Global Asymptotic Stabilization for Nonminimum Phase Nonlinear Systems Admitting a Strict Normal Form

V. Andrieu and Laurent Praly

Abstract—**In this paper, we address the problem of global asymptotic stabilization by output feedback for nonminimum phase nonlinear systems which admit a strict normal form. We assume the knowledge of an observer and, depending on its properties, we propose various approaches to design the control law. Each of these approaches needs a different stabilizability assumption on the inverse dynamics. In this way, within a unified framework, we recover and extend some already published results and we establish new ones.**

Index Terms—**Backstepping, nonmimum phase system, observer, output feedback.**

I. Introduction

We address the problem of global asymptotic stabilization by output feedback for systems whose dynamics can be written in the following strict[1] normal form:

$$\begin{cases} \dot{z} = F(z, \xi_1) \\ \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = \xi_3 \\ \quad \vdots \\ \dot{\xi}_n = f(z, \xi_1, \dots, \xi_n) + g(\xi_1)\, u \\ y = \xi_1 \end{cases} \tag{1}$$

with state $(z, \xi_1, \dots, \dots, \xi_n)$ where z is in $\mathbb{R}^m$ and ξ_i is in $\mathbb{R}$, and where the function F is in C^{n+1}, the functions f and g are in C^1 and we have:

$$F(0,0) = 0, \quad f(0,0,\dots,0) = 0$$
$$g(\xi_1) > 0 \quad \forall\, \xi_1 \in \mathbb{R}.$$

Systems whose dynamics can be written in the form (1) have been fully characterized, upon input scaling, by a coordinate free condition by Byrnes and Isidori in ([9, Corollary 5.7]). It is trivially satisfied by any system whose dynamics can be written as

$$\begin{cases} \dot{y} = b_1(y) + a_1(y)\, y_2 \\ \dot{y}_2 = b_2(y, y_2) + a_2(y)\, y_3 \\ \quad \vdots \\ \dot{y}_{n-1} = b_{n-1}(y, y_2, \dots, y_{n-1}) + a_{n-1}(y)\, y_n \\ \dot{y}_n = b_n(y, y_2, \dots, y_n) + c_0(y)\, z_1 + a_n(y)\, u \\ \dot{z}_1 = f_1(y, z_1) + c_1(y, z_1)\, z_2 \\ \quad \vdots \\ \dot{z}_{m-1} = f_{m-1}(y, z_1, \dots, z_{m-1}) \\ \quad +c_{m-1}(y, z_1, \dots, z_{m-1})\, z_m \\ \dot{z}_m = f_m(y, z_1, \dots, z_m) \\ \quad +c_m(y, z_1, \dots, z_m)\, y \end{cases} \tag{2}$$

where the a_i's and c_j's take positive values. This is one of the most general (nominal) form for which we know how to design a globally asymptotically stabilizing output feedback and whose study has been initiated by Kanellakopoulos, Kokotović and Morse in [16] and Marino and Tomei in [22]. In these works, the problem has been solved by imposing some restriction on the nonlinearities (in [16] and [22], the b_j's in (2) depend only of y) and by assuming that the z dynamics, the inverse dynamics, are linear in z and with an asymptotic stability property—the minimum phase assumption. From these original publications, many other results have been obtained, relaxing more and more the restriction on the functions b_j's but, for most of them, still preserving the minimum-phase assumption (see for instance [30], [28], [18], [31], [13], and [2] and references therein).

Until recently the only significant results concerning nonminimum phase nonlinear systems were about semiglobal stability, invoking high gain observers (see [8], [33] for instance). But fortunately, the minimum-phase assumption in the global stability case has been relaxed now, in particular by Karagiannis, Jiang, Ortega and Astolfi in [17], Marino and Tomei in [21] and by ourselves in the preliminary version [3] of this paper. In these contributions, the authors replace the minimum phase assumption by some specific form of state-stabilizability of the inverse dynamics. In other words, they assume (explicitly in [17] and [3] and implicitly in [21]) the existence of a function ϕ_z such that the origin of the following system:

$$\dot{z} = F(z, \phi_z(z)) \tag{3}$$

is globally asymptotically stable. In [3], it is shown that, up to a regularity assumption, the existence of ϕ_z is necessary for the solvability of the output feedback stabilization problem for the

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[1]Only ξ_1 is present in the expression of $\dot{z}$.

system (1). Actually, more is required in [17], [21], and [3]. Not only the origin should be asymptotically stable for (3) but this should be in a robust way with respect to disturbances which may act differently, depending on the context.

The unifying formalism we propose here allows us to rephrase and/or obtain output feedback stabilizers for the system (1) without a minimum-phase assumption and under various sets of assumptions. In Section II, by exploiting a result of Freeman and Kokotović [11], we obtain a new result by following what we call the *state disturbance* or *direct* approach. In this case, the assumption is, in spirit, about the Input-to-State Stability (ISS) property of the following auxiliary system:

$$\dot{z} = F(z, \phi_z(z + d_z) + d_u) \tag{4}$$

where the disturbances d_z and d_u act as measurement error and input disturbance respectively. In Section III, another result (encompassing the ones by Marino and Tomei [21] and Andrieu and Praly [3]) is obtained by following the *dynamics error* or *indirect* approach. There, the assumption is on:

$$\dot{z} = F(z, \phi_z(z)) + d$$

where the disturbance d acts externally. Finally, in Section IV, we show how by combining the two previous approaches and relying on an assumption on the system (4), we can recover the result of Karagiannis, Jiang, Ortega and Astolfi [17] in the case with no disturbances (or actually when they are part of the known system). Section V is devoted to illustrating examples and Section VI contains our conclusions.

Above, we have quoted only the references in very direct relation with our topic and in particular with the non minimum phase case. Many other results are available, among which the most recent ones approach the design of output feedback via domination where the dominant model is mainly a simple chain of integrators but for which the output feedback is designed to cope with large disturbances. For this, it incorporates dynamically updated high gain controller and observer as in [13], [18] for instance or terms of higher order as dictated by weighted homogeneity as in [2], [31] for instance.

In the following proofs and examples, we focus on the ideas and concepts. Instead we do not detail the computations in particular when they could become heavy without bringing anymore light on our topic.

II. The State Disturbance or Direct Approach

A. The Context

The popular "separation principle" is not true in general for global asymptotic stabilization. Nevertheless the following separation recipe is appropriate:

If we have
1) an observer that provides boundedness and asymptotic convergence to zero of the state estimation error, independently of the control;
2) a state feedback that renders the system ISS with respect to additive error in its argument;

Then we can cook up a globally asymptotically stable output feedback.

This recipe, which may not have been written per se before, has already been followed by several authors (see, for instance, [6], [10], [34], and [31]).

The first step in following this recipe is to introduce a state observer. For this, we rewrite the dynamics of the system (1) in other coordinates. Given $2(n-1)$ arbitrary but sufficiently differentiable functions $(a_i)_{1\leq i\leq n-1}$, which take positive values, and $(b_i)_{1\leq i\leq n-1}$, there exist two other functions a_n, which takes positive values, and b_n, and a diffeomorphism

$$(z, \xi_1, \dots, \xi_n)^T \mapsto (z, y, \dots, y_n)^T \tag{5}$$

such that the dynamics of the system (1) can be rewritten in

$$\begin{cases} \dot{z} = F(z, y) \\ \dot{y} = a_1(z, y) y_2 + b_1(z, y) \\ \dot{y}_2 = a_2(z, y, y_2) y_3 + b_2(z, y, y_2) \\ \quad \vdots \\ \dot{y}_{n-1} = a_{n-1}(z, y, y_2, \dots, y_{n-1}) y_n \\ \qquad + b_{n-1}(y, y_2, \dots, y_{n-1}) \\ \dot{y}_n = a_n(y) u + b_n(z, y, y_2, \dots, y_n). \end{cases} \tag{6}$$

We insist here for having a_n to depend only on y.

With collecting z and y_2 to y_n into a single state vector x in $\mathbb{R}^{n+m-1}$, the dynamics (6) take the following form:

$$\begin{cases} \dot{x} = A(x, y) + B(y) u \\ \dot{y} = C(x, y). \end{cases} \tag{7}$$

Our detectability assumption is expressed as follows:

Assumption SD-D (State Disturbance, Detectability): The coordinates for z and the functions $(a_i)_{1\leq i\leq n-1}$, and $(b_i)_{1\leq i\leq n-1}$ can be chosen in such a way that there exist a C^{n+1} function $K : \mathbb{R} \to \mathbb{R}^{n+m-1}$ of y and a positive-definite symmetric matrix P satisfying

$$P \frac{\partial A - KC}{\partial x}(x, y) + \frac{\partial A - KC}{\partial x}(x, y)^T P < 0 \qquad \forall (x, y) \in \mathbb{R}^{n+m-1} \times \mathbb{R}. \tag{8}$$

This assumption is discussed in the next section.

