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Asymptotic Stabilization via Output Feedback for Lower Triangular Systems With Output Dependent Incremental Rate

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***Abstract*—We study the global asymptotic stabilization by output feedback for systems whose dynamics are in a feedback form and where the nonlinear terms admit an incremental rate depending only on the measured output. The output feedback we consider is of the observer-controller type where the design of the controller follows from standard robust backstepping. The novelty is in the observer which is high-gain such as with a gain coming from a Riccati equation.**

***Index Terms*—Backstepping, high-gain nonlinear observer, output nonlinear feedback, Riccati equation.**

I. INTRODUCTION

We consider a nonlinear system for which we can find coordinates y_1 to y_n and z_1 to z_m such that its dynamics can be written as

$$\begin{cases} \dot{y}_1 = f_1(y_1) + y_2 \\ \dot{y}_2 = f_2(y_1, y_2) + y_3 \\ \quad\vdots \\ \dot{y}_n = f_n(y_1, \ldots, y_n) + z_1 + u \\ \dot{z}_1 = h_1(y_1, \ldots, y_n, z_1, u) + z_2 \\ \dot{z}_2 = h_2(y_1, \ldots, y_n, z_1, z_2, u) + z_3 \\ \quad\vdots \\ \dot{z}_m = h_m(y_1, \ldots, y_n, z_1, \ldots, z_m, u) \end{cases} \tag{1}$$

where y_1 is the measured output in $\mathbb{R}$, u is the input in $\mathbb{R}$, the functions f_is are $n+1$ times continuously differentiable and zero at the origin, the functions h_is are continuously differentiable and zero at the origin and, for all i, u, y, z, ψ, and φ, we have

$$\begin{aligned} &|f_i(y_1, y_2+\psi_2, \ldots, y_i+\psi_i) - f_i(y_1, y_2, \ldots, y_i)| \\ &\quad \le \gamma(y_1)\,(|\psi_2| + \cdots + |\psi_i|) \end{aligned} \tag{2}$$

$$\begin{aligned} &|h_i(y_1, y_2+\psi_2, \ldots, y_n+\psi_n, z_1+\varphi_1, \ldots, z_i+\varphi_i, u) \\ &\; - h_i(y_1, y_2, \ldots, y_n, z_1, \ldots, z_i, u)| \\ &\quad \le \gamma(y_1)\,(|\psi_2| + \cdots + |\psi_n| + |\varphi_1| + \cdots + |\varphi_i|) \end{aligned} \tag{3}$$

where γ is a $n+1$ times continuously differentiable strictly positive function.

We address the problem of global asymptotic stabilization of the origin with output feedback.

This problem has received a lot of attention. But until recently, the contributions were assuming that the f_is at least are linear in y_2 to y_n,

Manuscript received August 29, 2002; revised January 26, 2003. Recommended by Associate Editor J. Huang.
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Digital Object Identifier 10.1109/TAC.2003.812819

as in [8, Sec. 7], [11, Sec. 6.3], [3, Ch. 7], or [16], [15], or [2],[1] for instance, or that γ is a constant as in [7] and [4].

Actually, for (1), if we have an observer leading to an error system with a state independent error Lyapunov function (see [13]) then we know how to get a controller from the observer dynamics, with robustification to the observation error. This design is based on the technique of robust backstepping, tackling with the observation errors via nonlinear damping (see [8, Sec. 7.1.2]), via interlacing (see [8, Sec. 7.4.1]) or by propagating an ISS property through integrators (see [6, Cor. 2.3]). Such a design allows us to deal with error structures more intricate than those obtained with the linearity or constant γ assumption. In particular it makes possible to take advantage of some sign or gain margin in the observer. The sign margin property for instance has been used in [1] for systems exhibiting a monotonicity property.

