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Necessary conditions for stability and attractivity of continuous systems

R. ORSI†*, L. PRALY‡ and I. MAREELS§

Considered in this paper are control systems of the form $\dot{x} = f(x, u)$. For such systems a number of related necessary conditions for various forms of stability and attractivity are presented. The paper starts by showing that Brockett's necessary condition for stabilizability via smooth feedback still persists if f is continuous and the class of allowable u increased to include continuous feedbacks. Using similar ideas to those used to prove the continuous Brockett result, again only assuming continuity of f and u, necessary conditions are then derived for global attractivity and for ultimate boundedness.

1. Introduction

Let $\mathbb{R}$ denote the real numbers. Let f: $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be a continuous function with $f(0,0) = 0$ and consider the system

$$\dot{x} = f(x, u(x)) \tag{1}$$

where u: $\mathbb{R}^n \to \mathbb{R}^m$ is an as yet unspecified function. A common problem in control theory is to find a function u with $u(0) = 0$ that makes the zero solution of (1) locally asymptotically stable. This problem is often referred to as the *local stabilizability problem*. From the start it is unclear whether such a control u exists and hence necessary conditions for the existence of such a control are of interest. One now famous necessary condition was given in Brockett (1983). Brockett's result states that if f is continuously differentiable a necessary condition for the existence of a continuously differentiable feedback control u that renders 0 locally asymptotically stable is that the image of f contain an open neighbourhood of 0.

An example of an f not satisfying Brockett's condition is

$$f: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2, \qquad (x, u) = (x_1, x_2, u_1, u_2) \mapsto (u_1, u_1 u_2)$$

The condition fails to be satisfied as no point of the form $(0, \epsilon)$, $\epsilon \neq 0$, is in the image of f.

Since the appearance of Brockett's result, a number of related papers have been published, see for example Zabczyk (1989), Coron (1990), Ryan (1994), Kappos (1995) and Clarke *et al.* (1998). In one of these, Zabczyk (1989), it was shown that if f and u are continuous and the solutions of the closed loop system unique then Brockett's condition that the image of f contain an open neighbourhood of 0 is still necessary for the existence of a zero solution which is locally asymptotically stable.

In this paper a number of related necessary conditions for various forms of stability and attractivity are presented and we start by giving a relatively elementary proof that Brockett's necessary condition still holds if f and u are continuous. Importantly the proof of this result does not assume that solutions of the closed loop system (1) are unique nor that solutions depend continuously on initial conditions.

While the ingredients for the proof of this result exist in various guises in the literature, to the best of our knowledge, no simple complete proof of the result exists. Note however that it was shown by Ryan that if f is continuous and has the property that

$$K \subset \mathbb{R}^m \text{ convex} \quad \Rightarrow \quad f(x, K) \subset \mathbb{R}^n \text{ convex} \tag{2}$$

then the Brockett result still holds if u is a member of a certain class of discontinuous functions (Ryan 1994). Ryan remarks that if f and u are continuously differentiable then condition (2) can be removed and hence that his result is a generalization of Brockett's. Though it is not stated in the paper, if f and u are continuous the same remark holds.

We believe the continuous case is especially important and hope that the relatively simple proof of the continuous Brockett result given here will bring the result to the attention of a much larger collection of researchers and practitioners.

Using similar ideas to those used in the proof of the continuous Brockett result we next derive two necessary conditions for the existence of a zero solution which is global attractive. The first of these is an extension of a result from Zabczyk (1989) which states that Brockett's condition for local stabilizability is also necessary for global attractivity. This result was shown to hold for

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* Author for correspondence. e-mail: robert.orsi@nicta.com.au

† National ICT Australia, C/- Research School of Information Sciences and Engineering, The Australian National University, Canberra ACT 0200, Australia.

‡ Centre Automatique et Systèmes, École des Mines de Paris, 77305 Fontainebleau Cédex, France. e-mail: Laurent.Praly@ensmp.fr

§ Department of Electrical and Electronic Engineering, University of Melbourne, Parkville VIC 3052, Australia. e-mail: i.mareels@unimelb.edu.au

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continuous f and u under the assumption that closed loop solutions were unique. We show that the same result is true without requiring uniqueness of closed loop solutions.

It is also shown that global attractivity implies that the closed loop vector field must take on all non-zero directions on all spheres centred about the origin. The second condition shown to be necessary for global attractivity is based on this fact. Unlike the Brockett condition, it is non-local in nature.

As we show, a modified version of this latter condition is also necessary for ultimate boundedness. We show that the closed loop vector field must take on all non-zero directions on all origin centred spheres of sufficiently large radius. The modified necessary condition is based on this fact.

As in Zabczyk (1989) and Ryan (1994), proofs of results in this paper are degree-theoretic in nature and do not rely on converse Lyapunov results.

