of degree 2,

$$V_\infty(x_1,x_2) \;=\; \frac{2-p}{2}\,|x_1|^{\frac{2}{2-p}} \;+\; \frac{1}{2}\left|x_2 \,+\, x_1^{\frac{1}{2-p}}\right|^2 \;,$$

we get

$$\frac{\partial V_\infty}{\partial x}(x)\, f_\infty(x) \;=\; -|x_1|^{\frac{p+1}{2-p}} \;-\; \left|x_2 \,+\, x_1^{\frac{1}{2-p}}\right|^{p+1} \;.$$

It follows that the control law (2.6) guarantees boundedness of the solutions of the closed-loop system. Furthermore, boundedness of solutions is preserved in the presence of any perturbation which does not change the approximating homogeneous function in the ∞-limit, i.e., in the presence of perturbations which are negligible with respect to the dominant homogeneous part (see P2 in section 2.2).

The key step in the proof of Propositions 2.16 and 2.18 is the converse Lyapunov theorem given by Rosier in [29]. This result can also be extended to the case of homogeneity in the bi-limit.

THEOREM 2.20 (homogeneous in the bi-limit Lyapunov functions). *Consider a homogeneous in the bi-limit vector field $f : \mathbb{R}^n \to \mathbb{R}^n$, with associated triples $(r_\infty, \mathfrak{d}_\infty, f_\infty)$ and $(r_0, \mathfrak{d}_0, f_0)$ such that the origins of the systems*

$$\dot{x} = f(x), \quad \dot{x} = f_\infty(x), \quad \dot{x} = f_0(x) \tag{2.7}$$

are globally asymptotically stable equilibria. Let d_{V_∞} and d_{V_0} be real numbers such that $d_{V_\infty} > \max_{1\leq i\leq n} r_{\infty,i}$ and $d_{V_0} > \max_{1\leq i\leq n} r_{0,i}$. Then there exists a C^1, positive definite, and proper function $V : \mathbb{R}^n \to \mathbb{R}_+$ such that, for each i in $\{1,\ldots,n\}$,

[3]See [34] for the definition of global asymptotical stability for invariant compact sets.

the function $x \mapsto \frac{\partial V}{\partial x_i}$ *is homogeneous in the bi-limit with associated triples* $(r_0, d_{V_0} - r_{0,i}, \frac{\partial V_0}{\partial x_i})$ *and* $(r_\infty, d_{V_\infty} - r_{\infty,i}, \frac{\partial V_\infty}{\partial x_i})$, *and the functions* $x \mapsto \frac{\partial V}{\partial x}(x)\, f(x)$, $x \mapsto \frac{\partial V_0}{\partial x}(x)\, f_0(x)$, *and* $x \mapsto \frac{\partial V_\infty}{\partial x}(x)\, f_\infty(x)$ *are negative definite.*

The proof is given in Appendix E. A direct consequence of this result is an input-to-state stability (ISS) property with respect to disturbances (see [31]). To illustrate this property, consider the system with exogenous disturbance $\delta = (\delta_1, \dots, \delta_m)$ in $\mathbb{R}^m$,

$$\dot{x} = f(x, \delta) \ , \tag{2.8}$$

with $f : \mathbb{R}^n \times \mathbb{R}^m$ a continuous vector field homogeneous in the bi-limit with associated triples $(\mathfrak{d}_0, (r_0, \mathfrak{r}_0), f_0)$ and $(\mathfrak{d}_\infty, (r_\infty, \mathfrak{r}_\infty), f_\infty)$, where $\mathfrak{r}_0$ and $\mathfrak{r}_\infty$ in $(\mathbb{R}_+ \setminus \{0\})^m$ are the weights associated with the disturbance δ.

COROLLARY 2.21 (ISS property). *If the origins of the systems*

$$\dot{x} = f(x, 0), \qquad \dot{x} = f_0(x, 0), \qquad \dot{x} = f_\infty(x, 0)$$

are globally asymptotically stable equilibria, then under the hypotheses of Theorem 2.20 *the function* V *given by Theorem* 2.20 *satisfies,*[4] *for all* $\delta = (\delta_1, \dots, \delta_m)$ *in* $\mathbb{R}^m$ *and* x *in* $\mathbb{R}^n$,

$$\begin{aligned} \frac{\partial V}{\partial x}(x)\, f(x, \delta) \ \le \ -c_V\, \mathfrak{H}\left(V(x)^{\frac{d_{V_0}+d_0}{d_{V_0}}},\ V(x)^{\frac{d_{V_\infty}+d_\infty}{d_{V_\infty}}} \right) & \\ + c_\delta \sum_{j=1}^{m} \mathfrak{H}\left(|\delta_j|^{\frac{d_{V_0}+d_0}{\mathfrak{r}_{0,j}}},\ |\delta_j|^{\frac{d_{V_\infty}+d_\infty}{\mathfrak{r}_{\infty,j}}} \right) & \ , \end{aligned} \tag{2.9}$$

where c_V *and* c_δ *are positive real numbers.*

In other words, system (2.8) with δ as input satisfies an ISS property. The proof of this corollary is given in Appendix F.

Finally, we have also the following small-gain result for homogeneous in the bi-limit vector fields.

COROLLARY 2.22 (small-gain). *Under the hypotheses of Corollary* 2.21, *there exists a real number* $c_G > 0$ *such that, for each class* $\mathcal{K}$ *function* γ_z *and* $\mathcal{KL}$ *function* β_δ, *there exists a class* $\mathcal{KL}$ *function* β_x *such that, for each function* $t \in [0, T) \mapsto (x(t), \delta(t), z(t))$, $T \le +\infty$, *with* x C^1 *and* δ *and* z *continuous, which satisfy* (2.8) *on* $[0, T)$ *and, for all* $0 \le s \le t \le T$,

$$|z(t)| \le \max\left\{ \beta_\delta\Big(|z(s)|, t-s\Big),\ \sup_{s \le \kappa \le t} \gamma_z(|x(\kappa)|) \right\} \ , \tag{2.10}$$

$$|\delta_i(t)| \le \max\left\{ \beta_\delta\Big(|z(s)|, t-s\Big),\ c_G \sup_{s \le \kappa \le t} \left\{ \mathfrak{H}\left(|x(\kappa)|_{r_0}^{\mathfrak{r}_{0,i}}, |x(\kappa)|_{r_\infty}^{\mathfrak{r}_{\infty,i}} \right) \right\} \right\} \ , \tag{2.11}$$

we have

$$|x(t)| \le \beta_x(|(x(s), z(s))|, t-s), \qquad 0 \le s \le t \le T \ . \tag{2.12}$$

[4]The function $\mathfrak{H}$ is defined in (1.6).

The proof is given in Appendix G.

Example 2.23. An interesting case, which can be dealt with by Corollary 2.22, is when the δ_i's are outputs of auxiliary systems with state z_i in $\mathbb{R}^{n_i}$, i.e.,

$$\delta_i(t) := \delta_i(z_i(t), x(t)), \qquad \dot{z}_i = g_i(z_i, x) . \tag{2.13}$$

It can be checked that the bounds (2.11) and (2.10) are satisfied by all the solutions of (2.8) and (2.13) if there exist positive definite and radially unbounded functions $Z_i : \mathbb{R}^{n_i} \to \mathbb{R}_+$; class $\mathcal{K}$ functions ω_1, ω_2, and ω_3; and a positive real number ϵ in $(0, 1)$ such that for all x in $\mathbb{R}^n$, for all i in $\{1, \dots, m\}$, and z_i in $\mathbb{R}^{n_i}$, we have

$$|\delta_i(z_i, x)| \leq \omega_1(x) + \omega_2(Z_i(z_i)), \qquad \frac{\partial Z_i}{\partial z_i}(z_i)\, g_i(z_i, x) \leq -Z_i(z_i) + \omega_3(|x|) ,$$
$$\omega_1(x) + \omega_2\left([1+\epsilon]\, \omega_3(|x|)\right) \leq c_G \mathfrak{H}\left(|x|_{r_0}^{\mathfrak{r}_{0,i}}, |x|_{r_\infty}^{\mathfrak{r}_{\infty,i}}\right) .$$

Another important result exploiting Theorem 2.20 deals with finite time convergence of solutions toward a globally asymptotically stable equilibrium (see [4]). It is well known that when the origin of the homogeneous approximation in the 0-limit is globally asymptotically stable with a strictly negative degree, then solutions converge to the origin in finite time (see [3]). We extend this result by showing that if, furthermore, the origin of the homogeneous approximation in the ∞-limit is globally asymptotically stable with strictly positive degree, then the convergence time doesn't depend on the initial condition. This is expressed by the following corollary.

COROLLARY 2.24 (uniform and finite time convergence). *Under the hypotheses of Theorem* 2.20*, if we have $\mathfrak{d}_\infty > 0 > \mathfrak{d}_0$, then all solutions of the system $\dot{x} = f(x)$ converge in finite time to the origin, uniformly in the initial condition.*

The proof is given in Appendix H.

3. Recursive observer design for a chain of integrators.

The notion of homogeneity in the bi-limit is instrumental in introducing a new observer design method. Throughout this section we consider a chain of integrators, with state $\mathfrak{X}_n = (\mathcal{x}_1, \dots, \mathcal{x}_n)$ in $\mathbb{R}^n$, namely,

$$\dot{\mathcal{x}}_1 = \mathcal{x}_2 , \ \dots\ , \dot{\mathcal{x}}_n = u, \quad \text{or in compact form,} \quad \dot{\mathfrak{X}}_n = \mathcal{S}_n \mathfrak{X}_n + B_n u , \tag{3.1}$$

where $\mathcal{S}_n$ is the shift matrix of order n, i.e., $\mathcal{S}_n \mathfrak{X}_n = (\mathcal{x}_2, \dots, \mathcal{x}_n, 0)^T$ and $B_n = (0, \dots, 0, 1)^T$. By selecting arbitrary vector field degrees $\mathfrak{d}_0$ and $\mathfrak{d}_\infty$ in $(-1, \frac{1}{n-1})$, we see that, to possibly obtain homogeneity in the bi-limit of the associated vector field, we must choose the weights $r_0 = (r_{0,1}, \dots, r_{0,n})$ and $r_\infty = (r_{\infty,1}, \dots, r_{\infty,n})$ as

$$\begin{aligned} r_{0,n} &= 1 , \quad r_{0,i} = r_{0,i+1} - \mathfrak{d}_0 = 1 - \mathfrak{d}_0\,(n-i) , \\ r_{\infty,n} &= 1 , \quad r_{\infty,i} = r_{\infty,i+1} - \mathfrak{d}_\infty = 1 - \mathfrak{d}_\infty\,(n-i) . \end{aligned} \tag{3.2}$$

The goal of this section is to introduce a global homogeneous in the bi-limit observer for the system (3.1). This design follows a recursive method, which constitutes one of the main contributions of this paper.

The idea of designing an observer recursively starting from $\mathcal{x}_n$ and going backwards towards $\mathcal{x}_1$ is not new. It can be found, for instance, in [28, 26, 23, 30, 35] and in [7, Lemma 6.2.1]. Nevertheless, the procedure we propose is new and extends the results in [23, Lemmas 1 and 2] to the homogeneous in the bi-limit case.

Also, as opposed to what is proposed in [28, 26],[5] this observer is an exact observer (with any input u) for a chain of integrators. The observer is given by the system[6]

$$\dot{\hat{\mathfrak{X}}}_n \;=\; \mathcal{S}_n\,\hat{\mathfrak{X}}_n \,+\, B_n\,u \;+\, K_1(\hat{x}_1 - x_1)\;, \tag{3.3}$$

with state $\hat{\mathfrak{X}}_n \;=\; (\hat{x}_1,\ldots,\hat{x}_n)$, and where $K_1 \;: \mathbb{R}^n \to \mathbb{R}^n$ is a homogeneous in the bi-limit vector field with weights r_0, r_∞ and degrees $\mathfrak{d}_0$, $\mathfrak{d}_\infty$. The output injection vector field K_1 has to be selected such that the origin is a globally asymptotically stable equilibrium for the system

$$\dot{E}_1 \;=\; \mathcal{S}_n\,E_1 \,+\, K_1(e_1), \qquad E_1 \;=\; (e_1,\ldots,e_n)^T\;, \tag{3.4}$$

and also for its homogeneous approximations. The construction of K_1 is performed via a recursive procedure whose induction argument is as follows.

Consider the system on $\mathbb{R}^{n-i}$ given by

$$\dot{E}_{i+1} \;=\; \mathcal{S}_{n-i}\,E_{i+1} + K_{i+1}(e_{i+1}), \qquad E_{i+1} \;=\; (e_{i+1},\ldots,e_n)^T\;, \tag{3.5}$$

with $\mathcal{S}_{n-i}$ the shift matrix of order $n-i$, i.e., $\mathcal{S}_{n-i}\,E_{i+1} \;=\; (e_{i+2},\ldots,e_n,0)^T$, and $K_{i+1} \;: \mathbb{R}^{n-i} \to \mathbb{R}^{n-i}$ a homogeneous in the bi-limit vector field, whose associated triples are $((r_{0,i+1},\ldots,r_{0,n}),\mathfrak{d}_0,K_{i+1,0})$ and $((r_{\infty,i+1},\ldots,r_{\infty,n}),\mathfrak{d}_\infty,K_{i+1,\infty})$.

Theorem 3.1 (homogeneous in the bi-limit observer design). *Consider the system* (3.5) *and its homogeneous approximation at infinity and around the origin,*

$$\dot{E}_{i+1} = \mathcal{S}_{n-i}\,E_{i+1} + K_{i+1,0}(e_{i+1}), \quad \dot{E}_{i+1} = \mathcal{S}_{n-i}\,E_{i+1} + K_{i+1,\infty}(e_{i+1})\;.$$

Suppose the origin is a globally asymptotically stable equilibrium for these systems. Then there exists a homogeneous in the bi-limit vector field $K_i \;: \mathbb{R}^{n-i+1} \to \mathbb{R}^{n-i+1}$, with associated triples $((r_{0,i},\ldots,r_{0,n}),\mathfrak{d}_0,K_{i,0})$ and $((r_{\infty,i},\ldots,r_{\infty,n}),\mathfrak{d}_\infty,K_{i,\infty})$, such that the origin is a globally asymptotically stable equilibrium for the systems

$$\begin{aligned} &\dot{E}_i = \mathcal{S}_{n-i+1}\,E_i + K_i(e_i)\;,\\ &\dot{E}_i = \mathcal{S}_{n-i+1}\,E_i + K_{i,0}(e_i), \qquad E_i \;=\; (e_i,\ldots,e_n)^T\;,\\ &\dot{E}_i = \mathcal{S}_{n-i+1}\,E_i + K_{i,\infty}(e_i)\;. \end{aligned} \tag{3.6}$$

Proof. We prove this result in two steps. First, we define a homogeneous in the bi-limit Lyapunov function. Then we construct the vector field K_i, depending on a parameter ℓ which, if sufficiently large, renders negative definite the derivative of this Lyapunov function along the solutions of the system.

Step 1. *Definition of the Lyapunov function.* Let d_{W_0} and d_{W_∞} be positive real numbers satisfying

$$d_{W_0} > 2\,\max_{1\leq j\leq n} r_{0,j} + \mathfrak{d}_0, \qquad d_{W_\infty} > 2\,\max_{1\leq j\leq n} r_{\infty,j} + \mathfrak{d}_\infty\;, \tag{3.7}$$

and

$$\frac{d_{W_\infty}}{r_{\infty,i}} \;\geq\; \frac{d_{W_0}}{r_{0,i}}\;. \tag{3.8}$$

[5]Note the term x_i in (3.15) of [28], for instance.

[6]To simplify the presentation, we use the compact notation $K_1(\hat{x}_1 - x_1)$ for $K_1(\hat{x}_1 - x_1, 0, \ldots, 0)$.

