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A relaxed condition for stability of nonlinear observer-based controllers

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Abstract

We study certainty-equivalence design of observer-based controllers, and present a new stability condition that exploits state-dependent convergence properties of observers. This result eliminates the conservatism of treating the observer error as an exogeneous disturbance. As an application, we show that the reduced-order variant of the class of observers in [Automatica 37 (2001) 9231] preserves global asymptotic stability in a certainty-equivalence implementation. In another application we study a nonminimum phase system, which cannot be stabilized with the existing output-feedback designs.

Keywords: Observer-based control; Certainty-equivalence

1. Introduction

One of the major difficulties for nonlinear output-feedback design is the absence of a controller–observer separation property. Even when a nonlinear observer is available, the certainty-equivalence implementation of a state-feedback control law may lead to severe forms of instability, including finite escape time. Stability of certainty-equivalence designs has been established under restrictive assumptions on the growth of nonlinearities, such as the global Lipschitz condition in [16]. With less restrictive assumptions, only *local* [16,3] and *semi-global* [12,15] stability results have been achieved. Other designs depart from certainty-equivalence, and either modify the control design as in [10,5,7], or the observer design as in [9,10].

This paper presents a new stability condition for certainty-equivalence designs. The conservatism of earlier results is eliminated with two key ingredients in our analysis. First, rather than treat the observer error as an exogenous disturbance, we exploit its dependence on the plant trajectories. Indeed, for several classes of nonlinear observers, the convergence speed of the observer error depends critically on the magnitude of the plant states. To take into account this interplay between the plant and the observer, we formulate an assumption that allows the observer error convergence to be state-dependent. Next, to further relax our conditions, we pursue the stability analysis neither with the plant states nor with their estimates, but with a combination of the two. We illustrate with several examples that the resulting stability condition allows severe nonlinearities in the observer-based

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controller. We also show that, in contrast to the control redesigns in [10,5,7], this certainty-equivalence result may be applicable to nonminimum phase systems. Finally, we apply the main result to the class of observers introduced in [2], and show that a reduced-order variant of this design preserves global asymptotic stability (GAS) in a certainty-equivalence design, under mild assumptions on the underlying state-feedback controller.

Although we mainly address certainty-equivalence designs, our result also gives guidelines for a redesign of controllers and observers. Among the examples where we pursue such a redesign is the system

$$\dot{x}_1 = -x_1 + x_2 x_1^2 + u, \quad \dot{x}_2 = -x_2 + x_1^2, \quad y = x_1 \quad (1)$$

which admits the globally asymptotically stabilizing state-feedback

$$u = \phi(x_1, x_2) = -x_2 x_1^2. \quad (2)$$

However, as shown in [4], the certainty-equivalence controller

$$u = \phi(x_1, \hat{x}_2) \quad (3)$$

leads to finite escape time when implemented with the reduced-order observer

$$\dot{\hat{x}}_2 = -\hat{x}_2 + y^2. \quad (4)$$

In contrast, it follows from our result that the same feedback achieves global asymptotic stability with an alternative observer construction presented in Section 2, Example 1.

The paper is organized as follows: Section 2 gives the main result (Theorem 1), followed by an illustration on system (1) above. Section 3 shows that the assumptions of Theorem 1 are tight, and proceeds with a certainty-equivalence design for a nonminimum phase example. Section 4 studies certainty-equivalence design with the class of observers introduced in [2]. Conclusions are given in Section 5.

2. Main result

We consider the system

$$\dot{x} = f(x, u), \qquad y = h(x), \quad (5)$$

where f and h are locally Lipschitz functions, which are zero at the origin, u in $\mathbb{R}^m$ is the control input, x in $\mathbb{R}^n$ is the state, and y in $\mathbb{R}^p$ is the measured output. Our problem is to design a globally asymptotically stabilizing observer-based controller. We first assume that the system is stabilizable via full-state feedback.

Assumption 1 (Stabilizability). There exist a locally Lipschitz state-feedback control ϕ and a C^1, non-negative radially unbounded function V, both zero at the origin, such that

$$\frac{\partial V}{\partial x}(x) f(x, \phi(x)) \leqslant 0 \quad \forall x \in \mathbb{R}^n, \quad (6)$$

and such that the origin is a globally asymptotically stable equilibrium of

$$\dot{x} = f(x, \phi(x)). \quad (7)$$

Next, we assume we are given an observer of the form

$$\dot{z} = \mu(z, u, y),$$

$$\widehat{x} = v(z, u, y), \quad (8)$$

where the functions μ and v are locally Lipschitz, z in $\mathbb{R}^q$, and $\widehat{x}$ is a (possibly partial) estimate of x. To express our restrictions on this observer, we introduce an auxiliary variable $\mathscr{X}$ in $\mathbb{R}^n$ which is made up of components of the state x of the system and state z of the observer, i.e.,

$$\mathscr{X} = K(x, z). \quad (9)$$

The choice of this $\mathscr{X}$ should be such that the control u in the output-feedback is

$$u = \phi(\mathscr{X}). \quad (10)$$

As an example, for the second-order system (1) with the control (3), $\mathscr{X}$ must be

$$\mathscr{X} = (x_1, \widehat{x}_2). \quad (11)$$

For our analysis we rewrite the dynamics of $\mathscr{X}$ in the form

$$\dot{\mathscr{X}} = f(\mathscr{X}, u) + \ell(x, z, u), \quad (12)$$

where $f(\mathscr{X}, u)$ represents the nominal state-feedback system, and $\ell(x, z, u)$ represents the perturbation due to the observer error. We further factorize this term as

$$\ell(x, z, u) = k(x, z, u)\varepsilon(x, z, u), \quad (13)$$

where ε is to satisfy the following assumption:

Assumption 2 (Uniform detectability).