Following the recipe, the second step is to find a state feedback $\phi : \mathbb{R}^{n+m} \to \mathbb{R}$ such that the system

$$\begin{cases} \dot{x} = A(x, y) + B(y)\phi(x + e, y) \\ \dot{y} = C(x, y) \end{cases} \tag{9}$$

is ISS with input e in $\mathbb{R}^{n+m-1}$. As far as we know, this problem has not been solved for systems of the form (6). However, Freeman and Kokotović have given a solution in [11] for the particular case where the z-dynamics can also be written in a strict feedback form. Specifically the appropriate assumption is as follows.

Assumption SD-S (State Disturbance, Stabilizability): The z dynamics have a strict feedback form, i.e., they are

$$\begin{cases} \dot{z}_1 = f_1(z_1) + c_1(z_1) z_2 \\ \quad \vdots \\ \dot{z}_{m-1} = f_{m-1}(z_1, \dots, z_{m-1}) \\ \qquad + c_{m-1}(z_1, \dots, z_{m-1}) z_m \\ \dot{z}_m = f_m(z_1, \dots, z_m) + c_m(z_1, \dots, z_m) y \end{cases} \tag{10}$$

where the functions c_j take strictly positive values.

Designing a globally stabilizing output feedback under assumptions SD-D and SD-S is an easy task in principle by following the procedure proposed by Freeman and Kokotović in [11] and by invoking the ISS formalism. Precisely, we have the following.

Theorem 1 (State Disturbance Approach): If the assumptions SD-D and SD-S hold then there exists a globally stabilizing dynamic output feedback of dimension $m+n-1$.

Proof: With assumption SD-S, the dynamics of the system (6) have a strict feedback form. Thus we can apply the design given in [11] to get a C^1 function $\phi : \mathbb{R}^{n+m} \to \mathbb{R}$ and a C^1, positive-definite and proper function $V : \mathbb{R}^{n+m} \to \mathbb{R}_+$ such that along the solutions of the system (9) we get, for some function γ of class $\mathcal{K}_\infty$ and for all (x, y) in $\mathbb{R}^{n+m}$ and e in $\mathbb{R}^{n+m-1}$

$$\frac{\partial V}{\partial x}(x,y)[A(x,y)+B(y)\phi(x-e,y)] + \frac{\partial V}{\partial y}(x,y)C(x,y) \le -V(x,y)+\gamma(|e|). \quad (11)$$

The output feedback is then defined as $u = \phi(\hat{x}, y)$ where $\hat{x} = (\hat{z}, \hat{y}_2, \ldots, \hat{y}_n)$ is given by the following (reduced order) observer:

$$\begin{cases} \hat{x} = w + M(y) \\ \dot{w} = A(\hat{x},y) + B(y)u - K(y)C(\hat{x},y) \end{cases} \quad (12)$$

with $M(y) = \int_0^y K(s)\,ds$.

With this feedback, the dynamics of the closed-loop system can be written as

$$\begin{cases} \dot{x} = A(x,y) + B(y)\phi(x-e,y) \\ \dot{y} = C(x,y), \\ \dot{e} = A(x,y) - A(x-e,y) \\ \qquad -K(y)[C(x,y)-C(x-e,y)] \end{cases} \quad (13)$$

where $e = x - \hat{x}$. It is seen as the interconnection of the system to be controlled and the error system. We have

$$\begin{aligned} &A(x,y) - A(x-e,y) \\ &\quad - K(y)[C(x,y)-C(x-e,y)] \\ &\quad = \left[\int_0^1 \frac{\partial A - KC}{\partial x}(x+(1-s)e,y)\,ds\right] e. \end{aligned} \quad (14)$$

So, with (8), to any compact subset $\mathcal{C}$ of $\mathbb{R}^{2(n+m)-1}$, we can associate a strictly positive real number c satisfying

$$\dot{\overbrace{e^T P e}} \le -c e^T P e \quad \forall (x,y,e) \in \mathcal{C}. \quad (15)$$

Inequalities (15) and (11) imply successively that, along the solutions of the closed-loop system, $|e|$ and $V(x, y)$ are bounded. Specifically, we get, for all $t \ge 0$

$$\begin{aligned} &e(t)^T P e(t) \le e(0)^T P e(0), \\ &V(x(t),y(t)) \le V(x(0),y(0)) + \gamma\left(\frac{e(0)^T P e(0)}{\lambda_{\min}(P)}\right) \end{aligned}$$

where $\lambda_{\min}(P)$ is the smallest eigenvalue of P and the argument t represents the time for the evaluation of the argument of the functions along the solution. These inequalities imply the global stability of the origin. Therefore, with (15), for each initial condition, there exists a strictly positive real number c such that

$$|e(t)| \le \exp(-ct)|e(0)|\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}. \quad (16)$$

But, with the variation of constant formula and by splitting the integration interval $[0, t]$ in $[0, (t/2)] \cup ((t/2), t]$, (11) gives, still for any solution and for any $t \ge 0$

$$\begin{aligned} V(x(t),y(t)) \le{} & \exp(-t)V(x(0),y(0)) \\ & + \exp\left(-\frac{t}{2}\right) \sup_{0\le s\le \frac{t}{2}} \gamma(|e(s)|) + \sup_{\frac{t}{2}\le s} \gamma(|e(s)|). \end{aligned}$$

With (16), this implies that $V(x(t), y(t))$ and therefore $x(t)$ and $y(t)$ converge to 0 as t goes to infinity. This establishes the global attractivity of the origin. ■

B. Discussion

1) On the Detectability Assumption SD-D: Guaranteeing the existence of a reduced order observer from assumption SD-D is a triviality. We have given ourselves this derogation in writing this assumption since sufficient conditions for it to hold are known. Specifically, the following:

- ***Monotonic nonlinearities:*** Following Arcak and Kokotović [5], consider the case where we can find a function K leading to the following decomposition:

$$A(x,y) - K(y)C(x,y) = Fx + Q(y) + \sum_{i=1}^{n+m-1} G_i \gamma_i\left(L_i^T x, y\right)$$

where, $\forall (s, y) \in \mathbb{R}^2$

$$-\infty < a_i \le \frac{\partial \gamma_i}{\partial s}(s,y) \le b_i \le +\infty.$$

Proposition 1 ([5]): In this context, the inequality (8) holds if there exists a positive-definite matrix symmetric P and real numbers $\lambda_i \ne 0$ satisfying

$$\begin{aligned} &\sum_{i=1}^{n+m-1} \frac{b_i}{4}\left|\lambda_i L_i + \frac{P}{\lambda_i} G_i\right|^2 \\ &\quad - \sum_{i=1}^{n+m-1} \frac{a_i}{4}\left|\lambda_i L_i - \frac{P}{\lambda_i} G_i\right|^2 \\ &\quad + PF + F^T P \le -I. \end{aligned}$$

- ***Output-dependent incremental rate:*** Following Krishnamurthy and Khorrami [19], we consider systems admitting the representation (2). To simplify our presentation, we introduce the functions $\phi_{i,j}$ and ψ_i as

$$\phi_{i,j} = \begin{cases} \frac{\partial b_i}{\partial y_j} & 2 \le j \le i \le n \\ 0 & n+1 \le i \le m, \quad 2 \le j \le n \\ \frac{\partial f_{i-n}}{\partial z_{j-n}} & n+1 \le j \le i \le n+m \end{cases} \quad (17)$$

$$\psi_i = \begin{cases} a_i & 2 \le i \le n-1 \\ c_{i-n} & n \le i \le n+m-1. \end{cases} \quad (18)$$

Proposition 2 ([19, Th. 2]): If there exists a positive real number ρ, such that, for all $(\mathcal{X}, y)$ in $\mathbb{R}^{n+m}$, we have

$$\begin{aligned} \rho &\leq \psi_i, \quad 2 \leq i \leq n+m-1 \\ \rho\psi_i &\leq \psi_{i-1}, \quad 3 \leq i \leq n+m-1 \\ \rho|\phi_{i,j}| &\leq \sqrt{\psi_i \psi_{j-1}}, \quad 3 \leq i \leq n+m-1 \\ & \qquad\qquad\qquad 3 \leq j \leq i \\ \rho|\phi_{n+m,j}| &\leq \sqrt{\psi_{n+m-1}\psi_{j-1}}, \quad 3 \leq j \leq n+m \end{aligned}$$

then there exists a continuous function K and a matrix P such that (8) is satisfied.

