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Linear output feedback with dynamic high gain for nonlinear systems☆

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Abstract

We propose a linear output feedback with dynamic high gain for global regulation of a class of nonlinear systems. The uncertain nonlinearities are assumed to be bounded by a polynomial function of the output multiplied by unmeasured states. The crucial point made in this paper is that a linear observer-based output feedback can globally regulate an equilibrium of strongly nonlinear systems, provided that a single high gain is appropriately tuned.

Keywords: Nonlinear systems; Output feedback; Global regulation; Lyapunov functions; Input-to-state stability; Small-gain

1. Introduction

1.1. Motivation

Output feedback of nonlinear systems is a problem of paramount importance in control engineering. Some well-known challenging facts are the lack of a global "Separation Principle" and a systematic observer design for genuinely nonlinear systems. The main contribution of this paper is at the *conceptual* level. Namely, we will show that linear control can still deal with strongly nonlinear systems provided that a single dynamic gain is appropriately tuned.

Specifically, we approach our objective by considering a class of nonlinear systems whose dynamics are described by

$$\begin{aligned}
\dot z &= q(z,y),\\
\dot x_1 &= x_2 + \delta_1(z,x_1)\\
&\vdots\\
\dot x_i &= x_{i+1} + \delta_i(z,x_1,\dots,x_i)\\
&\vdots\\
\dot x_n &= u - u^* + \delta_n(z,x_1,\dots,x_n),\\
y &= x_1,
\end{aligned} \tag{1}$$

where $u, y \in \mathbb{R}$ are the input and output, u^* is an unknown constant and $(z,x) \in \mathbb{R}^{n_0} \times \mathbb{R}^n$ is the state. Only the output y is available for feedback. The

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presence of u^* is motivated by the fact that, in some cases, the value of the control, related to a desired equilibrium point, may be unknown, such as in the case of set-point regulation or in the presence of sensor disturbance [1].

Throughout this paper, the following two hypotheses are imposed.

Hypothesis (H1):

(H1.1) The z-system in (1) is input-to-state stable (ISS) [14,12]. Namely, there exist a positive definite and radially unbounded function V_z and a class $\mathscr{K}$ function γ satisfying

$$\frac{\partial V_z}{\partial z}(z)q(z,y) \leqslant -V_z(z) + \gamma(|y|). \quad (2)$$

(H1.2) There exist a locally Lipschitz nonnegative function L and a class $\mathscr{K}$ function κ, satisfying

$$|\delta_i(z,x_1,\dots,x_i)| \leqslant L(|y|)(|x_1| + \dots + |x_i|) + \kappa(V_z(z)) \quad \forall i \in \{1,\dots,n\}. \quad (3)$$

(H1.3) There exist strictly positive real numbers k and s_0 such that

$$\kappa(2\gamma(s)) \leqslant ks \quad \forall s \in [0,s_0]. \quad (4)$$

Hypothesis (H2):

(H2.1) There exist an integer $m \geqslant 1$ and a positive real number p satisfying

$$L(s) + \frac{\kappa(2\gamma(s))}{s} \leqslant p + s^m \quad \forall s \geqslant 0. \quad (5)$$

(H2.2) The functions γ and κ are C^1 on $(0,+\infty)$ and there exists a real number $\ell \geqslant 1$ such that

$$\ell s\kappa'(s) \geqslant \kappa(s) \quad \forall s > 0. \quad (6)$$

Under the above hypotheses, the origin $(z,x)=(0,0)$ is an equilibrium point for the open-loop system (1) with $u = u^*$. The control problem of interest is to design an output feedback controller to make the (x,z) components of the closed-loop solutions bounded and converging to this equilibrium.

1.2. About the class of systems (1)

Output feedback stabilization or tracking for nonlinear systems with a (dominant) triangular structure like (1) has received a lot of attention during the last decade; see [6,7,15] and references therein. In most of these works, stringent restrictions are imposed on the functions δ_i's and q. For instance, the popular class of output feedback form systems, with or without uncertainties, only includes output nonlinearities. In other words, unmeasured state components appear linearly. Fortunately, thanks to the use of nonlinear small-gain techniques [3], this restriction can be completely removed for the state z of the inverse dynamics when they are ISS, as expressed by (H1.1) (see [11]).

Actually, as shown in [8], for some systems with only polynomial growth nonlinearities with respect to unmeasured state components, the problem of global output feedback stabilization may not be solvable. So expectedly some growth conditions are necessary on the way the nonlinearities depend on the unmeasured states. This leads to the recent work [10], where the restriction is that the nonlinearities are globally Lipschitz but with a Lipschitz rate allowed to depend on the output. To deal with such a class of systems a standard high gain observer is introduced but whose gain is updated through a Riccati equation driven by the Lipschitz rate. This result has been extended in [5] to an even broader class of systems. This has been made possible by completing the high gain observer with a high gain controller, although involving also an output nonlinear term. The use of a linear high gain controller in combination with a linear high gain observer was already proposed in [4] to deal with globally Lipschitz nonlinear systems. A somewhat related result, presented in [13], tackles with the case where the nonlinearities are linearly bounded. There, a linear high gain observer is used but the controller has a much more complex structure although this is not necessary.