The selection (3.2) implies $r_{0,j} + \mathfrak{d}_0 > 0$ and $r_{\infty,j} + \mathfrak{d}_\infty > 0$ for each j in $\{1, \ldots, n\}$. Hence,

$$d_{W_0} > \max_{1 \le j \le n} r_{0,j}, \qquad d_{W_\infty} > \max_{1 \le j \le n} r_{\infty,j} \ ,$$

and we can invoke Theorem 2.20 for the system (3.4) and its homogeneous approximations given in (3.5). This implies that there exists a C^1, positive definite, and proper function $W_{i+1} \ : \mathbb{R}^{n-i} \to \mathbb{R}_+$ such that, for each j in $\{i+1, \ldots, n\}$, the function $\frac{\partial W_{i+1}}{\partial e_j}$ is homogeneous in the bi-limit with associated triples

$$\left((r_{0,i+1}, \ldots, r_{0,n}), d_{W_0} - r_{0,j}, \frac{\partial W_{i+1,0}}{\partial e_j} \right) \quad \text{and}$$

$$\left((r_{\infty,i+1}, \ldots, r_{\infty,n}), d_{W_\infty} - r_{\infty,j}, \frac{\partial W_{i+1,\infty}}{\partial e_j} \right).$$

Moreover, for all $E_{i+1} \in \mathbb{R}^{n-i} \setminus \{0\}$, we have

$$\begin{aligned} \frac{\partial W_{i+1}}{\partial E_{i+1}}(E_{i+1}) \ (\mathcal{S}_{n-i} \, E_{i+1} + K_{i+1}(e_{i+1})) &< 0 \ , \\ \frac{\partial W_{i+1,0}}{\partial E_{i+1}}(E_{i+1}) \ (\mathcal{S}_{n-i} \, E_{i+1} + K_{i+1,0}(e_{i+1})) &< 0 \ , \\ \frac{\partial W_{i+1,\infty}}{\partial E_{i+1}}(E_{i+1}) \ (\mathcal{S}_{n-i} \, E_{i+1} + K_{i+1,\infty}(e_{i+1})) &< 0 \ . \end{aligned} \tag{3.9}$$

Consider the function $q_i : \mathbb{R} \to \mathbb{R}$ defined as

$$q_i(s) \ = \left\{ \begin{array}{ll} \frac{r_{0,i}}{r_{0,i}+\mathfrak{d}_0} \ s^{\frac{r_{0,i}+\mathfrak{d}_0}{r_{0,i}}}, & |s| \le 1 \ , \\ \frac{r_{\infty,i}}{r_{\infty,i}+\mathfrak{d}_\infty} s^{\frac{r_{\infty,i}+\mathfrak{d}_\infty}{r_{\infty,i}}} + \frac{r_{0,i}}{r_{0,i}+\mathfrak{d}_0} - \frac{r_{\infty,i}}{r_{\infty,i}+\mathfrak{d}_\infty}, & |s| \ge 1 \ . \end{array} \right. \tag{3.10}$$

Since we have $0 < r_{0,i} + \mathfrak{d}_0$ and $0 < r_{\infty,i} + \mathfrak{d}_\infty$, this function is well defined and continuous on $\mathbb{R}$, strictly increasing and onto, and C^1 on $\mathbb{R} \setminus \{0\}$. Furthermore, it is by construction homogeneous in the bi-limit with approximating continuous functions $\frac{r_{0,i}}{r_{0,i}+\mathfrak{d}_0} s^{\frac{r_{0,i}+\mathfrak{d}_0}{r_{0,i}}}$ and $\frac{r_{\infty,i}+\mathfrak{d}_\infty}{r_{\infty,i}} s^{\frac{r_{\infty,i}+\mathfrak{d}_\infty}{r_{\infty,i}}}$. The inverse function q_i^{-1} of q_i is defined as

$$q_i^{-1}(s) \ = \left\{ \begin{array}{ll} \left(\frac{r_{0,i}+\mathfrak{d}_0}{r_{0,i}} \ s \right)^{\frac{r_{0,i}}{r_{0,i}+\mathfrak{d}_0}}, & |s| \le \frac{r_{0,i}+\mathfrak{d}_0}{r_{0,i}} \ , \\ \left(\left(s - \frac{r_{0,i}}{r_{0,i}+\mathfrak{d}_0} + \frac{r_{\infty,i}}{r_{\infty,i}+\mathfrak{d}_\infty} \right) \frac{r_{\infty,i}+\mathfrak{d}_\infty}{r_{\infty,i}} \right)^{\frac{r_{\infty,i}}{r_{\infty,i}+\mathfrak{d}_\infty}}, & |s| \ge \frac{r_{0,i}+\mathfrak{d}_0}{r_{0,i}} \ . \end{array} \right.$$

By (3.8), the function

$$s \ \mapsto \ q_i^{-1}(s)^{\frac{d_{W_0}-r_{0,i}}{r_{0,i}}} \ + \ q_i^{-1}(s)^{\frac{d_{W_\infty}-r_{\infty,i}}{r_{\infty,i}}} \tag{3.11}$$

is homogeneous in the bi-limit with associated approximating functions $(\frac{r_{0,i}+\mathfrak{d}_0}{r_{0,i}} s)^{\frac{d_{W_0}-r_{0,i}}{r_{0,i}+\mathfrak{d}_0}}$ and $(\frac{r_{\infty,i}+\mathfrak{d}_\infty}{r_{\infty,i}} s)^{\frac{d_{W_\infty}-r_{\infty,i}}{r_{\infty,i}+\mathfrak{d}_\infty}}$. Furthermore, by (3.7), it is C^1 on $\mathbb{R}$, and its derivative is homogeneous in the bi-limit with continuous approximating functions

$$s \mapsto \frac{d_{W_0}-r_{0,i}}{r_{0,i}} \left| \frac{d_{W_0}-r_{0,i}}{r_{0,i}+\mathfrak{d}_0} s \right|^{\frac{d_{W_0}-2r_{0,i}-\mathfrak{d}_0}{r_{0,i}+\mathfrak{d}_0}} \quad \text{and} \quad s \mapsto \frac{d_{W_\infty}-r_{\infty,i}}{r_{\infty,i}} \left| \frac{d_{W_\infty}-r_{\infty,i}}{r_{\infty,i}+\mathfrak{d}_\infty} s \right|^{\frac{d_{W_\infty}-2r_{\infty,i}-\mathfrak{d}_\infty}{r_{\infty,i}+\mathfrak{d}_\infty}} \ .$$

Let $\mathfrak{W}_i : \mathbb{R}^{n-i+1} \to \mathbb{R}_+$ be defined by

$$\mathfrak{W}_i(E_{i+1}, s) \;=\; W_{i+1}(E_{i+1}) \;+\; \int_{q_i^{-1}(e_{i+1})}^{s} \left(h^{\frac{d_{W_0} - r_{0,i}}{r_{0,i}}} \;+\; h^{\frac{d_{W_\infty} - r_{\infty,i}}{r_{\infty,i}}} \right) dh$$
$$- \int_{q_i^{-1}(e_{i+1})}^{s} \left(q_i^{-1}(e_{i+1})^{\frac{d_{W_0} - r_{0,i}}{r_{0,i}}} + q_i^{-1}(e_{i+1})^{\frac{d_{W_\infty} - r_{\infty,i}}{r_{\infty,i}}} \right) dh \; .$$

This function is C^1, and by (3.8), Proposition 2.12 yields that it is homogeneous in the bi-limit with weights $(r_{0,i+1}, \dots, r_{0,n})$ and $(r_{\infty,i+1}, \dots, r_{\infty,n})$ for E_{i+1}, $r_{0,i}$ and $r_{\infty,i}$ for s, and degrees d_{W_0} and d_{W_∞}. Furthermore, for each j in $\{i+1, \dots, n\}$, the functions $\frac{\partial \mathfrak{W}_i}{\partial e_j}(E_{i+1}, s)$ are also homogeneous in the bi-limit with the same weights and with degrees $d_{W_0} - r_{0,j}$ and $d_{W_\infty} - r_{\infty,j}$.

Step 2. *Construction of the vector field K_i.* Given a positive real number ℓ, we define the vector field $K_i : \mathbb{R}^{n-i} \to \mathbb{R}^{n-i}$ as

$$K_i(e_i) = \begin{pmatrix} -q_i(\ell e_i) \\ K_{i+1}(q_i(\ell e_i)) \end{pmatrix} .$$

By Proposition 2.10 and the properties we have established for q_i, K_i is a homogeneous in the bi-limit vector field. We show now that selecting ℓ large enough yields the asymptotic stability properties. To begin with, note that for all $E_i = (E_{i+1}, e_i)$ in $\mathbb{R}^{n-i}$,

$$\frac{\partial \mathfrak{W}_i(E_{i+1}, \ell e_i)}{\partial E_i}(E_i)\,(\mathcal{S}_{n-i+1}\, E_i + K_i(e_i)) \;\le\; T_1(E_{i+1}, \ell e_i) \;-\; \ell\, T_2(E_{i+1}, \ell e_i) \; ,$$

with the functions T_1 and T_2 defined as

$$T_1(E_{i+1}, \vartheta_i) \;=\; \frac{\partial \mathfrak{W}_i}{\partial E_{i+1}}(E_{i+1}, \vartheta_i) \;\; (\mathcal{S}_{n-i}\, E_{i+1} + K_{i+1}(q_i(\vartheta_i))) \;\; ,$$
$$T_2(E_{i+1}, \vartheta_i) \;=\; \left(\vartheta_i^{\frac{d_{W_0} - r_{0,i}}{r_{0,i}}} - q_i^{-1}(e_{i+1})^{\frac{d_{W_0} - r_{0,i}}{r_{0,i}}} + \vartheta_i^{\frac{d_{W_\infty} - r_{\infty,i}}{r_{\infty,i}}} - q_i^{-1}(e_{i+1})^{\frac{d_{W_\infty} - r_{\infty,i}}{r_{\infty,i}}} \right)$$
$$\times (q_i(\vartheta_i) - e_{i+1}) \; .$$

These functions are homogeneous in the bi-limit with weights $(r_{\infty,i}, \dots, r_{\infty,n})$ and $(r_{0,i}, \dots, r_{0,n})$, degrees $\mathfrak{d}_0 + d_{W_0}$ and $\mathfrak{d}_\infty + d_{W_\infty}$, and continuous approximating functions

$$T_{1,0}(E_{i+1}, \vartheta_i) = \frac{\partial \mathfrak{W}_{i,0}}{\partial E_{i+1}}(E_{i+1}, \vartheta_i) \;\; (\mathcal{S}_{n-i}\, E_{i+1} + K_{i+1,0}(q_{i,0}(\vartheta_i))) \;\; ,$$
$$T_{1,\infty}(E_{i+1}, \vartheta_i) = \frac{\partial \mathfrak{W}_{i,\infty}}{\partial E_{i+1}}(E_{i+1}, \vartheta_i) \;\; (\mathcal{S}_{n-i}\, E_{i+1} + K_{i+1,\infty}(q_{i,\infty}(\vartheta_i))) \;\; ,$$
$$T_{2,0}(E_{i+1}, \vartheta_i) = \left(\vartheta_i^{\frac{d_{W_0} - r_{0,i}}{r_{0,i}}} - q_{i,0}^{-1}(e_{i+1})^{\frac{d_{W_0} - r_{0,i}}{r_{0,i}}} \right) (q_{i,0}(\vartheta_i) - e_{i+1}) \; ,$$

and

$$T_{2,\infty}(E_{i+1}, \vartheta_i) = \left(\vartheta_i^{\frac{d_{W_\infty} - r_{\infty,i}}{r_{\infty,i}}} - q_{i,\infty}^{-1}(e_{i+1})^{\frac{d_{W_\infty} - r_{\infty,i}}{r_{\infty,i}}} \right) (q_{i,\infty}(\vartheta_i) - e_{i+1}) \; .$$

As the function q_i^{-1} is continuous, strictly increasing and onto, the function

$$\vartheta_i^{\frac{d_{W_0}-r_{0,i}}{r_{0,i}}} - q_i^{-1}(e_{i+1})^{\frac{d_{W_0}-r_{0,i}}{r_{0,i}}} + \vartheta_i^{\frac{d_{W_\infty}-r_{\infty,i}}{r_{\infty,i}}} - q_i^{-1}(e_{i+1})^{\frac{d_{W_\infty}-r_{\infty,i}}{r_{\infty,i}}}$$

has a unique zero at $q_i(\vartheta_i) = e_{i+1}$ and has the same sign as $q_i(\vartheta_i) - e_{i+1}$. It follows that

$$\begin{array}{ll} T_2(E_{i+1}, \vartheta_i) \geq 0 & \forall\ (E_{i+1}, \vartheta_i) \in \mathbb{R}^{n-i} , \\ T_2(E_{i+1}, \vartheta_i) = 0 & \Rightarrow \quad q_i(\vartheta_i) = e_{i+1} . \end{array}$$

On the other hand, for all $E_i \neq 0$,

$$T_1(E_{i+1}, q_i^{-1}(e_{i+1})) = \frac{\partial W_{i+1}}{\partial E_{i+1}}(E_{i+1})\ (\mathcal{S}_{n-i}\, E_{i+1} + K_{i+1}(e_{i+1})) < 0 .$$

Hence (3.9) yields

$$\begin{aligned} &\left\{(E_{i+1}, \vartheta_i) \in \mathbb{R}^{n-i+1} \setminus \{0\} \,:\, T_2(E_{i+1}, \vartheta_i) = 0\right\} \\ &\qquad\qquad \subseteq \quad \left\{(E_{i+1}, \vartheta_i) \in \mathbb{R}^{n-i+1} \,:\, T_1(E_{i+1}, \vartheta_i) < 0\right\} . \end{aligned}$$

By following the same argument, it can be shown that this property holds also for the homogeneous approximations, i.e.,

$$\begin{aligned} &\left\{(E_{i+1}, \vartheta_i) \in \mathbb{R}^{n-i+1} \setminus \{0\} \,:\, T_{2,0}(E_{i+1}, \vartheta_i) = 0\right\} \\ &\qquad\qquad \subseteq \quad \left\{(E_{i+1}, \vartheta_i) \in \mathbb{R}^{n-i+1} \,:\, T_{1,0}(E_{i+1}, \vartheta_i) < 0\right\} , \\ &\left\{(E_{i+1}, \vartheta_i) \in \mathbb{R}^{n-i+1} \setminus \{0\} \,:\, T_{2,\infty}(E_{i+1}, \vartheta_i) = 0\right\} \\ &\qquad\qquad \subseteq \quad \left\{(E_{i+1}, \vartheta_i) \in \mathbb{R}^{n-i+1} \,:\, T_{1,\infty}(E_{i+1}, \vartheta_i) < 0\right\} . \end{aligned}$$

Therefore, by Lemma 2.13, there exists ℓ^* such that, for all $\ell \geq \ell^*$ and all $(E_{i+1}, \vartheta_i) \neq 0$,

$$\begin{aligned} T_1(E_{i+1}, \vartheta_i) - \ell\, T_2(E_{i+1}, \vartheta_i) &< 0 , \\ T_{1,0}(E_{i+1}, \vartheta_i) - \ell\, T_{2,0}(E_{i+1}, \vartheta_i) &< 0 , \\ T_{1,\infty}(E_{i+1}, \vartheta_i) - \ell\, T_{2,\infty}(E_{i+1}, \vartheta_i) &< 0 . \end{aligned}$$

This implies that the origin is a globally asymptotically stable equilibrium of the systems (3.6), which concludes the proof. ∎

To construct the function K_1, which defines the observer (3.3), it is sufficient to iterate the construction proposed in Theorem 3.1 starting from

$$K_n(e_n) = -\begin{cases} \frac{1}{1+\mathfrak{d}_0}\,(\ell_n e_n)^{1+\mathfrak{d}_0}, & |\ell_n e_n| \leq 1 , \\ \frac{1}{1+\mathfrak{d}_\infty}(\ell_n e_n)^{1+\mathfrak{d}_\infty} + \frac{1}{1+\mathfrak{d}_0} - \frac{1}{1+\mathfrak{d}_\infty}, & |\ell_n e_n| \geq 1 , \end{cases}$$

where ℓ_n is any strictly positive real number. Indeed, K_n is a homogeneous in the bi-limit vector field with approximating functions $K_{n,0}(e_n) = \frac{1}{1+\mathfrak{d}_0}\,(\ell_n e_n)^{1+\mathfrak{d}_0}$ and $K_{n,\infty}(e_n) = \frac{1}{1+\mathfrak{d}_\infty}(\ell_n e_n)^{1+\mathfrak{d}_\infty}$. This selection implies that the origin is a globally asymptotically stable equilibrium for the systems $\dot{e}_n = K_n(e_n)$, $\dot{e}_n = K_{n,0}(e_n)$, and $\dot{e}_n = K_{n,\infty}(e_n)$.

Consequently the assumptions of Theorem 3.1 are satisfied for $i+1=n$. We can apply it recursively up to $i=1$, obtaining the vector field K_1.

As a result of this procedure we obtain a homogeneous in the bi-limit observer, which globally asymptotically observes the state of the system (3.1), and also the state for its homogeneous approximations around the origin and at infinity. In other words, the origin is a globally asymptotically stable equilibrium of the systems

$$\dot{E}_1 = \mathcal{S}_n E_1 + K_1(e_1) \ , \quad \dot{E}_1 = \mathcal{S}_n E_1 + K_{1,0}(e_1) \ , \quad \dot{E}_1 = \mathcal{S}_n E_1 + K_{1,\infty}(e_1) \ . \tag{3.12}$$

Remark 3.2. Note that when $0 \leq \mathfrak{d}_0 \leq \mathfrak{d}_\infty$, we have $1 \leq \frac{r_{0,i}+\mathfrak{d}_0}{r_{0,i}} \leq \frac{r_{\infty,i}+\mathfrak{d}_\infty}{r_{\infty,i}}$ for $i=1,\ldots,n$ and we can replace the function q_i in (3.10) by the simpler function

$$q_i(s) \;=\; s^{\frac{r_{0,i}+\mathfrak{d}_0}{r_{0,i}}} \;+\; s^{\frac{r_{\infty,i}+\mathfrak{d}_\infty}{r_{\infty,i}}} \ ,$$

which has been used already in [1].