Another point about the detectability assumption SD-D is that, very often, the degree of freedom left in the definition of the functions a_i's and b_i's is forgotten in the literature. To illustrate it, consider the following second order system with no inverse dynamics

$$\begin{cases} \dot{\xi}_1 = \xi_2, \\ \dot{\xi}_2 = f(\xi_1, \xi_2) + g(\xi_1)u \\ y = \xi_1. \end{cases} \tag{19}$$

Then the functions a_1, a_2, b_1, and b_2 are free up to satisfying the following constraints:

$$\begin{aligned} &g(y) = a_1(y)a_2(y) > 0 \\ &f(y, a_1(y)y_2 + b_1(y)) \\ &\quad = [a_1'(y)y_2 + b_1'(y)][a_1(y)y_2 + b_1(y)] \\ &\qquad + a_1(y)\, b_2(y, y_2). \end{aligned}$$

It follows that (8) holds if we can find a positive C^1 function a_1 and a C^1 function ℓ such that we have

$$\frac{\partial f}{\partial \xi_2}(\xi_1, \xi_2) < \ell(\xi_1) + 2\frac{a_1'(\xi_1)}{a_1(\xi_1)}\xi_2 \quad \forall(\xi_1, \xi_2). \tag{20}$$

Indeed, in this case, we pick

$$\begin{aligned} a_2(y) &= \frac{g(y)}{a_1(y)}, \quad b_1(y) = 0 \\ b_2(y, y_2) &= \frac{f(y, a_1(y)y_2) - a_1(y)a_1'(y)y_2^2}{a_1(y)} \end{aligned}$$

and (8) holds with

$$P = 1, \quad k_2(y) = \frac{\ell(y)}{a_1(y)}.$$

2) On the Stabilizability Assumption SD-S: The specific strict feedback form imposed on the z-dynamics by the stabilizability assumption SD-S implies the existence of a function ϕ_z such that the origin of (3) is globally asymptotically stable. We have mentioned in the introduction that the existence of ϕ_z is "almost" necessary. So the main restriction imposed by assumption SD-S is the fact that, as proved by Freeman and Kokotović [11], it allows us to get a function ϕ_z which not only stabilizes asymptotically the origin of the z-subsystem but also ensures the ISS property of the following auxiliary system:

$$\dot{z} = F(z, \phi_z(z + d_z) + d_u) \tag{21}$$

with input (d_z, d_u) (see assumption SD-S' in Section IV). It would be very useful to know whether or not, assumption SD-S can be replaced by this ISS property, i.e., whether or not the recursive Lyapunov design of [11] applies or can be modified to get V and ϕ satisfying (11).

III. The Dynamics Error or Indirect Approach

This section is a reproduction of our conference paper [3].

A. The Context

Another usual approach to design an output feedback is again to design the observer first but then to design the state feedback for this observer and not for the system to be controlled as done in the previous section. Specifically, the state feedback is designed for the following system with state $(\hat{\mathcal{X}}, y)$ given by the observer (12)

$$\begin{cases} \dot{y} = C(\hat{\mathcal{X}}, y) + \Delta C \\ \dot{\hat{\mathcal{X}}} = A(\hat{\mathcal{X}}, y) + B(y)u + K(y)\Delta C \end{cases} \tag{22}$$

where, the term $\Delta C = C(\mathcal{X}, y) - C(\hat{\mathcal{X}}, y)$ is the correction term. Despite, this term is a good term for the observer, it is considered as a disturbance in the design of the state feedback. This approach that we call the *dynamics error* or *indirect* approach, is therefore the application of another separation recipe as follows.

If we have
1) An observer providing L^2-correction terms;
2) A state feedback making the system L^2-ISS;

Then we can cook up a globally asymptotically stable output feedback law.

Again, this recipe may not have been formalized in this way previously (see, however, [4], [29]) but it is certainly not new. Most of the published results on output feedback stabilization, starting from [16], [22], can be reinterpreted along its lines (see, for instance, [30], [28], [13], [21], [2]). To follow this recipe, we propose the following set of assumptions.

Assumption DE-D2 (Dynamics Error, L^2-Detectability):

DE-D2.1: The coordinates for z and the functions $(a_i)_{1\leq i\leq n-1}$ and $(b_i)_{1\leq i\leq n-1}$ can be chosen in such a way that there exist a C^{n+1} function K of y and a positive-definite symmetric matrix P satisfying

$$\begin{aligned} P\frac{\partial A - KC}{\partial \mathcal{X}}(\mathcal{X}, y) &+ \frac{\partial A - KC}{\partial \mathcal{X}}(\mathcal{X}, y)^T P \\ &\leq -\frac{\partial C}{\partial \mathcal{X}}(\mathcal{X}, y)^T \frac{\partial C}{\partial \mathcal{X}}(\mathcal{X}, y) \\ &\qquad \forall(\mathcal{X}, y) \in \mathbb{R}^{n+m-1} \times \mathbb{R}. \end{aligned} \tag{23}$$

DE-D2.2: The system (7) is zero-state detectable, i.e., any solution $\mathcal{X}(\mathcal{X}, t)$ of

$$\dot{\mathcal{X}} = A(\mathcal{X}, 0) \quad , C(\mathcal{X}, 0) = 0 \tag{24}$$

is defined on $[0, +\infty)$ and converges to 0 as t tends to infinity.

Assumption DE-S2 (Dynamics Error, L^2-Stabilizability): There exists a C^{n+1} function ϕ_z, zero at the origin, and such that the following system is L^2-ISS :

$$\dot{z} = F(z, \phi_z(z)) + K_z(\phi_z(z))d$$

where K_z collects all the z-components of the function K. Specifically, there exist a C^{n+1}, positive-definite and proper function V_z and a positive-definite continuous function α_z such that we have

$$\frac{\partial V_z}{\partial z}(z)[F(z, \phi_z(z)) + K_z(\phi_z(z))d] \le -\alpha_z(z) + |d|^2. \quad (25)$$

These two assumptions are discussed in the next section.

Again, designing a globally stabilizing output feedback under this set of assumptions is an easy task by relying on the observer backstepping technique. Precisely, we have the following.

Theorem 2 (Dynamics Error, L^2 Case): If the assumptions DE-D2 and DE-S2 hold then there exists a globally stabilizing dynamic output feedback of dimension $m + n - 1$.

Proof: Consider again the observer (12). We have

$$\begin{aligned} &|C(\hat{\mathcal{X}}, y) - C(\mathcal{X}, y)|^2 \\ &\quad = \left| \left[\int_0^1 \frac{\partial C}{\partial \mathcal{X}}(\mathcal{X} + s[\hat{\mathcal{X}} - \mathcal{X}], y)\, ds \right] [\hat{\mathcal{X}} - \mathcal{X}] \right|^2 \\ &\quad \le \int_0^1 \left| \left[\frac{\partial C}{\partial \mathcal{X}}(\mathcal{X} + s[\hat{\mathcal{X}} - \mathcal{X}], y) \right] [\hat{\mathcal{X}} - \mathcal{X}] \right|^2 ds. \end{aligned}$$

So, with (14) and (23), we get

$$\begin{aligned} \overbrace{(\hat{\mathcal{X}} - \mathcal{X})^T P (\hat{\mathcal{X}} - \mathcal{X})}^{\cdot} &\le -|C(\hat{\mathcal{X}}, y) - C(\mathcal{X}, y)|^2 \\ &= -|\Delta C|^2. \end{aligned} \quad (26)$$

This establishes that the observer makes the correction term ΔC an L^2 function along the solutions of the closed-loop system. So, according to the above separation recipe, it remains to design a state feedback making L^2-ISS the following system with input d:

$$\begin{cases} \dot{\hat{z}} = F(\hat{z}, y) + K_z(y)d \\ \dot{y} = a_1(\hat{z}, y)\hat{y}_2 + b_1(\hat{z}, y) + d, \\ \dot{\hat{y}}_2 = a_2(\hat{z}, y, \hat{y}_2)\hat{y}_3 + b_2(\hat{z}, y, \hat{y}_2) + K_2(y)d \\ \quad \vdots \\ \dot{\hat{y}}_n = a_n(y)u + b_n(\hat{z}, y, \hat{y}_2, \ldots, \hat{y}_n) + K_n(y)d. \end{cases} \quad (27)$$

Using assumption DE-S2, we have a C^{n+1}, positive-definite and proper function V_z satisfying

$$\frac{\partial V_z}{\partial \hat{z}}(\hat{z})(F(\hat{z}, \phi_z(\hat{z})) + K_z(\phi_z(\hat{z}))d) \le -\alpha_z(\hat{z}) + |d|^2.$$

By applying recursively Lemma 1 given in the Appendix, we can propagate this property up to getting a C^1, positive-definite and proper function V_n and a C^1 function ϕ_n such that $u = \phi_n(\hat{z}, y, \hat{y}_2, \ldots, \hat{y}_n)$ gives for the system (27)

$$\overbrace{V_n(\hat{z}, y, \hat{y}_2, \ldots, \hat{y}_n)}^{\cdot} \le -\alpha_n(\hat{z}, y, \hat{y}_2, \ldots, \hat{y}_n) + |d|^2 \quad (28)$$

where α_n is a positive-definite continuous function.