So we arrive naturally to the question: how much can be achieved with a combination of a high gain linear controller and a high gain linear observer? We show, in this paper, that, if we accept to update on-line a single gain, then such a combination solves the output feedback regulation problem for systems (1) with hypotheses (H1) and (H2).

1.3. About hypotheses (H1) and (H2)

1.3.1. About (H1)

As far as we are aware, the hypothesis (H1) is without any further structural restriction, the weakest assumption, known at the time this paper is written, under which global asymptotic stabilization by output feedback can be established (see [5]).

(H1.1) is equivalent to asking that the inverse dynamics be ISS (see [14,12]). As explained in our earlier paper [11], this ISS assumption on the inverse z-system is already implicitly assumed in most of the work in robust/adaptive output feedback control of nonlinear systems [6,7].

(H1.2) puts a restriction on the nonlinearities that lead the speed of the state components x_i which are not in the inverse dynamics. These nonlinearities are captured in the uncertain functions δ_i's. They have to have a linear growth in those components but with the growth rate $L(y)$ allowed to depend on the output y. They are also allowed to depend on the state z of the inverse dynamics.

In the following only the functions γ, L and κ will be used. It follows that the functions δ_i's and q are not required to be known precisely. So, actually, they can depend on other state components and even external signals, as long as Hypotheses (H1.1) and (H1.2) still hold.

(H1.3) imposes a linear growth, close to 0, of the nonlinear gain $\kappa(2(\gamma(|y|)))$ of the system with input y, state z and output κ. We find this restriction in all the previous work on nonlinear output feedback, starting from [11]. It is needed as soon as we impose the control law to be locally Lipschitz in y. It is satisfied in the usual case where the trivial solution of the unforced system $\dot{z} = q(z, 0)$ is locally exponentially stable and the vector fields of the system are locally Lipschitz.

1.3.2. About (H2)

The restriction imposed by Hypothesis (H2) is the extra price we have to pay for imposing a linear high gain output feedback. As written above, if we were to accept to use a truly nonlinear controller, these restrictions could be removed as shown in [5].

(H2.1) is putting a polynomial growth restriction at infinity on the growth rate $L(y)$ and the nonlinear gain $\kappa(2(\gamma(|y|)))$. It is linked to the high gain structure of the controller.

(H2.2) imposes that the function κ increases at least like powers. For instance, close to the origin it can behave like $s^{1/\ell}$ but not like $1/(-\log(s))$. We know, from [12, Proposition 8], that, without loss of generality, the function γ can be assumed to be C^1 on $[0, +\infty)$ and convex and the function κ to be C^1 on $(0, +\infty)$ and concave. So (H2.2) says that κ should not "start" too fast at the origin nor "settle" too fast at infinity.

2. Design of the output feedback

The output feedback we propose is made of a linear high gain (partial) observer and a linear high gain controller.

For the design of these two blocks, we select a real number a and compute a set of data (d_0, d_1, Q, P, K, F) according to the following Lemma which was already announced in [9] and whose proof is given in the Appendix.

Let I_i be the identity matrix of order i, and set

$$A_i = \begin{pmatrix} 0 & \\ \vdots & I_{i-1} \\ 0 & 0 \cdots 0 \end{pmatrix}, \qquad B = \mathrm{col}(0, \ldots, 0, 1) \in \mathbb{R}^n,$$

$$C = \mathrm{col}(1, 0, \ldots, 0) \in \mathbb{R}^{n+1},$$

$$D_i = \mathrm{diag}(0, 1, \ldots, i-1). \tag{7}$$

We have:

Lemma 1. *For any strictly positive real number a, there exist real numbers d_0 and d_1, symmetric matrices P and Q, and column and row vectors K and F satisfying the following set of inequalities*:

$$0 < d_0, \quad 0 \leqslant d_1, \quad 0 < P, \quad 0 < Q,$$

$$P(A_{n+1} - KC^{\mathrm{T}}) + (A_{n+1} - KC^{\mathrm{T}})^{\mathrm{T}} P \leqslant -d_0 P,$$

$$Q(A_n - BF) + (A_n - BF)^{\mathrm{T}} Q \leqslant -d_0 Q,$$

$$-aP \leqslant PD_{n+1} + D_{n+1} P \leqslant d_1 P,$$

$$-aQ \leqslant QD_n + D_n Q \leqslant d_1 Q. \tag{8}$$

Remark 1. As a consequence of Lemma 1, the set (d_0, d_1, Q, P, K, F) is dependent on the design parameter a, as well as all the real numbers d_i's to be introduced later on.

2.1. Observer design

We adopt here the high gain observer of [10].