1. There exist a locally Lipschitz function K satisfying

$$K(0,0)=0 \tag{14}$$

and a continuous function α_1, where $\alpha_1((x,z),\cdot)$ is of class K_∞ for each (x,z) in $\mathbb{R}^n\times\mathbb{R}^q$, such that, whatever the control u and the initial condition (x_0,z_0) are, the corresponding solution $(x(t),z(t))$ of

$$\dot{x}=f(x,u),$$
$$\dot{z}=\mu(z,u,h(x)) \tag{15}$$

is such that, on its right maximal interval of definition $[0,T)$, we have

$$|x(t)-\hat{x}(t)|+|x(t)-\mathscr{X}(t)|+\int_0^t|\varepsilon(s)|\,\mathrm{d}s \leqslant \alpha_1((x_0,z_0),|\mathscr{X}_0-x_0|) \tag{16}$$

with

$$\mathscr{X}=K(x,z),\quad \hat{x}=v(z,u,h(x)). \tag{17}$$

Moreover, if $T=+\infty$, then

$$\lim_{t\to+\infty}|x(t)-\mathscr{X}(t)|=0. \tag{18}$$

2. There exists a class K_∞ function α_2 such that for each z in $\mathbb{R}^q$, u in $\mathbb{R}^m$, y in $\mathbb{R}^p$, we have

$$|z|\leqslant\alpha_2(|v(z,u,y)|+|u|+|y|). \tag{19}$$

The inequality (16) characterizes the convergence properties of the observer. It expresses the dependence of the observer convergence on the plant state through the fact that the function $\varepsilon(x,z,u)$ is L^1 along the solutions. We prove GAS for the certainty equivalence design $u=\phi(\mathscr{X})$ by studying the auxiliary system

$$\dot{\mathscr{X}}=f(\mathscr{X},u)+k(x,z,u)\varepsilon(t) \tag{20}$$

with the following restriction on the coupling term $k(x,z,u)$:

Theorem 1. *Assume the stabilizability and uniform detectability assumptions hold. If there exists a C^1, class-K_∞ function $\mathscr{L}$ and a positive real number M satisfying, for all (x,z) in $\mathbb{R}^n\times\mathbb{R}^q$,*

$$|\mathscr{L}'(V(\mathscr{X}))|\left|\frac{\partial V}{\partial x}(\mathscr{X})k(x,z,\phi(\mathscr{X}))\right|\leqslant M \tag{21}$$

with $\mathscr{X}=K(x,z)$, then the certainty-equivalent output-feedback

$$\dot{z}=\mu(z,u,y),$$
$$\hat{x}=v(z,u,y),$$
$$u=\phi(K(x,z)) \tag{22}$$

is globally asymptotically stabilizing.

Proof. See Appendix A.

Example 1. We illustrate Theorem 1 on the system (1), which satisfies our stabilizability assumption with

$$V(x_1,x_2)=\frac{x_1^4}{4}+\frac{x_2^2}{2},\qquad \phi(x_1,x_2)=-x_2x_1^2. \tag{23}$$

To satisfy condition (21), the function $\mathscr{L}$ should have the smallest possible derivative. With the choice

$$\mathscr{L}(v)=\log(1+v) \tag{24}$$

we get

$$\frac{\partial\mathscr{L}(V)}{\partial x_1}=\frac{x_1^3}{1+x_1^4/4+x_2^2/2},\qquad \frac{\partial\mathscr{L}(V)}{\partial x_2}=\frac{x_2}{1+x_1^4/4+x_2^2/2}, \tag{25}$$

in which the derivative with respect to x_1 is bounded when multiplied by x_1 or $\sqrt{|x_2|}$. This means that the first component k_1 of k in (21) can grow as $|x_1|$ or $\sqrt{|x_2|}$. Likewise, k_2 can grow as x_1^2 or x_2.