So now, instead of viewing the dynamics of the closed-loop system as the interconnection of the system to be controlled and the error system, as in the state disturbance approach (see (13)), we view them as the interconnection of the observer (27) with input

$$d = \Delta C = C(e + \hat{\mathcal{X}}, y) - C(\hat{\mathcal{X}}, y)$$

and the error system

$$\dot{e} = A(\hat{\mathcal{X}} + e, y) - A(\hat{\mathcal{X}}, y) - K(y)\Delta C \quad (29)$$

with output ΔC and input y and

$$\hat{\mathcal{X}} = (\hat{z}, \hat{y}_2, \ldots, \hat{y}_n).$$

As proved above, the latter generates a function ΔC which is square-integrable along the solutions of the closed-loop system and the former is L^2-ISS with this function as input. From here proving global asymptotic stability is easy. Indeed, with (26) and (28), we get readily

$$\overbrace{e^T P e + V_n(\hat{\mathcal{X}}, y)}^{\cdot} \le -\alpha_n(\hat{\mathcal{X}}, y).$$

Since α_n is a positive-definite continuous function of its arguments, this establishes global stability of the origin as well as the convergence of any solution to the largest invariant set contained in the set $\{(e, \hat{\mathcal{X}}, y) : \hat{\mathcal{X}} = y = 0\}$. In this set, we have

$$\dot{\mathcal{X}} = A(\mathcal{X}, 0) \quad , C(\mathcal{X}, 0) = 0.$$

So, by following the same arguments as in [12, p. 44], we can conclude with assumption DE-D2.2, that each solution converges to the origin, i.e., we have global attractivity. ■

B. Discussion

1) On Assumption DE-S2: Again, the main restriction imposed by assumption DE-S2 is the fact that the function ϕ_z not only stabilizes asymptotically the origin of the z-subsystem but also that it provides the L^2-ISS property of the following auxiliary system:

$$\dot{z} = F(z, \phi_z(z)) + d$$

where the disturbance d acts externally instead of internally as we had with the state disturbance approach (see (21)).

2) On Assumption DE-D2: Assumption DE-D2 is very similar to assumption SD-D. We have only replaced < 0 in (8) by $\le -(\partial C)/(\partial \mathcal{X})^T (\partial C)/(\partial \mathcal{X})$ in (23). However, even with strengthening (8) as

$$P\frac{\partial A-KC}{\partial \mathcal{X}}(\mathcal{X},y)+\frac{\partial A-KC}{\partial \mathcal{X}}(\mathcal{X},y)^TP\leq -I \quad \times(\mathcal{X},y)\in\mathbb{R}^{n+m-1}\times\mathbb{R}$$

it is a difficult task to go from this stronger version of SD-D to DE-D2.1. For this we need an extra property on the function C. Typically it is that $|(\partial C)/(\partial \mathcal{X})(\mathcal{X},y)|$ is bounded or more specifically that a_1 does not depend on z and $|(\partial b_1)/(\partial z)(z,y)|$ is bounded. Without such a property, a possibility is to redesign the observer by augmenting the gain K. This idea has been exploited already in the literature (see [14], [27], and [7], for instance). Here it can be exploited at least in the case where C is affine in $\mathcal{X}$, i.e.

$$C(\mathcal{X},y)=C_0(y)+C_1(y)\mathcal{X}.$$

In this case, (8) reads

$$\begin{aligned}&\left[P\frac{\partial A}{\partial \mathcal{X}}(\mathcal{X},y)+\frac{\partial A}{\partial \mathcal{X}}(\mathcal{X},y)^TP\right]\\&\times[PK(y)C_1(y)+C_1(y)^TK(y)^TP]<0\\&\qquad\forall(\mathcal{X},y)\in\mathbb{R}^{n+m-1}\times\mathbb{R}.\end{aligned}$$

So, by augmenting $K(y)$ with $(1/2)P^{-1}C_1(y)^T$, we get

$$\begin{aligned}&\left[P\frac{\partial A}{\partial \mathcal{X}}(\mathcal{X},y)+\frac{\partial A}{\partial \mathcal{X}}(\mathcal{X},y)^TP\right]\\&-\left[P\left(K(y)+\frac{1}{2}P^{-1}C_1(y)^T\right)C_1(y)\right.\\&\qquad\left.+C_1(y)^T\left(K(y)+\frac{1}{2}P^{-1}C_1(y)^T\right)^TP\right]\\&<-C_1(y)^TC_1(y)=-\frac{\partial C}{\partial \mathcal{X}}(\mathcal{X},y)^T\frac{\partial C}{\partial \mathcal{X}}(\mathcal{X},y)\\&\qquad\forall(\mathcal{X},y)\in\mathbb{R}^{n+m-1}\times\mathbb{R}\end{aligned}$$

which is (23). For example, for the system (19), the assumption (23) still holds under the constraint (20). For this, it is sufficient to modify k_2 as:

$$k_2(y)=\frac{\ell(y)}{a_1(y)}+\frac{a_1(y)}{2}.$$

Another possibility of relaxing assumption DE-D2 is offered when the correction term can be decomposed as (see [29])

$$C(\mathcal{X},y)-C(\hat{\mathcal{X}},y)=m(\hat{z},y)\varepsilon(z,\hat{z},y). \tag{30}$$

In this case it is sufficient that the observer ensures that the term $\varepsilon(z,\hat{z},y)$ is in L^2 along the solutions. But then the stabilizability assumption DE-S2 is about the following system:

$$\dot{z}=F(z,\phi_z(z))+K_z(\phi_z(z))m(\hat{z},\phi_z(z))d. \tag{31}$$

This extension is used for the system (44) studied below.

If none of the above succeeds, we abandon the L^2 framework and try the following L^1 one. The assumptions we need then are (see also [29], [7]).

Assumption DE-D1 (Dynamics Error, L^1-Detectability):

DE-D1.1: The coordinates for z and the functions $(a_i)_{1\leq i\leq n-1}$ and $(b_i)_{1\leq i\leq n-1}$ can be chosen in such a way that there exist a C^{n+1} function K and a positive semidefinite symmetric matrix P and k vectors v_i satisfying

$$P+\sum_{i=1}^{k}v_iv_i^T>0 \tag{32}$$

$$\begin{aligned}&\frac{e^TP\frac{\partial A-KC}{\partial \mathcal{X}}(\mathcal{X},y)e}{\sqrt{e^TPe}}\\&\quad+\sum_{i=1}^{k}\operatorname{sign}(v_i^Te)v_i^T\frac{\partial A-KC}{\partial \mathcal{X}}(\mathcal{X},y)e\\&\leq-\left|\frac{\partial C}{\partial \mathcal{X}}(\mathcal{X},y)e\right|\\&\qquad\forall(\mathcal{X},y,e)\in\mathbb{R}^{n+m-1}\times\mathbb{R}\times\mathbb{R}^{n+m-1}.\end{aligned} \tag{33}$$

Moreover the K_z component of K is bounded.

DE-D1.2. Same as assumption DE-D2.2

Assumption DE-S1 (Dynamics Error, L^1-Stabilizability): There exist a C^{n+1} function ϕ_z, zero at the origin, and a C^{n+1}, positive-definite and proper function V_z and a positive-definite continuous function α_z satisfying[2]

$$\frac{\partial V_z}{\partial z}(z)[F(z,\phi_z(z))+d]\leq-\alpha_z(z)+|d|.$$

Theorem 3 (Dynamics Error, L^1 Case): If the assumptions DE-D1 and DE-S1 hold then there exists a globally stabilizing dynamic output feedback of dimension $m+n-1$.

Proof: For a locally Lipschitz function W and a system $\dot{e}=f(e,\mathcal{X})$, we denote by D^+W, the Dini derivative of W along the solutions of this system, i.e.,

$$D^+W(e)=\limsup_{t\searrow 0}\frac{W(e+tf(e,\mathcal{X}))-W(e)}{t}.$$

With this notation, the proof follows exactly the same lines as the one of Theorem 2.

Since we have

$$\begin{aligned}&|C(\mathcal{X}+e,y)-C(\mathcal{X},y)|\\&\quad=\left|\int_0^1\frac{\partial C}{\partial \mathcal{X}}(\mathcal{X}+se,y)e\,ds\right|\\&\quad\leq\int_0^1\left|\frac{\partial C}{\partial \mathcal{X}}(\mathcal{X}+se,y)e\right|ds\end{aligned}$$

with (14) and (33) and e satisfying (29), we get[3]

$$D^+\left(\sqrt{e^TPe}+\sum_{i=1}^{k}\left|v_i^Te\right|\right)\leq-|\Delta C|.$$

This establishes that the observer makes the correction term ΔC an L^1 function along the solutions of the closed-loop system.

[2]This is equivalent to writing that V_z has a bounded gradient.

[3]The use of polyhedral Lyapunov functions has a long history in control theory (see [26] and the references therein, for instance).

Moreover, the function K_z being bounded, the same integrability property holds for $K_z \Delta C$.

Then we follow exactly the same lines as the ones after (26), except that, to propagate the L^1-ISS property, we apply recursively Lemma 2 given in the Appendix. In this way, we get a C^1, positive-definite and proper function V_n and a C^1 function ϕ_n such that $u = \phi_n(\hat{z}, y, \hat{y}_2, \ldots, \hat{y}_n)$ gives for the system (27)

$$\overbrace{V_n(\hat{z}, y, \hat{y}_2, \ldots, \hat{y}_n)}^{\cdot} \leq -\alpha_n(\hat{z}, y, \hat{y}_2, \ldots, \hat{y}_n) + |\Delta C|$$

where α_n is a positive-definite continuous function.