The objective of this note, whose preliminary version can be found in [14], is to use a gain margin property. This leads us to use a high gain like observer. For such observers, it is known (see [7]. for instance) that, at least locally around the true state, the value of the gain is dictated by the global Lipschitz constant of the non linearities if it exists. Here, this Lipschitz "constant" is not constant but depends on the output. This forces us to modify the gain on line. This creates some resemblance with the adapted high-gain observers used typically in universal controllers for (perturbed) linear systems (see [5] for a survey or [19] for a more recent contribution for instance). However, there is an important difference since our gain up date law depends on the increments of the nonlinearities and not on the nonlinearities themselves. Actually, our update law is a Riccati equation and, for this reason, we view our observer more something like a Kalman filter (compare with [15]) than an adapted high-gain observer.

Unfortunately, as all the previous results for the class of systems (1), we do require a "minimum phase" assumption for the inverse dynamics which we phrase as follows.

Minimum-Phase Assumption: The system

$$\left\{\begin{array}{rcl} \dot z_1 &=& h_1(v_1,\,\ldots,\,v_n,\,z_1,\,v_0-z_1)+z_2 \\ \dot z_2 &=& h_2(v_1,\,\ldots,\,v_n,\,z_1,\,z_2,\,v_0-z_1)+z_3 \\ &\vdots& \\ \dot z_m &=& h_1(v_1,\,\ldots,\,v_n,\,z_1,\,\ldots,\,z_m,\,v_0-z_1) \end{array}\right. \tag{4}$$

with input $(v_0,\,\ldots,\,v_n)$ and state $(z_1,\,\ldots,\,z_m)$ is Input-to-State Stable (see [17]).

The dynamic output feedback controller we propose has the structure of an observer-controller. The observer is high-gain like but with an on-line adapted gain. Its design is given in Section II. The controller, presented in Section III, is derived with the observer backstepping technique. In Section IV, we analyze the behavior of the closed-loop system.

II. Observer Design

To express the observer more easily, we rewrite the system (1) in the following more compact form:

$$\left\{\begin{array}{rcl} \dot x_1 &=& g_1(x_1,\,u)+x_2 \\ &\vdots& \\ \dot x_{p-1} &=& g_{p-1}(x_1,\,\ldots,\,x_{p-1},\,u)+x_p \\ \dot x_p &=& g_p(x_1,\,\ldots,\,x_p,\,u) \\ y_1 &=& x_1 \end{array}\right. \tag{5}$$

[1]In [2], an extra assumption in terms of structure and growth of Lyapunov functions is used.

where $p=n+m$, x in $\mathbb{R}^{n+m}$ collects the n components y_is and m components z_is and the functions g_is are the f_js or h_ls respectively. From (2) and (3) on the increments, we have, for all i, u, x and ξ

$$\begin{aligned} |g_i(x_1,\,x_2+\xi_2,\,\ldots,\,x_i+\xi_i,\,u)-g_i(x_1,\,x_2,\,\ldots,\,x_i,\,u)| \\ \le \gamma(y_1)\,(|\xi_2|+\cdots+|\xi_i|). \end{aligned} \tag{6}$$

The observer we propose is

$$\left\{\begin{array}{rcl} \dot{\hat x}_1 &=& g_1(y_1)+\hat x_2+k_1 r[y_1-\hat x_1] \\ &\vdots& \\ \dot{\hat x}_{p-1} &=& g_{p-1}(y_1,\,\hat x_2,\ldots,\hat x_{p-1},\,u)+\hat x_p+k_{p-1}r^{p-1}[y_1-\hat x_1] \\ \dot{\hat x}_p &=& g_p(y_1,\,\hat x_2,\ldots,\hat x_p,\,u)+k_p r^p[y_1-\hat x_1] \\ \dot r &=& \ell(r,\,y_1) \end{array}\right. \tag{7}$$

where r is an extra state, ℓ is a $n+1$ times continuously differentiable function to be defined below and the k_is are constant chosen such that (always possible) there exist strictly positive real numbers q and a and a symmetric matrix Q satisfying

$$Q\mathcal{O}+\mathcal{O}^T Q \le -aQ, \qquad qI \le Q \le I \tag{8}$$

where

$$\mathcal{O}=\begin{pmatrix} -k_1 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \\ -k_{p-1} & 0 & \cdots & 0 & 1 \\ -k_p & 0 & \cdots & \cdots & 0 \end{pmatrix}. \tag{9}$$

