control design presented in this work are applicable to systems (1) with multiple unstable eigenvalues with the Dirichlet, Neumann, or Robin type boundary controllers.

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Norm Estimators and Global Output Feedback Stabilization of Nonlinear Systems With ISS Inverse Dynamics

Georgia Kaliora, Alessandro Astolfi, and Laurent Praly

***Abstract*—A preliminary result on the construction of norm estimators for general nonlinear systems that do not necessarily admit a input output to state stable (IOSS)-Lyapunov characterization is given. Furthermore, an output feedback stabilization scheme is presented that makes use of norm estimators. This construction extends some previous results allowing for more general nonlinearities. Two examples complete the work.**

***Index Terms*—Input-output-to-state stability, nonlinear systems, norm estimators, output feedback.**

I. INTRODUCTION

It has been clear for years now that, for nonlinear systems, global uniform observability alone does not imply the existence of a convergent observer, or even more so, the existence of a (globally) stabilizing (dynamic) output feedback control law. On the contrary, it has been shown in [5] that globally observable systems that do not however possess the *unboundedness observability* property cannot be stabilized by any dynamic output feedback scheme.

It is now a growing trend to use high-gain observers as part of an output feedback stabilization architecture for a variety of nonlinear systems that exhibit a triangular structure [1]. In [6], a high-gain technique was introduced where the gain was time varying, i.e., tuned on line. Motivated by the above reference the authors of [4] considered an output feedback made of the combination of high, variable-gain observer and controller. Both the previous output feedbacks are inherently nonlinear, while in [8] a linear observer/controller (again with varying high gain) proves to be sufficient for the output feedback stabilization for a class of nonlinear systems.

On the other hand, for nonlinear systems written in observability canonical form and that are *input output to state stable* (IOSS) globally convergent observers can be designed via the idea of *norm estimators* (see [3] for this and other related definitions), as shown in [7], where again a "high-gain" idea is used, but the gain is this time tuned via the norm estimator.

Motivated by [7] in this note we provide an approach toward the design of norm estimators for systems that are not necessarily IOSS. As applications, we examine a nonlinear system that—in open loop—exhibits finite escape time and nonlinear systems that are linear in the unmeasured state.

Finally, we extend the result of [8]. Namely, under the assumption of existence of a norm estimator, it is shown that the restrictions on the growth of the system nonlinearities can be relaxed, allowing, thus, for a larger class of nonlinear systems to be stabilized with this approach.

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II. On Norm Estimators

Consider a single-input–single-output nonlinear system of the form

$$\begin{aligned}\dot{x} &= f(x,u)\\ y &= h(x)\end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the input, and $y \in \mathbb{R}$ is the output, respectively. In [3] it is explained how the assumption that system (1) is input output to state stable (IOSS) can be used for the design of a first order dynamical system $\dot{\omega} = \alpha(\omega, u, y)$ such that a function of $\omega(t)$ serves asymptotically as an upper limit of the norm of x. In this design it is instrumental to assume the existence of a IOSS-Lyapunov function $V(x)$ that satisfies the dynamic estimate

$$\dot{V}(x,u) = \frac{\partial V}{\partial x}(x) f(x,u) \le -V(x) + \gamma_1(|h(x)|) + \gamma_2(|u|) \tag{2}$$

for class $\mathcal{K}$ functions $\gamma_1(|y|)$ and $\gamma_2(|u|)$. In practice, even for systems that are knowingly IOSS it might be difficult to compute such functions V, γ_1 and γ_2. Even though this difficulty does not necessarily hinder the construction of the norm estimator, it is however desirable to investigate whether norm estimators can be built under the assumption that an (IOSS-)Lyapunov function exhibits a "good enough" dynamical estimate, that is different from the exponential decaying one in (2). In this section such an alternative is presented.