All results in this paper are first derived for systems of the form

$$\dot{x} = f(x) \tag{3}$$

where $f: \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function and then extended to systems of the form (1).

It should be noted that a result quite close to Brockett's for systems of the form (3) was provided independently in Krasnosel'skiĭ and Zabreĭko (1984, p. 340). The Russian original edition of this work was published in 1975, a number of years prior to Brockett's paper.

In the remainder of this section we outline some of the main ideas used in the paper and give an overview of what follows.

Let S_a be a sphere of radius a centred at the origin and let $\bar{B}(0;b)$ be a closed ball of radius $b < a$, also centered at the origin. A fundamental result used throughout the paper is the following. If f is a continuous vector field and there exists a constant T such that all solutions of the system $\dot{x} = f(x)$ starting on S_a are confined to the set $\bar{B}(0;b)$ for all time greater than or equal to T, then the vector field, amongst other properties, must take on all non-zero directions on S_a. Indeed the conclusion of this result is true under weaker conditions and the paper starts with a proper statement and proof of this result in § 2. The result is first proved for the class of continuous vector fields with unique solutions. This is done using a homotopy argument. The general case is then proved by appropriately approximating an arbitrary continuous vector field with one that has unique solutions.

The next section of the paper, § 3, contains a proof of Brockett's result for continuous f and u and is a direct consequence of the results of § 2. Section 4 contains proofs of the global attractivity conditions. The ultimate boundedness result is given in § 5. This essentially completes the paper and it ends with some concluding remarks.

Stability plays a major role in control theory and is an area of much current research. For further background and motivation the reader is referred to Bacciotti (1992).

2. Main result

The main aim of this section is to prove Theorem 2. This result is used repeatedly in the rest of the paper including § 3 where it is used to extend Brockett's necessary condition for local stabilizability.

Let S_r denote the set $\{y \in \mathbb{R}^n \mid |y| = r\}$, $B(x;r)$ the set $\{y \in \mathbb{R}^n \mid |y - x| < r\}$ and $\bar{B}(x;r)$ the set $\{y \in \mathbb{R}^n \mid |y - x| \le r\}$.

Before proceeding any further, let us introduce some facts from topology that will be needed throughout the rest of the paper. Suppose f_1 and f_2 are two continuous functions from S_a into S_1. Then f_1 and f_2 are *homotopic* if there exists a continuous function $\Phi: [0,c] \times S_a \to S_1$ such that

$$\Phi(0,x) = f_1(x) \quad \text{and} \quad \Phi(c,x) = f_2(x) \quad \text{for all } x \in S_a$$

(That is, f_1 and f_2 are homotopic if it is possible to continuously interpolate in S_1 between f_1 and f_2.) In addition, recall that for each continuous function $f: S_a \to S_1$ there is an associated integer called the *degree* of f. Loosely speaking, the degree of f equals the number of times f winds S_a around S_1. Degree is rather complicated to define and a precise definition will not be given here.† As well as some other properties of degree which we will recall as required, we will need the following fact. If two continuous functions $f_1: S_a \to S_1$ and $f_2: S_a \to S_1$ are homotopic then they have the same degree. For more details about degree theory and a proof of the result just mentioned see, e.g. Dugundji (1966). Readers interested in learning more about degree theory are also referred to Lloyd (1978).

† Readers not familiar with degree theory should keep in mind the $n = 2$ dimensional case. In this case, S_a and S_1 are circles in $\mathbb{R}^2$ and the degree of f equals the number times the image point $f(x)$ rotates around S_1 as x makes one rotation of S_a. In greater detail, one method of calculating the degree of f is to define $\gamma: [0, 2\pi] \to S_a$, $\theta \mapsto a(\cos\theta, \sin\theta)$, and to consider the function $f \circ \gamma: [0, 2\pi] \to S_1$. The function $f \circ \gamma$ is a closed curved in S_1. (By closed we mean that $f \circ \gamma(0) = f \circ \gamma(2\pi)$.) As a result it will encircle the origin a whole number of times as it argument goes from 0 to 2π. The degree of f equals the total number of times $f \circ \gamma$ encircles the origin in a counter-clockwise direction minus the total number of times it encircles the origin in a clockwise direction.

Lemma 1: *Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be continuous and suppose that for each $x_0 \in \mathbb{R}^n$ there is an unique solution $x(\cdot, x_0)$ to the initial value problem*

$$\dot{\xi} = f(\xi), \qquad \xi(0) = x_0$$

If there exist constants a and b, $a > b > 0$, and $T > 0$ such that for each $x_0 \in S_a$, $x(\cdot, x_0)$ exists over the interval $0 \le t \le 2T$ and satisfies

$$|x(t, x_0)| \le b < a \qquad \text{for all } t \in [T, 2T] \tag{4}$$

then the function

$$g\colon S_a \to S_1, \qquad x \mapsto \frac{f(x)}{|f(x)|}$$

has non-zero degree.