Example 3.3. Consider a chain of integrators of dimension two, with the following weights and degrees:

$$(r_0,\, \mathfrak{d}_0) \;=\; \Big((2-q,1),\, q-1\Big), \qquad (r_\infty,\, \mathfrak{d}_\infty) \;=\; \Big((2-p,1),\, p-1\Big) \ .$$

When $q \geq p$ (i.e., $\mathfrak{d}_0 \leq \mathfrak{d}_\infty$), by following the above recursive observer design we obtain two positive real numbers ℓ_1 and ℓ_2 such that the system

$$\dot{\hat{x}}_1 \;=\; \hat{x}_2 - q_1(\ell_1 e_1) \ , \quad \dot{\hat{x}}_2 \;=\; u - q_2(\ell_2\, q_1(\ell_1 e_1)) \ , \ e_1 \;=\; \hat{x}_1 - y \ ,$$

with

$$q_2(s) = \begin{cases} \frac{1}{q}\ s^q, \ |s|\leq 1, \\ \frac{1}{p} s^p + \frac{1}{q} - \frac{1}{p}, \ |s| \geq 1, \end{cases} \qquad q_1(s) = \begin{cases} (2-q)\ s^{\frac{1}{2-q}}, \ |s| \leq 1, \\ (2-p) s^{\frac{1}{2-p}} + p - q, \ |s| \geq 1, \end{cases} \tag{3.13}$$

is a global observer for the system $\dot{x}_1 = x_2,\ \dot{x}_2 = u,\ y = x_1$. Furthermore, its homogeneous approximations around the origin and at infinity are also global observers for the same system.

4. Recursive design of a homogeneous in the bi-limit state feedback. It is well known that the system (3.1) can be rendered homogeneous by using a stabilizing homogeneous state feedback which can be designed by backstepping (see [21, 25, 19, 26, 33, 10], for instance). We show in this section that this property can be extended to the case of homogeneity in the bi-limit. More precisely, we show that there exists a homogeneous in the bi-limit function ϕ_n such that the system (3.1) with $u = \phi_n(\mathfrak{X}_n)$ is homogeneous in the bi-limit, with weights r_0 and r_∞ and degrees $\mathfrak{d}_0$ and $\mathfrak{d}_\infty$. Furthermore, its origin and the origin of the approximating systems in the 0-limit and in the ∞-limit are globally asymptotically stable equilibria.

To design the state feedback we follow the approach of Praly and Mazenc [25]. To this end, consider the auxiliary system with state $\mathfrak{X}_i = (x_1,\ldots,x_i)$ in $\mathbb{R}^i$, $1 \leq i < n$, and dynamics

$$\dot{x}_1 \;=\; x_2, \ldots, \dot{x}_i \;=\; u \quad \text{or in compact form} \quad \dot{\mathfrak{X}}_i \;=\; \mathcal{S}_i\, \mathfrak{X}_i \;+\; B_i\, u \ , \tag{4.1}$$

where u is the input in $\mathbb{R}$, $\mathcal{S}_i$ is the shift matrix of order i, i.e., $\mathcal{S}_i\, \mathfrak{X}_i = (x_2,\ldots,x_i,0)^T$, and $B_i = (0,\ldots,1)^T$ is in $\mathbb{R}^i$. We show that, if there exists a homogeneous in the bi-limit stabilizing control law for the origin of the system (4.1), then there is one for the origin of the system with state $\mathfrak{X}_{i+1} = (x_1, \ldots, x_{i+1})$ in $\mathbb{R}^{i+1}$ defined by

$$\dot{x}_1 = x_2, \ldots, \dot{x}_{i+1} = u, \text{ i.e.,} \qquad \dot{\mathfrak{X}}_{i+1} = \mathcal{S}_{i+1}\,\mathfrak{X}_{i+1} + B_{i+1}\,u \,. \tag{4.2}$$

Let $\mathfrak{d}_0$ and $\mathfrak{d}_\infty$ be in $(-1, \frac{1}{n-1})$ and consider the weights and degrees defined in (3.2).

THEOREM 4.1 (homogeneous in the bi-limit backstepping). *Suppose there exists a homogeneous in the bi-limit function $\phi_i : \mathbb{R}^i \to \mathbb{R}$ with associated triples $(r_0, \mathfrak{d}_0 + r_{0,i}, \phi_{i,0})$ and $(r_\infty, \mathfrak{d}_\infty + r_{\infty,i}, \phi_{i,\infty})$ such that the following hold:*

1. *There exists $\alpha_i \geq 1$ such that the function $\psi_i(\mathfrak{X}_i) = \phi_i(\mathfrak{X}_i)^{\alpha_i}$ is C^1 and for each j in $\{1, \ldots, i\}$ the function $\frac{\partial \psi_i}{\partial x_j}$ is homogeneous in the bi-limit with weights $(r_{0,1}, \ldots, r_{0,i})$, $(r_{\infty,1}, \ldots, r_{\infty,i})$, degrees $\alpha_i(r_{0,i} + \mathfrak{d}_0) - r_{0,j}$, $\alpha_i(r_{\infty,i} + \mathfrak{d}_\infty) - r_{\infty,j}$, and approximating functions $\frac{\partial \psi_{i0}}{\partial x_j}$, $\frac{\partial \psi_{i\infty}}{\partial x_j}$.*
2. *The origin is a globally asymptotically stable equilibrium of the systems*

$$\dot{\mathfrak{X}}_i = \mathcal{S}_i\,\mathfrak{X}_i + B_i\,\phi_i(\mathfrak{X}_i)\ , \quad \dot{\mathfrak{X}}_i = \mathcal{S}_i\,\mathfrak{X}_i + B_i\,\phi_{i,0}(\mathfrak{X}_i)\ , \quad \dot{\mathfrak{X}}_i = \mathcal{S}_i\,\mathfrak{X}_i + B_i\,\phi_{i,\infty}(\mathfrak{X}_i)\ . \tag{4.3}$$

Then there exists a homogeneous in the bi-limit function $\phi_{i+1} : \mathbb{R}^{i+1} \to \mathbb{R}$ with associated triples $(r_0, \mathfrak{d}_0 + r_{0,i+1}, \phi_{i+1,0})$ and $(r_\infty, \mathfrak{d}_\infty + r_{\infty,i+1}, \phi_{i+1,\infty})$ such that the same properties hold, i.e.,

1. *there exists a real number $\alpha_{i+1} > 1$ such that the function $\psi_{i+1}(\mathfrak{X}_{i+1}) = \phi_{i+1}(\mathfrak{X}_{i+1})^{\alpha_{i+1}}$ is C^1 and for each j in $\{1, \ldots, i+1\}$ the function $\frac{\partial \psi_{i+1}}{\partial x_j}$ is homogeneous in the bi-limit with weights $(r_{0,1}, \ldots, r_{0,i+1})$, $(r_{\infty,1}, \ldots, r_{\infty,i+1})$, degrees $\alpha_{i+1}(r_{0,i+1} + \mathfrak{d}_0) - r_{0,j}$, $\alpha_{i+1}(r_{\infty,i+1} + \mathfrak{d}_\infty) - r_{\infty,j}$, and approximating functions $\frac{\partial \psi_{i+1,0}}{\partial x_j}$, $\frac{\partial \psi_{i+1,\infty}}{\partial x_j}$;*
2. *the origin is a globally asymptotically stable equilibrium of the systems*

$$\begin{aligned} \mathfrak{X}_{i+1} &= \mathcal{S}_{i+1}\,\mathfrak{X}_{i+1} + B_{i+1}\,\phi_{i+1}(\mathfrak{X}_{i+1})\ , \\ \mathfrak{X}_{i+1} &= \mathcal{S}_{i+1}\,\mathfrak{X}_{i+1} + B_{i+1}\,\phi_{i+1,0}(\mathfrak{X}_{i+1})\ , \\ \mathfrak{X}_{i+1} &= \mathcal{S}_{i+1}\,\mathfrak{X}_{i+1} + B_{i+1}\,\phi_{i+1,\infty}(\mathfrak{X}_{i+1})\ . \end{aligned} \tag{4.4}$$

Proof. We prove this result in three steps. First, we construct a homogeneous in the bi-limit Lyapunov function; then we define a control law parametrized by a real number k. Finally, we show that there exists k such that the time derivative, along the trajectories of systems (4.4), of the Lyapunov function and of its approximating functions is negative definite.

Step 1. *Construction of the Lyapunov function.* Let d_{V_0} and d_{V_∞} be positive real numbers satisfying

$$d_{V_0} > \max_{j \in \{1,\ldots,n\}} \{r_{0,j}\}, \qquad d_{V_\infty} > \max_{j \in \{1,\ldots,n\}} \{r_{\infty,j}\}\ , \tag{4.5}$$

and

$$\frac{d_{V_\infty}}{r_{\infty,i+1}} \geq \frac{d_{V_0}}{r_{0,i+1}} > 1 + \alpha_i\ . \tag{4.6}$$

With this selection, Theorem 2.20 gives the existence of a C^1, proper, and positive definite function $V_i : \mathbb{R}^i \to \mathbb{R}_+$ such that, for each j in $\{1, \ldots, n\}$, the function $\frac{\partial V_i}{\partial x_j}$

is homogeneous in the bi-limit with weights $(r_{0,1},\ldots,r_{0,i})$, $(r_{\infty,1},\ldots,r_{\infty,i})$, degrees $d_{V_0}-r_{0,j}$, $d_{V_\infty}-r_{\infty,j}$, and approximating functions $\frac{\partial V_{i,0}}{\partial x_j}$, $\frac{\partial V_{i,\infty}}{\partial x_j}$. Moreover, we have for all $\mathfrak{X}_i\in\mathbb{R}^i\setminus\{0\}$,

$$\begin{aligned}\frac{\partial V_i}{\partial \mathfrak{X}_i}(\mathfrak{X}_i)\ [\mathcal{S}_i\,\mathfrak{X}_i+B_i\,\phi_i(\mathfrak{X}_i)]<0\ ,\\ \frac{\partial V_{i,0}}{\partial \mathfrak{X}_i}(\mathfrak{X}_i)\ [\mathcal{S}_i\,\mathfrak{X}_i+B_i\,\phi_{i,0}(\mathfrak{X}_i)]<0\ ,\\ \frac{\partial V_{i,\infty}}{\partial \mathfrak{X}_i}(\mathfrak{X}_i)\ [\mathcal{S}_i\,\mathfrak{X}_i+B_i\,\phi_{i,\infty}(\mathfrak{X}_i)]<0\ .\end{aligned}\tag{4.7}$$

Following [21], consider the Lyapunov function $V_{i+1}\ :\mathbb{R}^{i+1}\to\mathbb{R}_+$ defined by

$$\begin{aligned}V_{i+1}(\mathfrak{X}_{i+1})\ =\ V_i(\mathfrak{X}_i)\ +\ &\int_{\phi_i(\mathfrak{X}_i)}^{x_{i+1}}\left(h^{\frac{d_{V_0}-r_{0,i+1}}{r_{0,i+1}}}\ -\ \phi_i(\mathfrak{X}_i)^{\frac{d_{V_0}-r_{0,i+1}}{r_{0,i+1}}}\right)\,dh\\ &+\int_{\phi_i(\mathfrak{X}_i)}^{x_{i+1}}\left(h^{\frac{d_{V_\infty}-r_{\infty,i+1}}{r_{\infty,i+1}}}\ -\ \phi_i(\mathfrak{X}_i)^{\frac{d_{V_\infty}-r_{\infty,i+1}}{r_{\infty,i+1}}}\right)\,dh\ .\end{aligned}$$

This function is positive definite and proper. Furthermore, as d_{V_∞} and d_{V_0} satisfy (4.6), we have

$$\frac{d_{V_\infty}-r_{\infty,i+1}}{r_{\infty,i+1}}\geq\frac{d_{V_0}-r_{0,i+1}}{r_{0,i+1}}>\alpha_i\geq 1\ .$$

Since the function $\psi_i(\mathfrak{X}_i)=\phi_i(\mathfrak{X}_i)^{\alpha_i}$ is C^1, this inequality yields that the function V_{i+1} is C^1 . Finally, for each j in $\{1,\ldots,n\}$, the function $\frac{\partial V_{i+1}}{\partial x_j}$ is homogeneous in the bi-limit with associated triples

$$\left((r_{0,1},\ldots,r_{0,i+1}),d_{V_0}-r_{0,j},\frac{\partial V_{i+1,0}}{\partial x_j}\right),\quad\left((r_{\infty,1},\ldots,r_{\infty,i+1}),d_{V_\infty}-r_{\infty,j},\frac{\partial V_{i+1,\infty}}{\partial x_j}\right).$$

Step 2. *Definition of the control law.* Recall (1.6) and consider the function $\psi_{i+1}\ :\mathbb{R}^{i+1}\to\mathbb{R}$ defined by

$$\psi_{i+1}(\mathfrak{X}_{i+1})=-k\int_0^{x_{i+1}^{\alpha_i}-\phi_i(\mathfrak{X}_i)^{\alpha_i}}\mathfrak{H}\left(|s|^{\alpha_{i+1}\frac{\mathfrak{d}_0+r_{0,i+1}}{\alpha_i\,r_{0,i+1}}-1},\ |s|^{\alpha_{i+1}\frac{\mathfrak{d}_\infty+r_{\infty,i+1}}{\alpha_i\,r_{\infty,i+1}}-1}\right)\,ds\ ,$$

where k in $\mathbb{R}_+$ is a design parameter and α_{i+1} is selected as

$$\alpha_{i+1}\ \geq\ \max\left\{\frac{\alpha_i\,r_{0,i+1}}{\mathfrak{d}_0+r_{0,i+1}},\ \frac{\alpha_i\,r_{\infty,i+1}}{\mathfrak{d}_\infty+r_{\infty,i+1}},1\right\}\ .$$

ψ_{i+1} takes values with the same sign as $x_{i+1}-\phi_i(\mathfrak{X}_i)$, is C^1, and, by Proposition 2.12, is homogeneous in the bi-limit. Furthermore, by Proposition 2.10, for each j in $\{1,\ldots,i+1\}$, the function $\frac{\partial\psi_{i+1}}{\partial x_j}$ is homogeneous in the bi-limit, with weights $(r_{0,1},\ldots,r_{0,i+1})$, $(r_{\infty,1},\ldots,r_{\infty,i+1})$, degrees $\alpha_{i+1}(r_{0,i+1}+\mathfrak{d}_0)-r_{0,j}$, $\alpha_{i+1}(r_{\infty,i+1}+\mathfrak{d}_\infty)-r_{\infty,j}$, and approximating functions $\frac{\partial\psi_{i+1,0}}{\partial x_j}$, $\frac{\partial\psi_{i+1,\infty}}{\partial x_j}$. With this at hand, we choose the control law ϕ_{i+1} as

$$\phi_{i+1}(\mathfrak{X}_{i+1})\ =\ \psi_{i+1}(\mathfrak{X}_{i+1})^{\frac{1}{\alpha_{i+1}}}\ .$$

Step 3. *Selection of* k. Note that

$$\frac{\partial V_{i+1}}{\partial \mathfrak{X}_{i+1}}(\mathfrak{X}_{i+1})\ [\mathcal{S}_{i+1}\,\mathfrak{X}_{i+1} + B_{i+1}\,\phi_{i+1}(\mathfrak{X}_{i+1})] \;=\; T_1(\mathfrak{X}_{i+1}) - k\,T_2(\mathfrak{X}_{i+1})\ , \tag{4.8}$$

with the functions T_1 and T_2 defined as

$$T_1(\mathfrak{X}_{i+1}) \;=\; \frac{\partial V_{i+1}}{\partial \mathfrak{X}_i}(\mathfrak{X}_{i+1})\ [\mathcal{S}_i\,\mathfrak{X}_i + B_i x_{i+1})]$$
$$T_2(\mathfrak{X}_{i+1}) \;=\; \left(x_{i+1}^{\frac{d_{V_0}-r_{0,i+1}}{r_{0,i+1}}} - \phi_i(\mathfrak{X}_i)^{\frac{d_{V_0}-r_{0,i+1}}{r_{0,i+1}}} + x_{i+1}^{\frac{d_{V_\infty}-r_{\infty,i+1}}{r_{\infty,i+1}}} - \phi_i(\mathfrak{X}_i)^{\frac{d_{V_\infty}-r_{\infty,i+1}}{r_{\infty,i+1}}} \right) \phi_{i+1}(\mathfrak{X}_{i+1})\ .$$

By the definition of homogeneity in the bi-limit and Proposition 2.10, these functions are homogeneous in the bi-limit with weights $(r_{0,1},\dots,r_{0,i+1})$ and $(r_{\infty,1},\dots,r_{\infty,i+1})$ and degrees $d_{V_0}+\mathfrak{d}_0$ and $d_{V_\infty}+\mathfrak{d}_\infty$. Moreover, since $\phi_{i+1}(\mathfrak{X}_{i+1})$ has the same sign as $x_{i+1}-\phi_i(\mathfrak{X}_i)$, $T_2(\mathfrak{X}_{i+1})$ is nonnegative for all $\mathfrak{X}_{i+1}$ in $\mathbb{R}^{i+1}$ and, as $\phi_{i+1}(\mathfrak{X}_{i+1})=0$ only if $x_{i+1}-\phi_i(\mathfrak{X}_i)=0$, we get

$$\begin{aligned} T_2(\mathfrak{X}_{i+1})=0 \quad &\Longrightarrow \quad x_{i+1} \;=\; \phi_i(\mathfrak{X}_i)\ ,\\ x_{i+1} \;=\; \phi_i(\mathfrak{X}_i) \quad &\Longrightarrow \quad T_1(\mathfrak{X}_{i+1}) \;=\; \frac{\partial V_i}{\partial \mathfrak{X}_i}(\mathfrak{X}_i)\,[\mathcal{S}_i\,\mathfrak{X}_i + B_i\phi_i(\mathfrak{X}_i)]\ . \end{aligned}$$