The corresponding observation error

$$\xi = x-\hat x \tag{10}$$

satisfies the following equation:

$$\left\{\begin{array}{rcl} \dot\xi_1 &=& \xi_2-k_1 r\xi_1 \\ &\vdots& \\ \dot\xi_{p-1} &=& [g_{p-1}(y_1,\,x_2,\,\ldots,\,x_{p-1},\,u) \\ && \quad -\,g_{p-1}(y_1,\,x_2-\xi_2,\,\ldots,\,x_{p-1}-\xi_{p-1},\,u)] \\ && \quad +\,\xi_p-k_{p-1}r^{p-1}\xi_1 \\ \dot\xi_p &=& [g_p(y_1,\,x_2,\,\ldots,\,x_p,\,u) \\ && \quad -\,g_p(y_1,\,x_2-\xi_2,\,\ldots,\,x_p-\xi_p,\,u)]-k_p r^p\xi_1. \end{array}\right. \tag{11}$$

To go further, we want to make sure that the observer state component r stays bounded away from 0, say larger than 1. For this we impose to the function ℓ to satisfy, for all y_1,

$$\ell(1,\,y_1)>0 \tag{12}$$

and we choose the initial condition $r(0)$ strictly larger than 1. Then, as by now routine in the analysis of error dynamics of high-gain observers (see [7], for instance), we introduce the following change of coordinates:

$$\varepsilon_i=\frac{\xi_i}{r^{i-1+b}}. \tag{13}$$

The novelty here is that b is not taken as 0 or 1 as usual. Instead, it is a strictly positive real number chosen (sufficiently large) to satisfy

$$bQ \ge QD+DQ \ge -bQ \tag{14}$$

where D is the diagonal matrix

$$D=\mathtt{diag}(0,\,\ldots,\,p-1). \tag{15}$$

Actually, we could impose $b = 1$ and choose[2] the gains k_i's so that (8) and

$$Q(D + bI) + Q(D + bI)^T Q > 0 \tag{16}$$

hold. However, such a design restricts the choice of observer poles.

For the ε_is coordinates, we have (17), as shown at the bottom of the page. With (8) and (6), we get the inequality (if $r \geq 1$)

$$\begin{aligned}\overbrace{\varepsilon^T Q \varepsilon}^{\cdot} \leq & -ar\varepsilon^T Q\varepsilon - 2\,\frac{\dot{r}}{r}\,\varepsilon^T Q(D + bI)\varepsilon \\ & + 2\gamma(y_1)\sum_{i=2}^{p}\left|\varepsilon^T Q_i\right| \frac{r^{1+b}|\varepsilon_2| + \cdots + r^{i-1+b}|\varepsilon_i|}{r^{i-1+b}}\end{aligned} \tag{18}$$

$$\begin{aligned}\leq & -\left(ar + 2b\,\frac{\dot{r}}{r}\right)\varepsilon^T Q\varepsilon - 2\,\frac{\dot{r}}{r}\varepsilon^T QD\varepsilon \\ & + 2\gamma(y_1)\sum_{i=2}^{p}\left|\varepsilon^T Q_i\right|\sum_{j=2}^{i}|\varepsilon_j|\end{aligned} \tag{19}$$

$$\begin{aligned}\leq & -\left(ar + 2b\,\frac{\dot{r}}{r}\right)\varepsilon^T Q\varepsilon - 2\,\frac{\dot{r}}{r}\varepsilon^T QD\varepsilon \\ & + 2\gamma(y_1)(p-1)\left|\varepsilon^T Q\right||\varepsilon|\end{aligned} \tag{20}$$

$$\leq -\left(ar + 2b\,\frac{\dot{r}}{r} - \frac{2(p-1)}{\sqrt{q}}\,\gamma(y_1)\right)\varepsilon^T Q\varepsilon - 2\frac{\dot{r}}{r}\varepsilon^T QD\varepsilon. \tag{21}$$