One consequence of g having non-zero degree is that g must be a surjective function. (This will be shown later in Theorem 2.) In the more intuitive terminology used in the introduction of this paper, f must take on all non-zero directions on S_a.

Proof: As the antipodal map $h\colon S_a \to S_1$, $x \mapsto -x/|x|$, has degree $(-1)^n$ (Dugundji 1966, p. 339), it is sufficient to show that g and h are homotopic.

Before proceeding recall that uniqueness of solutions implies $x(t, x_0)$ depends continuously on (t, x_0) (Hartman 1964, p. 94). Also note that (4) implies that $f(x) \neq 0$ for any $x \in S_a$ and hence that g is a well-defined function.

Define $\Phi\colon [0, 2T] \times S_a \to S_1$ as

$$\Phi(t, x_0) = \begin{cases} g(x_0), & t = 0 \\ \dfrac{x(t, x_0) - x_0}{|x(t, x_0) - x_0|}, & t \in (0, T] \\ \dfrac{\frac{2T - t}{T} x(T, x_0) - x_0}{\left|\frac{2T - t}{T} x(T, x_0) - x_0\right|}, & t \in (T, 2T] \end{cases}$$

Let us verify that Φ is well defined by first checking that $|x(t, x_0) - x_0| \neq 0$ on $(0, T] \times S_a$. Suppose that there exists $x_0 \in S_a$ and $\tau > 0$ such that $x(\tau, x_0) = x_0$. Uniqueness of solutions implies that the orbit passing through x_0 must be periodic. As $|x_0| = a$ and $|x(t, x_0)| \le b < a$ for all $t \in [T, 2T]$, τ must be greater than T and hence we can conclude that $|x(t, x_0) - x_0| \neq 0$ on $(0, T] \times S_a$. That $|x_0| = a$ and $|x(t, x_0)| \le b < a$ for all $t \in [T, 2T]$ also implies that

$$\left|\frac{2T - t}{T} x(T, x_0) - x_0\right| \neq 0$$

on $(T, 2T] \times S_a$ and hence Φ is indeed a well-defined function.

We now show that Φ is continuous. Note that the fact that

$$\lim_{(t, x_0') \to (0^+, x_0)} \frac{x(t, x_0') - x_0'}{|x(t, x_0') - x_0'|} = g(x_0) \quad \text{for all } x_0 \in S_a$$

follows from the fact that

$$\lim_{(t, x_0') \to (0^+, x_0)} \frac{x(t, x_0') - x_0'}{t} = f(x_0) \quad \text{for all } x_0 \in S_a \tag{5}$$

and the fact that $f(x) \neq 0$ for any $x \in S_a$. That (5) is true follows by applying the triangle inequality to

$$\left|\frac{x(t, x_0') - x_0'}{t} - \frac{x(t, x_0) - x_0}{t} + \frac{x(t, x_0) - x_0}{t} - f(x_0)\right|,$$

noting that

$$\lim_{t \to 0^+} \frac{x(t, x_0) - x_0}{t} = f(x_0),$$

that

$$\begin{aligned} &\left|\frac{x(t, x_0') - x_0'}{t} - \frac{x(t, x_0) - x_0}{t}\right| \\ &\quad = \left|\frac{\int_0^t [f(x(s, x_0')) - f(x(s, x_0))]\,\mathrm{d}s}{t}\right| \\ &\quad \le \max_{0 \le s \le t} |f(x(s, x_0')) - f(x(s, x_0))| \end{aligned}$$

and finally that $f(x(t, x_0))$ is continuous in (t, x_0). From these facts it follows that Φ is a well-defined continuous deformation of g to h and hence that g and h are homotopic. □

In order to prove our main result, we will need the following theorem taken in modified form from Filippov (1988, p. 90). It says that even when solutions are not unique, they still in a certain sense depend continuously on the vector field and on the initial condition.

Theorem 1: *Let $G \subset \mathbb{R}^n$ be open, $f\colon G \to \mathbb{R}^n$ continuous and x_0 be a given point in G. Suppose there exists $T > 0$ such that all solutions of the problem*

$$\dot{\xi} = f(\xi), \qquad \xi(0) = x_0 \tag{6}$$

exist for $0 \le t \le 2T$ and remain in G during this time.

Then given $\epsilon > 0$ there exists $\delta > 0$ such that for any $x_0^* \in G$ and any continuous function $f^*\colon G \to \mathbb{R}^n$ satisfying the conditions

$$|x_0 - x_0^*| \le \delta, \qquad \sup_{x \in G} |f(x) - f^*(x)| \le \delta$$

each solution of the problem

$$\dot{\xi} = f^*(\xi), \qquad \xi(0) = x_0^*$$

exists for $0 \le t \le 2T$ and differs for these t from a certain solution of problem (6) by not more than ϵ.