Consequently, equations (4.7) yield

$$\left\{\mathfrak{X}_{i+1}\in\mathbb{R}^{i+1}\setminus\{0\}\ :\ T_2(\mathfrak{X}_{i+1})=0\right\} \quad\subseteq\quad \left\{\mathfrak{X}_{i+1}\in\mathbb{R}^{i+1}\ :\ T_1(\mathfrak{X}_{i+1})<0\right\}\ .$$

The same implication holds for the homogeneous approximations of the two functions at infinity and around the origin, i.e.,

$$\begin{aligned} \left\{\mathfrak{X}_{i+1}\in\mathbb{R}^{i+1}\setminus\{0\}\ :\ T_{2,0}(\mathfrak{X}_{i+1})=0\right\} \quad&\subseteq\quad \left\{\mathfrak{X}_{i+1}\in\mathbb{R}^{i+1}\ :\ T_{1,0}(\mathfrak{X}_{i+1})<0\right\}\ ,\\ \left\{\mathfrak{X}_{i+1}\in\mathbb{R}^{i+1}\setminus\{0\}\ :\ T_{2,\infty}(\mathfrak{X}_{i+1})=0\right\} \quad&\subseteq\quad \left\{\mathfrak{X}_{i+1}\in\mathbb{R}^{i+1}\ :\ T_{1,\infty}(\mathfrak{X}_{i+1})<0\right\}\ . \end{aligned}$$

Hence, by Lemma 2.13, there exists $k^*>0$ such that, for all $k\geq k^*$, we have for all $\mathfrak{X}_{i+1}\neq 0$,

$$\begin{aligned} \frac{\partial V_{i+1}}{\partial \mathfrak{X}_{i+1}}(\mathfrak{X}_{i+1})\ [\mathcal{S}_{i+1}\,\mathfrak{X}_{i+1} + B_{i+1}\phi_{i+1}(\mathfrak{X}_{i+1})] &< 0\ ,\\ \frac{\partial V_{i+1,0}}{\partial \mathfrak{X}_{i+1}}(\mathfrak{X}_{i+1})\ [\mathcal{S}_{i+1}\,\mathfrak{X}_{i+1} + B_{i+1}\phi_{i+1,0}(\mathfrak{X}_{i+1})] &< 0\ ,\\ \frac{\partial V_{i+1,\infty}}{\partial \mathfrak{X}_{i+1}}(\mathfrak{X}_{i+1})\,[\mathcal{S}_{i+1}\,\mathfrak{X}_{i+1} + B_{i+1}\phi_{i+1,\infty}(\mathfrak{X}_{i+1})] &< 0\ . \end{aligned}$$

This implies that the origin is a globally asymptotically stable equilibrium of the systems (4.4). ◻

To construct the function ϕ_n it is sufficient to iterate the construction in Theorem 4.1 starting from

$$\phi_1(x_1) \;=\; \psi_1(x_1)^{\frac{1}{\alpha_1}},\qquad \psi_1(x_1) = -k_1\int_0^{x_1} \mathfrak{H}\left(|s|^{\alpha_1\frac{r_{0,2}}{r_{0,1}}-1}, |s|^{\alpha_1\frac{r_{\infty,2}}{r_{\infty,1}}-1}\right)\,ds\ ,$$

with $k_1>0$.

At the end of the recursive procedure, we have that the origin is a globally asymptotically stable equilibrium of the systems

$$\begin{aligned} \mathfrak{X}_n &= \mathcal{S}_n\, \mathfrak{X}_n \,+\, B_n\, \phi_n(\mathfrak{X}_n)\ , \\ \mathfrak{X}_n &= \mathcal{S}_n\, \mathfrak{X}_n \,+\, B_n\, \phi_{n,0}(\mathfrak{X}_n)\ , \\ \mathfrak{X}_n &= \mathcal{S}_n\, \mathfrak{X}_n \,+\, B_n\, \phi_{n,\infty}(\mathfrak{X}_n)\ . \end{aligned} \tag{4.9}$$

Remark 4.2. Note that if $\mathfrak{d}_0 \geq 0$ and $\mathfrak{d}_\infty \geq 0$, then we can select $\alpha_i = 1$ for all $1 \leq i \leq n$, and if $\mathfrak{d}_0 \leq 0$ and $\mathfrak{d}_\infty \geq \mathfrak{d}_0$, then we can select $\alpha_i = \frac{r_{0,1}}{r_{0,i+1}}$. Finally, if $\mathfrak{d}_\infty \leq 0$ and $\mathfrak{d}_0 \geq \mathfrak{d}_\infty$, then we can select $\alpha_i = \frac{r_{\infty,1}}{r_{\infty,i+1}}$.

Remark 4.3. As in the observer design, when $\mathfrak{d}_0 \leq \mathfrak{d}_\infty$, we have $\frac{r_{0,i+1}+\mathfrak{d}_0}{r_{0,i+1}} \leq \frac{r_{\infty,i+1}+\mathfrak{d}_\infty}{r_{\infty,i+1}}$ for $i = 1, \dots, n$ and we can replace the function ψ_i by the simpler function

$$\begin{aligned} \psi_{i+1}(\mathfrak{X}_{i+1}) \;=\; -k \Bigg(& \left|x_{i+1}^{\alpha_i} - \phi_i(\mathfrak{X}_i)^{\alpha_i}\right|^{\alpha_{i+1}\frac{\mathfrak{d}_0 + r_{0,i+1}}{\alpha_i\, r_{0,i+1}}} \\ & + \left|x_{i+1}^{\alpha_i} - \phi_i(\mathfrak{X}_i)^{\alpha_i}\right|^{\alpha_{i+1}\frac{\mathfrak{d}_\infty + r_{\infty,i+1}}{\alpha_i\, r_{\infty,i+1}}} \Bigg)\ . \end{aligned} \tag{4.10}$$

Finally, if $0 \leq \mathfrak{d}_0 \leq \mathfrak{d}_\infty$, then by taking $\alpha_i = 1$ (see Remark 4.2) and $\phi(\mathfrak{X}_{i+1}) = \psi_{i+1}(\mathfrak{X}_{i+1})$ as defined in (4.10), we recover the design in [1].

Example 4.4. Consider a chain of integrators of dimension two with weights and degrees

$$(r_0,\, \mathfrak{d}_0) \;=\; \Big((2-q, 1),\, q-1\Big), \qquad (r_\infty,\, \mathfrak{d}_\infty) \;=\; \Big((2-p, 1),\, p-1\Big)\ ,$$

with $2 > p > q > 0$. Given $k_1 > 0$, using the proposed backstepping procedure we obtain a positive real number k_2 such that the feedback

$$\phi_2(x_1, x_2) = -k_2 \int_0^{x_1 - \phi_i(x_1)} \mathfrak{H}\left(|s|^{q-1}, |s|^{p-1}\right)\, ds\ , \tag{4.11}$$

with $\phi_1(x_1) = -k_1 \int_0^{x_1} \mathfrak{H}(|s|^{\frac{q-1}{2-q}}, |s|^{\frac{p-1}{2-p}})\, ds$, renders the origin a globally asymptotically stable equilibrium of the closed-loop system. Furthermore, as a consequence of the robustness result in Corollary 2.22, there is a positive real number c_G such that, if the positive real numbers $|c_0|$ and $|c_\infty|$ associated with δ_i in (1.2) are smaller than c_G, then the control law ϕ_2 globally asymptotically stabilizes the origin of system (1.1).

5. Application to nonlinear output feedback design.

5.1. Results on output feedback. The tools presented in the previous sections can be used to derive two new results on stabilization by output feedback for the origin of nonlinear systems. The output feedback is designed for a simple chain of integrators,

$$\dot{x} \;=\; \mathcal{S}_n\, x \,+\, B_n\, u, \qquad y = x_1\ , \tag{5.1}$$

where x is in $\mathbb{R}^n$, y is the output in $\mathbb{R}$, and u is the control input in $\mathbb{R}$. It is then shown to be adequate to solve the output feedback stabilization problem for the origin of systems for which this chain of integrators can be considered as the dominant part of the dynamics.

Such a domination approach has a long history. It is the cornerstone of the results in [13] (see also [27] and [24]), where a linear controller was introduced to deal with nonlinear systems. This approach has also been followed with nonlinear controllers in [22] and more recently in combination with weighted homogeneity in [35, 26, 28] and the references therein.

In the context of homogeneity in the bi-limit, we use this approach exploiting the proposed backstepping and recursive observer designs. Following the idea introduced by Qian in [26] (see also [27]), the output feedback we proposed is given by

$$\dot{\hat{\mathfrak{x}}}_n = L\left(\mathcal{S}_n\,\hat{\mathfrak{x}}_n + B_n\phi_n(\hat{\mathfrak{x}}_n) + K_1(x_1 - \hat{x}_1)\right), \quad u = L^n\,\phi_n(\hat{\mathfrak{x}}_n)\ , \tag{5.2}$$

with $\hat{\mathfrak{x}}_n$ in $\mathbb{R}^n$ and where ϕ_n and K_1 are continuous functions and L is a positive real number. Employing the recursive procedure given in sections 3 and 4, we get the following theorem, whose proof is in section 5.2.

THEOREM 5.1. *For all real numbers $\mathfrak{d}_0$ and $\mathfrak{d}_\infty$ in $(-1, \frac{1}{n-1})$, there exists a homogeneous in the bi-limit function $\phi_n : \mathbb{R}^n \to \mathbb{R}$ with associated triples $(r_0, 1+\mathfrak{d}_0, \phi_{n,0})$ and $(r_\infty, 1+\mathfrak{d}_\infty, \phi_{n,\infty})$ and a homogeneous in the bi-limit vector field $K_1 : \mathbb{R}^n \to \mathbb{R}^n$ with associated triples $(r_0, \mathfrak{d}_0, K_{1,0})$ and $(r_\infty, \mathfrak{d}_\infty, K_{1,\infty})$ such that for all real numbers $L > 0$ the origin is a globally asymptotically stable equilibrium of the systems* (5.1) *and* (5.2) *and their homogeneous approximations.*

We can then apply Corollary 2.22 to get an output feedback result for nonlinear systems described by

$$\dot{x} \;=\; \mathcal{S}_n\,x \;+\; B_n\,u \;+\; \delta(t), \qquad y \;=\; x_1\ , \tag{5.3}$$

where $\delta : \mathbb{R}_+ \to \mathbb{R}^n$ is a continuous function related to the solutions as described in the two corollaries below and proved in section 5.2. Depending on whether $\mathfrak{d}_0 \le \mathfrak{d}_\infty$ or $\mathfrak{d}_\infty \le \mathfrak{d}_0$, we get an output feedback result for systems in feedback or feedforward form.

COROLLARY 5.2 (feedback form). *If, in the design of ϕ_n and K_1, we select $\mathfrak{d}_0 \le \mathfrak{d}_\infty$, then for all positive real numbers c_0 and c_∞ there exists a real number $L^* > 0$ such that for every L in $[L^*, +\infty)$, the following holds:*

- *For every class $\mathcal{K}$ function γ_z and class $\mathcal{KL}$ function β_δ, we can find two class $\mathcal{KL}$ functions β_x and $\beta_{\hat{x}}$ such that, for each function $t \in [0,T) \mapsto (x(t), \hat{\mathfrak{x}}_n(t), \delta(t), z(t))$, $T \le +\infty$, with $(x, \hat{\mathfrak{x}}_n)$ C^1 and δ and z continuous, which satisfies* (5.3), (5.2)*, and for i in $\{1,\dots,n\}$ and $0 \le s \le t < T$,*

$$|z(t)| \;\le\; \max\left\{\beta_\delta\Big(|z(s)|, t-s\Big),\ \sup\nolimits_{s\le\kappa\le t}\gamma_z(|x(\kappa)|)\right\}\ ,$$

$$|\delta_i(t)| \;\le\; \max\left\{\beta_\delta\Big(|z(s)|, t-s\Big)\ ,\right.$$

$$\left.\sup_{s\le\kappa\le t}\left\{c_0\sum_{j=1}^{i}|x_j(\kappa)|^{\frac{1-\mathfrak{d}_0(n-i-1)}{1-\mathfrak{d}_0(n-j)}} \;+\; c_\infty\sum_{j=1}^{i}|x_j(\kappa)|^{\frac{1-\mathfrak{d}_\infty(n-i-1)}{1-\mathfrak{d}_\infty(n-j)}}\right\}\right\}\ , \tag{5.4}$$

we have for all $0 \le s \le t \le T$,

$$|x(t)| \le \beta_x(|(x(s), \hat{\mathfrak{x}}_n(s), z(s))|, t-s), \qquad |\hat{\mathfrak{x}}_n(t)| \le \beta_{\hat{x}}(|(x(s), \hat{\mathfrak{x}}_n(s), z(s))|, t-s)\ .$$

COROLLARY 5.3 (feedforward form). *If, in the design of ϕ_n and K_1, we select $\mathfrak{d}_\infty \leq \mathfrak{d}_0$, then for all positive real numbers c_0 and c_∞ there exists a real number $L^* > 0$ such that for every L in $(0, L^*]$, the following holds:*

- *For every class $\mathcal{K}$ function γ_z and class $\mathcal{KL}$ function β_δ, we can find two class $\mathcal{KL}$ functions β_x and $\beta_{\hat{x}}$ such that, for each function $t \in [0,T) \mapsto (x(t), \hat{\mathfrak{X}}_n(t), \delta(t), z(t))$, $T \leq +\infty$, with $(x, \hat{\mathfrak{X}}_n)$ C^1 and δ and z continuous, which satisfies* (5.3), (5.2), *and for i in $\{1, \dots, n\}$ and $0 \leq s \leq t < T$,*

$$|z(t)| \leq \max\left\{\beta_\delta\Big(|z(s)|, t-s\Big), \sup_{s\leq\kappa\leq t} \gamma_z(|x(\kappa)|)\right\} ,$$

$$|\delta_i(t)| \leq \max\left\{ \begin{array}{l} \beta_\delta(|z(s)|, t-s) , \\ \displaystyle\sup_{s\leq\kappa\leq t}\left\{c_0 \sum_{j=i+2}^{n} |x_j(\kappa)|^{\frac{1-\mathfrak{d}_0(n-i-1)}{1-\mathfrak{d}_0(n-j)}} + c_\infty \sum_{j=i+2}^{n} |x_j(\kappa)|^{\frac{1-\mathfrak{d}_\infty(n-i-1)}{1-\mathfrak{d}_\infty(n-j)}}\right\}\end{array}\right\} , \tag{5.5}$$

we have for all $0 \leq s \leq t \leq T$,

$$|x(t)| \leq \beta_x(|(x(s), \hat{\mathfrak{X}}_n(s), z(s))|, t-s), \qquad |\hat{\mathfrak{X}}_n(t)| \leq \beta_{\hat{x}}(|(x(s), \hat{\mathfrak{X}}_n(s), z(s))|, t-s) .$$

Example 5.4. Following Example 2.23, we can consider the case where the δ_i's are outputs of auxiliary systems given in (2.13). Suppose there exist n positive definite and radially unbounded functions $Z_i : \mathbb{R}^{n_i} \to \mathbb{R}_+$, three class $\mathcal{K}$ functions ω_1, ω_2, ω_3, and a positive real number ϵ in $(0,1)$ such that

$$|\delta_i(z_i, x)| \leq \omega_1(x) + \omega_2(Z_i(z_i)), \qquad \frac{\partial Z_i}{\partial z_i}(z_i)\, g_i(z_i, x) \leq -Z_i(z_i) + \omega_3(|x|) ;$$

then, if there exist two real numbers $\mathfrak{d}_0$ and $\mathfrak{d}_\infty$ satisfying $-1 < \mathfrak{d}_0 \leq \mathfrak{d}_\infty < \frac{1}{n-1}$ and

$$\omega_1(x) + \omega_2\left([1+\epsilon]\,\omega_3(|x|)\right) \leq \left(\sum_{j=1}^{i} |x_j|^{\frac{1-\mathfrak{d}_0(n-i-1)}{1-\mathfrak{d}_0(n-j)}} + \sum_{j=1}^{i} |x_j|^{\frac{1-\mathfrak{d}_\infty(n-i-1)}{1-\mathfrak{d}_\infty(n-j)}}\right) , \tag{5.6}$$

then Corollary 5.2 gives $L^* > 0$ such that for all L in $[L^*, +\infty)$, the output feedback (5.2) is globally asymptotically stabilizing. Compared to already published results (see [13] and [26], for instance), the novelty of this case is in the simultaneous presence of the terms $|x_j|^{\frac{1-\mathfrak{d}_0(n-i-1)}{1-\mathfrak{d}_0(n-j)}}$ and $|x_j|^{\frac{1-\mathfrak{d}_\infty(n-i-1)}{1-\mathfrak{d}_\infty(n-j)}}$.