However, with (14), we have

$$-2\,\frac{\dot{r}}{r}\,\varepsilon^T QD\varepsilon \leq b\,\frac{|\dot{r}|}{r}\,\varepsilon^T Q\varepsilon. \tag{22}$$

So, we obtain

$$\overbrace{\varepsilon^T Q \varepsilon}^{\cdot} \leq -\left(ar + b\left(2\,\frac{\dot{r}}{r} - \frac{|\dot{r}|}{r}\right) - \frac{2(p-1)}{\sqrt{q}}\,\gamma(y_1)\right)\varepsilon^T Q\varepsilon. \tag{23}$$

From here, our idea to choose the function ℓ, i.e., $\dot{r}$, is to make the term in parenthesis negative. For instance, let us pick

$$\dot{r} = \ell(r, y_1) = -\frac{1}{b}\,r\left(\frac{a}{3}\,[r-1] - \frac{2(p-1)}{\sqrt{q}}\,\gamma(y_1)\right). \tag{24}$$

Since $\gamma(y_1)$ is strictly positive, (12) holds. Also, this yields if $\dot{r} \geq 0$ (if $r \geq 1$)

$$\overbrace{\varepsilon^T Q \varepsilon}^{\cdot} \leq -\left(ar + b\,\frac{\dot{r}}{r} - \frac{2(p-1)}{\sqrt{q}}\,\gamma(y_1)\right)\varepsilon^T Q\varepsilon \tag{25}$$

$$\leq -\frac{a}{3}\,[2r+1]\varepsilon^T Q\varepsilon \tag{26}$$

$$\leq -a\varepsilon^T Q\varepsilon. \tag{27}$$

[2]This choice is always possible has already remarked in [12].

If $\dot{r} \leq 0$, we obtain (if $r \geq 1$)

$$\overbrace{\varepsilon^T Q \varepsilon}^{\cdot} \leq -\left(ar + 3b\,\frac{\dot{r}}{r} - \frac{2(p-1)}{\sqrt{q}}\,\gamma(y_1)\right)\varepsilon^T Q\varepsilon \tag{28}$$

$$\leq -\left(a + \frac{4(p-1)}{\sqrt{q}}\,\gamma(y_1)\right)\varepsilon^T Q\varepsilon \tag{29}$$

$$\leq -a\varepsilon^T Q\varepsilon. \tag{30}$$

To summarize, for any choice for the k_i's so that (8) holds, we can express the function ℓ in such a way that, at each point in the closed-loop state space where $r \geq 1$, we have

$$\overbrace{\varepsilon^T Q \varepsilon}^{\cdot} \leq -a\varepsilon^T Q\varepsilon. \tag{31}$$

III. Controller Design

To design the controller, we work from a part of the observer equation (7) rewritten with the coordinates $(r, y_1, \hat{y}_2, \ldots, \hat{y}_n)$

$$\begin{cases} \dot{r} = -\frac{1}{b}\,r\left(\frac{a}{3}\,[r-1] - \frac{2(p-1)}{\sqrt{q}}\,\gamma(y_1)\right) \\ \dot{y}_1 = f_1(y_1) + \hat{y}_2 + r^{1+b}\varepsilon_2 \\ \dot{\hat{y}}_2 = f_2(y_1, \hat{y}_2) + \hat{y}_3 + k_2 r^{2+b}\varepsilon_1 \\ \quad\vdots \\ \dot{\hat{y}}_n = f_n(y_1, \hat{y}_2, \ldots, \hat{y}_n) + v + k_n r^{n+b}\varepsilon_1 \end{cases} \tag{32}$$