It deals with the case[1] where we have a C^1 function W and two continuous functions $\bar{\alpha}$, upperbounded in its first argument, and β, nondecreasing in its first argument, satisfying

$$\dot{W}(x,u) \le \bar{\alpha}(W(x), u, h(x)) \quad \forall (x,u) \tag{3}$$

and:

$$|x| \le \beta(W(x), h(x)) \qquad \forall x. \tag{4}$$

Let α be a locally Lipschitz function, upperbounded in its first argument and c_1 to c_3 be strictly positive real numbers satisfying

$$\alpha(0,u,y) \ge 0 \quad \forall (u,y) \tag{5}$$

$$\bar{\alpha}(W,u,y) \le \alpha(\omega,u,y) \qquad \forall W \ge \omega \quad \forall (u,y) \tag{6}$$

$$\alpha((1+c_1)\omega + c_2, u, y) + c_3 \le [1+c_1]\alpha(\omega,u,y) \qquad \forall (\omega,u,y). \tag{7}$$

Consider the augmented system

$$\begin{aligned}\dot{x} &= f(x,u)\\ \dot{\omega} &= \alpha(\omega,u,h(x)), \qquad \omega(0) \ge 0.\end{aligned} \tag{8}$$

Then the following fact holds.

Lemma 1: For any locally essentially bounded input function u, the right maximal interval of definition $[0,T)$ of any corresponding solution $(x(t),\omega(t))$ of (8) is not larger than the one of the corresponding solution $x(t)$ of (1) and there exists T^* such that

$$|x(t)| \le \beta([1+c_1]\omega(t) + c_2 + |y(t)|, y(t)) \qquad \forall t \in [T^*, T). \tag{9}$$

Proof: Since the $\dot{x}$ equation is the same in (1) and (8), the $x(t)$ solution of (1) is necessarily defined at least on the right maximal interval of definition $[0,T)$ of the corresponding solution $(x(t),\omega(t))$ of (8). Also, remark that (5) implies

$$\omega(t) \ge 0 \qquad \forall t \in [0,T).$$

Then, (3) and (6) give

$$\overbrace{\max\{W(x)-\omega,0\}^2}^{\cdot} \le 0.$$

Hence

$$W(x(t)) \le \omega(t) + \max\{W(x(0)) - \omega(0), 0\} \qquad \forall t \in [0,T). \tag{10}$$

Note now that, as α is upperbounded in its first argument ω can become unbounded in finite time only if u and/or y become unbounded, and this is in turn possible only if x is unbounded. Now, if T is finite, by maximality, we have

$$\lim_{t\to T} c_1\omega(t) + |x(t)| = +\infty.$$

By (10) and (4), where β is nondecreasing in its first argument, this implies

$$\lim_{t\to T} c_1\omega(t) + \beta(\omega(t) + \max\{W(x(0)) - \omega(0), 0\}, y(t)) = +\infty.$$

The function β being continuous, $\omega(t)$ and/or $|y(t)|$ must go to infinity. We deduce

$$\lim_{t\to T} c_1\omega(t) + |y(t)| = +\infty. \tag{11}$$

So there exists a real number T^* in $[0,T)$ such that:

$$\max\{W(x(0)) - \omega(0), 0\} \le c_1\omega(t) + |y(t)| \quad \forall t \in [T^*, T). \tag{12}$$

With (4) and (10), (9) follows.

If T is infinite, with (6) and (7), we get:

$$\overbrace{\max\{W(x) - [1+c_1]\omega - c_2, 0\}^2}^{\cdot} \le -c_3 \max\{W(x) - [1+c_1]\omega - c_2, 0\}.$$

So there exists

$$T^* \le \frac{2\sqrt{\max\{W(x(0)) - [1+c_1]\omega(0) - c_2, 0\}}}{c_3}$$

such that

$$W(x(t)) \le [1+c_1]\omega(t) + c_2 \qquad \forall t \in [T^*, +\infty). \tag{13}$$

So again (9) follows. ■

III. Examples

Example 1: Consider the third-order system

$$\begin{aligned}\dot{z} &= -z + \psi_o(x_1)\\ \dot{x}_1 &= x_2\\ \dot{x}_2 &= u + x_2^2 + \psi_2(z)\\ y_o &= x_1\end{aligned} \tag{14}$$

where ψ_2 satisfies, for some real number δ

$$|\psi_2(z)| \le \delta z^2 \qquad \forall z. \tag{15}$$

The first equation of (14) represents the inverse dynamics, and it is clear that these are ISS with respect to their input x_1. Nonetheless, this system possesses solutions escaping to infinity in (positive) finite time. However, system (14) is iIOSS. To see this, consider the partial coordinates and feedback transformation

$$\begin{aligned}\xi_1 &= \exp(-x_1) - 1\\ \xi_2 &= -\exp(-x_1)x_2\\ v &= -\exp(-x_1)u\end{aligned} \tag{16}$$