Lemma 2: *Let $f:\mathbb{R}^n \to \mathbb{R}^n$ be continuous and $K \subset \mathbb{R}^n$ compact. Suppose that for each $x_0 \in K$ all solutions of the equation*

$$\dot{\xi} = f(\xi), \qquad \xi(0) = x_0 \tag{7}$$

exist for $0 \le t \le 2T$. Then there exists a compact set K' such that for any $x_0 \in K$ and any solution $x(\cdot, x_0)$ of (7), $x(t, x_0) \in K'$ for all $0 \le t \le 2T$.

Proof: See Filippov (1988, p. 9). □

We will also need the following approximation result. It will allow us to approximate an arbitrary continuous vector field, whose solutions may not be unique, by a continuously differentiable vector field, whose solutions will necessarily be unique.

Lemma 3: *Given a compact set $K \subset \mathbb{R}^n$, a continuous function $f: K \to \mathbb{R}^n$, and $\delta > 0$, there exists a continuously differentiable function $f^*: \mathbb{R}^n \to \mathbb{R}^n$ satisfying*

$$\max_{x \in K} |f(x) - f^*(x)| \le \delta$$

Proof: By the Weierstrass approximation theorem (Dieudonné 1970, p. 139), for each coordinate function f_i of f there exists a polynomial function $f_i^*: \mathbb{R}^n \to \mathbb{R}$ such that $\max_{x \in K} |f_i(x) - f_i^*(x)| \le \delta/n$. The desired function is $f^* = (f_1^*, \dots, f_n^*)$. □

The following theorem is the main result of this section. Its proof starts by showing that if f satisfies the requirements of the theorem, then it can be approximated by a vector field f^* which (i) still satisfies the requirements of the theorem, and (ii) has unique solutions. Lemma 1 then implies that g^* in (10), has non-zero degree. Using a homotopy argument, it is then shown that g and g^* have the same degree. Finally, we show that non-zero degree implies surjectivity.

Theorem 2: *Let $f: \mathbb{R}^n \to \mathbb{R}^n$ be continuous. If there exist constants a and b, $a > b > 0$, and $T > 0$ such that for each $x_0 \in S_a$ all solutions $x(\cdot, x_0)$ of the differential equation $\dot{\xi} = f(\xi)$, $\xi(0) = x_0$, exist over the interval $0 \le t \le 2T$ and satisfy*

$$|x(t, x_0)| \le b < a \qquad \text{for all } t \in [T, 2T]$$

then the function

$$g: S_a \to S_1, \qquad x \mapsto \frac{f(x)}{|f(x)|} \tag{8}$$

has non-zero degree and is surjective.

Proof: Taking $K = S_a$ in Lemma 2 implies there exists a compact set K' that contains all solutions of the initial value problems $\dot{\xi} = f(\xi)$, $\xi(0) = x_0 \in S_a$ for $0 \le t \le 2T$. Let G be an open bounded set containing K' and let ϵ be a constant that satisfies $0 < \epsilon < a - b$. Using the f and T of the theorem statement and the G and ϵ described above, for each $x_0 \in S_a$ let δ_{x_0} denote the value of δ in Theorem 1.

The sets $B(x_0, \delta_{x_0})$, $x_0 \in S_a$, form an open cover of S_a. As S_a is compact this cover has a finite subcover consisting say of the sets $B(x_0^1, \delta_{x_0^1}), \dots, B(x_0^m, \delta_{x_0^m})$. As f is continuous and non-zero on S_a there exists $B > 0$ such that $|f(x)| \ge B$ for all $x \in S_a$. Let δ satisfy $0 < \delta < \min\{\delta_{x_0^1}, \dots, \delta_{x_0^m}, B\}$ and let $\bar{G}$ denote the closure of G. Taking $K = \bar{G}$ in Lemma 3, it follows that there exists a continuously differentiable function $f^*: \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\max_{x \in \bar{G}} |f(x) - f^*(x)| \le \delta$$

Let x_0^* be an arbitrary but fixed point in S_a. As the sets $B(x_0^1, \delta_{x_0^1}), \dots, B(x_0^m, \delta_{x_0^m})$ cover S_a, $x_0^* \in B(x_0^i, \delta_{x_0^i})$ for some $i \in \{1, \dots, m\}$. From Theorem 1 it now follows that $x^*(\cdot, x_0^*)$, the unique solution of the differential equation

$$\dot{\xi} = f^*(\xi), \qquad \xi(0) = x_0^*$$

exists for $0 \le t \le 2T$ and differs from a certain solution of

$$\dot{\xi} = f(\xi), \qquad \xi(0) = x_0^i$$

by no more than ϵ over the interval $[0, 2T]$. Hence it follows that

$$|x^*(t, x_0^*)| \le b + \epsilon < a \qquad \text{for all } t \in [T, 2T] \tag{9}$$

As $x_0^* \in S_a$ was arbitrary, equation (9) holds for all $x_0^* \in S_a$ and Lemma 1 now implies that the function

$$g^*: S_a \to S_1, \qquad x \mapsto \frac{f^*(x)}{|f^*(x)|} \tag{10}$$

has non-zero degree.