For the observer we study two designs. The reduced-order design

$$\dot{\hat{x}}_2=-\hat{x}_2+y^2 \tag{26}$$

ensures that $\hat{x}_2-x_2$ exponentially decaying; i.e., it is an L^1 function of time. The full-order design

$$\dot{\hat{x}}_1=-\hat{x}_1+\hat{x}_2y^2+u-y^2(\hat{x}_1-x_1),$$
$$\dot{\hat{x}}_2=-\hat{x}_2+y^2-y^2(\hat{x}_1-x_1) \tag{27}$$

ensures L^1 for not only $(\hat{x}_1-x_1,\ \hat{x}_2-x_2)$, but also for

$$y^2(|\hat{x}_1-x_1|+|\hat{x}_2-x_2|). \tag{28}$$

Indeed, the function

$$U = \sqrt{(\widehat{x}_1 - x_1)^2 - (\widehat{x}_1 - x_1)(\widehat{x}_2 - x_2) + (\widehat{x}_2 - x_2)^2}$$

satisfies, for almost all $t \geqslant 0$,

$$\dot{U} \leqslant -U - \tfrac{1}{2} U y^2 \tag{29}$$

which, when integrated in the maximal interval of definition for $y(t)$, implies that Uy^2 is L^1 as in Assumption 2. Because $U \leqslant |\widehat{x}_1 - x_1| + |\widehat{x}_2 - x_2|$ it follows that (28) is also L^1.

To build an output-feedback, we can combine the state-feedback evaluated at $(x_1, \widehat{x}_2)$ or at $(\widehat{x}_1, \widehat{x}_2)$ with the reduced- or the full-order observer. Let us explore each choice:

1. If we employ the controller

$$u = -\widehat{x}_2 x_1^2 \tag{30}$$

then the $\mathscr{X}$ vector is

$$\mathscr{X} = (x_1, \widehat{x}_2). \tag{31}$$

(a) If we combine this control with the reduced order observer, the dynamics of $\mathscr{X}$ is

$$\dot{\mathscr{X}}_1 = -\mathscr{X}_1 + \mathscr{X}_2 \mathscr{X}_1^2 + u - \mathscr{X}_1^2[\mathscr{X}_2 - x_2],$$
$$\dot{\mathscr{X}}_2 = -\mathscr{X}_2 + \mathscr{X}_1^2, \tag{32}$$

where the function ℓ is

$$\ell(\mathscr{X}_1, \mathscr{X}_2, x_2) = \begin{pmatrix} -\mathscr{X}_1^2[\mathscr{X}_2 - x_2] \\ 0 \end{pmatrix}. \tag{33}$$

Because the reduced-order observer only ensures that $\mathscr{X}_2 - x_2$ is L^1 we factorize ℓ as

$$k = -\begin{pmatrix} \mathscr{X}_1^2 \\ 0 \end{pmatrix}, \quad \varepsilon = \mathscr{X}_2 - x_2. \tag{34}$$

Condition (21) of Theorem 1 does not hold because, (25) multiplied with this k is not bounded. Indeed, the feedback (30) with the reduced-order observer leads to solutions which escape in finite time, as discussed in Section 1.

(b) If we combine the control (30) with the full-order observer, the dynamics of $\mathscr{X}$ is

$$\dot{\mathscr{X}}_1 = -\mathscr{X}_1 + \mathscr{X}_2 \mathscr{X}_1^2 + u - \mathscr{X}_1^2(\mathscr{X}_2 - x_2),$$
$$\dot{\mathscr{X}}_2 = -\mathscr{X}_2 + \mathscr{X}_1^2 - \mathscr{X}_1^2(\widehat{x}_1 - \mathscr{X}_1), \tag{35}$$

where, because $\mathscr{X}_1^2(|\widehat{x}_1 - \mathscr{X}_1| + |\mathscr{X}_2 - x_2|)$ is L^1, we can use the factorization

$$k = 1, \quad \varepsilon = -\mathscr{X}_1^2 \begin{pmatrix} \mathscr{X}_2 - x_2 \\ \widehat{x}_1 - \mathscr{X}_1 \end{pmatrix}. \tag{36}$$

With this k, condition (21) is indeed satisfied, and Theorem 1 guarantees that the output-feedback

$$\dot{\widehat{x}}_1 = -\widehat{x}_1 + \widehat{x}_2 y^2 + u - y^2(\widehat{x}_1 - x_1),$$
$$\dot{\widehat{x}}_2 = -\widehat{x}_2 + y^2 - y^2(\widehat{x}_1 - x_1),$$
$$u = -\widehat{x}_2 x_1^2 \tag{37}$$

is globally asymptotically stabilizing.

2. If we employ the controller

$$u = \widehat{x}_2 \widehat{x}_1^2 \tag{38}$$

with the full-order observer, then $\mathscr{X}$ is

$$\mathscr{X} = (\widehat{x}_1, \widehat{x}_2), \tag{39}$$

which satisfies

$$\dot{\mathscr{X}}_1 = -\mathscr{X}_1 + \mathscr{X}_2 \mathscr{X}_1^2 + u - [\mathscr{X}_2(\mathscr{X}_1 + x_1) + x_1)^2](\mathscr{X}_1 - x_1),$$
$$\dot{\mathscr{X}}_2 = -\mathscr{X}_2 + \mathscr{X}_1^2 - [\mathscr{X}_1 + x_1 + x_1^2](\mathscr{X}_1 - x_1). \tag{40}$$