[1] This case encompasses, among others, Unboundedness Observability (UO), integral input output to state stability (iIOSS) and input output to state stability (IOSS).

yielding

$$\begin{aligned}\dot{z} &= -z + \psi(\xi_1)\\ \dot{\xi}_1 &= \xi_2\\ \dot{\xi}_2 &= v - (1+\xi_1)\psi_2(z)\\ y &= \xi_1\end{aligned} \tag{17}$$

with $\psi(\xi_1) = \psi_o(-\log(1+\xi_1))$. As the linear part of the ξ-subsystem of system (17) is a linear observable system there exist a matrix $P \in \mathbb{R}^{n\times n}$ with $P > 0$ and positive numbers γ_v and γ_y such that, with $V_\xi(\xi) = \xi' P \xi$, it holds that

$$\frac{\partial V_\xi}{\partial \xi}(\xi)\begin{bmatrix}\xi_2\\ v\end{bmatrix} \le -kV_\xi(\xi) + \gamma_v v^2 + \gamma_y y^2$$

for some $k \in (0,1]$. Along the trajectories of the ξ-subsystem of (17) we obtain

$$\dot{V}_\xi \le -kV_\xi(\xi) + \gamma_v v^2 + \gamma_y y^2 + p|1+\xi_1||\psi_2(z)||\xi| \tag{18}$$

for some positive real number p. Consider now the positive–definite and radially unbounded function

$$V(z,\xi) = \frac{\lambda}{2}z^2 + \log(1+V_\xi(\xi)) \tag{19}$$

with a positive real number λ to be defined. This yields

$$\dot{V} \le -\lambda z^2 + \lambda z\psi(y) + p\frac{|1+\xi_1||\xi|}{1+V_\xi(\xi)}|\psi_2(z)| - \frac{kV_\xi(\xi)}{1+V_\xi(\xi)} + \frac{\gamma_y}{1+V_\xi(\xi)}y^2 + \frac{\gamma_v}{1+V_\xi(\xi)}v^2.$$

Since the quantity $(|1+\xi_1||\xi|)/(1+V_\xi(\xi))$ is bounded for all ξ, with (15), it can be seen that, by picking λ large enough, there exists λ_1 such that we have

$$\begin{aligned}\dot{V} &\le -\frac{\lambda z^2}{2} - \frac{kV_\xi(\xi)}{1+V_\xi(\xi)} + \lambda_1\psi^2(y) + \gamma_y y^2 + \gamma_v v^2\\ &\le -k\frac{V(z,\xi)}{1+V(z,\xi)} + \lambda_1\psi^2(y) + \gamma_y y^2 + \gamma_v v^2.\end{aligned}$$

We have also

$$z^2 + |\xi|^2 \le \frac{2V(z,\xi)}{\lambda} + q(\exp(V(z,\xi)) - 1)$$

for some real number q. So we do have the iIOSS property. It follows that Lemma 1 applies with

$$\begin{aligned}\alpha(\omega,v,y) &= -k\frac{\omega}{1+\omega} + \lambda_1\psi^2(y) + \gamma_y y^2 + \gamma_v v^2\\ \beta(\omega,y) &= \sqrt{\frac{2\omega}{\lambda} + q(\exp(\omega)-1)}.\end{aligned}$$

Example 2: Consider nonlinear systems described by equations of the form

$$\begin{aligned}\dot{x} &= Ax + \Delta(x,y,u) + B(y,u)\\ y &= Cx\end{aligned} \tag{20}$$

where the functions Δ and B are continuous, $|\Delta(x,y,u)| \le \kappa(y,u)(1+|x|)$, and the pair $\{C,A\}$ is observable.

By observability of the pair $\{C,A\}$ there exist a matrix $P > 0$ and a row vector L, satisfying, for some positive real number k

$$x'(A'P + PA)x \le -kx'Px + 2x'PLy.$$

The following also holds for some continuous function $v(y,u)$:

$$\Delta'(x,y,u)Px + x'P\Delta(x,y,u) \le |v(y,u)|(1+x'Px).$$

Consider the C^1 positive definite and radially unbounded function $V(x) = \log(1+x'Px)$. There exist positive real numbers γ_y and γ_u such that, along any solution of system (20), we have

$$\dot{V} \le -\frac{k}{2}\frac{x'Px}{1+x'Px} + |v(y,u)| + \gamma_y y^2 + \gamma_u|B(y,u)|^2.$$

So, Lemma 1 applies again.