We now show that g and g^* are homotopic. Consider the function

$$\Phi: [0,1] \times S_a \to S_1, \qquad (\lambda, x) \mapsto \frac{\lambda f^*(x) + (1-\lambda) f(x)}{|\lambda f^*(x) + (1-\lambda) f(x)|}$$

That Φ is indeed a well-defined function can be seen by noting that $|f^*(x) - f(x)| \le \delta < B \le |f(x)|$ for all $x \in S_a$ and hence that

$$\begin{aligned}|\lambda f^*(x) + (1-\lambda) f(x)| &= |f(x) + \lambda(f^*(x) - f(x))| \\ &\ge |f(x)| - |f^*(x) - f(x)| \\ &> 0\end{aligned}$$

for all $(\lambda, x) \in [0,1] \times S_a$. Φ is also continuous. It follows that g and g^* are homotopic and hence that g has the same degree as g^*.

Finally, if a function $h: S_a \to S_1$ is not surjective, it will be homotopic to a constant function (Dugundji 1966, p. 316). As constant functions have zero degree (Dugundji 1966, p. 339), g must be surjective. □

3. An extension of Brockett's necessary condition for local stabilizability

In this section we show that Brockett's necessary condition for local stabilizability still persists if f and u are only continuous.

Before proceeding let us introduce some further terminology. The following terms are taken from Rouche *et al.* (1977) and while they are fairly standard they have been altered to cover non-uniqueness of solutions.

Let $f: \mathbb{R}^n \to \mathbb{R}^n$ be a given continuous function and suppose $f(0) = 0$. Consider the system

$$\dot{\xi} = f(\xi)$$

The trivial solution $x \equiv 0$ will be called:

- *stable* if for each $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that if $|x_0| \le \delta$ and $x(\cdot, x_0)$ is a solution of $\dot{\xi} = f(\xi)$, $\xi(0) = x_0$, then
$$|x(t, x_0)| \le \epsilon \qquad \text{for all } t \ge 0$$
- *attractive* if there exists $\eta > 0$ such that if $|x_0| \le \eta$ and $x(\cdot, x_0)$ is a solution of $\dot{\xi} = f(\xi)$, $\xi(0) = x_0$, then $x(t, x_0)$ is defined for all $t \in [0, \infty)$ and
$$\lim_{t\to\infty} x(t, x_0) = 0$$
- *equi-attractive* if there exists $\eta > 0$ such that for each $\epsilon > 0$ there exists $\sigma = \sigma(\epsilon) > 0$ such that if $|x_0| \le \eta$ and $x(\cdot, x_0)$ is a solution of $\dot{\xi} = f(\xi)$, $\xi(0) = x_0$, then $x(t, x_0)$ is defined for all $t \in [0, \infty)$ and
$$|x(t, x_0)| \le \epsilon \qquad \text{for all } t \ge \sigma$$
- *asymptotically stable* if it is stable and attractive.
- *equi-asymptotically stable* if it is stable and equi-attractive.

Lemma 4: *Suppose $f: \mathbb{R}^n \to \mathbb{R}^n$ is continuous and that $x \equiv 0$ is an asymptotically stable solution of the system $\dot{\xi} = f(\xi)$. Then $x \equiv 0$ is equi-asymptotically stable.*

Proof: See Filippov (1988, p. 160). □

In order to extend Brockett's result we will need the follow lemma taken from Zabczyk (1989).

Lemma 5: *Suppose $f: \mathbb{R}^n \to \mathbb{R}^n$ is continuous, that $f(0) = 0$, and that there exists $a > 0$ such that $f(x) \ne 0$ for all $x \in \bar{B}(0; a) - \{0\}$. Furthermore suppose*

$$g: S_a \to S_1, \qquad x \mapsto \frac{f(x)}{|f(x)|}$$

has non-zero degree. Then the image under f of any open neighbourhood of 0 *contains an open neighbourhood of* 0.

Theorem 3: *If $f: \mathbb{R}^n \to \mathbb{R}^n$ is continuous and $x \equiv 0$ is an asymptotically stable solution of the system $\dot{\xi} = f(\xi)$, then the image under f of any open neighbourhood of* 0 *contains an open neighbourhood of* 0.