For notational convenience, we denote $x_{n+1} = -u^*$. Let the k_i's be the entries of the vector K given by Lemma 1, i.e.,

$$K = \mathrm{col}(k_1, \dots, k_{n+1}). \tag{9}$$

We introduce the following $(n+1)$th-order observer

$$\begin{aligned}\dot{\hat{x}}_i &= \hat{x}_{i+1} + k_i r^i (x_1 - \hat{x}_1), \quad 1 \leqslant i \leqslant n-1,\\ \dot{\hat{x}}_n &= \hat{x}_{n+1} + u + k_n r^n (x_1 - \hat{x}_1),\\ \dot{\hat{x}}_{n+1} &= k_{n+1} r^{n+1} (x_1 - \hat{x}_1).\end{aligned} \tag{10}$$

Following [10], the gain r, involved in this observer, is obtained as a solution of the system

$$\dot{r} = -r(br - \sigma(y, r)), \tag{11}$$

where the strictly positive real number b and the function σ are other design parameters to be made precise later on. However, we note at this time that, by imposing

$$\sigma(y, r) \geqslant b, \quad r(0) \geqslant 1, \tag{12}$$

we get that $r(t)$ is larger than or equal to 1, for all positive times t and for any solution.

For every $1 \leqslant i \leqslant n+1$, let

$$e_i = x_i - \hat{x}_i. \tag{13}$$

Then, with (1) and (10), it holds

$$\begin{aligned}\dot{e}_i &= e_{i+1} - k_i r^i e_1 + \delta_i, \quad 1 \leqslant i \leqslant n,\\ \dot{e}_{n+1} &= -k_{n+1} r^{n+1} e_1.\end{aligned} \tag{14}$$

We introduce the scaled estimation error ε in $\mathbb{R}^{n+1}$:

$$\varepsilon = \mathrm{col}(\varepsilon_1, \dots, \varepsilon_{n+1}), \tag{15}$$

as follows:

$$\varepsilon_i = \frac{e_i}{r^{i-1+a}}, \quad \forall 1 \leqslant i \leqslant n+1. \tag{16}$$

We have

$$\dot{\varepsilon} = r(A_{n+1} - KC^{\mathrm{T}})\varepsilon - (aI_{n+1} + D_{n+1})\frac{\dot{r}}{r}\varepsilon + \Delta_1, \tag{17}$$

where

$$\Delta_1 = \mathrm{col}\left(\frac{\delta_1}{r^a}, \dots, \frac{\delta_n}{r^{n-1+a}}, 0\right). \tag{18}$$

The main property of interest to us of the observer can be expressed with the help of the quadratic function

$$V_e = \varepsilon^{\mathrm{T}} P \varepsilon. \tag{19}$$

By means of (8) and (11), we can see that, along the solutions of (17), the time derivative of V_e satisfies

$$\dot{V}_e \leqslant -([d_0 - (2a + d_1)b]r + a\sigma)V_e + 2\varepsilon^{\mathrm{T}} P \Delta_1. \tag{20}$$

2.2. Controller design

Here, we follow the suggestion of [5]. But we restrict the controller to be linear in $(y, \hat{x}_2, \dots, \hat{x}_{n+1})$. For this, as usual, we consider the following auxiliary system:

$$\begin{aligned}\dot{y} &= \hat{x}_2 + e_2 + \delta_1,\\ \dot{\hat{x}}_2 &= \hat{x}_3 + k_2 r^2 e_1\\ &\vdots\\ \dot{\hat{x}}_n &= \hat{x}_{n+1} + u + k_n r^n e_1.\end{aligned} \tag{21}$$

We introduce the scaled state and input variables

$$\bar{x} = \mathrm{col}\left(\frac{y}{r^a}, \frac{\hat{x}_2}{r^{1+a}}, \dots, \frac{\hat{x}_n}{r^{n-1+a}}\right), \tag{22}$$

$$\bar{u} = \frac{\hat{x}_{n+1} + u}{r^n}. \tag{23}$$

Using the notation introduced previously, we have

$$\dot{\bar{x}} = rA_n\bar{x} + rB\bar{u} - (aI_n + D_n)\frac{\dot{r}}{r}\bar{x} + r\Delta_2 \tag{24}$$

where

$$\Delta_2 = \mathrm{col}\left(\frac{\delta_1}{r^{1+a}} + \varepsilon_2, k_2\varepsilon_1, \dots, k_n\varepsilon_1\right). \tag{25}$$

We compute the scaled input $\bar{u}$ as

$$\bar{u} = -F\bar{x} \tag{26}$$

with F given by Lemma 1.