On the other hand, if there exist two real numbers $\mathfrak{d}_0$ and $\mathfrak{d}_\infty$ satisfying $-1 < \mathfrak{d}_\infty \leq \mathfrak{d}_0 < \frac{1}{n-1}$ and

$$\omega_1(x) + \omega_2\left([1+\epsilon]\,\omega_3(|x|)\right) \leq \left(\sum_{j=i+2}^{n} |x_j|^{\frac{1-\mathfrak{d}_0(n-i-1)}{1-\mathfrak{d}_0(n-j)}} + \sum_{j=i+2}^{n} |x_j|^{\frac{1-\mathfrak{d}_\infty(n-i-1)}{1-\mathfrak{d}_\infty(n-j)}}\right) ,$$

then Corollary 5.3 gives $L^* > 0$ such that for all L in $(0, L^*]$, the output feedback (5.2) is globally asymptotically stabilizing.

Example 5.5. Consider the illustrative system (1.1). The bound (5.6) gives the condition

$$0 \ < \ q \ < \ p \ < \ 2 \ . \tag{5.7}$$

This is almost the least conservative condition we can obtain with the domination approach. Specifically, it is shown in [18] that, when $p > 2$, there is no stabilizing output feedback. However, when $p = 2$, (5.6) is not satisfied, although the stabilization problem is solvable (see [18]).

By Corollary 2.24, when (5.7) holds, the output feedback

$$u \ = \ L^2\phi_2(\hat{x}_1, \hat{x}_2), \qquad \left\{ \begin{array}{rcl} \dot{\hat{x}}_1 & = & L\,\hat{x}_2 - L\,q_1(\ell_1 e_1) \ , \\ \dot{\hat{x}}_2 & = & \dfrac{u}{L} - L\,q_2(\ell_2\,q_1(\ell_1 e_1)) \ , \\ e_1 & = & \hat{x}_1 - y \ , \end{array} \right.$$

with ℓ_1, ℓ_2, ϕ_2, q_1, and q_2 defined in (3.13) and (4.11) and with picking $\mathfrak{d}_0$ in $(-1, q-1]$ and $\mathfrak{d}_\infty$ in $[p-1, 1)$, globally asymptotically stabilizes the origin of the system (1.1), with L chosen sufficiently large. Furthermore, if $\mathfrak{d}_0$ is chosen strictly negative and $\mathfrak{d}_\infty$ strictly positive, by Corollary 2.24, convergence to the origin occurs in finite time, uniformly in the initial conditions.

Example 5.6. To illustrate the feedforward result consider the system[7]

$$\dot{x}_1 = x_2 + x_3^{\frac{3}{2}} + z^3 \ , \quad \dot{x}_2 = x_3 \ , \quad \dot{x}_3 = u \ , \quad \dot{z} = -z^4 + x_3 \ , \quad y = x_1 \ .$$

For any $\varepsilon > 0$, there exists a class $\mathcal{KL}$ function β_δ such that

$$|z(t)|^3 \ \leq \ \max\left\{\beta_\delta(|z(s)|, t-s), \quad (1+\varepsilon) \sup_{s \leq \kappa \leq t} |x_3(\kappa)|^{\frac{3}{4}}\right\}.$$

Therefore by letting $\delta_1 = x_3^{\frac{3}{2}} + z^3$ we get, for all $0 \leq s \leq t < T$ on the time of existence of the solutions,

$$|\delta_1(t)| \ \leq \ \max\left\{\beta_\delta(|z(s)|, t-s), \quad \sup_{s \leq \kappa \leq t} (1+\varepsilon)|x_3(\kappa)|^{\frac{3}{4}} \ + \ |x_3(\kappa)|^{\frac{3}{2}}\right\} \ .$$

This is inequality (5.5) with $\mathfrak{d}_0 = -\frac{1}{2}$ and $\mathfrak{d}_\infty \ = \ \frac{1}{4}$. Consequently, Corollary 5.3 says that it is possible to design a globally asymptotically stabilizing output feedback.

5.2. Proofs of output feedback results.

Proof of Theorem 5.1. The homogeneous in the bi-limit state feedback ϕ_n and the homogeneous in the bi-limit vector field K_1 involved in this feedback are obtained by following the procedures given in sections 3 and 4. They are such that the origin is a globally asymptotically stable equilibrium of the systems given in (4.9) and (3.12). To this end, as in [26], we write the dynamics of this system in the coordinates $\hat{\mathfrak{X}}_n = (\hat{x}_1, \ldots, \hat{x}_n)$ and $E_1 = (e_1, \ldots, e_n)$ and in the time τ defined by

$$e_i \ = \ \hat{x}_i \ - \ \frac{x_i}{L^{i-1}}, \qquad \frac{d}{d\tau} \ = \ \frac{1}{L}\,\frac{d}{dt} \ . \tag{5.8}$$

[7]Recall the notation (1.4).

This yields

$$\left\{ \begin{array}{rcl} \frac{d}{d\tau}\,\widehat{\mathfrak{X}}_n & = & \mathcal{S}_n\,\widehat{\mathfrak{X}}_n + B_n\phi_n(\widehat{\mathfrak{X}}_n)) \,+\, K_1(e_1)\ , \\ \frac{d}{d\tau}\,E_1 & = & \mathcal{S}_n\,E_1 + K_1(e_1) \end{array} \right. \tag{5.9}$$

with $E_1 = (e_1, \ldots, e_n)$, $\widehat{\mathfrak{X}}_n = (\hat{x}_1, \ldots, \hat{x}_n)$. The right-hand side of (5.9) is a vector field which is homogeneous in the bi-limit with weights $(r_0, r_0), (r_\infty, r_\infty)$.

Given $d_U > \max_j\{r_{0,j}, r_{\infty,j}\}$, by applying Theorem 2.20 twice, we get two C^1, proper, and positive definite functions $V\ :\mathbb{R}^n \to \mathbb{R}_+$ and $W\ :\mathbb{R}^n \to \mathbb{R}_+$ such that for each i in $\{1, \ldots, n\}$, the functions $\frac{\partial V}{\partial x_i}$ and $\frac{\partial W}{\partial e_i}$ are homogeneous in the bi-limit, with weights r_0 and r_∞, degrees $d_U - r_{0,i}$ and $d_U - r_{\infty,i}$, and approximating functions $\frac{\partial V_0}{\partial \hat{x}_j}$, $\frac{\partial V_\infty}{\partial \hat{x}_j}$ and $\frac{\partial W_0}{\partial e_j}$, $\frac{\partial W_\infty}{\partial e_j}$. Moreover, for all $\widehat{\mathfrak{X}}_n \neq 0$,

$$\begin{array}{c} \frac{\partial V}{\partial \widehat{\mathfrak{X}}}(\widehat{\mathfrak{X}}_n)\left[\mathcal{S}_n\,\widehat{\mathfrak{X}}_n + B_n\phi_n(\widehat{\mathfrak{X}}_n)\right] < 0\ , \\ \frac{\partial V_0}{\partial \widehat{\mathfrak{X}}_n}(\widehat{\mathfrak{X}}_n)\ \left[\mathcal{S}_n\,\widehat{\mathfrak{X}}_n + B_n\phi_{n,0}(\widehat{\mathfrak{X}}_n)\right] < 0\ , \\ \frac{\partial V_\infty}{\partial \widehat{\mathfrak{X}}_n}(\widehat{\mathfrak{X}}_n)\left[\mathcal{S}_n\,\widehat{\mathfrak{X}}_n + B_n\phi_{n,\infty}(\widehat{\mathfrak{X}}_n)\right] < 0\ , \end{array} \tag{5.10}$$

and for all $E_1 \neq 0$,

$$\begin{array}{c} \frac{\partial W}{\partial E_1}(E_1)\,(\mathcal{S}_n\,E_1 + K_1(e_1)) < 0\ , \\ \frac{\partial W_0}{\partial E_1}(E_1)\,(\mathcal{S}_n\,E_1 + K_{1,0}(e_1)) < 0\ , \\ \frac{\partial W_\infty}{\partial E_1}(E_1)\,(\mathcal{S}_n\,E_1 + K_{1,\infty}(e_1)) < 0\ . \end{array} \tag{5.11}$$

Consider now the Lyapunov function candidate

$$U(\hat{\mathfrak{X}}_n, E_1) \;=\; V(\hat{\mathfrak{X}}_n) \,+\, \mathfrak{c}\,W(E_1)\ , \tag{5.12}$$

where $\mathfrak{c}$ is a positive real number to be specified. Let

$$\begin{aligned} \eta(\hat{\mathfrak{X}}_n, E_1) &= \frac{\partial V}{\partial \widehat{\mathfrak{X}}_n}(\widehat{\mathfrak{X}}_n)\left(\mathcal{S}_n\,\hat{\mathfrak{X}}_n + B_n\phi_n(\hat{\mathfrak{X}}_n) \,+\, K_1(e_1)\right)\ , \\ \gamma(E_1) &= \quad -\frac{\partial W}{\partial E_1}(E_1)\,(\mathcal{S}_n\,E_1 \,+\, K_1(e_1))\ . \end{aligned}$$

These two functions are continuous and homogeneous in the bi-limit with associated triples $((r_0, r_0), d_U + \mathfrak{d}_0, \eta_0)$, $((r_\infty, r_\infty), d_U + \mathfrak{d}_\infty, \eta_\infty)$ and $((r_0, r_0), d_U + \mathfrak{d}_0, \gamma_0)$, $((r_\infty, r_\infty), d_U + \mathfrak{d}_\infty, \gamma_\infty)$, where γ_0, γ_∞ and η_0, η_∞ are continuous functions. Furthermore, by (5.11), $\gamma(E_1)$ is negative definite. Hence, by (5.10), we have

$$\begin{array}{c} \left\{(\hat{\mathfrak{X}}_n, E_1) \in \mathbb{R}^{2n} \setminus \{0\}\ :\ \gamma(E_1) \,=\, 0\right\} \;\subseteq\; \left\{(\hat{\mathfrak{X}}_n, E_1) \in \mathbb{R}^{2n}\ :\ \eta(\hat{\mathfrak{X}}_n, E_1) \,<\, 0\right\}\ , \\ \left\{(\hat{\mathfrak{X}}_n, E_1) \in \mathbb{R}^{2n} \setminus \{0\}\ :\ \gamma_0(E_1) \,=\, 0\right\} \;\subseteq\; \left\{(\hat{\mathfrak{X}}_n, E_1) \in \mathbb{R}^{2n}\ :\ \eta_0(\hat{\mathfrak{X}}_n, E_1) \,<\, 0\right\}\ , \\ \left\{(\hat{\mathfrak{X}}_n, E_1) \in \mathbb{R}^{2n} \setminus \{0\}\ :\ \gamma_\infty(E_1) \,=\, 0\right\} \;\subseteq\; \left\{(\hat{\mathfrak{X}}_n, E_1) \in \mathbb{R}^{2n}\ :\ \eta_\infty(\hat{\mathfrak{X}}_n, E_1) \,<\, 0\right\}\ . \end{array}$$

Consequently, by Lemma 2.13, there exists a positive real number $\mathfrak{c}^*$ such that, for all $\mathfrak{c} > \mathfrak{c}^*$ and all $(\hat{\mathfrak{X}}_n, E_1) \neq (0,0)$, the Lyapunov function U, defined in (5.12), satisfies

$$\frac{\partial U}{\partial \hat{\mathfrak{X}}_n}(\hat{\mathfrak{X}}_n, E_1)\left(\mathcal{S}_n \hat{\mathfrak{X}}_n + B_n\phi_n(\hat{\mathfrak{X}}_n) + K_1(e_1)\right) + \frac{\partial U}{\partial E_1}(\hat{\mathfrak{X}}_n, E_1)(E_1)\left(\mathcal{S}_n E_1 + K_1(e_1)\right) < 0$$

and the same holds for the homogeneous approximations in the 0-limit and in the ∞-limit; hence the claim. □

Proof of Corollary 5.2. We write the dynamics of the system (5.3) in the coordinates $\hat{\mathfrak{X}}_n$ and E_1 and in the time τ given in (5.8). This yields

$$\left\{\begin{array}{rcl} \dfrac{d}{d\tau}\,\widehat{\mathfrak{X}}_n & = & \mathcal{S}_n \widehat{\mathfrak{X}}_n + B_n\phi_n(\hat{\mathfrak{X}}_n)) + K_1(e_1), \\ \dfrac{d}{d\tau}\,E_1 & = & \mathcal{S}_n E_1 + K_1(e_1) + \mathfrak{D}(L) \end{array}\right. \tag{5.13}$$

with

$$\mathfrak{D}(L) = \left(\frac{\delta_1}{L}, \dots, \frac{\delta_n}{L^n}\right).$$

We denote the solution of this system, starting from $(\widehat{\mathfrak{X}}_n(0), E_1(0))$ in $\mathbb{R}^{2n}$ at time τ, by $(\widehat{\mathfrak{X}}_{\tau,n}(\tau), E_{\tau,1}(\tau))$. We have

$$x_i(t) = L^{i-1}\left(\hat{x}_{\tau,i}(Lt) - e_{\tau,i}(Lt)\right). \tag{5.14}$$

The right-hand side of (5.13) is a vector field which is homogeneous in the bi-limit with weights $(r_0, r_0), (r_\infty, r_\infty)$ for $(\widehat{\mathfrak{X}}_n, E_1)$ and $(\mathfrak{r}_0, \mathfrak{r}_\infty)$ for $\mathfrak{D}(L)$, where $\mathfrak{r}_{0,i} = r_{0,i} + \mathfrak{d}_0$ and $\mathfrak{r}_{\infty,i} = r_{\infty,i} + \mathfrak{d}_\infty$ for each i in $\{1, \dots, n\}$.

The time function $\tau \mapsto \delta(\frac{\tau}{L})$ is considered as an input, and when $\mathfrak{D}(L) = 0$, Theorem 5.1 implies global asymptotic stability of the origin of the system (5.13) and of its homogeneous approximations. To complete the proof we show that there exists L^* such that the "input" $\mathfrak{D}(L)$ satisfies the small-gain condition (2.11) of Corollary 2.22 for all $L > L^*$. Using (5.8) and (5.14), assumption (5.4) becomes, for all $0 \leq \sigma \leq \tau < LT$ and all i in $\{1, \dots, n\}$,

$$\frac{\left|\delta_i\left(\frac{\tau}{L}\right)\right|}{L^i} \leq \max\left\{\frac{1}{L^i}\beta_\delta\left(\left|z\left(\frac{\sigma}{L}\right)\right|, \frac{\tau-\sigma}{L}\right), \right. \\ L^{-i}\sup_{\sigma\leq\kappa\leq\tau}\left\{c_0\sum_{j=1}^{i}\left|L^{(j-1)}(\hat{x}_{\tau,j}(\kappa) - e_{\tau,j}(\kappa))\right|^{\frac{1-\mathfrak{d}_0(n-i-1)}{1-\mathfrak{d}_0(n-j)}} \right. \\ \left.\left. + c_\infty\sum_{j=1}^{i}\left|L^{(j-1)}(\hat{x}_{\tau,j}(\kappa) - e_{\tau,j}(\kappa))\right|^{\frac{1-\mathfrak{d}_\infty(n-i-1)}{1-\mathfrak{d}_\infty(n-j)}}\right\}\right\}. \tag{5.15}$$

Note that when $1 \leq j \leq i \leq n$, the function $s \mapsto \frac{1-(n-i-1)\,s}{1-(n-j)\,s}$ is strictly increasing, mapping $(-1, \frac{1}{n-1})$ in $(\frac{n-i}{n+1-j}, \frac{i}{j-1})$. As $\mathfrak{d}_0 \leq \mathfrak{d}_\infty < \frac{1}{n-1}$, we have for all $1 \leq j \leq i \leq n$,

$$\frac{1-\mathfrak{d}_0(n-i-1)}{1-\mathfrak{d}_0(n-j)} \leq \frac{1-\mathfrak{d}_\infty(n-i-1)}{1-\mathfrak{d}_\infty(n-j)} < \frac{i}{j-1}.$$

Hence, selecting $L \geq 1$, there exists a real number $\epsilon > 0$ such that

$$L^{-\epsilon} \;\geq\; L^{(j-1)\frac{1-\mathfrak{d}_\infty(n-i-1)}{1-\mathfrak{d}_\infty(n-j)}-i} \;\geq\; L^{(j-1)\frac{1-\mathfrak{d}_0(n-i-1)}{1-\mathfrak{d}_0(n-j)}-i} .$$