Because $\mathscr{X}_1 - x_1$ is bounded and $\mathscr{X}_1^2|\mathscr{X}_1 - x_1|$ is L^1, condition (21) is satisfied with the choice

$$k = -\begin{pmatrix} \mathscr{X}_2/(1 + |\mathscr{X}_1|) & 1 \\ 1/(1 + |\mathscr{X}_1|) & 1 \end{pmatrix},$$
$$\varepsilon = -\begin{pmatrix} (2\mathscr{X}_1 - [\mathscr{X}_1 - x_1])(1 + |\mathscr{X}_1|) \\ (\mathscr{X}_1 - [\mathscr{X}_1 - x_1])^2 \end{pmatrix} \times (\mathscr{X}_1 - x_1). \tag{41}$$

It follows that

$$\dot{\widehat{x}}_1 = -\widehat{x}_1 + \widehat{x}_2 y^2 + u - y^2(\widehat{x}_1 - x_1),$$
$$\dot{\widehat{x}}_2 = -\widehat{x}_2 + y^2 - y^2(\widehat{x}_1 - x_1),$$
$$u = -\widehat{x}_2 \widehat{x}_1^2 \tag{42}$$

is also a globally asymptotically stabilizing output-feedback.

3. Discussion

3.1. About inequality (21)

The main idea in Theorem 1 is to factor out an L^1 term ε from $\ell(x,z,u)$ in (12) using observer convergence properties, and to study robustness of the resulting system (20) against this ε. Because it is not known a priori whether the closed-loop solutions exist for all $t \geqslant 0$, ε may be defined only on $[0,T)$, with T finite. To make sure that, for any initial condition $\mathscr{X}_0$, there is a bounded solution $\mathscr{X}$ defined on $[0,T)$, we assume that $\frac{\partial V}{\partial \mathscr{X}}k$ is a bounded function, possibly after rescaling of V by $\mathscr{L}$. Such a condition is standard when dealing with boundedness of solutions (see e.g. [1, Theorem 2], [6, (b) p. 109, Section 23], [17, Example 10.2]). In particular, [1, Theorem 2] indicates that this assumption is not conservative, because forward completeness is equivalent to the existence of a radially unbounded function whose time derivative is bounded. To show that the boundedness assumption for $\frac{\partial V}{\partial \mathscr{X}}k$ cannot be relaxed, we consider the system

$$\dot{\mathscr{X}} = -\mathscr{X} + \mathscr{X}^2\varepsilon(t), \tag{43}$$

where ε is an exponentially decaying function. No nonnegative radially unbounded function V exists such that $\frac{\mathrm{d}V}{\mathrm{d}\mathscr{X}}\mathscr{X}^2$ is bounded on $\mathbb{R}$. In fact, this system has solutions which escape to $+\infty$ in finite time.

Our boundedness assumption for $\frac{\mathrm{d}V}{\mathrm{d}\mathscr{X}}k$ does not make use of the stabilizing term $-\mathscr{X}$ in (43). It may appear that a less conservative condition could be derived when this term is stronger than the coupling term, as in the input-to-state stable [13] system

$$\dot{\mathscr{X}} = -\mathscr{X}^3 + \mathscr{X}^2\varepsilon(t). \tag{44}$$

However, because ε is not guaranteed a priori to be bounded on its interval of definition, we may still get unbounded solutions for $\mathscr{X}$. This is the case in (44) if $\varepsilon(t)$ is generated as a solution of

$$\dot{\varepsilon} = \varepsilon^3. \tag{45}$$

Each such ε, although escaping in finite time, is in L^1 in its domain of definition. Despite this L^1 property, $\varepsilon - 2\mathscr{X} \geqslant 0$ is an invariant set and, in this set $\dot{\mathscr{X}} \geqslant \mathscr{X}^3$, which means that $\mathscr{X}$ exhibits finite escape time.

In the examples above, the unboundedness phenomena are due to the coupling of the L^1 disturbance ε with the explosive term $k = \mathscr{X}^2$. When k is a bounded function, our theorem only stipulates the existence of a Lyapunov function with bounded gradient $\frac{\mathrm{d}V}{\mathrm{d}\mathscr{X}}$, which is much less restrictive. However, this is still an 'assumption' as shown by Sontag and Krichman [14], who constructed a second-order example of the form

$$\dot{\mathscr{X}} = f(\mathscr{X}) + \varepsilon(t) \tag{46}$$

in which $\mathscr{X} = 0$ is GAS for the unperturbed system and yet, a bounded and L^1 disturbance $\varepsilon(t)$ gives rise to an unbounded solution $\mathscr{X}(t)$. This proves that no nonnegative and radially unbounded $V(\mathscr{X})$ exists with bounded gradient because, otherwise, $\mathscr{X}$ would remain bounded.

3.2. Nonminimum phase systems

As we have discussed in the introduction, one way to depart from certainty-equivalence is to modify the control design as in [10,5,7]. This requires a gain assignment from the estimation error to the measured output, which is achieved under a minimum phaseness assumption. Our result does not rely on this assumption, as we now illustrate with an example.