The main property of interest to us of the controller can be expressed with the help of the quadratic function

$$V_c = \bar{x}^{\mathrm{T}} Q \bar{x}. \tag{27}$$

By means of (8) and (11), we can see that, along the solutions of (24), the time derivative of V_c satisfies

$$\dot V_c \leqslant -([d_0-(2a+d_1)b]r+a\sigma)V_c + 2r\bar{x}^{\mathrm{T}}Q\Delta_2. \quad (28)$$

3. Main result

Theorem 1. *Consider system* (1). *Under hypotheses* (H1) *and* (H2), *the following output feedback makes the solutions of the closed loop system bounded and their components z and x_i, $1\leqslant i\leqslant n$, to converge to the origin*:

$$\dot{\hat x}_i = \hat x_{i+1} + k_i r^i (x_1-\hat x_1), \quad 1\leqslant i\leqslant n-1,$$

$$\dot{\hat x}_n = \hat x_{n+1} + u + k_n r^n (x_1-\hat x_1),$$

$$\dot{\hat x}_{n+1} = k_{n+1} r^{n+1}(x_1-\hat x_1),$$

$$u = -\hat x_{n+1} - r^n F \,\mathrm{col}\left(\frac{y}{r^a}, \frac{\hat x_2}{r^{1+a}}, \dots, \frac{\hat x_n}{r^{n-1+a}}\right),$$

$$\dot r = -r(br-\sigma(y,r)), \quad (29)$$

where the scalars k_i's, a and b, the matrix F and the locally Lipschitz function σ are the appropriately chosen feedback parameters. [1]

Proof of Theorem 1. From the hypothesis (H1.2) and the fact that r can be imposed to stay larger than 1, we get

$$\frac{|\delta_i|}{r^{i-1+a}} \leqslant L(|y|)[(|\bar x_1|+\cdots+|\bar x_i|) + (|\varepsilon_2|+\cdots+|\varepsilon_i|)] + \frac{\kappa(V_z)}{r^{i-1+a}}. \quad (30)$$

By completing the squares, this yields

$$2\varepsilon^{\mathrm{T}}P\Delta_1 \leqslant L(|y|)V_c + d_2 L(|y|)V_e + d_3\sqrt{V_e}\,\frac{\kappa(V_z)}{r^a}, \quad (31)$$

with some nonnegative real numbers d_2, d_3, depending on a. So, it holds

$$\dot V_e \leqslant -([d_0-(2a+d_1)b]r + a\sigma - d_2L(|y|))V_e + L(|y|)V_c + d_3\sqrt{V_e}\,\frac{\kappa(V_z)}{r^a}. \quad (32)$$

[1] See Eqs. (8), (37) and (59).

Similarly, we get

$$2r\bar x^{\mathrm{T}}Q\Delta_2 \leqslant \left(d_4 L(y) + \frac{d_0}{2}r\right)V_c + rd_5V_e + d_6\sqrt{V_c}\,\frac{\kappa(V_z)}{r^a} \quad (33)$$

with some strictly positive real numbers d_4 to d_6, depending on a. This, in turn, implies

$$\dot V_c \leqslant -\left(\left[\frac{d_0}{2}-(2a+d_1)b\right]r + a\sigma - d_4L(|y|)\right)V_c + rd_5V_e + d_6\sqrt{V_c}\,\frac{\kappa(V_z)}{r^a}. \quad (34)$$

Now, consider the function

$$V_{ec} = \frac{2d_5}{d_0}V_e + V_c. \quad (35)$$

Its time derivative satisfies

$$\dot V_{ec} \leqslant -\frac{2d_5}{d_0} \times \left(\left[\frac{d_0}{2}-(2a+d_1)b\right]r + a\sigma - d_2L(|y|)\right)V_e - \left(\left[\frac{d_0}{2}-(2a+d_1)b\right]r + a\sigma - \left[d_4+\frac{2d_5}{d_0}\right] \times L(|y|)\right)V_c + d_7\sqrt{V_{ec}}\,\frac{\kappa(V_z)}{r^a} \quad (36)$$

with some nonnegative real number d_7, depending on a. So let us select our design parameters b and σ to satisfy

$$\frac{d_0(a)}{4(2a+d_1(a))} \geqslant b > 0, \quad (37)$$

$$\sigma(y,r) = \sigma_1(y) + \sigma_2(y,r), \quad (38)$$

$$\sigma_1(y) = \frac{\max\{d_2(a), [d_4(a)+2d_5(a)/d_0(a)]\}}{a} \times L(|y|), \quad (39)$$

where σ_2 is a function, lower bounded by b, to be defined in (55) below.