IV. Global Output Feedback Stabilization

In this section, we show how the existence of a norm estimator can be used in an output feedback stabilization scheme.

Specifically, the property to be exploited is that, knowing how to get a bound for the norm of the system state after a finite time, we can also evaluate any bounding functions.

Consider nonlinear systems in the form

$$\begin{aligned}\dot{z} &= q(z,y)\\ \dot{x}_1 &= x_2 + \delta_1(z,x_1)\\ &\vdots\\ \dot{x}_i &= x_{i+1} + \delta_i(z,x_1,\ldots,x_i)\\ &\vdots\\ \dot{x}_{n-1} &= x_n + \delta_{n-1}(z,x_1,\ldots,x_{n-1})\\ \dot{x}_n &= u + \delta_n(z,x_1,\ldots,x_n)\\ y &= x_1\end{aligned} \tag{21}$$

where $y \in \mathbb{R}$ is the available output, and the z-subsystem represents the inverse dynamics. It is useful to rewrite this system in the compact form

$$\dot{X} = F(X) + Gu \tag{22}$$

with $X = (x,z)$.

Complementing the work of [8] we present a result that relaxes the assumptions made in this reference. In particular, we use the following set of assumptions.

A1) The subsystem $\dot{z} = q(z,y)$ is ISS with respect to y, i.e., there exist a positive definite and radially unbounded function $V_z(z)$ and a class $\mathcal{K}$ function γ such that

$$\frac{\partial V_z(z)}{\partial z}q(z,y) \le -V_z(z) + \gamma(|y|). \tag{23}$$

A2) There exist a continuous nonnegative function L and a class $\mathcal{K}$ function κ such that for all $i = 1,\ldots,n$

$$|\delta_i(z,x_1,\ldots,x_i)| \le L(x,z)(|x_1| + \cdots + |x_i|) + \kappa(V_z(z)). \tag{24}$$

The key novelty here is that L may depend on both x and z and not only on $y = x_1$. However, in this case, we need an estimate of an upperbound for L. Specifically, (compare with Lemma 1), consider the following assumption.

A3) There exist locally Lipschitz functions α and β such that the system

$$\dot{\omega} = \alpha(\omega,u,y) \quad \hat{L} = \beta(\omega,y) \tag{25}$$

is ISS with input (u, y) and in particular, there exists a class $\mathcal{KL}$ function $\mathfrak{B}$ and class $\mathcal{K}$ functions χ_u and χ_y so that, for any positive t for which $(\omega(t), u(t), y(t))$ makes sense, we have

$$|\hat{L}(t)| \leq \max\{\mathfrak{B}(\omega(0), \mathfrak{t}), \sup_{\mathfrak{s}\in[0,\mathfrak{t})} \{\chi_{\mathfrak{y}}(|\mathfrak{y}(\mathfrak{s})|), \chi_{\mathfrak{u}}(|\mathfrak{u}(\mathfrak{s})|)\}\}. \tag{26}$$

Moreover, for any solution $(X(t), \omega(t))$ of the augmented system (22) and (25), right maximally defined on $[0, T)$, there exists $T^* \in [0, T)$ such that

$$L(x(t), z(t)) \leq \hat{L}(t) \qquad \forall t \in [T^*, T). \tag{27}$$

In the above, we need further restrictions on the functions γ, κ, χ_u, and χ_y.

A4) The functions γ and κ are $\mathcal{C}^1$ on $(0, +\infty)$ and

1) there exists a real number $\theta \geq 1$ such that

$$\theta s \frac{d\kappa(s)}{ds} \geq \kappa(s) \qquad \forall s > 0. \tag{28}$$

2) There exist strictly positive real numbers k and s_0 such that

$$\kappa(2\gamma(s)) \leq ks \qquad \forall s \in [0, s_0].$$

A5)

1) There exists an integer $m \geq 1$ and a positive real number p satisfying

$$\chi_y(s) + \frac{\kappa(2\gamma(s))}{s} \leq p + s^m \qquad \forall s \geq 0. \tag{29}$$

2) There exists a real number η in $(0, 1)$ and a positive real number q satisfying

$$\chi_u(s) \leq q + s^{\frac{1-\eta}{n}} \quad \forall s \geq 0. \tag{30}$$

Remark 1: The set of assumptions given here are a generalization of the assumptions given in [8], where it is assumed that the nonlinearities $\delta_i(\cdots)$ are linearly bounded—in growth—with a rate which is output dependent. This situation can be recovered in the present set up by letting $\chi_u(s) = 0$. Assumptions A1 and A2 describe a class of systems which is significantly enlarged. This is made possible by the existence of the bounding function estimator, described in Assumption A3. The cost of this generalization is having to satisfy conditions (30).