This implies

$$\begin{aligned}\frac{\left|\delta_i\left(\frac{\tau}{L}\right)\right|}{L^i} \leq \max\Bigg\{ & \frac{1}{L^i}\beta_\delta\left(\left|z\left(\frac{\sigma}{L}\right)\right|, \frac{\tau-\sigma}{L}\right), \\ & L^{-\epsilon}\sup_{\sigma\leq\kappa\leq\tau}\Bigg\{c_0\sum_{j=1}^{i}\left|(\hat{x}_{\tau,j}(\kappa)-e_{\tau,j}(\kappa))\right|^{\frac{1-\mathfrak{d}_0(n-i-1)}{1-\mathfrak{d}_0(n-j)}} \\ & \qquad + c_\infty\sum_{j=1}^{i}\left|(\hat{x}_{\tau,j}(\kappa)-e_{\tau,j}(\kappa))\right|^{\frac{1-\mathfrak{d}_\infty(n-i-1)}{1-\mathfrak{d}_\infty(n-j)}}\Bigg\}\Bigg\} .\end{aligned}$$

On the other hand, the function

$$(\widehat{\mathfrak{X}}_n, E_1) \;\mapsto\; c_0\sum_{j=1}^{i}|\hat{x}_j - e_j|^{\frac{1-\mathfrak{d}_0(n-i-1)}{1-\mathfrak{d}_0(n-j)}} + c_\infty\sum_{j=1}^{i}|\hat{x}_j - e_j|^{\frac{1-\mathfrak{d}_\infty(n-i-1)}{1-\mathfrak{d}_\infty(n-j)}}$$

is homogeneous in the bi-limit with weights (r_0, r_0) and (r_∞, r_∞) and degrees $1 - \mathfrak{d}_0(n-i-1) = r_{0,i} + \mathfrak{d}_0$ and $1 - \mathfrak{d}_\infty(n-i-1) = r_{\infty,i} + \mathfrak{d}_\infty$ (see (3.2)). Hence, by Corollary 2.15, there exists a positive real number c_1 such that

$$\begin{aligned}& c_0\sum_{j=1}^{i}|\hat{x}_j - e_j|^{\frac{1-\mathfrak{d}_0(n-i-1)}{1-\mathfrak{d}_0(n-j)}} + c_\infty\sum_{j=1}^{i}|\hat{x}_j - e_j|^{\frac{1-\mathfrak{d}_\infty(n-i-1)}{1-\mathfrak{d}_\infty(n-j)}} \\ & \qquad \leq\; c_1\,\mathfrak{H}\left(|(\widehat{\mathfrak{X}}_n, E_1)|_{(r_0,r_0)}^{\mathfrak{d}_0+r_{0,i}}, |(\widehat{\mathfrak{X}}_n, E_1)|_{(r_\infty,r_\infty)}^{\mathfrak{d}_\infty+r_{\infty,i}}\right) .\end{aligned} \tag{5.16}$$

Hence, by Corollary 2.22 (applied in the τ time-scale), there exists c_G such that for any L^* large enough such that $c_1 L^{*-\varepsilon} \leq c_G$, the conclusion holds. ∎

Proof of Corollary 5.3. The proof is similar to the previous one with the only difference being that, when i and j satisfy $3 \leq i+2 \leq j \leq n$, the function $s \mapsto \frac{1-(n-i-1)\,s}{1-(n-j)\,s}$ is strictly decreasing, mapping $(-1, \frac{1}{n-1})$ in $(\frac{i}{j-1}, \frac{n-i}{n+1-j})$. Moreover the condition $-1 < \mathfrak{d}_\infty \leq \mathfrak{d}_0 < \frac{1}{n-1}$ gives the inequalities

$$\frac{1-\mathfrak{d}_\infty(n-i-1)}{1-\mathfrak{d}_\infty(n-j)} \;\geq\; \frac{1-\mathfrak{d}_0(n-i-1)}{1-\mathfrak{d}_0(n-j)} \;>\; \frac{i}{j-1} .$$

Hence (5.16) holds, and by selecting $L < 1$ we obtain the existence of a positive real number ϵ such that

$$L^{\epsilon} \;\geq\; L^{(j-1)\frac{1-\mathfrak{d}_0(n-i-1)}{1-\mathfrak{d}_0(n-j)}-i} \;\geq\; L^{(j-1)\frac{1-\mathfrak{d}_\infty(n-i-1)}{1-\mathfrak{d}_\infty(n-j)}-i} .$$

From (5.5), this yields, for all $0 \leq \sigma \leq \tau < LT$ and all i in $\{1, \ldots, n\}$,

$$\begin{aligned}\frac{\left|\delta_i\left(\frac{\tau}{L}\right)\right|}{L^i} \leq \max\Bigg\{ & \frac{1}{L^i}\beta_\delta\left(\left|z\left(\frac{\sigma}{L}\right)\right|, \frac{\tau-\sigma}{L}\right), \\ & L^{\epsilon}\sup_{\sigma\leq\kappa\leq\tau}\Bigg\{c_0\sum_{j=i+2}^{n}\left|(\hat{x}_{\tau,j}(\kappa)-e_{\tau,j}(\kappa))\right|^{\frac{1-\mathfrak{d}_0(n-i-1)}{1-\mathfrak{d}_0(n-j)}} \\ & \qquad + c_\infty\sum_{j=i+2}^{n}\left|(\hat{x}_{\tau,j}(\kappa)-e_{\tau,j}(\kappa))\right|^{\frac{1-\mathfrak{d}_\infty(n-i-1)}{1-\mathfrak{d}_\infty(n-j)}}\Bigg\}\Bigg\} .\end{aligned}$$

From Corollary 2.22, the result holds for all L^* small enough to satisfy $c_1 L^{*\varepsilon} \leq c_G$. $\Box$

6. Conclusion. We have presented two new tools that can be useful in nonlinear control design. The first one is introduced to formalize the notion of homogeneous approximation valid both at the origin and at infinity. With this formalism we have given several novel results concerning asymptotic stability, robustness analysis, and also finite time convergence (uniformly in the initial conditions). The second one is a new recursive design for an observer for a chain of integrators. The combination of these two tools allows us to obtain a new result on stabilization by output feedback for systems whose dominant homogeneous in the bi-limit part is a chain of integrators.

Appendix A. Proof of Proposition 2.10. We give the proof only in the 0-limit case since the ∞-limit case is similar. Let C be an arbitrary compact subset of $\mathbb{R}^n \setminus \{0\}$ and ϵ any strictly positive real number. By the definition of homogeneity in the 0-limit, there exists $\lambda_1 > 0$ such that we have

$$\left|\frac{\phi(\lambda^{r_{\phi,0}} \diamond x)}{\lambda^{d_{\phi,0}}} - \phi_0(x)\right| \leq 1 \quad \forall\, x \in C\,, \quad \forall\, \lambda \in (0, \lambda_1]\,.$$

Hence, as ϕ_0 is a continuous function on $\mathbb{R}^n$, for all λ in $(0, \lambda_1]$, the function $x \mapsto \frac{\phi(\lambda^{r_0} \diamond x)}{\lambda^{d_{\phi,0}}}$ takes its values in a compact set $C_\phi = \phi_0(C) + B_1$, where B_1 is the unity ball.

Now, as ζ_0 is continuous on the compact subset C_ϕ, it is uniformly continuous; i.e., there exists $\nu > 0$ such that

$$|z_1 - z_2| < \nu \qquad \Longrightarrow \qquad |\zeta_0(z_1) - \zeta_0(z_2)| < \epsilon\,.$$

Also there exists $\mu_\epsilon > 0$ satisfying

$$\left|\frac{\zeta(\mu^{r_{\zeta,0}} z)}{\mu^{d_{\zeta,0}}} - \zeta_0(z)\right| \leq \epsilon \qquad \forall\, z \in C_\phi\,, \quad \forall\, \mu \in (0, \mu_\epsilon]\,,$$

or equivalently, since $d_{\phi,0} > 0$,

$$\left|\frac{\zeta(\lambda^{d_{\phi,0}} z)}{\lambda^{\frac{d_{\phi,0}\, d_{\zeta,0}}{r_{\zeta,0}}}} - \zeta_0(z)\right| \leq \epsilon \qquad \forall\, z \in C_\phi\,, \quad \forall\, \lambda \in \left(0, \mu_\epsilon^{\frac{r_{\zeta,0}}{d_{\phi,0}}}\right]\,.$$

Similarly, there exists λ_ν such that

$$\left|\frac{\phi(\lambda^{r_{\phi,0}} \diamond x)}{\lambda^{d_{\phi,0}}} - \phi_0(x)\right| \leq \nu \quad \forall\, x \in C\,, \quad \forall\, \lambda \in (0, \lambda_\nu]\,.$$

It follows that

$$\begin{aligned}
\left|\frac{\zeta(\phi(\lambda^{r_{\phi,0}} \diamond x))}{\lambda^{\frac{d_{\phi,0}\, d_{\zeta,0}}{r_{\zeta,0}}}} - \zeta_0\left(\phi_0(x)\right)\right| &\leq \left|\frac{\zeta(\phi(\lambda^{r_{\phi,0}} \diamond x))}{\lambda^{\frac{d_{\phi,0}\, d_{\zeta,0}}{r_{\zeta,0}}}} - \zeta_0\left(\frac{\phi(\lambda^{r_{\phi,0}} \diamond x)}{\lambda^{d_{\phi,0}}}\right)\right| \\
&\quad + \left|\zeta_0\left(\frac{\phi(\lambda^{r_{\phi,0}} \diamond x)}{\lambda^{d_{\phi,0}}}\right) - \zeta_0\left(\phi_0(x)\right)\right| \\
&\leq 2\,\epsilon \quad \forall\, x \in C\,, \quad \forall\, \lambda \in \min\left\{\lambda_1, \lambda_\nu, \mu_\epsilon^{\frac{r_{\zeta,0}}{d_{\phi,0}}}\right\}\,.
\end{aligned}$$

This establishes homogeneity in the 0-limit of the function $\zeta \circ \phi$.

Appendix B. Proof of Proposition 2.11. We give the proof only in the 0-limit case since the ∞-limit case is similar. The function ϕ being a bijection, we can assume without loss of generality that it is a strictly increasing function (otherwise we take $-\phi$). This, together with homogeneity in the 0-limit, implies that φ_0 is strictly positive. Moreover, for each $\delta > 0$, there exists $t_0(\delta) > 0$ such that

$$\left|\frac{\phi(t)}{t^{d_0}} - \varphi_0\right| \leq \delta \qquad \forall\, t \in (0, t_0(\delta)]\ .$$

By letting $\lambda = \phi(t)$, this gives

$$\varphi_0 - \delta \leq \frac{\lambda}{\phi^{-1}(\lambda)^{d_0}} \leq \varphi_0 + \delta \qquad \forall\, \lambda \in (0, \phi(t_0(\delta))]\ , \quad \forall\, \delta > 0\ .$$

Since for $\delta < \varphi_0$ the term on the left is strictly positive, these inequalities give

$$\left(\frac{1}{\varphi_0 + \delta}\right)^{\frac{1}{d_0}} \leq \frac{\phi^{-1}(\lambda)}{\lambda^{\frac{1}{d_0}}} \leq \left(\frac{1}{\varphi_0 - \delta}\right)^{\frac{1}{d_0}} \qquad \forall\, \lambda \in (0, \phi^{-1}(t_0(\delta))],\ \forall\, \delta \in (0, \varphi_0)\ .$$

Then since the function $\delta \mapsto (\frac{1}{\varphi_0 - \delta})^{\frac{1}{d_0}}$ is continuous at zero, for every $\epsilon_1 > 0$ there exists $\delta_1(\epsilon_1) > 0$ satisfying

$$\left(\frac{1}{\varphi_0}\right)^{\frac{1}{d_0}} - \epsilon_1 \leq \left(\frac{1}{\varphi_0 + \delta_1(\epsilon_1)}\right)^{\frac{1}{d_0}} \leq \left(\frac{1}{\varphi_0 - \delta_1(\epsilon_1)}\right)^{\frac{1}{d_0}} \leq \left(\frac{1}{\varphi_0}\right)^{\frac{1}{d_0}} + \epsilon_1\ .$$

This yields

$$\left|\frac{\phi^{-1}(\lambda)}{\lambda^{\frac{1}{d_0}}} - \left(\frac{1}{\varphi_0}\right)^{\frac{1}{d_0}}\right| \leq \epsilon_1 \qquad \forall\, \lambda \in (0, \lambda_-(\epsilon_1)]\ ,$$

with $\lambda_-(\epsilon_1) = \phi(t_0(\delta_1(\epsilon_1)))$. With a similar argument, we get

$$\left|\frac{\phi^{-1}(-\lambda)}{\lambda^{\frac{1}{d_0}}} + \left(\frac{1}{\varphi_0}\right)^{\frac{1}{d_0}}\right| \leq \epsilon_1 \qquad \forall\, \lambda \in (0, \lambda_+(\varepsilon_1)]$$

for some $\lambda_+ > 0$. Let $\lambda_0 = \min\{\lambda_-, \lambda_+\}$.

Now, for $x \neq 0$ and $\lambda > 0$, we have

$$\left|\frac{\phi^{-1}(\lambda x)}{\lambda^{\frac{1}{d_0}}} - \left(\frac{x}{\varphi_0}\right)^{\frac{1}{d_0}}\right| = |x|^{\frac{1}{d_0}} \left|\frac{\phi^{-1}(\lambda x)}{(x\lambda)^{\frac{1}{d_0}}} - \left(\frac{1}{\varphi_0}\right)^{\frac{1}{d_0}}\right|\ .$$

Therefore, for any compact set C of $\mathbb{R}\backslash\{0\}$ and any $\epsilon > 0$, by letting $\epsilon_1 = \frac{\epsilon}{\max_{x\in C} |x|^{\frac{1}{d_0}}}$, we have

$$|x|^{\frac{1}{d_0}} \epsilon_1 \leq \epsilon, \qquad 0 < |\lambda x| \leq \lambda_0(\epsilon_1) \qquad \forall\, \lambda \in \left(0, \frac{\lambda_0(\epsilon_1)}{\max_{x\in C} |x|}\right]\ , \ \forall\, x \in C,$$

and therefore

$$\left|\frac{\phi^{-1}(\lambda x)}{\lambda^{\frac{1}{d_0}}} - \left(\frac{x}{\varphi_0}\right)^{\frac{1}{d_0}}\right| \leq \epsilon \qquad \forall\, \lambda \in \left(0, \frac{\lambda_0(\epsilon_1)}{\max_{x\in C} |x|}\right]\ , \ \forall\, x \in C\ .$$

This establishes homogeneity in the 0-limit of the function ϕ^{-1}.

Appendix C. Proof of Lemma 2.13. The proof of this lemma is divided into three parts.

1. We first show, by contradiction, that there exists a real number c_0 satisfying

$$\eta_0(\theta) \;-\; c\,\gamma_0(\theta) \;<\; 0 \qquad \forall\,\theta \in S_{r_0}\;, \quad \forall\, c \,\geq\, c_0\;.$$

Suppose there is no such c_0. This means there is a sequence $(\theta_i)_{i\in\mathbb{N}}$ in S_{r_0} which satisfies

$$\eta_0(\theta_i) \;-\; i\,\gamma_0(\theta_i) \;\geq\; 0 \quad \forall\, i \;\in\; \mathbb{N}\;.$$

The sequence $(\theta_i)_{i\in\mathbb{N}}$ lives in a compact set. Thus we can extract a convergent subsequence $(\theta_{i_\ell})_{\ell\in\mathbb{N}}$ which converges to a point denoted θ_∞.
As the functions η_0 and γ_0 are bounded on S_{r_0} and γ_0 takes nonnegative values,[8] $\gamma_0(\theta_{i_\ell})$ must go to 0 as i_ℓ goes to infinity. Since the functions η_0 and γ_0 are continuous, we get $\gamma_0(\theta_\infty) = 0$ and $\eta_0(\theta_\infty) \geq 0$, which is impossible.
Consequently, there exist c_0 and $\varepsilon_0 > 0$ such that

$$\eta_0(\theta) \;-\; c\,\gamma_0(\theta) \;\leq\; -\varepsilon_0 \;<\; 0 \qquad \forall\,\theta \in S_{r_0}\;, \quad \forall\, c \,\geq\, c_0\;. \tag{C.1}$$

Moreover, since the functions η_0 and γ_0 are homogeneous in the standard sense (see Remark 2.6), we have the second inequality in (2.4).
Following the same argument, we can find positive real numbers c_∞ and ε_∞ such that

$$\eta_\infty(\theta) \;-\; c\,\gamma_\infty(\theta) \;<\; -\varepsilon_\infty \qquad \forall\,\theta \in S_{r_\infty}\;, \quad \forall\, c \,\geq\, c_\infty, \tag{C.2}$$

and the third inequality in (2.4) holds.
In the rest of the proof, let

$$c_1 \;=\; \max\{c_0, c_\infty\}, \qquad \varepsilon_1 \;=\; \min\{\varepsilon_0, \varepsilon_\infty\}\;.$$

2. Since η and γ are homogeneous in the 0-limit, there exists λ_0 such that, for all $\lambda \in (0, \lambda_0]$ and all $\theta \,\in\, S_{r_0}$, we have

$$\eta(\lambda^{r_0} \diamond \theta) \;\leq\; \lambda^{d_0}\,\eta_0(\theta) \;+\; \lambda^{d_0}\,\frac{\varepsilon_1}{4}\;, \quad \lambda^{d_0}\,\gamma_0(\theta) \;-\; \lambda^{d_0}\,\frac{\varepsilon_1}{4c_1} \;\leq\; \gamma(\lambda^{r_0} \diamond \theta)\;,$$

which readily gives

$$\eta(\lambda^{r_0} \diamond \theta) \;-\; c_1\,\gamma(\lambda^{r_0} \diamond \theta) \;\leq\; \lambda^{d_0}\,\eta_0(\theta) \;+\; \lambda^{d_0}\,\frac{\varepsilon_1}{2} - c_1\lambda^{d_0}\,\gamma_0(\theta)\;.$$

Using (C.1), we get

$$\eta(\lambda^{r_0} \diamond \theta) \;-\; c_1\,\gamma(\lambda^{r_0} \diamond \theta) \;\leq\; -\lambda^{d_0}\,\frac{\varepsilon_1}{2} \qquad \forall\,\lambda \in (0, \lambda_0]\,,\; \forall\,\theta \,\in\, S_{r_0}\;,$$

and therefore, since γ takes nonnegative values,

$$\eta(\lambda^{r_0} \diamond \theta) - c\,\gamma(\lambda^{r_0} \diamond \theta) \;\leq\; -\lambda^{d_0}\frac{\varepsilon_1}{2} \quad \forall\,\lambda \in (0, \lambda_0]\,,\; \forall\,\theta \,\in\, S_{r_0}\;,\; \forall\, c \geq c_1\;.$$

[8]Indeed, if we had $\gamma_0(x) < 0$ for some x in $\mathbb{R}^n \setminus \{0\}$, by letting $\epsilon = -\frac{\gamma_0(x)}{2}$, the homogeneity in the 0-limit of γ would give a real number $\lambda > 0$ satisfying $\frac{\gamma(\lambda^{r_0}\diamond x)}{\lambda^{d_0}} \leq \gamma_0(x) + \epsilon = \frac{\gamma_0(x)}{2} < 0$. This contradicts the fact that γ takes nonnegative values only. Also by continuity we have $\gamma_0(0) \geq 0$.