Proof: Lemma 4 implies that the solution $x \equiv 0$ is equi-asymptotically stable and hence that there exists $\eta > 0$ such that for arbitrary a and b satisfying $0 < b < a \le \eta$, there exists $T > 0$ such that if $|x_0| = a$ and $x(\cdot, x_0)$ is solution of $\dot{\xi} = f(\xi)$, $\xi(0) = x_0$, then

$$|x(t, x_0)| \le b \qquad \text{for all } t \ge T$$

Theorem 2 now implies that the function g, see (8), has non-zero degree. The result now follows from Lemma 5. □

As an immediate corollary we have the following extension of Brockett's result.

Corollary 1: *Suppose $f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a given continuous function satisfying $f(0, 0) = 0$. Furthermore suppose $u: \mathbb{R}^n \to \mathbb{R}^m$ is a continuous function satisfying $u(0) = 0$ and that $x \equiv 0$ is an asymptotically stable solution of the system $\dot{\xi} = f(\xi, u(\xi))$. Then the image under f of any open neighbourhood of $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^m$ contains an open neighbourhood of $0 \in \mathbb{R}^n$.*

Proof: Let $P \subset \mathbb{R}^n \times \mathbb{R}^m$ be an open neighbourhood of $(0, 0)$. Then necessarily there exist open sets $N \subset \mathbb{R}^n$ and $M \subset \mathbb{R}^m$ such that $N \times M \subset P$ and $(0, 0) \in N \times M$. Let $N_1 = N \cap u^{-1}(M)$. As u is continuous and $u(0) = 0$, it follows that N_1 is an open neighbourhood of 0. Applying Theorem 3 to the continuous function $f(\cdot, u(\cdot))$ implies $f(N_1, u(N_1))$ contains an open neighbourhood of 0. Noting that $f(N_1, u(N_1)) \subset f(N, M)$ completes the proof. □

4. Global attractivity

In this section we show that a system that is globally attractive must also necessarily satisfy the conclusion of Brockett's result (Corollary 2(i)). In addition, it will be shown that the global attractivity of a system also implies that the closed loop vector field of the system must take on all non-zero directions on all spheres about the origin. This latter condition will be used to derive an additional necessary condition for global attractivity (Corollary 2(ii)).

Let us first start with a precise definition of global attractivity. Let $f: \mathbb{R}^n \to \mathbb{R}^n$ be a given continuous function and suppose $f(0) = 0$. Consider again the system $\dot{\xi} = f(\xi)$. The trivial solution $x \equiv 0$ will be called *globally attractive* if for each $x_0 \in \mathbb{R}^n$, each solution $x(\cdot, x_0)$ of $\dot{\xi} = f(\xi)$, $\xi(0) = x_0$, is defined for all $t \in [0, \infty)$ and satisfies

$$\lim_{t\to\infty} x(t, x_0) = 0$$

(Rouche *et al.* 1977).

Lemma 6: *Suppose $f:\mathbb{R}^n \to \mathbb{R}^n$ is continuous and that $x \equiv 0$ is a globally attractive solution of the system $\dot{\xi} = f(\xi)$. Let a and b be constants satisfying $0 < b < a$. Then there exists $\tau > 0$ such that if $x_0 \in S_a$ and $x(\cdot, x_0)$ is a solution of $\dot{\xi} = f(\xi)$, $\xi(0) = x_0$, then*

$$x([0,\tau], x_0) \cap \bar{B}(0; b) \neq \varnothing$$

Proof: The proof will be by contradiction. Suppose the conclusion of the lemma does not hold. This implies that for each integer $n \geq 1$ there exists $x_0^n \in S_a$ and a solution $x^n(\cdot, x_0^n)$ of $\dot{\xi} = f(\xi)$, $\xi(0) = x_0^n$, such that $|x^n(t, x_0^n)| > b$ for all $0 \leq t \leq n$. We will now show that this implies there exists a solution $x(\cdot, x_0)$, $x_0 \in S_a$, such that $|x(t, x_0)| \geq b$ for all $t \geq 0$. As $x \equiv 0$ is globally attractive, this will provide the desired contradiction.