Note that, as L, σ_1 is a locally Lipschitz function of the state of the closed loop system. Our motivation for this choice is that it leads to

$$\dot{V}_{ec} \leqslant -\left(\frac{d_0 r}{4} + a\sigma_2\right) V_{ec} + d_7 \frac{\sqrt{V_{ec}}}{r^a} \kappa(V_z). \tag{40}$$

Now, with $\ell \geqslant 1$ the real number given by the hypothesis (H2.2), let ρ_z be the function, defined on $(0,+\infty)$, as

$$\rho_z(s) = \frac{\kappa(s)^\ell}{s}. \tag{41}$$

As κ, this function ρ_z is C^1 on $(0,+\infty)$. Precisely, we have

$$\rho_z'(s) = \frac{\kappa(s)^{\ell-1}}{s^2}[\ell\kappa'(s)s - \kappa(s)] \geqslant 0. \tag{42}$$

So ρ_z is a nondecreasing function. Being nonnegative, it has a limit ρ_{z0} as z goes to 0. So we can extend the definition of ρ_z to $[0,+\infty)$ by letting

$$\rho_z(0) = \rho_{z0}. \tag{43}$$

This way we get a nondecreasing continuous function on $[0,+\infty)$. It follows that $\int_0^v ((\kappa(s)^\ell)/s)\,\mathrm{d}s$ defines a C^1 function on $[0,\infty)$ which is radially unbounded. Also, because ρ is non-decreasing it holds

$$\int_0^v \frac{\kappa(s)^\ell}{s}\,\mathrm{d}s \leqslant \kappa(v)^\ell. \tag{44}$$

With this at hand and with the function V_z given by hypothesis (H1.1), introduce the function

$$U = c\int_0^{V_z} \frac{\kappa(s)^\ell}{s}\,\mathrm{d}s + \frac{1}{\ell}\left(2\sqrt{V_{ec}}\right)^\ell, \tag{45}$$

where c is another design parameter to be chosen as a strictly positive real number. This function U is positive definite and radially unbounded in $(\bar{x},\varepsilon,z)$. It is also differentiable at any point except at $(0,0,z)$. But it is locally Lipschitz. So it admits an upper right Dini derivative along any solution. We denote this derivative $\dot{U}$. With hypothesis (H1.1), it satisfies

$$\dot{U} \leqslant -c\frac{\kappa(V_z)^\ell}{V_z}\left[V_z - \gamma(|y|)\right] - \left(2\sqrt{V_{ec}}\right)^{\ell-1} \times\left[\left(\frac{d_0 r}{8} + \frac{a}{2}\sigma_2\right)\left(2\sqrt{V_{ec}}\right) - \frac{d_7}{r^a}\kappa(V_z)\right]. \tag{46}$$

Since ρ_z is a nondecreasing function, by considering successively the two cases $V_z \geqslant 2\gamma(|y|)$ and $V_z < 2\gamma(|y|)$, we get the inequality

$$\frac{\kappa(V_z)^\ell}{V_z}\left[V_z - \gamma(|y|)\right] \geqslant \tfrac{1}{2}\kappa(V_z)^\ell - \tfrac{1}{2}\kappa(2\gamma(|y|))^\ell. \tag{47}$$

Similarly, with the two cases $\kappa(V_z) \geqslant (2\sqrt{V_{ec}})$ and $\kappa(V_z) < (2\sqrt{V_{ec}})$, and using the fact that we can guarantee that r remains larger than 1, all along any solution, we get

$$\left(2\sqrt{V_{ec}}\right)^{\ell-1}\frac{d_7}{r^a}\kappa(V_z) \leqslant d_7\kappa(V_z)^\ell + \frac{d_7}{r^a}\left(2\sqrt{V_{ec}}\right)^\ell. \tag{48}$$

All this gives

$$\dot{U} \leqslant -\left(\frac{c}{2} - d_7\right)\kappa(V_z)^\ell + \frac{c}{2}\kappa(2\gamma(|y|))^\ell - \left(2\sqrt{V_{ec}}\right)^\ell\left(\frac{d_0 r}{8} + \frac{a}{2}\sigma_2 - \frac{d_7}{r^a}\right). \tag{49}$$

So we choose c satisfying

$$c \geqslant 4d_7(a). \tag{50}$$

Then let d_8, depending on a, be the square root of the minimum eigenvalue of Q. We have

$$d_8|y| \leqslant r^a\sqrt{V_{ec}} \tag{51}$$

By imposing the constraint $a/2\sigma_2 \geqslant d_7/r^a$ in the choice of the function σ_2 to come later on, we get

$$\dot{U} \leqslant -\frac{c}{4}\kappa(V_z)^\ell - \frac{d_0 r}{8}\left(2\sqrt{V_{ec}}\right)^\ell + \frac{c}{2}\kappa(2\gamma(|y|))^\ell - \left(\frac{a}{2}\sigma_2 - \frac{d_7}{r^a}\right)\frac{1}{r^{a\ell}}(2d_8|y|)^\ell. \tag{52}$$

So our idea is to select the function σ_2 in order to get the inequality

$$\frac{c}{2}\kappa(2\gamma(|y|))^\ell - \left(\frac{a}{2}\sigma_2 - \frac{d_7}{r^a}\right)\frac{1}{r^{a\ell}}(2d_8|y|)^\ell \leqslant 0. \tag{53}$$

From hypothesis (H1.3), such a choice is possible since, when $|y| \leqslant s_0$, it is sufficient to have

$$\frac{a/2\sigma_2 - d_7/r^a}{r^{a\ell}}(2d_8)^\ell \geqslant \frac{c}{2}k^\ell. \tag{54}$$