Similarly, there exists λ_∞ satisfying

$$\eta(\lambda^{r_\infty}\diamond\theta) - c\,\gamma(\lambda^{r_\infty}\diamond\theta) \;\leq\; -\lambda^{d_\infty}\frac{\varepsilon_1}{2} \quad \forall\,\lambda\in[\lambda_\infty,+\infty)\;,\;\forall\,\theta\,\in\,S_{r_\infty}\;,\;\forall\,c\geq c_1\;.$$

Consequently, for each $c\,\geq\,c_1$, the set

$$\{x\,\in\,\mathbb{R}^n\setminus\{0\}\mid\eta(x)\;-\;c\,\gamma(x)\;\geq\;0\}\;,$$

if not empty, must be a subset of

$$C\;=\;\{x\,\in\,\mathbb{R}^n\;:\;|x|_{r_0}\;\geq\;\lambda_0\}\bigcup\{x\,\in\,\mathbb{R}^n\;:\;|x|_{r_\infty}\;\leq\;\lambda_\infty\}\;,$$

which is compact and does not contain the origin.

3. Suppose now that for all c the first inequality in (2.4) is not true. This means that, for all integers c larger than c_1, there exists x_c in $\mathbb{R}^n$ satisfying

$$\eta(x_c)\;-\;c\,\gamma(x_c)\;\geq\;0\;,$$

and therefore x_c is in C. Since C is a compact set, there is a convergent subsequence $(x_{c_\ell})_{\ell\in\mathbb{N}}$ which converges to a point denoted x^* different from zero. Also as above, we must have $\gamma(x^*)\,=\,0$ and $\eta(x^*)\,\geq\,0$. But this contradicts the assumption, namely,

$$\{\;x\in\mathbb{R}^n\setminus\{0\}\;,\quad\gamma(x)=0\;\}\qquad\Rightarrow\qquad\eta(x)<0\;.$$

Appendix D. Proof of Proposition 2.18. Because the vector field f is homogeneous in the ∞-limit, its approximating vector field f_∞ is homogeneous in the standard sense (see Remark 2.6). Let d_{V_∞} be a positive real number larger than $r_{\infty,i}$ for all i in $\{1,\ldots,n\}$. Following Rosier [29], there exists a C^1, positive definite, proper, and homogeneous function $V_\infty\;:\mathbb{R}^n\to\mathbb{R}_+$, with weight r_∞ and degree d_{V_∞}, satisfying

$$\frac{\partial V_\infty}{\partial x}(x)f_\infty(x)\;<\;0\quad\forall\,x\,\neq\,0\;. \tag{D.1}$$

From P1 in section 2.2, we know that the function $x\mapsto\frac{\partial V_\infty}{\partial x}(x)f(x)$ is homogeneous in the ∞-limit with associated triple $\left(r_\infty,\mathfrak{d}_\infty+d_{V_\infty},\frac{\partial V_\infty}{\partial x}(x)f_\infty(x)\right)$. Let

$$\epsilon_\infty\;=\;-\frac{1}{2}\max_{\theta\,\in\,S_{r_\infty}}\left\{\frac{\partial V_\infty}{\partial x}(\theta)f_\infty(\theta)\right\}\;,$$

and note that, by inequality (D.1), ϵ_∞ is a strictly positive real number. By the definition of homogeneity in the ∞-limit, there exists λ_∞ such that

$$\left|\frac{\frac{\partial V_\infty}{\partial x}(\lambda^{r_\infty}\diamond\theta)f(\lambda^{r_\infty}\diamond\theta)}{\lambda^{d_{V_\infty}+\mathfrak{d}_\infty}}-\frac{\partial V_\infty}{\partial x}(\theta)f_\infty(\theta)\right|\;\leq\;\epsilon_\infty\qquad\forall\,\theta\in S_{r_\infty}\;,\;\forall\,\lambda\geq\lambda_\infty\;.$$

This yields

$$\begin{aligned}\frac{\partial V_\infty}{\partial x}(\lambda^{r_\infty}\diamond\theta)f(\lambda^{r_\infty}\diamond\theta)&\leq\lambda^{d_{V_\infty}+\mathfrak{d}_\infty}\left(\frac{\partial V_\infty}{\partial x}(\theta)f_\infty(\theta)\;+\;\epsilon_\infty\right)\\&\leq-\,\lambda^{d_{V_\infty}+\mathfrak{d}_\infty}\,\epsilon_\infty\qquad\forall\,\theta\in S_{r_\infty}\;,\;\forall\,\lambda\geq\lambda_\infty\;,\end{aligned}$$

or in other words,

$$\frac{\partial V_\infty}{\partial x}(x)\, f(x) \;<\; 0 \quad \forall\, x \,:\, |x|_{r_\infty} \;\geq\; \lambda_\infty \;. \tag{D.2}$$

This establishes global asymptotic stability of the compact set

$$\mathcal{C}_\infty \;=\; \{x \,:\, V_\infty(x) \;\leq\; v_\infty\} \;,$$

where v_∞ is given by

$$v_\infty \;=\; \max_{|x|_{r_\infty} = \lambda_\infty} \{V_\infty(x)\} \;.$$

Appendix E. Proof of Theorem 2.20. The proof is divided into three steps. First, we define three Lyapunov functions V_0, V_m, and V_∞. Then we build another Lyapunov function V from these three. Finally, we show that its derivative along the trajectories of the system (2.7) and its homogeneous approximations are negative definite.

1. As established in the proof of Proposition 2.18, there exist a positive real number λ_∞ and a C^1 positive definite, proper, and homogeneous function $V_\infty \;: \mathbb{R}^n \to \mathbb{R}_+$, with weight r_∞ and degree d_{V_∞} satisfying (D.2). Similarly, there exist a number $\lambda_0 > 0$ and a C^1 positive definite, proper, and homogeneous function $V_0 \;: \mathbb{R}^n \to \mathbb{R}_+$, with weight r_0 and degree d_{V_0}, satisfying

$$\frac{\partial V_0}{\partial x}(x)\, f(x) \;<\; 0 \qquad \forall\, x \;:\; 0 \;<\; |x|_{r_0} \;\leq\; \lambda_0 \;. \tag{E.1}$$

Finally, the global asymptotic stability of the origin of the system $\dot{x} = f(x)$ implies the existence of a C^1, positive definite, and proper function $V_m \;: \mathbb{R}^n \to \mathbb{R}_+$ satisfying

$$\frac{\partial V_m}{\partial x}(x)\, f(x) \;<\; 0 \qquad \forall\, x \;\neq\; 0 \;. \tag{E.2}$$

2. Now we build a function V from the functions V_m, V_∞, and V_0. For this, we follow a technique used by Mazenc in [17] (see also [15]). Let v_∞ and v_0 be two strictly positive real numbers such that $v_0 < v_\infty$ and

$$v_\infty \;\geq\; \max_{x:\,|x|_{r_\infty} \leq \lambda_\infty} V_m(x), \qquad v_0 \;\leq\; \min_{x:\,|x|_{r_0} \geq \lambda_0} V_m(x) \;.$$

This implies

$$\begin{aligned} \{x \in \mathbb{R}^n \;:\; V_m(x) \;\geq\; v_\infty\} \;&\subseteq\; \{x \in \mathbb{R}^n \;:\; |x|_{r_\infty} \;\geq\; \lambda_\infty\} \;, \\ \{x \in \mathbb{R}^n \;:\; V_m(x) \;\leq\; v_0\} \;&\subseteq\; \{x \in \mathbb{R}^n \;:\; |x|_{r_0} \;\leq\; \lambda_0\} \;. \end{aligned}$$

Let ω_0 and ω_∞ be defined as

$$\omega_0 \;=\; \min_{x\,:\,\frac{1}{2}v_0 \leq V_m(x) \leq v_0} \frac{V_m(x)}{V_0(x)}, \qquad \omega_\infty \;=\; \max_{x\,:\,v_\infty \leq V_m(x) \leq 2\,v_\infty} \frac{V_m(x)}{V_\infty(x)} \;.$$

We have

$$\begin{aligned} \omega_\infty\, V_\infty(x) \;-\; V_m(x) \geq 0 \quad &\forall\, x \;:\; v_\infty \leq V_m(x) \leq 2\,v_\infty \;, \\ V_m(x) \;-\; \omega_0\, V_0(x) \geq 0 \quad &\forall\, x \;:\; \frac{1}{2}\,v_0 \leq V_m(x) \leq v_0 \;. \end{aligned}$$

Let

$$V(x) \;=\; \omega_\infty\,\varphi_\infty(V_m(x))V_\infty(x) + [1-\varphi_\infty(V_m(x))]\,\varphi_0(V_m(x))\,V_m(x) + \omega_0\,[1-\varphi_0(V_m(x))]\,V_0(x)\ ,$$

where φ_0 and φ_∞ are C^1 nondecreasing functions satisfying

$$\text{(E.3)}\qquad \varphi_0(s) \;=\; 0 \qquad \forall\, s \le \frac{1}{2}\,v_0, \qquad \varphi_0(s) \;=\; 1 \qquad \forall\, s \ge v_0\ ,$$

$$\text{(E.4)}\qquad \varphi_\infty(s) \;=\; 0 \qquad \forall\, s \le v_\infty, \qquad \varphi_\infty(s) \;=\; 1 \qquad \forall\, s \ge 2v_\infty\ .$$

Then V is C^1, positive definite, and proper. Moreover, by construction,

$$V(x) = \begin{cases} \omega_0\,V_0(x) & \forall\ x\ :\ V_m(x)\ \le\ \frac{1}{2}v_0\ ,\\ \varphi_0(V_m(x))\,V_m(x) + \omega_0\,[1-\varphi_0(V_m(x))]\,V_0(x) & \\ & \forall\ x\ :\ \frac{1}{2}v_0\ \le\ V_m(x)\ \le\ v_0\ ,\\ V_m(x) & \forall\ x\ :\ v_0\ \le\ V_m(x)\ \le\ v_\infty\ ,\\ \omega_\infty\,\varphi_\infty(V_m(x))V_\infty(x) + [1-\varphi_\infty(V_m(x))]\ V_m(x) & \\ & \forall\ x\ :\ v_\infty\ \le\ V_m(x)\ \le\ 2\,v_\infty\ ,\\ \omega_\infty\,V_\infty(x) & \forall\ x\ :\ V_m(x)\ \ge\ 2\,v_\infty\ . \end{cases}$$

Thus for each i in $\{1,\ldots,n\}$,

$$\text{(E.5)}\qquad \frac{\partial V}{\partial x_i}(x) \;=\; \omega_\infty\,\frac{\partial V_\infty}{\partial x_i}(x) \qquad \forall\ x\ :\ V_m(x)\ >\ 2\,v_\infty$$

and

$$\text{(E.6)}\qquad \frac{\partial V}{\partial x_i}(x) \;=\; \omega_0\,\frac{\partial V_0}{\partial x_i}(x) \qquad \forall\ x\ :\ V_m(x)\ <\ \frac{1}{2}v_0\ .$$

Since $\frac{\partial V_\infty}{\partial x_i}$ and $\frac{\partial V_0}{\partial x_i}$ are homogeneous in the standard sense, this proves that for each i in $\{1,\ldots,n\}$, $\frac{\partial V}{\partial x_i}$ is homogeneous in the bi-limit, with weights r_0 and r_∞ and degrees $d_{V_0} - r_{0,i}$ and $d_{V_\infty} - r_{\infty,i}$.

3. It remains to show that the Lie derivative of V along f is negative definite. To this end note that, for all x such that $\frac{1}{2}v_0\ \le\ V_m(x)\ \le\ v_0$,

$$\frac{\partial V}{\partial x}(x)f(x) \;=\; \varphi_0'(V_m(x))\,[V_m(x) - \omega_0\,V_0(x)]\frac{\partial V_m}{\partial x}(x)f(x) + \omega_0\,[1-\varphi_0(V_m(x))]\,\frac{\partial V_0}{\partial x}(x)f(x) + \varphi_0(V_m(x))\frac{\partial V_m}{\partial x}(x)f(x)$$

and, for all x such that $v_\infty\ \le\ V_m(x)\ \le\ 2\,v_\infty$,

$$\frac{\partial V}{\partial x}(x)f(x) \;=\; \varphi_\infty'(V_m(x))\,[\omega_\infty\,V_\infty(x) - V_m(x)]\frac{\partial V_m}{\partial x}(x)f(x) + \omega_\infty\,\varphi_\infty(V_m(x))\frac{\partial V_\infty}{\partial x}(x)f(x) + [1-\varphi_\infty(V_m(x))]\,\frac{\partial V_m}{\partial x}(x)f(x)\ .$$

By (D.2), (E.1), (E.2), (E.3), and (E.4), these inequalities imply

$$\frac{\partial V}{\partial x}(x)\,f(x)\ <\ 0 \qquad \forall x\ \neq\ 0\ ,$$

which proves the claim.

Appendix F. Proof of Corollary 2.21. Recall (1.6) and consider the functions $\eta_1 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $\gamma_1 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+$ defined as

$$\eta_1(x,\delta) = \frac{\partial V}{\partial x}(x)\left[f(x,\,\delta) - \frac{1}{2}\,f(x,\,0)\right], \quad \gamma_1(x,\delta) = \sum_{j=1}^{m} \mathfrak{H}\left(|\delta_j|^{\frac{d_{V_0}+\mathfrak{d}_0}{\mathfrak{r}_{0,j}}}, |\delta_j|^{\frac{d_{V_\infty}+\mathfrak{d}_\infty}{\mathfrak{r}_{\infty,j}}}\right).$$

These functions are homogeneous in the bi-limit with weights r_0 and r_∞ for x and $\mathfrak{r}_0$ and $\mathfrak{r}_\infty$ for δ and degrees $d_{V_0}+\mathfrak{d}_0$ and $d_{V_\infty}+\mathfrak{d}_\infty$. Since the function $x \mapsto \frac{\partial V}{\partial x}(x)\,f(x,0)$ is negative definite, then

$$\{(x,\delta) \in \mathbb{R}^{n+m} \setminus \{0\} \;:\; \gamma_1(x,\delta) \;=\; 0\} \quad \subseteq \{ \quad (x,\delta) \in \mathbb{R}^{n+m} \;:\; \eta_1(x,\delta) \;<\; 0\}\ .$$

Moreover, since the homogeneous approximations of η are negative definite, we get

$$\{(x,\delta) \in \mathbb{R}^{n+m} \setminus \{0\} \;:\; \gamma_{1,0}(x,\delta) \;=\; 0\} \quad \subseteq \{ \quad (x,\delta) \in \mathbb{R}^{n+m} \;:\; \eta_{1,0}(x,\delta) \;<\; 0\}\ ,$$
$$\{(x,\delta) \in \mathbb{R}^{n+m} \setminus \{0\} \;:\; \gamma_{1,\infty}(x,\delta) \;=\; 0\} \quad \subseteq \{ \quad (x,\delta) \in \mathbb{R}^{n+m} \;:\; \eta_{1,\infty}(x,\delta) \;<\; 0\}\ .$$

Hence, by Lemma 2.13, there exists a positive real number c_δ such that

$$\text{(F.1)} \qquad \frac{\partial V}{\partial x}(x)\left[f(x,\,\delta) - \frac{1}{2}\,f(x,\,0)\right] \;\le\; c_\delta \sum_{j=1}^{m} \mathfrak{H}\left(|\delta_j|^{\frac{d_{V_0}+\mathfrak{d}_0}{\mathfrak{r}_{0,j}}}, |\delta_j|^{\frac{d_{V_\infty}+\mathfrak{d}_\infty}{\mathfrak{r}_{\infty,j}}}\right)\ .$$

Consider now the functions $\eta_2 : \mathbb{R}^n \to \mathbb{R}_+$ and $\gamma_2 : \mathbb{R}^n \to \mathbb{R}_+$ defined as

$$\eta_2(x) \;=\; \mathfrak{H}\left(V(x)^{\frac{d_{V_0}+\mathfrak{d}_0}{d_{V_0}}}, V(x)^{\frac{d_{V_\infty}+\mathfrak{d}_\infty}{d_{V_\infty}}}\right), \qquad \gamma_2(x) = -\frac{1}{2}\,\frac{\partial V}{\partial x}(x)\,f(x,0)\ .$$

They are homogeneous in the bi-limit with weights r_0 and r_∞ and degrees $d_{V_0}+\mathfrak{d}_0$ and $d_{V_\infty}+\mathfrak{d}_\infty$. Since γ_2 and its homogeneous approximations are positive definite, by Corollary 2.15 there exists a positive real number c_V such that

$$\text{(F.2)} \qquad \frac{1}{2}\,\frac{\partial V}{\partial x}(x)\,f(x,0) \;\le\; -c_V\,\mathfrak{H}\left(V(x)^{\frac{d_{V_0}+\mathfrak{d}_0}{d_{V_0}}}, V(x)^{\frac{d_{V_\infty}+\mathfrak{d}_\infty}{d_{V_\infty}}}\right)\ .$$

The two inequalities (F.1) and (F.2) yield the claim.