This yields the following for the closed-loop system:

$$D^+ \left(\sqrt{e^T P e} + \sum_{i=1}^{k} \left| v_i^T e \right| + V_n(\hat{z}, y, \hat{y}_2, \ldots, \hat{y}_n) \right) \leq -\alpha_n(\hat{z}, y, \hat{y}_2, \ldots, \hat{y}_n).$$

With (32) and ([32, Ths. II.6.2 and VII.3.2]), we conclude that we have global stability of the origin and convergence of any solution to the largest invariant set contained in the set $\{(e, \hat{\mathcal{X}}, y) : \hat{\mathcal{X}} = y = 0\}$. From this point, the proof is completed as the one of Theorem 2. ■

We end this section by mentioning that the very specific structure of the system studied by Marino and Tomei in [21] is such that the assumptions invoked in that work imply that, for all $p \geq 1$, both L^p-detectability and L^p-stabilizability are satisfied and so in particular assumptions DE-D1, DE-D2, DE-S1 and DE-S2 (see [1, Ex. 4.2.3]).

IV. Combined Approach

A. The Context

In this section, we rephrase part of the result obtained by Karagiannis, Jiang, Ortega, and Astolfi in [17] by viewing its proof as a combination of the state disturbance and the dynamics error approaches.

In this context, the stabilizability assumption is given as the following.

Assumption SD-S': There exists a C^{n+1} function ϕ_z zero at the origin, and such that the following system is ISS:

$$\dot{z} = F(z, \phi_z(z + d_z) + d_u).$$

Specifically, there exist a C^{n+1}, positive-definite and proper function V_z and two continuous non-negative functions γ_1 and γ_2 which are zero at zero and such that we have

$$\frac{\partial V_z}{\partial z}(z) \left[F(z, \phi_z(z + d_z) + d_u) \right] \leq -V_z(z) + \gamma_1(|d_z|) + \gamma_2(|d_u|). \quad (34)$$

Theorem 4 (Combined Approach): If the assumptions DE-D2 and SD-S' hold then there exists a globally stabilizing dynamic output feedback of dimension $m + n - 1$.

Proof: From (34), we get readily

$$\frac{\partial V_z}{\partial z}(z) F(z, y) \leq -V_z(z) + \gamma_1(|z - \hat{z}|) + \gamma_2(|y - \phi_z(\hat{z})|). \quad (35)$$

This establishes that the z-subsystem is ISS with respect to $z - \hat{z}$ and $y - \phi_z(\hat{z})$. The observer takes care of making the disturbance $z - \hat{z}$ "small". It remains to design a state feedback taking care of the other disturbance $y - \phi_z(\hat{z})$. To do so, we consider the coordinate

$$\mu = y - \phi_z(\hat{z})$$

and write its dynamics as

$$\begin{cases} \dot{\mu} = a_1(\hat{z}, y)\hat{y}_2 + b_1(\hat{z}, y) + d \\ \quad - \frac{\partial \phi_z}{\partial z}(\hat{z})[F(\hat{z}, y) + K_z(y) d] \\ \vdots \\ \dot{\hat{y}}_n = a_n(y) u + b_n(\hat{z}, \ldots, \hat{y}_n) + K_n(y) d \end{cases} \quad (36)$$

where, $\hat{z}$ is considered as a measured exogenous input for which we know its time derivative satisfies

$$\dot{\hat{z}} = F(\hat{z}, \mu + \phi_z(\hat{z})) + K_z(\mu + \phi_z(\hat{z})) d. \quad (37)$$

Here d is a seen as a disturbance but is actually the correction term associated to the observer

$$d = C(\mathcal{X}, y) - C(\hat{\mathcal{X}}, y). \quad (38)$$

We start the design by observing that the function ϕ_1 defined as

$$\phi_1(\hat{z}, \mu) = \frac{1}{a_1(\hat{z}, y)} \left[-2\mu - \left(\frac{\partial \phi_z}{\partial z}(\hat{z}) K_z(y) \right)^2 \mu - b_1(\hat{z}, y) + \frac{\partial \phi_z}{\partial z}(\hat{z})(F(\hat{z}, y)) \right]$$

where $y = \mu + \phi_z(\hat{z})$, gives, for the μ-subsystem,

$$\overbrace{\mu^2}^{\cdot} \leq -\mu^2 + 2|d|^2 \quad \forall (\hat{z}, \mu, \hat{y}_2) : \hat{y}_2 = \phi_1(\mu, \hat{z}).$$

This establishes an L^2-ISS property for this μ-subsystem. Then, by applying recursively Lemma 1 of the Appendix, we can propagate this L^2-ISS property to get a C^1 function ϕ_n and a C^1, positive-definite and proper function V_n such that $u = \phi_n(\hat{z}, \mu, \hat{y}_2, \ldots, \hat{y}_n)$ gives, for the system (36)

$$\overbrace{V_n(\mu, \hat{y}_2, \ldots, \hat{y}_n)}^{\cdot} \leq -\alpha_n(V_n(\mu, \hat{y}_2, \ldots, \hat{y}_n)) + |d|^2 \quad (39)$$

where, α_n is a positive-definite continuous function.

Note that here the recursive procedure starts with μ (equivalent to y) instead of $\hat{z}$ as in the dynamics error approach [see (27)].

With all these preliminaries, we can write the dynamics of the closed-loop system as

$$\begin{cases} \dot{z} = F(z, \phi_z(\hat{z}) + \mu) \\ \dot{\mu} = a_1(\hat{z}, y)\hat{y}_2 + b_1(\hat{z}, y) + d \\ \quad -\frac{\partial \phi_z}{\partial z}(\hat{z})[F(\hat{z}, y) + K_z(y)d] \\ \quad \vdots \\ \dot{\hat{y}}_n = a_n(y)\phi_n(\hat{z}, \mu, \hat{y}_2, \dots, \hat{y}_n) \\ \quad +b_n(\hat{z}, y, \hat{y}_2, \dots, \hat{y}_n) + K_n(y)d \\ \dot{e} = A(\hat{\mathcal{X}} + e, y) - A(\hat{\mathcal{X}}, y) - K(y)d \end{cases} \tag{40}$$

with (38), $\hat{z} = z - e_z$, $y = \mu + \phi_z(\hat{z})$ and

$$\hat{\mathcal{X}} = (\hat{z}, \hat{y}_2, \dots, \hat{y}_n), \quad e = \mathcal{X} - \hat{\mathcal{X}},$$
$$K(y) = (K_z(y), K_2(y), \dots, K_n(y)).$$

It is seen as the interconnection of the error system and a system which combines the z-part of the system to be controlled and the $(\mu, \hat{y}_2, \dots, \hat{y}_n)$-part of the observer.

With (35), (39), (26), (34), and (37), we get along the solutions of this system (40)

$$\overbrace{2e^T Pe + V_n(\mu, \hat{y}_2, \dots, \hat{y}_n)}^{\cdot}$$
$$\le -\alpha_n(V_n(\mu, \hat{y}_2, \dots, \hat{y}_n)) - |d|^2, \tag{41}$$
$$\overbrace{V_z(z)}^{\cdot} \le -V_z(z) + \rho(2e^T Pe + V_n(\mu, \hat{y}_2, \dots, \hat{y}_n)) \tag{42}$$
$$\overbrace{V_z(\hat{z})}^{\cdot} \le -V_z(\hat{z}) + \gamma_2(|\mu|) + \mathfrak{k}(y, \hat{z})|d| \tag{43}$$

where ρ is any class $\mathcal{K}_\infty$ function and $\mathfrak{k}$ is the continuous function satisfying

$$\rho(s) \ge 2\max\left\{\gamma_1\left(\left|\frac{1}{2}P^{-\frac{1}{2}}\right|\sqrt{s}\right)\right.$$
$$\left.\max_{(\mu,\hat{y}_2,\dots,\hat{y}_n):V_n(\mu,\hat{y}_2,\dots,\hat{y}_n)\le s} \gamma_2(2|\mu|)\right\}$$
$$\mathfrak{k}(y, \hat{z}) = \left|\frac{\partial V_z}{\partial z}(\hat{z})K_z(y)\right|.$$

Inequalities (41) and (42) imply successively that, along the solutions of the closed-loop system, $2e^T Pe + V_n(\mu, \hat{y}_2, \dots, \hat{y}_n)$ and $V_z(z)$ are bounded. Specifically, we get, for all $t \ge 0$

$$2e(t)^T Pe(t) + V_n(\mu(t), \hat{y}_2(t), \dots, \hat{y}_n(t))$$
$$\le 2e(0)^T Pe(0) + V_n(\mu(0), \hat{y}_2(0), \dots, \hat{y}_n(0))$$
$$V_z(z(t)) \le V_z(0)$$
$$+ \rho(2e(0)^T Pe(0) + V_n(\mu(0), \hat{y}_2(0), \dots, \hat{y}_n(0))$$

where the argument t represents the times for the evaluation of the argument of the functions along the solution. These inequalities imply the global stability of the origin.