It will be shown that, even with this set of relaxed assumptions, boundedness of solutions as well as convergence to the desired equilibrium can be achieved by means of a linear dynamic output feedback, with a dynamic high gain, following the ideas in [8].

Proposition 1: Suppose that Assumptions A1 to A5 hold. Then there exist a function $\sigma(\hat{\omega}, y, r)$, matrices $F \in \mathbb{R}^{1\times n}$ and $K \in \mathbb{R}^{n\times 1}$, with $K = [k_1, k_2, \ldots, k_n]'$ and a positive real number b such that the dynamic output feedback control law composed as follows:

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + k_1 r(y - \hat{x}_1) \\ &\vdots \\ \dot{\hat{x}}_i &= \hat{x}_{i+1} + k_i r^i (y - \hat{x}_1) \\ &\vdots \\ \dot{\hat{x}}_n &= u + k_n r^n (y - \hat{x}_1) \end{aligned} \tag{31}$$

$$u = -r^n F \begin{bmatrix} y \\ \frac{\hat{x}_2}{r^1} \\ \vdots \\ \frac{\hat{x}_n}{r^{n-1}} \end{bmatrix} \tag{32}$$

$$\begin{aligned} \dot{\omega} &= \alpha(\omega, u, y) \\ \hat{L} &= \beta(\omega, y) \\ \dot{r} &= -r(br - \sigma(\hat{L}, y, r)) \end{aligned} \tag{33}$$

with $r(0) \geq 1$, is such that all trajectories of system (21) are bounded and converge to the origin.

Before continuing with the proof of Proposition 1, we define the matrices $A,\ D \in \mathbb{R}^{n\times n}$ and $B, C \in \mathbb{R}^{n\times 1}$ as follows:

$$A = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & & \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & n-1 \end{bmatrix}$$

and recall the following lemma from [8], which is instrumental in the design of the control scheme (31)–(33).

Lemma 2: For any strictly positive real number a, there exist real numbers d_0 and d_1, symmetric matrices $P \in \mathbb{R}^{n\times n}$ and $Q \in \mathbb{R}^{n\times n}$, and matrices $K \in \mathbb{R}^{n\times 1}$ and $F \in \mathbb{R}^{1\times n}$ satisfying the following set of inequalities:

$$\begin{aligned} &d_0 > 0 \quad d_1 \geq 0 \quad P > 0 \quad Q > 0 \\ &P(A - KC') + (A - KC')'P \leq -d_0 P \\ &Q(A - BF) + (A - BF)'Q \leq -d_0 Q \\ &\quad -aP \leq PD + DP \leq d_1 P \\ &\quad -aQ \leq QD + DQ \leq d_1 Q. \end{aligned} \tag{34}$$

Remark 2: Lemma 2 implies that the controller gains F and K are function of the positive parameter a.

Sketch of the Proof of Proposition 1: Let the matrices K and F in (31) and (32), respectively, be chosen according to Lemma 2. In (33), choose the function $\sigma(\hat{L}, y, r)$ satisfying at least

$$\sigma(\hat{L}, y, r) \geq b \geq 0. \tag{35}$$

By constraining the initial condition of the varying gain r to be larger than one, i.e., $r(0) \geq 1$, we guarantee that for each solution and all t where it makes sense, $r(t) \geq 1$. The choice of the real number b will be dictated later.

Considering the observer given by (31) define the error variables

$$e_i = x_i - \hat{x}_i$$

the *normalized* error variables

$$\varepsilon_i = \frac{e_i}{r^{i-1+a}}$$

for $i = 1, \ldots, n$, and the corresponding error vector $\varepsilon = \mathrm{col}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$. Following straightforward manipulations, one gets

$$\dot{\varepsilon} = r(A - KC')\varepsilon - (aI + D)\frac{\dot{r}}{r}\varepsilon + \Delta_1 \tag{36}$$

where

$$\Delta_1 = \mathrm{col}\left\{\frac{\delta_1}{r^a}, \frac{\delta_2}{r^{1+a}}, \ldots, \frac{\delta_n}{r^{n-1+a}}\right\}.$$