Taking $K = S_a$ in Lemma 2 implies that the functions $x^n(\cdot, x_0^n)$, $n = 1, 2, \ldots$, are uniformly bounded on $[0,1]$. By standard properties of ordinary differential equations, these functions are also equicontinuous on $[0,1]$ (Filippov 1988, p. 7). Arzela's theorem now implies that the sequence of functions $x^n(\cdot, x_0^n)$, $n = 1, 2, \ldots$, has a subsequence, $x^{n_i}(\cdot, x_0^{n_i})$, $i = 1, 2, \ldots$, which is uniformly convergent on $[0,1]$ to some function $y_1: [0,1] \to \mathbb{R}^n$. By properties of ordinary differential equations, $y_1(\cdot)$ will satisfy $\dot{\xi} = f(\xi)$, $\xi(0) = x_0$, for some $x_0 \in S_a$ (Filippov 1988, p. 8). In addition, as $|x^n(t, x_0^n)| > b$ for all $t \in [0,1]$ and all $n \geq 1$, $|y_1(t)| \geq b$ for all $t \in [0,1]$. By a similar argument, there exists a subsequence of the $x^{n_i}(\cdot, x_0^{n_i})$'s which is uniformly convergent on $[0,2]$ to some function $y_2: [0,2] \to \mathbb{R}^n$. Necessarily, $y_2(t)$ will equal $y_1(t)$ for all $t \in [0,1]$ and $y_2(\cdot)$ will satisfy $\dot{\xi} = f(\xi)$, $\xi(0) = x_0$ (over the interval $[0,2]$). As $|x^n(t, x_0^n)| > b$ for all $t \in [0,2]$ and all $n \geq 2$, $|y_2(t)| \geq b$ for all $t \in [0,2]$. The solution y_2 can be thought of as an extension of y_1 and it is clear that this process can be continued and implies the existence of a solution $x(\cdot, x_0)$, $x_0 \in S_a$, which satisfies $|x(t, x_0)| \geq b$ for all $t \geq 0$. □

Theorem 4: *If $f:\mathbb{R}^n \to \mathbb{R}^n$ is continuous and $x \equiv 0$ is a globally attractive solution of the system $\dot{\xi} = f(\xi)$, then*

(i) *the image under f of any open neighbourhood of 0 contains an open neighbourhood of 0,*

(ii) *for each $a > 0$ the function*

$$g: S_a \to S_1, \; x \mapsto \frac{f(x)}{|f(x)|}$$

is surjective.

Proof: Let a_1 and b_1 be arbitrary numbers satisfying $0 < b_1 < a_1$ and let τ_1 be the τ in Lemma 6 for $a = a_1$ and $b = b_1$. Let y be an element of the set C if and only if for some $x_0 \in S_{a_1}$, there exists a solution $x(\cdot, x_0)$ of $\dot{\xi} = f(\xi)$, $\xi(0) = x_0$, such that $y = x(t, x_0)$ for some $t \in [0, \tau_1]$. Let $D = \bar{B}(0; a_1) \cup C$. Note that D is an invariant set. Lemma 2 implies that C is bounded and hence that D is bounded. Let a_2 and b_2 be constants such that $0 < b_2 < a_2$ and $D \subset B(0; b_2)$. Taking $a = a_2$ and $b = a_1$ in Lemma 6 implies there exists $\tau_2 > 0$ such that all solutions starting on S_{a_2} enter D within time τ_2. As D is an invariant set and $D \subset B(0; b_2)$, Theorem 2 is satisfied with $a = a_2$, $b = b_2$ and $T = \tau_2$, and it follows that the function

$$g_{a_2}: S_{a_2} \to S_1, \qquad x \mapsto \frac{f(x)}{|f(x)|}$$

has non-zero degree. The first part of the result now follows from Lemma 5.

Let $a > 0$ be an arbitrary but fixed number and define the function

$$\tilde{g}: S_{a_2} \to S_1, \qquad x \mapsto \frac{f((a/a_2)x)}{|f((a/a_2)x)|}$$

As $f(x) \neq 0$ for all non-zero $x \in \mathbb{R}^n$, the function

$$\Phi: [0,1] \times S_{a_2} \to S_1, \qquad (t,x) \mapsto \frac{f((1-t)x + t(a/a_2)x)}{|f((1-t)x + t(a/a_2)x)|}$$

is a continuous deformation of g_{a_2} to $\tilde{g}$ and g_{a_2} and $\tilde{g}$ are homotopic. Hence $\tilde{g}$ also has non-zero degree. The functions $\tilde{g}$ and g have the same degree and the surjectivity of g follows by the same argument used in the proof of Theorem 2. □

If f and u are as in the corollary statement below, Theorem 4 implies that $f(\cdot, u(\cdot))$ must take on all non-zero directions on all spheres centered about the origin. Condition (ii) of the corollary (which depends only on f) follows directly from this observation.†

Corollary 2: *Suppose $f:\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a given continuous function satisfying $f(0,0) = 0$. Furthermore suppose $u:\mathbb{R}^n \to \mathbb{R}^m$ is a continuous function satisfying $u(0) = 0$ and that $x \equiv 0$ is a globally attractive solution of the system $\dot{\xi} = f(\xi, u(\xi))$. Then*

(i) *the image under f of any open neighbourhood of $(0,0) \in \mathbb{R}^n \times \mathbb{R}^m$ contains an open neighbourhood of $0 \in \mathbb{R}^n$,*

(ii) *for each $a > 0$ and $\psi \in S_1$ there exists $(x,u) \in S_a \times \mathbb{R}^m$ such that*

$$\frac{f(x,u)}{|f(x,u)|} = \psi$$

† A technical note: if $(x,u) = (0,0)$ is the only point at which $f(x,u) = 0$, condition (ii) of Corollary 2 is equivalent to the statement 'for each $a > 0$ the function $g: S_a \times \mathbb{R}^m \to S_1$, $(x,u) \mapsto f(x,u)/|f(x,u)|$ is surjective'. This formulation cannot be used in general as if f is zero at points other than $(0,0)$, then there exists $a > 0$ for which g is not well defined.