Actually, since V_n is positive-definite and $\overbrace{2e^T Pe + V_n(\mu, \hat{y}_2, \dots, \hat{y}_n)}^{\cdot}$ is strictly negative if $V_n(\mu, \hat{y}_2, \dots, \hat{y}_n)$ and d are not zero, we have also

$$\lim_{t\to+\infty} \sup_{s\ge t} \left[|\mu(s)| + |d(s)| + \sum_{i=2}^{n} |\hat{y}_i(s)|\right] = 0.$$

Also, as we have done from (11), inequality (43) gives, for any bounded solution, a real number $\bar{\mathfrak{k}}$, such that we have for all $t \ge 0$

$$V_z(\hat{z}(t)) \le \exp(-t)V_z(\hat{z}(0))$$
$$+ \exp(-\frac{t}{2}) \sup_{0\le s\le \frac{t}{2}} [\gamma_2(|\mu(s)|) + \bar{\mathfrak{k}}|d(s)|]$$
$$+ \sup_{\frac{t}{2}\le s} \left[\gamma_2(|\mu(s)|) + \bar{\mathfrak{k}}|d(s)|\right].$$

We conclude that, for any closed-loop solution, there exists a real number c such that the corresponding ω-limit set is contained in the set $\{(\mathcal{X}, \hat{\mathcal{X}}, y) : e^T Pe = c, |\hat{\mathcal{X}}| = y = 0\}$. But, with Assumption DE-D2.2, we know that any solution in this set converges to the origin. So, by following the same arguments as in ([12], p. 44), we can conclude that we have global attractivity. ■

B. Discussion

Compared to the result in [17], in Theorem 4, we are less restrictive in allowing the terms b_i to depend also on $(y_2, \dots, y_i)$. But we are more restrictive in not dealing with the (unknown) disturbance terms. In doing so, we can work with a less demanding detectability assumption with replacing an ISS property by a simpler stability property. If we were to cope also with these disturbances, as in [17], in the detectability assumption, we would come back to an ISS property and, in the output feedback design, we would replace the propagation of the L^2-ISS property by the propagation of the ISS property with a gain assignment, a technique introduced in [30].

Also if we compare Theorems 1 and 4, we see that, for the former, the stabilizability assumption SD-S is more restrictive (strict feedback form) but the detectability assumption SD-D is weaker. However, if we succeed in proving that the procedure of Freeman and Kokotović proposed in [11] extends to the case of SD-S', then Theorem 1 would give a less restrictive result. Nevertheless, even in this case, Theorem 4 will remain very interesting since the design part of the state feedback is much simpler compared with what can be expected to be obtained from [11].

V. Examples

Example 1: Consider the system in $\mathbb{R}^2$

$$\begin{cases} \dot{z} = -z + y^2 z^2 \\ \dot{y} = u - z^2. \end{cases} \tag{44}$$

We have the following.

1) With the following reduced-order observer:

$$\hat{z} = w - \frac{y^3}{3}, \quad \dot{w} = -w + \frac{y^3}{3} + y^2 u$$

we obtain

$$\overbrace{\frac{1}{2}(\hat{z}-z)^2 + \frac{1}{4}(\hat{z}-z)^4}^{\cdot} = -(\hat{z}-z)^2 - (\hat{z}-z)^4. \quad (45)$$

To follow the dynamics error approach, we have to write the dynamics of $\hat{z}$ and y in such a way that the corresponding correction term is an L^2 function along the solutions. To do this here we decompose $\dot{\hat{z}}$ and $\dot{y}$ as

$$\begin{cases} \dot{\hat{z}} = -\hat{z} + y^2\hat{z}^2 + \underbrace{y^2}_{K_z(y)} \underbrace{(1 \quad 2\hat{z})}_{m(\hat{z},y)} \underbrace{\begin{pmatrix} (z-\hat{z})^2 \\ z-\hat{z} \end{pmatrix}}_{\varepsilon(z,\hat{z})} \\ \dot{y} = u - \hat{z}^2 + \underbrace{(1 \quad 2\hat{z})}_{m(\hat{z},y)} \underbrace{\begin{pmatrix} (z-\hat{z})^2 \\ z-\hat{z} \end{pmatrix}}_{\varepsilon(z,\hat{z})}. \end{cases}$$

According to the discussion following (30), the correction term is identified as being ε. With (45), we see it is an L^2 function along the solutions. This proves that the (modified) detectability assumption DE-S2 of the dynamics error approach is satisfied.

2) For any function ϕ_z, any $z > 2$, and d_z, we have, for all $d_u \geq 1$

$$\dot{z} = -z + (\phi_z(z+d_z))^2 z^2 + d_u z^2 \geq \frac{1}{2}z^2.$$

Hence the stabilizability assumptions SD-S and SD-S' are not satisfied. Hence only the dynamics error approach can be considered.

3) With $y = \phi_z(\hat{z}) = 0$, and $V_z(\hat{z}) = (1/2)\hat{z}^2$, we have:

$$\frac{\partial V_z}{\partial z}(\hat{z})\left[-\hat{z} + y^2\hat{z}^2 + y^2(1 \quad 2\hat{z})\begin{pmatrix} d_1 \\ d_2 \end{pmatrix}\right] = -2V_z(\hat{z}) \quad \forall(\hat{z}, d_1, d_2).$$

So the (modified) stabilizability assumptions DE-S2 of the dynamics error approach is satisfied.

From the above and by applying Lemma 1, we can conclude that the output feedback

$$\begin{cases} \dot{w} = -w + \dfrac{y^3}{3} + y^2 u. \\ \hat{z} = w - \dfrac{y^3}{3} \\ u = -y + \hat{z}^2 - y\hat{z}^3 - (y + y^3\hat{z}^2)(1 + 4\hat{z}^2) \end{cases}$$

is globally asymptotically stabilizing.

Example 2: Consider the system (see [7, Ex. 2])

$$\begin{cases} \dot{z} = 3z + 2z^3 + y \\ \dot{y} = z + z^3 + u. \end{cases} \quad (46)$$

We have the following.

1) From the inequality

$$\mathtt{sign}(e_z)\left(\frac{d}{dz}[3z + 2z^3] - 4\frac{d}{dz}[z + z^3]\right)e_z \leq -\left|\frac{d}{dz}[z + z^3]\right| |e_z|$$

we conclude that the inequalities (8) and (33) are satisfied with $P = 1$, $v = 0$, and $K = 4$. *Hence the detectability assumptions SD-D of the state disturbance approach and the detectability assumption DE-D1 of the dynamics error approach in the* L^1*-case are satisfied.*

2) The z dynamics are in strict feedback form, thus *the stabilizability assumption SD-S of the state disturbance approach is satisfied*. Also with

$$\phi_z(\hat{z}) = -4\hat{z} - 2\hat{z}^3, \quad V_z(\hat{z}) = \sqrt{1 + \hat{z}^2} - 1$$

we get

$$\frac{\partial V}{\partial z}(\hat{z})[3\hat{z} + 2\hat{z}^3 + \phi_z(\hat{z}) + d] \leq -\frac{\hat{z}^2}{\sqrt{1+\hat{z}^2}} + |d|.$$

This proves that *the stabilizability assumption DE-S1 of the dynamics error approach is satisfied* also.

From the above, and by applying Lemma 2, for instance, we can conclude that the output feedback

$$\begin{cases} \dot{w} = -\hat{z} - 2\hat{z}^3 - 4u + y, \\ \hat{z} = w + 4y, \\ u = -\hat{z} - \hat{z}^3 - (4 + 6\hat{z}^2)(3\hat{z} + 2\hat{z}^3 + y) \\ \quad +r(y,\hat{z})\left(-4\hat{z} - 2\hat{z}^3 - y - 24|1 + \hat{z}^3|\dfrac{\hat{z}}{\sqrt{1+\hat{z}^2}}\right). \end{cases}$$

where $r(y, \hat{z}) = 1 + (1/2)(y + 4\hat{z} + 2\hat{z}^3)^2$ is globally asymptotically stabilizing.

Example 3: Consider the system (see [29]):

$$\begin{cases} \dot{x}_1 = x_2 + u \\ \dot{x}_2 = f(x_1) + x_3 - u \\ \dot{x}_3 = -f(x_1) \\ y = x_1 \end{cases} \quad (47)$$

where f is a C^3 function such that $f(0) = 0$ and $f'(0) \neq 0$.

This system (47) is of the form studied by Marino and Tomei in [21]. But it is proved in [3] that the assumptions of [21] are not satisfied if f possesses another zero not at the origin.