Example 2. Consider the system

$$\begin{aligned}
\dot{x}_1 &= x_2 + u,\\
\dot{x}_2 &= f(x_1) + x_3 - u,\\
\dot{x}_3 &= -f(x_1),\\
y &= x_1,
\end{aligned} \tag{47}$$

where f is a C^1 function which is zero at the origin. This system is nonminimum phase with inverse dynamics

$$\begin{aligned}
\dot{x}_2 &= x_2 + x_3,\\
\dot{x}_3 &= 0.
\end{aligned} \tag{48}$$

Except for the presence of $f(x_1)$, this system is an observable linear system and, hence, a full-order observer of the form

$$\begin{aligned}
\dot{\hat{x}}_1 &= \hat{x}_2 + u - k_1(\hat{x}_1 - y),\\
\dot{\hat{x}}_2 &= f(y) + \hat{x}_3 - u - k_2(\hat{x}_1 - y),\\
\dot{\hat{x}}_3 &= -f(y) - k_3(\hat{x}_1 - y)
\end{aligned} \tag{49}$$

with appropriate coefficients k_1, k_2 and k_3, gives boundedness and exponential convergence of the estimation error. Thus, if we employ the control law

$$u = \phi(\widehat{x}_1, \widehat{x}_2, \widehat{x}_3) \tag{50}$$

condition (16) holds with

$$\mathscr{X} = \begin{pmatrix} \widehat{x}_1 \\ \widehat{x}_2 \\ \widehat{x}_3 \end{pmatrix}, \quad k = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}, \quad \varepsilon = \mathscr{X}_1 - x_1. \tag{51}$$

To satisfy the stabilizability assumption and condition (21), it remains to design a state-feedback controller which admits a Lyapunov function V with bounded gradient. To this end we introduce the new coordinates

$$\xi_3 = x_3, \quad \xi_2 = x_1 + x_2 + x_3, \quad \xi_1 = x_2 + x_3 \tag{52}$$

and obtain, via a backstepping design for the (ξ_1, ξ_2)-subsystem, followed by the *forwarding modulo* $L_g V$ procedure of [11]

$$V(\xi_3, \xi_2, \xi_1) = V_2(\xi_2) + \tfrac{1}{2}(\xi_1 + \xi_2)^2 + \left(\xi_3 - \int_0^{\xi_2} \frac{f(2s)}{s}\,\mathrm{d}s\right)^2, \tag{53}$$

$$\begin{aligned}\phi(x_1, x_2, x_3) &= x_3 + (\xi_2 + \xi_1) + \xi_1 + V_2'(\xi_2) \\ &\quad + \left(\xi_3 - \int_0^{\xi_2} \frac{f(2s)}{s}\,\mathrm{d}s\right) \\ &\quad \times \left(-\frac{f(2\xi_2)}{\xi_2} + \frac{f(2\xi_2) - f(\xi_2 - \xi_1)}{\xi_2 + \xi_1}\right),\end{aligned} \tag{54}$$

where V_2 is any C^1, positive definite, radially unbounded function. We observe that condition (21) of Theorem 1 holds with

$$\mathscr{L}(v) = \log(1 + v), \tag{55}$$

if f meets a growth condition such that we can find a function V_2 with the appropriate properties above and satisfying

$$\max\left\{\left|\frac{f(2\xi_2)}{\xi_2}\right|^2, |V_2'(\xi_2)|\right\} \leqslant c(1 + V_2(\xi_2)) \tag{56}$$

for some real number c. It is satisfied for instance by

$$f(\xi) = \xi \exp(\xi), \quad V_2(\xi) = \xi^2 \exp(4|\xi|). \tag{57}$$

In this case it follows from Theorem 1 that

$$\begin{aligned}\dot{\widehat{x}}_1 &= \widehat{x}_2 + u - k_1(\widehat{x}_1 - y), \\ \dot{\widehat{x}}_2 &= f(y) + \widehat{x}_3 - u - k_2(\widehat{x}_1 - y), \\ \dot{\widehat{x}}_3 &= -f(y) - k_3(\widehat{x}_1 - y), \\ u &= \phi(y, \widehat{x}_2, \widehat{x}_3)\end{aligned} \tag{58}$$

is a globally asymptotically stabilizing output-feedback.

4. Certainty-equivalence design with the class of observers in [2]

We now study the observer design introduced in [2] and prove that its reduced-order variant preserves GAS in a certainty-equivalence implementation, under mild assumptions on the state-feedback controller. This reduced-order design is applicable to a class of systems of the form

$$y = x_1, \tag{59}$$

$$\dot{x}_1 = A_1 x_2 + G_1 \gamma(H_1 x_1 + H_2 x_2) + \beta_1(y, u), \tag{60}$$

$$\dot{x}_2 = A_2 x_2 + G_2 \gamma(H_1 x_1 + H_2 x_2) + \beta_2(y, u), \tag{61}$$

where $H = [H_1 \; H_2]$ is a row vector, $\gamma(\cdot)$ is a nonlinearity of Hx, and $\beta_1(y, u)$ and $\beta_2(y, u)$ include all terms that do not depend on the unmeasured x_2. An estimate of x_2 is obtained with the help of the variable $\xi = x_2 + N x_1$, where $N \in \mathbb{R}^{(n-p)\times p}$ is to be designed. From (60)–(61), the derivative of ξ is

$$\begin{aligned}\dot{\xi} &= (A_2 + N A_1) x_2 + (G_2 + N G_1)\gamma(H_1 x_1 + H_2 x_2) \\ &\quad + [\beta_2(y, u) + N \beta_1(y, u)].\end{aligned} \tag{62}$$