A specific expression for σ_2 is for instance

$$\sigma_2 = \max\left\{b, \frac{2d_7(a)}{a}\right\} + \frac{c}{a(2d_8(a))^\ell} r^{a\ell} \times \max\left\{k^\ell, \left(\frac{\kappa(2\gamma(|y|))}{|y|}\right)^\ell\right\}. \tag{55}$$

This is a locally Lipschitz function. In this way, r being larger than 1, with (44), we obtain

$$\dot{U} \leqslant -\frac{c}{4}\int_0^{V_z} \frac{\kappa(s)^\ell}{s}\,\mathrm{d}s - \frac{d_0 r}{8}\left(2\sqrt{V_{ec}}\right)^\ell \tag{56}$$

$$\leqslant -\min\left\{\frac{1}{4}, \frac{\ell d_0}{8}\right\} U. \tag{57}$$

So it follows from [16, Section 13] for instance, that, along any solution, $U(t)$ is exponentially decreasing and therefore also bounded.

Now consider any closed-loop solution. Its state can be taken as $(r, \bar{x}, \varepsilon, z)$. Let it be right maximally defined on $[0, T)$. From the properties of $U(t)$, we know that $V_{ec}(t)$ and $V_z(t)$ are bounded on $[0, T)$. Therefore, for each solution, the components $\bar{x}(t)$, $\varepsilon(t)$ and $z(t)$ are bounded. This, in turn, gives that $\bar{x}_1(t) = y(t)/r^a(t)$ is bounded, i.e. we have

$$|\bar{x}_1(t)| \leqslant X_1 \quad \forall t \in [0, T) \tag{58}$$

for some real number X_1, depending on the solution.

Let us show that, by picking a small enough, we can guarantee that the component $r(t)$ is also bounded on $[0, T)$. We know that we have (see (11), (38), (39), (55))

$$\dot{r} = -r\left(b_1 r - b_2 - b_3 L(|y|) - b_4 r^{a\ell} \max\left\{k^\ell, \left(\frac{\kappa(2\gamma(|y|))}{|y|}\right)^\ell\right\}\right) \tag{59}$$

with

$$b_1 = b, \tag{60}$$

$$b_2 = \max\left\{b, \frac{2d_7}{a}\right\}, \tag{61}$$

$$b_3 = \frac{\max\{d_2, [d_4 + 2d_5/d_0]\}}{a}, \tag{62}$$

$$b_4 = \frac{c}{a(2d_8)^\ell}. \tag{63}$$

With the help of hypotheses (H2.1), and the inequality

$$|y(t)| \leqslant X_1 r^a \quad \forall t \in [0, T), \tag{64}$$

we get

$$\dot{r} \leqslant -r(b_1 r - [b_2 + b_3 p] - b_3 X_1^m r^{am} - b_4[2k + 2^{\ell-1} p^\ell] r^{a\ell} - 2^{\ell-1} b_4 X_1^m r^{am\ell}). \tag{65}$$

But if a is selected to satisfy the constraint

$$am\ell < 1, \tag{66}$$

with the help of Young's inequality, we can find a positive real number b_5 so that

$$b_1 r - [b_2 + b_3 p] - b_3 X_1^m r^{am} - b_4[2k + 2^{\ell-1} p^\ell] r^{a\ell} - 2^{\ell-1} b_4 X_1^m r^{am\ell} \geqslant \frac{b_1}{2} r - b_5. \tag{67}$$

This yields simply

$$\dot{r} \leqslant -r\left(\frac{b_1}{2} r - b_5\right). \tag{68}$$

It follows readily that the component $r(t)$ is also bounded on $[0, T)$.

So all the components being bounded, T must be infinite. This implies that the solution is bounded on $[0, +\infty)$ and $U(t)$ converges to 0 as t goes to ∞. So the state components $(x_i(t), e_i(t), z(t))$ converge to the origin.

4. Conclusion

In this paper, a high gain linear output feedback is proposed for global regulation of an equilibrium of a class of nonlinear systems where the nonlinearities have a linear growth in the (partial) unmeasured state components, with an output dependent growth rate and with ISS inverse dynamics. This feedback involves an on-line tuned gain.

We have seen that the main loss we have by imposing a linear structure, as compared to the best available results using truly nonlinear output feedback, is that the growth rate and the gain from the output to the perturbation via the inverse dynamics should have a growth which cannot be more than polynomial.

The stability analysis has been made possible by

1. the scaling of some components, as usual when high gain is used. However here we use the modification of the scaling introduced in [10].
2. the application of the Lyapunov version of the nonlinear small-gain techniques [2].