To be within the framework of this paper, we rewrite the dynamics of (47) as

$$\begin{cases} \dot{z}_1 = z_1 - y \\ \dot{z}_2 = -f(y) \\ \dot{y} = z_1 - z_2 - y + u. \end{cases}$$

We have

- From the inequalities

$$
\begin{aligned}
&\mathtt{sign}(2e_{z_1} - 3e_{z_2})\begin{pmatrix}2 & -3\end{pmatrix} \\
&\left[\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} - \begin{pmatrix}6\\ 2\end{pmatrix}\begin{pmatrix}1 & -1\end{pmatrix}\right] \\
&\begin{pmatrix}e_{z_1}\\ e_{z_2}\end{pmatrix} = -2|2e_{z_1} - 3e_{z_2}| \\
&\mathtt{sign}(e_{z_1} - 2e_{z_2})\begin{pmatrix}1 & -2\end{pmatrix} \\
&\left[\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} - \begin{pmatrix}6\\ 2\end{pmatrix}\begin{pmatrix}1 & -1\end{pmatrix}\right] \\
&\begin{pmatrix}e_{z_1}\\ e_{z_2}\end{pmatrix} = -\left|e_{z_1} - 2e_{z_2}\right|, \\
&|e_{z_1} - e_{z_2}| \leq |2e_{z_1} - 3e_{z_2}| + |2e_{z_2} - e_{z_1}|
\end{aligned}
$$

  we conclude that the inequality (33) is satisfied with $P = 0$, $v_1 = \begin{pmatrix}2\\ -3\end{pmatrix}$, $v_2 = \begin{pmatrix}1\\ -2\end{pmatrix}$, and $K = \begin{pmatrix}6\\ 2\end{pmatrix}$. Hence *the detectability assumption DE-D1 of the dynamics error approach in the L^1-case is satisfied.*

- Let a, σ, and δ be the real number and functions defined as

$$
\begin{aligned}
a &= \frac{1}{2\left(2\max_{|s|\leq 3}|f''(s)|\right)^2} \\
\sigma(z_1) &= \max_{|s|\leq 1}|f''(2z_1 + s)| \\
\delta(z_1, s) &= \sqrt{2a}|f(2z_1) + f'(2z_1)s - f(2z_1 + s)|.
\end{aligned}
$$

  Let also k be a C^2, positive-definite and proper functions whose derivative is nondecreasing and satisfies

$$
|k'(z_1)| \geq \max\left\{\sqrt{2a}|z_1|\sigma(z_1), \frac{f(2z_1)}{\sqrt{2a}}\right\}. \tag{48}
$$

  Since f is C^2, we have $\forall|s| \leq 1$

$$
\begin{aligned}
\delta(z_1, s) &\leq \frac{\sqrt{2a}\sigma(z_1)s^2}{2} \leq \frac{z_1 k'(z_1)}{2} \\
&\qquad\qquad\qquad\qquad \forall 1 \leq |z_1|, \\
&\leq \frac{s^2}{2} \quad \forall |z_1| \leq 1.
\end{aligned} \tag{49}
$$

  Now, inspired by a forwarding technique (see [25]), we consider V_{0z}, the C^2, positive-definite and proper function defined as

$$
V_{0z}(z_1, z_2) = k(z_1) + 2\left(\sqrt{1 + \frac{a}{2}\zeta_2^2} - 1\right)
$$

  where

$$
\zeta_2 = \left(z_2 - \int_0^{z_1}\frac{f(2s)}{s}ds\right).
$$

  With (49) and

$$
y = \phi_z(z_1, z_2) = 2z_1 + s
$$

  where s in $(-1, 1)$ is

$$
s = \frac{2}{\pi}\arctan\left(k'(z_1) - \frac{a\zeta_2}{\sqrt{1 + \frac{a}{2}\zeta_2^2}}\left(-f'(2z_1) + \frac{f(2z_1)}{z_1}\right)\right)
$$

  we get

$$
\begin{aligned}
\overbrace{V_{0z}(z_1, z_2)}^{\cdot} &\leq -z_1 k'(z_1) - s^2 + \delta(z_1, s) \\
&\leq -\frac{z_1 k'(z_1) + s^2}{2}
\end{aligned}
$$

  Also, with (48), we have

$$
\begin{aligned}
\left|\frac{\partial V_{0z}}{\partial z_2}(z_1, z_2)\right| &\leq \sqrt{2a}, \quad \forall (z_1, z_2) \\
\left|\frac{\partial V_{0z}}{\partial z_1}(z_1, z_2)\right| &\leq |k'(z_1)| + \sqrt{2a}\left|\frac{f(2z_1)}{z_1}\right| \\
&\leq 2|k'(z_1)|, \quad \forall (z_1, z_2) : 1 \leq |z_1| \\
&\leq b, \quad \forall (z_1, z_2) : |z_1| \leq 1
\end{aligned}
$$

  where $b = \max_{|z_1|\leq 1}\{|k'(z_1)| + \sqrt{2a}(f(2z_1)/z_1)\}$. Let $\ell : \mathbb{R}_+ \to \mathbb{R}_+$ be a C^2 and proper function which is zero at zero, has a strictly positive derivative and satisfies

$$
\ell(k(z_1)) = |z_1| \quad \forall z_1 : 1 \leq |z_1|.
$$

  So, finally we define a function V_z as

$$
V_z(z_1, z_2) = \ell(V_{0z}(z_1, z_2)).
$$

  It is C^2, positive-definite and proper with a bounded gradient. For instance, ℓ' being nonincreasing on $[k(1), +\infty)$, we have

$$
\begin{aligned}
\left|\frac{\partial V_z}{\partial z_1}(z_1, z_2)\right| &\leq 2\ell'(V_{0z}(z_1, z_2))k'(z_1) \\
&\leq 2\ell'(k(z_1))k'(z_1) \leq 2 \quad \forall (z_1, z_2) : 1 \leq |z_1| \\
&\leq b\ell'(V_{0z}(z_1, z_2)) \leq b\max_{|z_1|\leq 1}\ell'(k(z_1)) \\
&\qquad\qquad\qquad\qquad \forall (z_1, z_2) : |z_1| \leq 1.
\end{aligned}
$$

  Hence, with V_z and ϕ_z defined above*the stabilizability assumption DE-S1 of the dynamics error approach is satisfied.*

With Theorem 3, we conclude that there exists a globally asymptotically stabilizing output feedback.

Example 4: Consider the system

$$
\begin{cases}
\dot{z} = z^2 + y \\
\dot{y} = y_2 + z^3, \\
\dot{y}_2 = u + 2z^3 + z^2 + 2z.
\end{cases} \tag{50}
$$

We have the following.

1) Suppose there exists a reduced order observer which satisfies the detectability assumption DE-D2 of the dynamics error approach in the L^2-case. Then we have three real

numbers (p,q,r) and two functions K_z and K_2 such that, for each (e_z,e_2,z,y), we get

$$\begin{aligned}&\begin{pmatrix} e_z & e_2\end{pmatrix}\begin{pmatrix} p & q\\ q & r\end{pmatrix}\\ &\left[\begin{pmatrix} 2z & 0\\ 6z^2+2z+2 & 0\end{pmatrix} - \begin{pmatrix} K_z(y)\\ K_2(y)\end{pmatrix}\begin{pmatrix} 3z^2 & 1\end{pmatrix}\right]\begin{pmatrix} e_z\\ e_2\end{pmatrix}\\ &\leq -\left(\begin{pmatrix}3z^2 & 1\end{pmatrix}\begin{pmatrix} e_z\\ e_2\end{pmatrix}\right)^2.\end{aligned}$$

When $e_2=0$, this gives

$$9z^4 + q(6z^2+2z+2-3K_2(y)z^2) + p(2z-3K_z(y)z^2) \leq 0 \quad \forall (z,y).$$

For any given y, the left hand side of this inequality goes to $+\infty$ when z goes to infinity, which is impossible. This proves that *the detectability assumption DE-D2 of the dynamics error approach in the L^2-case cannot be satisfied*.

2) From the following inequalities:

$$\begin{aligned}&\mathrm{sign}(e_z)(1 \quad 0)\\ &\left[\begin{pmatrix} 2z & 0\\ 6z^2+2z+2 & 0\end{pmatrix} - \begin{pmatrix} 1\\ 3\end{pmatrix}(3z^2 \quad 1)\right]\begin{pmatrix} e_z\\ e_2\end{pmatrix}\\ &= -(3z^2-2z+1)|e_z|\mathrm{sign}(e_z)\,(e_z-e_2)\\ &\leq -(3z^2-2z+1)|e_z||e_z-e_2|\\ &\mathrm{sign}(e_z-e_2)(1 \quad -1)\\ &\left[\begin{pmatrix} 2z & 0\\ 6z^2+2z+2 & 0\end{pmatrix} - \begin{pmatrix} 1\\ 3\end{pmatrix}(3z^2 \quad 1)\right]\begin{pmatrix} e_z\\ e_2\end{pmatrix}\\ &= -2|e_z-e_2|\\ &\frac{1}{3}[3z^2e_z+e_2| \leq \frac{1}{3}(3z^2+1)|e_z| + \frac{1}{3}|e_z-e_2|\\ &\leq (3z^2-2z+1)|e_z| + |e_z-e_2|\end{aligned}$$

we conclude that the inequality (33) is satisfied with $P=0$, $v_1 = \binom{1}{0}$, $v_2 = \binom{1}{-1}$ and $K = \binom{1}{3}$. Hence the detectability assumption DE-D1 of the dynamics error approach in the L^1-case is satisfied.

3) With $V_z(\hat z) = \sqrt{1+\hat z^2}-1$ and $\phi_z(\hat z) = -\hat z^2 - \hat z$ we get

$$\frac{\partial V_z}{\partial \hat z}(\hat z)\left[\hat z^2 + \phi_z(\hat z) + d\right] \leq \frac{-\hat z^2}{\sqrt{1+\hat z^2}} + |d|.$$

It follows that the stabilizability assumption DE-S1 of the dynamics error approach in the L^1-case is satisfied.

With Theorem 3, we conclude that there exists a globally asymptotically stabilizing output feedback.