Appendix G. Proof of Corollary 2.22. Let d_{V_0} and d_{V_∞} be such that the assumption of Theorem 2.20 holds. For each i in $\{1,\dots,m\}$, let $\mu_i : \mathbb{R}_+ \to \mathbb{R}_+$ be the strictly increasing function defined as (see (1.6))

$$\text{(G.1)} \qquad \mu_i(s) \;=\; \mathfrak{H}\left(s^{q_i}, s^{p_i}\right)\ ,$$

where

$$p_i = \frac{\mathfrak{d}_\infty + d_{V_\infty}}{\mathfrak{r}_{\infty,i}}, \qquad q_i = \frac{\mathfrak{d}_0 + d_{V_0}}{\mathfrak{r}_{0,i}}\ .$$

We first prove that the inequality given by Corollary 2.21 implies that the system (2.8), with δ as input and x as output, is ISS with a linear gain between $\sum_{i=1}^m \mu_i(|\delta_i|)$ and $\mathfrak{H}(|x|_{r_0}^{\mathfrak{d}_0+d_{V_0}}, |x|_{r_\infty}^{\mathfrak{d}_\infty+d_{V_\infty}})$. To do so we introduce the function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ as

$$\alpha(s) \;=\; \mathfrak{H}\left(s^{\frac{\mathfrak{d}_0+d_{V_0}}{d_{V_0}}}, s^{\frac{\mathfrak{d}_\infty+d_{V_\infty}}{d_{V_\infty}}}\right), \qquad s \ge 0\ .$$

This function is a bijection, strictly increasing, and homogeneous in the bi-limit with approximating functions $s^{\frac{d_{V_0}+\mathfrak{d}_0}{d_{V_0}}}$ and $s^{\frac{d_{V_\infty}+\mathfrak{d}_\infty}{d_{V_\infty}}}$. Moreover, from Proposition 2.10, the function $x \mapsto \alpha(V(x))$ is positive definite and homogeneous in the bi-limit with associated weights r_0 and r_∞ and degrees $\mathfrak{d}_0 + d_{V_0}$ and $\mathfrak{d}_\infty + d_{V_\infty}$. Moreover, its approximating homogeneous functions $V_0(x)^{\frac{d_{V_0}+\mathfrak{d}_0}{d_{V_0}}}$ and $V_\infty(x)^{\frac{d_{V_\infty}+\mathfrak{d}_\infty}{d_{V_\infty}}}$ are positive definite as well. Hence, we get from Corollary 2.15 the existence of a positive real number c_1 satisfying

$$\mathfrak{H}\left(|x|_{r_0}^{\mathfrak{d}_0+d_{V_0}}, |x|_{r_\infty}^{\mathfrak{d}_\infty+d_{V_\infty}}\right) \le c_1\,\alpha(V(x)) \qquad \forall\, x \in \mathbb{R}^n . \tag{G.2}$$

On the other hand, from inequality (2.9) in Corollary 2.21, we have the property

$$\begin{aligned} &\left\{(x,\delta) \in \mathbb{R}^n \times \mathbb{R}^m \,:\, \alpha(V(x)) \ge 2\,\frac{c_\delta}{c_V} \sum_{i=1}^m \mu_i(|\delta_i|)\right\} \\ &\qquad \subseteq \left\{(x,\delta) \in \mathbb{R}^n \times \mathbb{R}^m \,:\, \frac{\partial V}{\partial x}(x)\, f(x,\delta) \le -\frac{c_V}{2}\,\alpha(V(x))\right\} . \end{aligned} \tag{G.3}$$

In the following, let $t \in [0,T) \mapsto (x(t),\delta(t),z(t))$ be any function which satisfies (2.8) on $[0,T)$ and (2.10) and (2.11) for all $0 \le s \le t \le T$. From [32], we know the inclusion (G.3) implies the existence of a class $\mathcal{KL}$ function β_V such that, for all $0 \le s \le t \le T$,

$$V(x(t)) \le \max\left\{\beta_V(V(x(s)), t-s)\,,\, \sup_{s\le\kappa\le t}\left\{\alpha^{-1}\left(\frac{2c_\delta}{c_V}\sum_{j=1}^m \mu_j(|\delta_j(\kappa)|)\right)\right\}\right\}. \tag{G.4}$$

With α acting on both sides of inequality (G.4), (G.2) gives, for all $0 \le s \le t \le T$,

$$\begin{aligned} &\mathfrak{H}\left(|x(t)|_{r_0}^{\mathfrak{d}_0+d_{V_0}}, |x(t)|_{r_\infty}^{\mathfrak{d}_\infty+d_{V_\infty}}\right) \\ &\quad \le \max\left\{c_1\,\alpha\circ\beta_V(V(x(s)), t-s)\,,\, \frac{2c_1c_\delta}{c_V}\sup_{s\le\kappa\le t}\left\{\sum_{j=1}^m \mu_j(|\delta_j(\kappa)|)\right\}\right\} . \end{aligned} \tag{G.5}$$

This is the linear gain property required. To conclude the proof it remains to show the existence of c_G such that a small gain property is satisfied.

First, note that the function $x \mapsto \mathfrak{H}(|x|_{r_0}^{\mathfrak{d}_0+d_{V_0}}, |x|_{r_\infty}^{\mathfrak{d}_\infty+d_{V_\infty}})$ is positive definite and homogeneous in the bi-limit with weights r_0 and r_∞ and degrees $\mathfrak{d}_0 + d_{V_0}$ and $\mathfrak{d}_\infty + d_{V_\infty}$. By Proposition 2.10, for i in $\{1,\ldots,m\}$ the same holds with the function $x \mapsto \mu_i\left(\mathfrak{H}\left(|x|_{r_0}^{\mathfrak{r}_{0,i}}, |x|_{r_\infty}^{\mathfrak{r}_{\infty,i}}\right)\right)$. Hence, by Corollary 2.15, there exists a positive real number c_2 satisfying

$$\mu_i\left(\mathfrak{H}\left(|x|_{r_0}^{\mathfrak{r}_{0,i}}, |x|_{r_\infty}^{\mathfrak{r}_{\infty,i}}\right)\right) \le c_2\,\mathfrak{H}\left(|x|_{r_0}^{\mathfrak{d}_0+d_{V_0}}, |x|_{r_\infty}^{\mathfrak{d}_\infty+d_{V_\infty}}\right) \quad \forall\, x \in \mathbb{R}^n . \tag{G.6}$$

Let C_i for i in $\{1,\ldots,m\}$ be the class $\mathcal{K}_\infty$ functions defined as

$$C_i(c) = \max\{c^{q_i}, c^{p_i}\} + c^{\frac{p_i q_i}{q_i+p_i}} + c^{p_i+q_i} .$$

From (G.1), we get, for each $s > 0$ and $c > 0$,

$$\frac{\mu_i(cs)}{\mu_i(s)} = c^{q_i}\frac{(1+s^{q_i})(1+c^{p_i}s^{p_i})}{(1+s^{p_i})(1+c^{q_i}s^{q_i})} \le c^{q_i}\left[\frac{1+c^{p_i}s^{p_i+q_i}}{1+c^{q_i}s^{p_i+q_i}} + \frac{s^{q_i}}{1+c^{q_i}s^{q_i+p_i}} + \frac{c^{p_i}s^{p_i}}{1+s^{p_i}}\right] ,$$

where

$$c^{q_i}\frac{1+c^{p_i}s^{p_i+q_i}}{1+c^{q_i}s^{p_i+q_i}} \leq \max\{c^{q_i},c^{p_i}\}\ , \quad \frac{c^{q_i}s^{q_i}}{1+c^{q_i}s^{q_i+p_i}} \leq c^{\frac{p_i q_i}{q_i+p_i}}\ , \quad \frac{c^{q_i}c^{p_i}s^{p_i}}{1+s^{p_i}} \leq c^{p_i+q_i}\ .$$

Hence, by continuity at 0, we have

$$\mu_i(c\,s)\ \leq\ C_i(c)\,\mu_i(s) \qquad \forall\ (c,s)\in\mathbb{R}_+^2\ . \tag{G.7}$$

Consider the positive real numbers c_1, c_2, c_δ, and c_V previously introduced, and select c_G in $\mathbb{R}_+$ satisfying

$$c_G\ <\ \min_{1\leq i\leq m} C_i^{-1}\left(\frac{c_V}{2\,m\,c_1\,c_2\,c_\delta}\right)\ . \tag{G.8}$$

To show that such a selection for c_G is appropriate, observe that by (G.6) and (G.7) and μ_i acting on both sides of the inequality (2.11), we get for each i in $\{1,\ldots,m\}$ and all $0\leq s\leq t\leq T$,

$$\begin{aligned}\mu_i(|\delta_i(t)|)\ \leq\ \max\Big\{&\mu_i\circ\beta_\delta(|z(s)|,t-s)\,,\\ &C_i(c_G)\,c_2\sup_{s\leq\kappa\leq t}\left\{\mathfrak{H}\left(|x(\kappa)|_{r_0}^{\mathfrak{d}_0+d_{V_0}},|x(\kappa)|_{r_\infty}^{\mathfrak{d}_\infty+d_{V_\infty}}\right)\right\}\Big\}\ .\end{aligned}$$

Consequently

$$\begin{aligned}\sum_{i=1}^m\mu_i(|\delta_i(t)|)\ \leq\ \max\Big\{&m\max_{1\leq i\leq m}\{\mu_i\circ\beta_\delta(|z(s)|,t-s)\}\,,\\ &(m\max\nolimits_{1\leq i\leq m}C_i(c_G)\,c_2)\sup\nolimits_{s\leq\kappa\leq t}\left\{\mathfrak{H}\left(|x(\kappa)|_{r_0}^{\mathfrak{d}_0+d_{V_0}},|x(\kappa)|_{r_\infty}^{\mathfrak{d}_\infty+d_{V_\infty}}\right)\right\}\Big\}\ .\end{aligned} \tag{G.9}$$

Since (G.8) yields

$$\frac{2c_1c_\delta}{c_V}\,m\max_{1\leq i\leq m}C_i(c_G)\,c_2\ <\ 1\ ,$$

the existence of the function β_x follows from (2.10), (G.5), (G.9), and the (proof of the) small-gain theorem [11].

Appendix H. Proof of Corollary 2.24. First, observe that the continuity of f_0, at least, on $\mathbb{R}^n\setminus\{0\}$ implies

$$|\mathfrak{d}_0|\ =\ -\mathfrak{d}_0\ \leq\ \min_{1\leq i\leq n} r_{0,i}\ \leq\ \max_{1\leq i\leq n} r_{0,i}\ <\ d_{V_0}\ .$$

Then, let V be the function given in Theorem 2.20 and, since $\mathfrak{d}_0<0<\mathfrak{d}_\infty$, the function $\phi(x)=V(x)^{\frac{d_{V_0}+\mathfrak{d}_0}{d_{V_0}}}\ +\ V(x)^{\frac{d_{V_\infty}+\mathfrak{d}_\infty}{d_{V_\infty}}}$ is homogeneous in the bi-limit with weights r_0 and r_∞, degrees $d_{V_0}+\mathfrak{d}_0$ and $d_{V_\infty}+\mathfrak{d}_\infty$, and approximating functions $V(x)^{\frac{d_{V_0}+\mathfrak{d}_0}{d_{V_0}}}$ and $V(x)^{\frac{d_{V_\infty}+\mathfrak{d}_\infty}{d_{V_\infty}}}$. Moreover, the function $\zeta(x)\ =\ -\frac{\partial V}{\partial x}(x)\,f(x)$ is homogeneous in the bi-limit with the same weights and degrees as ϕ. Furthermore, since the function ζ and its homogeneous approximations are positive definite, Corollary 2.15 yields a strictly positive real number c such that

$$\frac{\partial V}{\partial x}(x)\,f(x)\ \leq\ -c\left(V(x)^{\frac{d_{V_0}+\mathfrak{d}_0}{d_{V_0}}}\ +\ V(x)^{\frac{d_{V_\infty}+\mathfrak{d}_\infty}{d_{V_\infty}}}\right) \qquad \forall\ x\in\mathbb{R}^n\ . \tag{H.1}$$

Let x_{ic} in $\mathbb{R}^n \setminus \{0\}$ be the initial condition of a solution of the system $\dot{x} = f(x)$, and let $V_{x_{ic}} : \mathbb{R}_+ \to \mathbb{R}_+$ be the function of time given by the evaluation of V along this solution. Then

$$\dot{\overbrace{V_{x_{ic}}(t)}} \;\leq\; -c\, V_{x_{ic}}(t)^{\frac{d_{V_\infty}+\mathfrak{d}_\infty}{d_{V_\infty}}} \qquad \forall\, t \geq 0\ ,$$

from which we get

$$V_{x_{ic}}(t) \;\leq\; \frac{1}{\left(\frac{\mathfrak{d}_\infty}{d_{V_\infty}}\, ct + V(x_{ic})^{-\frac{\mathfrak{d}_\infty}{d_{V_\infty}}}\right)^{\frac{d_{V_\infty}}{\mathfrak{d}_\infty}}} \;\leq\; \frac{1}{\left(\frac{\mathfrak{d}_\infty}{d_{V_\infty}}\, ct\right)^{\frac{d_{V_\infty}}{\mathfrak{d}_\infty}}} \qquad \forall\, t > 0\ .$$

Therefore, setting $T_1 = \frac{d_{V_\infty}}{c\mathfrak{d}_\infty}$, we have

$$V_{x_{ic}}(t) \;\leq\; 1 \qquad \forall\, t \geq T_1, \qquad \forall\, x_{ic} \in \mathbb{R}^n$$

and

$$\dot{\overbrace{V_{x_{ic}}(t)}} \;\leq\; -c\, V_{x_{ic}}(t)^{\frac{d_{V_0}-|\mathfrak{d}_0|}{d_{V_0}}} \qquad \forall\, t \geq 0\ .$$

As a result, we get

$$\begin{aligned} V_{x_{ic}}(t) &\leq \max\left\{\left(-\frac{|\mathfrak{d}_0|}{d_{V_0}}\, c(t-T_1) + V_{x_{ic}}(T_1)^{\frac{|\mathfrak{d}_0|}{d_{V_0}}}\right)^{\frac{d_{V_0}}{|\mathfrak{d}_0|}},\, 0\right\}\ , \\ &\leq \max\left\{\left(1 - \frac{|\mathfrak{d}_0|}{d_{V_0}}\, c(t-T_1)\right)^{\frac{d_{V_0}}{|\mathfrak{d}_0|}},\, 0\right\} \qquad \forall\, t \geq T_1\ . \end{aligned}$$

Therefore, setting $T_2 = \frac{d_{V_0}}{c|\mathfrak{d}_0|}$ yields

$$V_{x_{ic}}(t) \;=0 \qquad \forall\, t \;\geq\; T_1 + T_2 = \frac{1}{c}\left(\frac{d_{V_\infty}}{\mathfrak{d}_\infty} + \frac{d_{V_0}}{|\mathfrak{d}_0|}\right), \qquad \forall\, x_{ic} \in \mathbb{R}^n\ ,$$

hence the claim.

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