Appendix A. Proof of Lemma 1

A.1. On the pair (Q, F)

The existence of the pair (Q, F) follows from the following lemma by letting:

$$\tau = -\frac{1}{n-1+a/2}. \tag{A.1}$$

Lemma 2. *Let R be the following diagonal matrix*:

$$R = -I - \tau\,\mathrm{diag}(n-1, n-2, \ldots, 0), \tag{A.2}$$

where $\tau > -1/(n-1)$. *Let* $A \in \mathbb{R}^{n\times n}$ *and* $B \in \mathbb{R}^n$ *be the matrices in the canonical controller form as in* (7). *Then, there exist matrices F and Q satisfying*:

$$Q(A-BF) + (A-BF)^\top Q < 0, \tag{A.3}$$

$$RQ + QR < 0, \tag{A.4}$$

$$Q > 0. \tag{A.5}$$

Proof. By letting

$$X = Q^{-1}, \quad Y^\top = FQ^{-1} \tag{A.6}$$

the statement becomes

There exist matrices Y and X of appropriate dimensions such that

$$[XA^\top + AX] - [YB^\top + BY^\top] < 0,$$

$$XR + RX < 0,$$

$$X > 0. \tag{A.7}$$

To prove this result, we proceed by induction on the dimension.

Case $n = 1$: We let

$$X_1 > 0, \quad A_1 = 0, \quad B_1 = 1,$$
$$R_1 = -1, \quad Y_1 > 0. \tag{A.8}$$

We have

$$[X_1 A_1^\top + A_1 X_1] - [Y_1 B_1^\top + B_1 Y_1^\top] = -2Y_1, \tag{A.9}$$

$$X_1 R_1 + R_1 X_1 = -2X_1. \tag{A.10}$$

It follows that

$$M_{11} = 2Y_1, \quad M_{21} = 2X_1 \tag{A.11}$$

and X_1 are strictly positive real numbers.

Case $n = j + 1$: We assume that we know X_j and Y_j solutions of

$$[X_j A_j^\top + A_j X_j] - [Y_j B_j^\top + B_j Y_j^\top] = -M_{1,j} < 0, \tag{A.12}$$

$$X_j R_j + R_j X_j = -M_{2,j} < 0, \tag{A.13}$$

$$X_j > 0. \tag{A.14}$$

where A_j, B_j and R_j are defined recursively as

$$A_{j+1} = \begin{pmatrix} A_j & B_j \\ 0 & 0 \end{pmatrix}, \quad B_{j+1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$
$$R_{j+1} = \begin{pmatrix} R_j - \tau I_j & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.15}$$

Let us define X_{j+1} and Y_{j+1} as

$$X_{j+1} = \begin{pmatrix} X_j & S_j \\ S_j^\top & T_j \end{pmatrix}, \quad Y_{j+1} = \begin{pmatrix} U_{j+1} \\ V_{j+1} \end{pmatrix}, \tag{A.16}$$

where S_j, T_j, U_{j+1} and V_{j+1} are to be chosen to solve (A.12) for $j+1$. We have

$$X_{j+1}A_{j+1}^\top - Y_{j+1}B_{j+1}^\top = \begin{pmatrix} X_j A_j^\top + S_j B_j^\top & -U_{j+1} \\ S_j^\top A_j^\top + T_j B_j^\top & -V_{j+1} \end{pmatrix}. \tag{A.17}$$

By taking the symmetric part and by comparing with (A.12), we see that, by choosing

$$S_j = -Y_j, \quad U_{j+1} = A_j S_j + B_j T_j, \tag{A.18}$$

we get simply

$$-M_{1,j+1} = \begin{pmatrix} -M_{1,j} & 0 \\ 0 & -2V_{j+1} \end{pmatrix}. \tag{A.19}$$

On the other hand, we compute

$$X_{j+1}R_{j+1} = \begin{pmatrix} X_j R_j - \tau X_j & -S_j \\ S_j^\top R_j - \tau S_j^\top & -T_j \end{pmatrix}. \tag{A.20}$$

It follows that

$$-M_{2,j+1} = \begin{pmatrix} -M_{2,j} - 2\tau X_j & R_j S_j^\top - (1+\tau)S_j \\ S_j^\top R_j - (1+\tau)S_j^\top & -2T_j \end{pmatrix} \tag{A.21}$$

$$= \begin{pmatrix} -M_{2,j} - 2\tau X_j & -R_j Y_j^\top + (1+\tau)Y_j \\ -Y_j^\top R_j + (1+\tau)Y_j^\top & -2T_j \end{pmatrix}. \tag{A.22}$$