Next, let P be the matrix given by Lemma 2. We consider the positive–definite and radially unbounded function

$$V_\varepsilon = \varepsilon' P \varepsilon.$$

With straightforward calculations, upper bounding, and using the inequalities (34) one obtains

$$\dot{V}_\varepsilon \leq -([d_0 - (2a + d_1)b]r + a\sigma(\hat{L}, y, r))V_\varepsilon + 2\varepsilon' P\Delta_1. \quad (37)$$

Next, consider the vector of *scaled estimated states* as:

$$\bar{x} = \text{col}\left\{\frac{y}{r^a}, \frac{\hat{x}_2}{r^{1+a}}, \ldots, \frac{\hat{x}_i}{r^{i-1+a}}, \ldots, \frac{\hat{x}_n}{r^{n-1+a}}\right\} \quad (38)$$

and a scaled input [recall (32)]

$$\bar{u} = \frac{u}{r^{n+a}} = -F\bar{x}. \quad (39)$$

This yields

$$\dot{\bar{x}} = r(A - BF)\bar{x} - (aI + D)\frac{\dot{r}}{r}\bar{x} + r\Delta_2 \quad (40)$$

where

$$\Delta_2 = \text{col}\left\{\frac{\delta_1}{r^{1+a}} + \varepsilon_2, k_2\varepsilon_1 \ldots, k_i\varepsilon_1, \ldots, k_n\varepsilon_1\right\}.$$

Consider the matrix Q that satisfies the inequalities (34) of Lemma 2 and define the positive–definite and radially unbounded function

$$V_c = \bar{x}' Q\bar{x}.$$

With straightforward computations, and using inequalities (34), it can be shown that V_c satisfies the estimate

$$\dot{V}_c \leq -([d_0 - (2a + d_1)b]r + a\sigma(\hat{L}, y, r))V_c + 2r\bar{x}' Q\Delta_2. \quad (41)$$

With Assumption A2 (and $r \geq 1$), we get

$$\left|\frac{\delta_i}{r^{i-1+a}}\right| \leq L(x, z)[(|\bar{x}_1| + \cdots + |\bar{x}_i|) + (|\varepsilon_1| + \cdots + |\varepsilon_i|)] + \frac{\kappa(V_z(z))}{r^{i-1+a}}.$$

By completing the squares, this yields

$$2\varepsilon' P\Delta_1 \leq L(x, z)[V_c + d_2 V_\varepsilon] + d_3\sqrt{V_\varepsilon}\frac{\kappa(V_z(z))}{r^a}$$

for some positive numbers d_2 and d_3 that depend on a. In a similar way we can obtain an estimate for the norm of the vector Δ_2 and

$$2r\bar{x}' Q\Delta_2 \leq \left(d_4 L(x, z) + \frac{d_0}{2}r\right)V_c + rd_5 V_\varepsilon + d_6\sqrt{V_c}\frac{\kappa(V_z(z))}{r^a}$$

for some strictly positive real numbers d_4, d_5 and d_6 depending on a. With the previous estimates we obtain

$$\dot{V}_\varepsilon \leq -([d_0 - (2a + d_1)b]r + a\sigma(\hat{L}, y, r) - d_2 L(x, z))V_\varepsilon + L(x, z)V_c + d_3\sqrt{V_\varepsilon}\frac{\kappa(V_z(z))}{r^a} \quad (42)$$

$$\dot{V}_c \leq -\left(\left[\frac{d_0}{2} - (2a + d_1)b\right]r + a\sigma(\hat{L}, y, r) - d_4 L(x, z)\right)V_c + rd_5 V_\varepsilon + d_6\sqrt{V_c}\frac{\kappa(V_z(z))}{r^a}. \quad (43)$$