To obtain the estimate

$$\hat{x}_2 = z - N x_1 \tag{63}$$

we employ the observer

$$\dot{z} = (A_2 + NA_1)\hat{x}_2 + (G_2 + NG_1)\,\gamma(H_1x_1 + H_2\hat{x}_2) + [\beta_2(y,u) + N\beta_1(y,u)], \tag{64}$$

which leads to the observer error system

$$\dot{e}_2 = (A_2 + NA_1)e_2 + (G_2 + NG_1) \times [\gamma(Hx + H_2e_2) - \gamma(Hx)], \tag{65}$$

where $e_2 := \hat{x}_2 - x_2 = z - \xi$. In [2] convergence of e_2 is established under two restrictions which allow (65) to satisfy the *circle criterion*: First, the nonlinearity $\gamma(\cdot)$ must be nondecreasing, which implies that $[\gamma(Hx + H_2e_2) - \gamma(Hx)]$ is a *sector* nonlinearity of H_2e_2, i.e.,

$$(H_2e_2)[\gamma(Hx + H_2e_2) - \gamma(Hx)] \geqslant 0. \tag{66}$$

The second restriction is that there exists a matrix $P = P^{\mathrm{T}} > 0$ such that

$$(A_2 + NA_1)^{\mathrm{T}}P + P(A_2 + NA_1) < 0, \tag{67}$$

$$P(G_2 + NG_1) + H_2^{\mathrm{T}} = 0 \tag{68}$$

which means that the system (65) is *strictly positive real* (SPR) from the input $-[\gamma(Hx + H_2e_2) - \gamma(Hx)]$ to the output H_2e_2. Under these two restrictions e_2 converges to zero exponentially, because (65) is the feedback interconnection of a sector nonlinearity and an SPR block as in the circle criterion.

To show that this observer satisfies Assumption 2, we let $\mathscr{X} = [x_1^{\mathrm{T}}\hat{x}_2^{\mathrm{T}}]^{\mathrm{T}}$ and rewrite its dynamics as in (12) with

$$\ell(x,z,u) = \begin{bmatrix} -A_1e_2 - G_1[\gamma(Hx + H_2e_2) - \gamma(Hx)] \\ NA_1e_2 + NG_1[\gamma(Hx + H_2e_2) - \gamma(Hx)] \end{bmatrix}. \tag{69}$$

As we prove in Proposition 2 below, this observer guarantees that $\ell(x,z,u)$ is integrable, i.e., in the factorization (13), we can take $\varepsilon = \ell(x,z,u)$. Global asymptotic stability of a certainty-equivalence controller is then established under the assumption that a function $\mathscr{L}$ exists as in Theorem 1 with $k = 1$.

Proposition 2. *Consider the system* (59)–(61) *where* $\gamma(\cdot)$ *is a nondecreasing function, and suppose the observer* (63) and (64) *has been designed such that* (67) and (68) *hold for some matrix* $P = P^{\mathrm{T}} > 0$. *If a state-feedback controller* $u = \phi(x_1, x_2)$ *satisfying Assumption* 1 *is available, and if there exists a* C^1, *class-*$\mathscr{K}_\infty$, *function* $\mathscr{L}$ *such that*

$$|\mathscr{L}'(V(\chi))| \left|\frac{\partial V}{\partial x}(\chi)\right| \tag{70}$$

is bounded, then the certainty-equivalence implementation $u = \phi(x_1, \hat{x}_2)$ *achieves GAS of the origin* $(x,z) = 0$.

Proof. The main task is to show that $\ell(x,z,u)$ defined in (69) is an integrable function of time. To prove this, we note that the derivative of the Lyapunov function $U = \frac{1}{2}e_2^{\mathrm{T}}Pe_2$ along the trajectories of (65) satisfies

$$\dot{U} = \tfrac{1}{2}\,e_2^{\mathrm{T}}[(A_2 + NA_1)^{\mathrm{T}}P + P(A_2 + NA_1)]e_2 + e_2^{\mathrm{T}}P(G_2 + NG_1)[\gamma(Hx + H_2e_2) - \gamma(Hx)]. \tag{71}$$