It remains to find the real numbers T_j and V_{j+1} so that the matrices

$$X_{j+1} = \begin{pmatrix} X_j & -Y_j \\ -Y_j^\top & T_j \end{pmatrix}, \tag{A.23}$$

$$M_{1,j+1} = \begin{pmatrix} M_{1,j} & 0 \\ 0 & 2V_{j+1} \end{pmatrix}, \tag{A.24}$$

$$M_{2,j+1} = \begin{pmatrix} M_{2,j} + 2\tau X_j & R_j Y_j^\top - (1+\tau)Y_j \\ Y_j^\top R_j - (1+\tau)Y_j^\top & 2T_j \end{pmatrix} \tag{A.25}$$

are positive definite. This is always possible if $M_{2,j} + 2\tau X_j$ is positive definite. Indeed, in this case, it is sufficient to take these two numbers large enough. But, since we have the following recurrence:

$$M_{2,j+1} + 2\tau X_{j+1} = \begin{pmatrix} M_{2,j} + 4\tau X_j & -R_j S_j^\top + (1+3\tau)S_j \\ -S_j^\top R_j + (1+3\tau)S_j^\top & 2(1+\tau)T_j \end{pmatrix} > 0, \tag{A.26}$$

our result holds for j in $\{1,\ldots,n\}$ if

$$M_{21} + 2\tau(j-1)X_1 = 2(1+\tau(j-1))X_1 > 0. \tag{A.27}$$

This condition is satisfied if, for all j in $\{1,\ldots,n\}$, we have

$$\tau > -1/(j-1). \tag{A.28}$$

A.2. On the pair (P,K)

The existence of the pair (P,K) follows from the same arguments by letting

$$R = -I - \frac{1}{a}\operatorname{diag}(0,\ldots,n-2,n-1), \tag{A.29}$$

$$X = P, \quad Y = PK \tag{A.30}$$

and doing a recursion with

$$A_{j+1} = \begin{pmatrix} 0 & C_j^{\mathrm{T}} \\ 0 & A_j \end{pmatrix}, \quad C_{j+1}^{\mathrm{T}} = \begin{pmatrix} 1 & 0 & \ldots & 0 \end{pmatrix},$$

$$R_{j+1} = \begin{pmatrix} -1 & 0 \\ 0 & R_j - \frac{1}{a} I_j \end{pmatrix} \tag{A.31}$$

$$X_{j+1} = \begin{pmatrix} T_j & -Y_j^{\mathrm{T}} \\ -Y_j & X_j \end{pmatrix}, \quad Y_{j+1} = \begin{pmatrix} V_{j+1} \\ U_{j+1} \end{pmatrix}. \tag{A.32}$$

References

[1] Z.P. Jiang, I. Mareels, Robust nonlinear integral control, IEEE Trans. Automat. Control 46 (8) (2001) 1336–1342.

[2] Z.P. Jiang, I. Mareels, Y. Wang, A Lyapunov formulation of the nonlinear small gain theorem for interconnected ISS systems, Automatica 32 (8) (1996) 1211–1215.

[3] Z.P. Jiang, A.R. Teel, L. Praly, Small-gain theorem for ISS systems and applications, Math. Control, Signals and Systems 7 (1994) 95–120.

[4] H. Khalil, A. Saberi, Adaptive stabilization of a class of nonlinear systems using high-gain feedback, IEEE Trans. Automat. Control 32 (11) (1987) 1031–1035.

[5] P. Krishnamurthy, F. Khorrami, Generalized adaptive output-feedback form with unknown parameters multiplying high output relative-degree states, Proceedings of the 41st IEEE Conference on Decision and Control, Las Vegas, NV, December 2002, pp. 1503–1508.

[6] M. Krstić, I. Kanellakopoulos, P.V. Kokotović, Nonlinear and Adaptive Control Design, Wiley, New York, 1995.

[7] R. Marino, P. Tomei, Nonlinear Control Design: Geometric, Adaptive and Robust, Prentice-Hall, London, 1995.
[8] F. Mazenc, L. Praly, W.P. Dayawansa, Global stabilization by output feedback: examples and counterexamples, Systems Control Lett. 23 (1994) 119–125.
[9] L. Praly, Generalized weighted homogeneity and state dependent time scale for linear controllable systems, Proceedings of the 36th IEEE Conference on Decision and Control, December 1997, pp. 4342–4347.
[10] L. Praly, Asymptotic stabilization via output feedback for lower triangular systems with output dependent incremental rate, IEEE Trans. Automat. Control 48 (6) (2003) 1103–1108.
[11] L. Praly, Z.P. Jiang, Stabilization by output feedback for systems with ISS inverse dynamics, Systems Control Lett. 21 (1993) 19–33.
[12] L. Praly, Y. Wang, Stabilization in spite of matched unmodeled dynamics and an equivalent definition of input-to-state stability, Math. Control, Signals, Systems 9 (1996) 1–33.
[13] C. Qian, W. Lin, Universal stabilization of a class of nonlinear systems by output feedback, Proc. American Control Conference, Anchorage, AK, May 8–10, 2002, pp. 122–127.
[14] E. Sontag, Y. Wang, On characterizations of the input-to-state stability property, Systems Control Lett. 24 (1995) 351–359.
[15] J.T. Spooner, M. Maggiore, R. Ordonez, K.M. Passino, Stable Adaptive Control and Estimation for Nonlinear Systems, Wiley, New York, 2002.
[16] J. Szarski, Differential Inequalities, PWN, Warsaw, 1965.