Proof: To prove (i), the proof of Corollary 1 carries over word for word if in that proof one replaces 'Theorem 3' by 'Theorem 4'.

To prove (ii), as we have already indicated, Theorem 4 implies that for each $a > 0$, the function

$$g: S_a \to S_1, \qquad x \mapsto \frac{f(x, u(x))}{|f(x, u(x))|}$$

is surjective. This implies the result. $\square$

Remark 1: A modified version of the second necessary condition of Theorem 4 exists for bounded globally attractive sets. Indeed suppose $A \subset \mathbb{R}^n$ is a bounded globally attractive set and that $c \geq 0$ is such that $A \subset \bar{B}(0; c)$. Then the conclusion of Lemma 6 still holds for a and b satisfying $c < b < a$. (The proof remains almost unchanged and is omitted.) Following the proof of Theorem 4, one can now show that for each $a > c$, g given by (11) is surjective. Similar comments hold for the second necessary condition of Corollary 2.

5. Ultimate boundedness

In this section we give a necessary condition for ultimate boundedness. This condition follows from the result that if solutions of a system are ultimately bounded, then the closed loop vector field of the system must take on all non-zero directions on all spheres about the origin of sufficiently large radius.

Let us first start with a precise definition of ultimate boundedness. Let $f: \mathbb{R}^n \to \mathbb{R}^n$ be continuous. The solutions of $\dot{\xi} = f(\xi)$ are *ultimately bounded* if there exists $c > 0$ such that if $x_0 \in \mathbb{R}^n$ and $x(\cdot, x_0)$ is solution of $\dot{\xi} = f(\xi)$, $\xi(0) = x_0$, then $x(t, x_0)$ is defined for all $t \in [0, \infty)$ and there exists a $T > 0$ (which may depend on $x(\cdot, x_0)$) such that

$$|x(t, x_0)| \leq c \qquad \text{for all } t \geq T$$

(Yoshizawa 1966). If the value of c is of interest, we will say that the solutions are ultimately bounded with bound c.

The proofs of the results of this section are very similar to those of the previous section and are omitted.

Lemma 7: *Suppose $f: \mathbb{R}^n \to \mathbb{R}^n$ is continuous and that the solutions of $\dot{\xi} = f(\xi)$ are ultimately bounded with bound c. Let a and b satisfy $c < b < a$. Then there exists $\tau > 0$ such that if $x_0 \in S_a$ and $x(\cdot, x_0)$ is a solution of $\dot{\xi} = f(\xi)$, $\xi(0) = x_0$, then*

$$x([0, \tau], x_0) \cap \bar{B}(0; b) \neq \varnothing$$

Theorem 5: *Suppose $f: \mathbb{R}^n \to \mathbb{R}^n$ is continuous and that the solutions of $\dot{\xi} = f(\xi)$ are ultimately bounded. Then there exists $\bar{a} > 0$ such that for each $a \geq \bar{a}$ the function*

$$g: S_a \to S_1, \qquad x \mapsto \frac{f(x)}{|f(x)|}$$

is surjective.

Corollary 3: *Suppose $f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $u: \mathbb{R}^n \to \mathbb{R}^m$ are continuous and that solutions of the system $\dot{\xi} = f(\xi, u(\xi))$ are ultimately bounded. Then there exists $\bar{a} > 0$ such that for each $a \geq \bar{a}$ and $\psi \in S_1$ there exists $(x, u) \in S_a \times \mathbb{R}^m$ such that*

$$\frac{f(x, u)}{|f(x, u)|} = \psi$$

6. Conclusion

In this paper we have presented a number of necessary conditions for various forms of stability and attractivity for systems of the form (1) and (3). All results were derived for arbitrary continuous f and u. The first result presented was a proof of Brockett's necessary condition for stabilizability. Next two necessary conditions for globally attractivity were given. The first of these was the same as Brockett's condition while the second followed from the fact that the closed loop vector field must take on all non-zero directions on all spheres about the origin. A necessary condition for the existence of a bounded globally attractive set was also mentioned (Remark 1). Lastly we proved that the requirement that a closed loop vector field take on all non-zero directions on all spheres about the origin of sufficiently large radius is necessary for ultimate boundedness. This requirement was then used to give a necessary condition on f.

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