Using (66)–(68), we can find a constant $\varepsilon > 0$ such that

$$\dot{U} \leqslant -\varepsilon|e_2|^2 - (H_2e_2)[\gamma(Hx + H_2e_2) - \gamma(Hx)] \leqslant -\varepsilon|e_2|^2 \tag{72}$$

which proves that, for all $t \in [0, t_f)$,

$$|e_2(t)| \leqslant \kappa|e_2(0)|\exp(-\beta t) \tag{73}$$

for some positive constants κ and β. It follows from (73) that, for any $T \in [0, t_f)$

$$\int_0^T |e_2|\,\mathrm{d}t \leqslant \frac{\kappa}{\beta}\,|e_2(0)| \tag{74}$$

and, if $t_f = \infty$, then $e_2(t) \to 0$ as $t \to \infty$. To obtain a similar integral bound for $\gamma(Hx + H_2e_2) - \gamma(Hx)$, we note from (65) that

$$\frac{\mathrm{d}}{\mathrm{d}t}(H_2e_2) = H_2(A_2 + NA_1)e_2 - h[\gamma(Hx + H_2e_2) - \gamma(Hx)], \tag{75}$$

where the constant $h := -H_2(G_2 + NG_1)$ is positive because, from (68), $-H_2(G_2 + NG_1) = H_2P^{-1}H_2^{\mathrm{T}} > 0$. Using (75) it is not difficult to show that, for almost all $t \in [0, t_f)$

$$\frac{\mathrm{d}}{\mathrm{d}t}|H_2 e_2| \leqslant \|H_2(A_2 + NA_1)\|\,|e_2| - h\,\mathrm{sgn}(H_2 e_2)[\gamma(Hx + H_2 e_2) - \gamma(Hx)]. \tag{76}$$

Next, using the nondecreasing property of $\gamma(\cdot)$, we substitute in (76)

$$\mathrm{sgn}(H_2 e_2) = \mathrm{sgn}\{\gamma(Hx + H_2 e_2) - \gamma(Hx)\}, \tag{77}$$

and obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\,|H_2 e_2| \leqslant \|H_2(A_2 + NA_1)\|\,|e_2| - h|\gamma(Hx + H_2 e_2) - \gamma(Hx)|. \tag{78}$$

Finally, integrating both sides from 0 to $T \in [0, t_f)$ and substituting (74), we get

$$\int_0^{\mathrm{T}} |\gamma(Hx + H_2 e_2) - \gamma(Hx)|\,\mathrm{d}t \leqslant \frac{1}{h}\,|H_2 e_2(0)| + \frac{1}{h}\,\|H_2(A_2 + NA_1)\| \int_0^{\mathrm{T}} |e_2|\,\mathrm{d}t \leqslant \frac{1}{h}\,\|H_2\|\,|e_2(0)| + \frac{\kappa}{h\beta}\,|e_2(0)|. \tag{79}$$

Combining (73), (74) and (79), we conclude that the observer (63) and (64) satisfies Assumption 2 with $\varepsilon = \ell(x, z, u)$. Because the controller $u = \phi(x_1, x_2)$ satisfies Assumption 1, and because the boundedness of (70), together with $k = 1$, implies (21), GAS of the certainty-equivalence controller follows from Theorem 1. □

Example 3. Consider the system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_2 - x_2^5 + u, \quad y = x_1 \tag{80}$$

which admits the state-feedback controller

$$u = \phi(x_1, x_2) = -x_1 - 2x_2 + x_2^5 \tag{81}$$

and the quadratic Lyapunov function

$$V(x) = x_1^2 + x_2^2. \tag{82}$$

This Lyapunov function satisfies the assumption of Proposition 2 because, with $\mathscr{L}(V) = \log(1 + V)$

$$|\mathscr{L}'(V(\chi))| \left|\frac{\partial V}{\partial x}(\chi)\right| \leqslant \frac{1}{1 + V}\, 2\sqrt{2V} \tag{83}$$

which is bounded. This means that the certainty-equivalence feedback $u = \phi(x_1, \hat{x}_2)$, implemented with the following observer designed as in [2], achieves GAS

$$\hat{x}_2 = z + 2y, \tag{84}$$

$$\dot{z} = -\hat{x}_2 - \hat{x}_2^5 + u. \tag{85}$$

This stability conclusion may be counter-intuitive because, due to the inexact cancellation of the nonlinearity x_2^5 with $\hat{x}_2^5$, the certainty-equivalence controller would be non-robust to the observer error $e_2 = x_2 - \hat{x}_2$. Indeed, in the closed-loop system

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = -x_1 - x_2 + 2e_2 + [(x_2 - e_2)^5 - x_2^5] \tag{86}$$

e_2 appears as a disturbance which is coupled with the state variable x_2 via $[(x_2 - e_2)^5 - x_2^5]$ and, as we prove in Appendix B, even an exponentially convergent observer results in finite escape time. However, the observer (85) preserves GAS because its error e_2 satisfies

$$\dot{e}_2 = -e_2 + [(x_2 - e_2)^5 - x_2^5], \tag{87}$$

where the term $[(x_2 - e_2)^5 - x_2^5]$ is opposite in sign to e_2 and, hence, plays a stabilizing role. This means that, as x_2 grows larger, the dynamics (87) are stiffer and the convergence of e_2 becomes faster than exponential, thus preventing finite escape time in (86).

5. Conclusions

We have derived a new stability condition for certainty-equivalence designs, which eliminates several conservative assumptions employed in earlier results. This is achieved by first taking into account the state-dependent convergence properties of the observer and next, by using the variable $\mathscr{X}$ which is different from the state estimate $\widehat{x}$ when the measured output y is used instead of its estimate $\hat{y}$ in the control. Among other examples, we have applied our certainty-equivalence design to a nonminimum phase system. This application is significant because the alternative approach of assigning an ISS gain from the observer error to the output relies on a minimum phaseness assumption, which is not required in our design. It would be of interest to further explore this research direction and to develop output-feedback designs for classes of nonminimum phase systems. Another promising direction is to pursue observer and controller redesign methods based on our new stability condition.