Following [8] again, we make the following choices for the real number b and the function $\sigma(\hat{L}, y, r)$

$$0 < b \leq \frac{d_0}{4(2a + d_1)}$$
$$\sigma(\hat{L}, y, r) = \sigma_1(\hat{L}) + \sigma_2(y, r)$$
$$\sigma_1(\hat{L}) = \frac{\hat{L}}{a}\max\left\{d_2, \left[d_4 + \frac{2d_5}{d_0}\right]\right\} \quad (44)$$

where the function $\sigma_2(y, r)$ is to be defined later on. Consider now the function

$$V_{\varepsilon c} = \frac{2d_5}{d_0}V_\varepsilon + V_c. \quad (45)$$

It can be shown that the choices given by the constraints (44) lead to the estimate

$$\dot{V}_{\varepsilon c} \leq -\left(\frac{d_0 r}{4} + a\sigma_2(y, r)\right)V_{\varepsilon c} + \frac{d_7}{r^a}\sqrt{V_{\varepsilon c}}\kappa(V_z(z)) + \max\{d_2, d_4\}(L(x, z) - \hat{L})V_{\varepsilon c}.$$

Consider now the positive–definite and radially unbounded function[2]

$$U(z, \varepsilon, \bar{x}) = c\int_0^{V_z(z)} \frac{\kappa^\theta(s)}{s}ds + \frac{1}{\theta}(2\sqrt{V_{\varepsilon c}})^\theta$$

where $c \geq 4d_7$ and θ is the positive real number of Assumption A4.1. It can be shown, see [8], that

$$\dot{U} \leq -\frac{c}{4}\kappa^\theta(V_z(z)) + \frac{c}{2}\kappa^\theta(2\gamma(y)) - \left[\left(\frac{d_0 r}{8} - \frac{d_7}{r^a} + \frac{a}{2}\sigma_2(y, r)\right) - \frac{\max\{d_2, d_4\}}{2}(L(x, z) - \hat{L})\right](2\sqrt{V_{\varepsilon c}})^\theta \quad (46)$$

which, if $(a/2)\sigma_2 \geq (d_7/r^a)$, can be upperbounded also as

$$\dot{U} \leq -\frac{c}{4}\kappa^\theta(V_z(z)) - \left[\frac{d_0 r}{8} - \frac{\max\{d_2, d_4\}}{2}(L(x, z) - \hat{L})\right](2\sqrt{V_{\varepsilon c}})^\theta + \frac{c}{2}\kappa^\theta(2\gamma(y)) - \left(\frac{a}{2}\sigma_2(y, r) - \frac{d_7}{r^a}\right)\left(\frac{2d_8}{r^a}\right)^\theta |y|^\theta$$

for a positive real number d_8 depending on a. Then, it is shown in [8] that, by picking the function σ_2 as[3]

$$\sigma_2(y, r) = \max\left\{b, \frac{2d_7}{a}\right\} + \frac{cr^{a\theta}}{a(2d_8)^\theta}\max\left\{k^\theta, \left(\frac{\kappa(2\gamma(|y|))}{|y|}\right)^\theta\right\} \quad (47)$$

we obtain

$$\dot{U} \leq -\min\left\{\frac{1}{4}, \frac{\theta d_0}{8}\right\}U + \theta\frac{\max\{d_2, d_4\}}{2}\max\{0, (L(x, z) - \hat{L})\}U. \quad (48)$$

Remark that, if we pick a small enough to get

$$a\theta < 1 \quad (49)$$

then, with (44) and (47), there exists continuous functions μ_1 and μ_2 such that we have

$$\dot{r} \leq \mu_1(\omega, y) + \mu_2(y)r. \quad (50)$$

Now, the state of the closed-loop system is made of $(X, \omega, (\hat{x}_i), r)$. Let $[0, T)$ be the right maximal interval of definition of some of its solution $(X(t), \omega(t), (\hat{x}_i(t)), r(t))$. It follows from (50) that $r(t)$ cannot escape

[2]Recall that, by Assumption A4.1, $\theta \geq 1$ and note that when $\theta < 2$, this function is in general only locally Lipschitz. The following still holds by considering $\dot{U}$ as the upper right Dini derivative, see [8].

[3]Assumption A4.1, implies that $\sigma_2(y, r)$ is a locally Lipschitz function when $r \geq 1$.