VI. Conclusion

We have investigated the problem of global asymptotic stabilization by output feedback for systems whose dynamics admit a strict normal form. By rephrasing and formalizing already known approaches we have been able to introduce several sets of assumptions that allow us to design an output feedback. As in the results given by Karagiannis, Jiang, Ortega and Astolfi in [17], and Marino and Tomei in [21] no minimum-phase assumptions are required but instead we ask for stabilizability by a state control of the inverse dynamics with various robustness properties depending on what can be achieved with a reduced order observer.

Appendix
Propagation of the L^p-ISS Property Through a Chain of Integrators

We consider a system in the form :

$$\begin{cases} \dot{x_1} = f(x_1,x_2) + K_1(x_1,x_2)d_1\\ \dot x_2 = a(x_1,x_2)u + b(x_1,x_2) + K_2(x_1,x_2)d_2.\end{cases} \tag{51}$$

where x_1 is in $\mathbb{R}^{n_1}$, x_2 is in $\mathbb{R}$, u is in $\mathbb{R}$, d_1 is in $\mathbb{R}^{n_1}$, d_2 in $\mathbb{R}$, $a(x_1,x_2)$ is strictly positive and the functions f, K_1, and K_2 are as many times differentiable as needed below.

Lemma 1: (L^2-ISS propagation): Suppose there exist a C^{q+1}, positive-definite and proper function $V_1:\mathbb{R}^{n_1}\to\mathbb{R}_+$, a C^{q+1} function $\phi_1:\mathbb{R}^{n_1}\to\mathbb{R}$, and a positive-definite continuous function $\alpha_1:\mathbb{R}^{n_1}\to\mathbb{R}_+$ such that, along the solutions of (51), we have

$$\overbrace{V_1(x_1)}^{\cdot} \leq -\alpha_1(x_1) + |d_1|^2 \forall (x_1,x_2,d_1) \;:\; x_2 = \phi_1(x_1).$$

Then, there exists a C^q, positive-definite and proper function $V_2:\mathbb{R}^{n_1+1}\to\mathbb{R}_+$, a C^q function $\phi_2:\mathbb{R}^{n_1+1}\to\mathbb{R}$, and a positive-definite continuous function $\alpha_2:\mathbb{R}^{n_1+1}\to\mathbb{R}_+$ such that, along the solutions of (51), we get

$$\overbrace{V_2(x_1,x_2)}^{\cdot} \leq -\alpha_2(x_1,x_2) + |d_1|^2 + |d_2|^2 \quad \forall (x_1,x_2,d_1,d_2,u) \;:\; u = \phi_2(x_1,x_2).$$

This result is well known. See [15], [30], [1] for a proof.

Lemma 2: (L^1-ISS propagation): Suppose the function K_1 does not depend on x_2 and there exist a continuous function $M:\mathbb{R}^{n_1}\to\mathbb{R}_+$, a C^{q+1}, positive-definite and proper function $V_1:\mathbb{R}^{n_1}\to\mathbb{R}_+$, a C^{q+1} function $\phi_1:\mathbb{R}^{n_1}\to\mathbb{R}$, and a positive-definite continuous function $\alpha_1:\mathbb{R}^{n_1}\to\mathbb{R}_+$ satisfying

$$|K_2(x_1,x_2)| \leq M(x_1)\,(1+|x_2|)\,. \tag{52}$$

and, along the solutions of (51)

$$\overbrace{V_1(x_1)}^{\cdot} \leq -\alpha_1(x_1) + |d_1| \forall (x_1,x_2,d_1) \;:\; x_2 = \phi_1(x_1).$$

Then, there exist a C^q, positive-definite and proper function $V_2:\mathbb{R}^{n_1+1}\to\mathbb{R}_+$, a C^q function $\phi_2:\mathbb{R}^{n_1+1}\to\mathbb{R}$, and a positive-definite continuous function $\alpha_2:\mathbb{R}^{n_1+1}\to\mathbb{R}_+$ such that, along the solutions of (51), we get

$$\overbrace{V_2(x_1,x_2)}^{\cdot} \leq -\alpha_2(x_1,x_2) + |d_1| + |d_2| \quad \forall (x_1,x_2,d_1,d_2,u) \;:\; u = \phi_2(x_1,x_2)$$

Proof: We follow here a suggestion of Frederic Mazenc who used a very similar argument in his dissertation [24, eq. (2.412)].

As the function V_1 is proper, we can find a C^q and increasing function $k' : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying for all x_1 in $\mathbb{R}^{n_1}$

$$k'(V(x_1)) \geq \max\left\{1, \left|\frac{\partial \phi_1}{\partial x_1}(x_1)\right| |K_1(x_1)| \right. \\ \left. \frac{1}{2}M(x_1)(3+|\phi_1(x_1)|)\right\}. \quad (53)$$

Let $W : \mathbb{R}^{n_1} \times \mathbb{R} \to \mathbb{R}_+$ be the C^q, positive-definite and proper function defined as

$$W(x_1,x_2) = k\big(V_1(x_1)\big) + \log\left(1 + \frac{1}{2}(x_2 - \phi(x_1))^2\right)$$

where k is the C^{q+1} and proper function defined as

$$k(s) = \int_0^s k'(u)du, \quad \forall s \in \mathbb{R}_+. \quad (54)$$

By differentiating W along the solutions of (51), we get

$$\overbrace{W(x_1,x_2)}^{\cdot} \leq -k'(V_1(x_1))\alpha_1(x_1) \\ + m_1(x_1,x_2)|d_1| + m_2(x_1,x_2)|d_2| \\ + (x_2 - \phi_1(x_1))\left[p(x_1,x_2) + q(x_1,x_2)u\right] \quad (55)$$

where $m_1 : \mathbb{R}^{n_1} \times \mathbb{R}$ and $m_2 : \mathbb{R}^{n_1} \times \mathbb{R}$ are continuous functions, and $p : \mathbb{R}^{n_1} \times \mathbb{R}$ is a C^q function defined as

$$m_1(x_1,x_2) = k'(V_1(x_1)) \\ + \frac{|x_2 - \phi_1(x_1)| \left|\frac{\partial \phi_1}{\partial x_1}(x_1)\right| |K_1(x_1)|}{1 + \frac{1}{2}(x_2 - \phi_1(x_1))^2} \quad (56)$$

$$m_2(x_1,x_2) = \frac{|x_2 - \phi_1(x_1)||K_2(x_1,x_2)|}{1 + \frac{1}{2}(x_2 - \phi_1(x_1))^2}$$

$$p(x_1,x_2) = \frac{b(x_1,x_2) - \frac{\partial \phi_1}{\partial x_1}(x_1) f(x_1,x_2)}{1 + \frac{1}{2}(x_2 - \phi_1(x_1))^2} \\ + k'(V_1(x_1))\frac{\partial V_1}{\partial x_1}(x_1) \cdot \int_0^1 \frac{\partial f}{\partial x_2}(x_1, \phi_1(x_1) \\ + s(x_2 - \phi_1(x_1)))ds$$

$$q(x_1,x_2) = \frac{a(x_1,x_2)}{1 + \frac{1}{2}(x_2 - \phi_1(x_1))^2} \quad (57)$$

where q is non zero from the same assumption on a. So let ϕ_2 be the following C^q function:

$$\phi_2(x_1,x_2) = \frac{1}{q(x_1,x_2)}(\phi_1(x_1) - x_2 - p(x_1,x_2)).$$

When $u = \phi_2(x_1,x_2)$, we obtain, along the solutions of (51)

$$\overbrace{W(x_1,x_2)}^{\cdot} \leq -k'(V_1(x_1))\alpha_1(x_1) - (x_2 - \phi_1(x_1))^2 \\ + m_1(x_1,x_2)|d_1| + m_2(x_1,x_2)|d_2|.$$

Furthermore, from (53), (56) and (57), and from the fact that k' is an increasing function, we get

$$m_1(x_1,x_2) \leq 2k'(V_1(x_1)) \leq 2k'(k^{-1}(W(x_1,x_2))) \\ m_2(x_1,x_2) \leq \frac{|x_2 - \phi_1(x_1)|}{1 + \frac{1}{2}(x_2 - \phi_1(x_1))^2} M(x_1) \\ (1 + |x_2 - \phi_1(x_1)| + |\phi_1(x_1)|) \\ \leq M(x_1)(3 + |\phi_1(x_1)|) \\ \leq 2k'(V_1(x_1)) \leq 2k'(k^{-1}(W(x_1,x_2))).$$

Thus, we have

$$\overbrace{W(x_1,x_2)}^{\cdot} \leq -k'(V_1(x_1))\alpha_1(x_1) - (x_2 - \phi_1(x_1))^2 \\ + 2k'(k^{-1}(W(x_1,x_2)))(|d_1| + |d_2|).$$

By taking, $V_2(x_1,x_2) = \ell(W(x_1,x_2))$ where ℓ is the C^q and proper function defined as $\ell(s) = (1/2)k^{-1}(s)$, we obtain finally

$$\overbrace{V_2(x_1,x_2)}^{\cdot} \leq -\ell'(W(x_1,x_2)) \\ \left[k'(V_1(x_1))\alpha_1(x_1) + (x_2 - \phi_1(x_1))^2\right] \\ + |d_1| + |d_2|\cdot$$

■

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