Appendix A. Proof of Theorem 1

The state of the closed-loop system is (x,z). By definition, $\mathscr{X}$ given by (9) satisfies (12). So, from the stabilizability assumption and (21), we get, for all (x,z) and with $\mathscr{X} = K(x,z)$

$$\overset{\cdot}{\overbrace{\mathscr{L}(\mathscr{V}(\mathscr{X}))}} \leqslant M|\varepsilon(x,z)|. \tag{A.1}$$

Let $(x(t),z(t))$ be any solution of the closed-loop system starting from (x_0,z_0). Let $[0,T(x_0,z_0))$ be its right maximal interval of definition. From (16) and (A.1), we get, for all $t \in [0,T(x_0,z_0))$

$$\mathscr{L}(V(\mathscr{X}(t))) \leqslant \mathscr{L}(V(K(x_0,z_0))) + M\alpha_1((x_0,z_0),\ |K(x_0,z_0)-x_0|). \tag{A.2}$$

With the properties of K, V and α_1, there exist a class $\mathscr{K}_\infty$ function α_3 such that we have, for all $t \in [0,T(x_0,z_0))$

$$\mathscr{X}(t) \leqslant \alpha_3(|x_0|+|z_0|). \tag{A.3}$$

With (16), this yields, for all $t \in [0,T(x_0,z_0))$

$$\begin{aligned} |\widehat{x}(t)| + |x(t)| &\leqslant 2\alpha_3(|x_0|+|z_0|) \\ &\quad + 3\alpha_1((x_0,z_0),|K(x_0,z_0)-x_0|) \\ &\leqslant \alpha_4(|x_0|+|z_0|), \end{aligned} \tag{A.4}$$

where α_4 is a class $\mathscr{K}_\infty$ function. This, with (19) and the properties of ϕ and h, implies the existence of a class K_∞ function α_5 satisfying, for all $t \in [0,T(x_0,z_0))$

$$z(t) \leqslant \alpha_4(|x_0|+|z_0|). \tag{A.5}$$

This implies that $T(x_0,z_0)$ is infinite for each pair (x_0,z_0) and that the origin is globally stable. So each solution has a non empty compact connected ω-limit set to which it converges. Also, with (18) and the continuity of ϕ, we have

$$\lim_{t\to+\infty} |u(t) - \phi(x(t))| = 0. \tag{A.6}$$

It follows that in the ω-limit set, the dynamics are

$$\begin{aligned} \dot{x} &= f(x,\phi(x)), \\ \dot{z} &= \mu(z,\phi(x),h(x)), \end{aligned} \tag{A.7}$$

where the origin is a globally asymptotically stable equilibrium of the x part. So, the ω-limit set being both invariant and compact, it is contained in $\{0\} \times \mathbb{R}^q$. This implies that $x(t)$ converges to the origin as time goes to infinity. With (18) and (19), the same property holds for $\widehat{x}(t)$ and $z(t)$. This proves that the origin is globally attractive. □

Appendix B. Proof of finite escape time in system (86)

Lemma 3. *Consider the system (86) and suppose e_2 satisfies*

$$e_2(t) = e_2(0)\exp(-\beta t) \tag{B.1}$$

for some $\beta > 0$. If $e_2(0) < 0$ and if $x_2(0)$ is sufficiently large, then the solutions exhibit finite escape time.

Proof. We use the following result from [8]:

Proposition 4. *Consider the system*

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= f_2(x_1,x_2,t), \end{aligned} \tag{B.2}$$

where $x_1,x_2 \in \mathbb{R}$, and $f_2 \in C^1$. If there exists a nonempty open set $U \subset \mathbb{R}$, and positive constants a, b, d, s and $n > 2$ such that, for all $t \in [0,s]$, $x_1 \in U$ and $x_2 > d$,

$$bx_2^n < f_2(x_1,x_2,t) < ax_2^n \tag{B.3}$$

then, for $x_1(0) \in U$ and sufficiently large $x_2(0) > d$, $x_2(t)$ escapes to infinity in finite time.

To continue with the proof of Lemma 3, we note that the system (86) is of the form (B.2) with

$$f_2(x_1,x_2,t) = -x_1 + x_2 + 2\exp(-\beta t)e_2(0) + [(x_2 - \exp(-\beta t)e_2(0))^5 - x_2^5]. \tag{B.4}$$

The leading term of $[(x_2 - \exp(-\beta t)e_2(0))^5 - x_2^5]$ is $-5\exp(-\beta t)e_2(0)x_2^4$. So the assumptions of Lemma 4 are satisfied. This means that, if $e_2(0) < 0$, then for sufficiently large x_2, and for x_1 belonging to a bounded set U, inequality (B.3) holds with $n = 4$. The assumptions of Proposition 4 being satisfied, we conclude that the system (86) exhibits finite escape time for large initial conditions $x_2(0)$. □

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