to infinity in finite time without $(\omega(t), y(t))$ doing the same. So, if T is finite then $|X(t)| + |\omega(t)| + |(\hat{x}_i(t))|$ goes to infinity when t goes to T. But if only $|(\hat{x}_i(t))|$ does so, then (48) and (50) imply that $U(t)$ and $r(t)$ and, therefore, $|(\hat{x}_i(t))|$ are bounded on $[0, T)$. This is a contradiction. So, Assumption A3 can be invoked. Hence there exists T^* in $[0, T)$ and a positive real number b_1, both depending on a and the initial condition, such that (27) holds and

$$|X(t)| + |\omega(t)| + r(t) \leq b_1 \qquad \forall t \in [0, T^*]. \tag{51}$$

However, inequality (48) proves that $U(t)$ and, therefore, $\bar{x}(t)$, $\varepsilon(t)$ and $z(t)$ are actually bounded on $[0, T)$. Since (25) is ISS, boundedness of the overall solution as well as $T = +\infty$ will be established if we prove that $r(t)$ is also bounded on $[0, T)$. So consider the dynamic equation for r, written here in a simplified form which can be obtained using Assumption A5.1

$$\dot{r} = -r(br - b_2 - b_3\hat{L} - b_4 r^{a\theta} - b_5 |y|^{\theta m} r^{a\theta}). \tag{52}$$

where the b_i's are positive real numbers depending both on a and the initial condition. By applying techniques which are standard for dealing with ISS systems, it can be shown that, with (49), we have

$$r(t) \leq b_6 + b_7 \sup_{s\in[0,t)} \left[|y(s)|^{\frac{\theta m}{1-a\theta}} + |\hat{L}(s)| \right] \qquad \forall t \in [0, T)$$

where b_6 and b_7 are other positive real numbers depending both on a and the initial condition. With Assumption A5 and (26) of Assumption A3, we can further refine this inequality in

$$r(t) \leq b_8 + b_9 \sup_{s\in[0,t)} \left[|y(s)|^{\frac{\theta m}{1-a\theta}} + |y(s)|^m + |u(s)|^{\frac{1-\eta}{n}} \right]. \tag{53}$$

Now, with (38) and (39) and the boundedness of $\bar{x}(t)$, we remark that, there exists a positive real number b_{10}, depending on a and the initial condition, satisfying:

$$|y(t)| \leq b_{10} r^a \quad |u(t)| \leq b_{10} r^{n+a} \qquad \forall t \in [0, T).$$

So, (53) becomes

$$r(t) \leq b_8 + b_{11} \sup_{s\in[0,t)} \left[r(s)^{\frac{a\theta m}{1-a\theta}} + r(s)^{am} + r(s)^{\frac{(n+a)(1-\eta)}{n}} \right].$$

From here, by using the same arguments as in the proof of the small gain theorem of [2], we can show that, if a is chosen small enough to satisfy

$$a\theta < 1 \quad \frac{a\theta m}{1-a\theta} < 1 \quad am < 1 \quad \frac{(n+a)(1-\eta)}{n} < 1$$

or, in short

$$a < \min\left\{ \frac{1}{\theta(m+1)}, \frac{n\eta}{1-\eta} \right\}$$

then $r(t)$ is bounded on $[0, T)$.

So, as mentioned before, the closed-loop solution is defined and bounded on $[0, +\infty)$. Then, from the inequality (48) we have that $U(t)$ converges to zero. This implies that both $z(t)$ and $x(t)$ converge to the origin as t tends to $+\infty$.

V. Conclusion

In this note, two issues have been addressed. First, it is shown that globally convergent norm estimators can be designed also for systems for which an exponentially decaying IOSS-Lyapunov function may not exist. In addition, the problem of output feedback stabilization for a class of nonlinear systems with ISS inverse dynamics has been addressed and solved by means of linear dynamic output feedback, with dynamic high gain. This result generalizes existing results by allowing for more general forms of nonlinearities.

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Output Regulation of Uncertain Nonlinear Systems With Nonlinear Exosystems

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***Abstract*—An adaptive control algorithm is proposed for output regulation of uncertain nonlinear systems in output feedback form under disturbances generated from nonlinear exosystems. A new nonlinear internal model is proposed to generate the desired input term for suppression of the disturbances. The proposed internal model design is based on boundedness of the disturbance, high gain design and saturation. It is capable to tackle disturbances in any specified initial conditions. Some uncertainties in the systems are allowed, provided that they do not affect the desired feedforward control term, and they are tackled by using nonlinear dominant functions and an adaptive control coefficient. The proposed control algorithm ensures the global convergence of the state variables to the invariant manifold, which implies that the measurement or the tracking error approaches to zero asymptotically.**

***Index Terms*—Disturbance rejection, nonlinear exosystems, nonlinear systems, output regulation, uncertainty.**

I. Introduction

The output regulation problem is well posed and solved for linear systems in [1], [2]. For nonlinear systems, an important contribution to

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