where we have

$$u = v - \hat{z}_1. \tag{33}$$

We follow exactly the same steps as in [8, Sec. 7.1.2] (see the Appendix for details). This way, we get recursively n functions $\alpha_i(r, y_1, \hat{y}_2, \ldots, \hat{y}_i)$ which are $n + 1 - i$ times continuously differentiable, respectively, and satisfy

$$\alpha_i(r, 0, 0, \ldots, 0) = 0. \tag{34}$$

In particular α_{i+1} is obtained from the gradient of α_i with respect to all its arguments. So it is in this process of getting these functions α_is that we need to differentiate may be up to n times the functions appearing in (32), i.e., the f_i and γ. Finally, we note that, for getting the nonlinear damping terms (see [8, p. 289]), we use (31) (which holds only if $r \geq 1$). This construction leads to the control

$$v = \alpha_n(r, y_1, \hat{y}_2, \ldots, \hat{y}_n) \tag{35}$$

and provides the variables

$$\zeta_1 = y_1 \tag{36}$$

$$\zeta_{i+1} = \hat{y}_{i+1} - \alpha_i(r, y_1, \hat{y}_2, \ldots, \hat{y}_i). \tag{37}$$

$$\begin{cases} \dot{\varepsilon}_1 = r\varepsilon_2 - rk_1\varepsilon_1 - b\,\frac{\dot{r}}{r}\,\varepsilon_1 \\ \quad\vdots \\ \dot{\varepsilon}_{p-1} = \frac{g_{p-1}(y_1, x_2, \ldots, x_{p-1}, u) - g_{p-1}(y_1, x_2 - r^{1+b}\varepsilon_2, \ldots, x_{p-1} - r^{p-2+b}\varepsilon_{p-1}, u)}{r^{p-2+b}} + r\varepsilon_p - rk_{p-1}\varepsilon_1 - (p-2+b)\,\frac{\dot{r}}{r}\,\varepsilon_{p-1} \\ \dot{\varepsilon}_p = \frac{g_p(y_1, x_2, \ldots, x_p, u) - g_p(y_1, x_2 - r^{1+b}\varepsilon_2, \ldots, x_p - r^{p-1+b}\varepsilon_p, u)}{r^{p-1+b}} - rk_p\varepsilon_1 - (p-1+b)\,\frac{\dot{r}}{r}\,\varepsilon_p. \end{cases} \tag{17}$$

It gives also the inequality (if $r \geq 1$):

$$\overbrace{y_1^2 + \sum_{i=2}^{n} \zeta_i^2 + \varepsilon^T Q \varepsilon}^{\cdot} \leq -y_1^2 - \sum_{i=2}^{n} \zeta_i^2 - \frac{a}{2}\,\varepsilon^T Q \varepsilon. \tag{38}$$

Finally, our output feedback controller is

$$\begin{cases} \dot{r} \;=\; -\dfrac{1}{b}\, r\left(\dfrac{a}{3}\,[r-1] - \dfrac{2(p-1)}{\sqrt{q}}\,\gamma(y_1)\right), \qquad r(0) > 1 \\ \dot{\hat{y}}_1 \;=\; f_1(y_1) + \hat{y}_2 + k_1 r (y_1 - \hat{y}_1) \\ \dot{\hat{y}}_2 \;=\; f_2(y_1, \hat{y}_2) + \hat{y}_3 + k_2 r^2 (y_1 - \hat{y}_1) \\ \quad \vdots \\ \dot{\hat{y}}_n \;=\; f_n(y_1, \hat{y}_2, \ldots, \hat{y}_n) + \hat{z}_1 + u + k_n r^n (y_1 - \hat{y}_1) \\ \dot{\hat{z}}_1 \;=\; h_1(y_1, \hat{y}_2, \ldots, \hat{y}_n, \hat{z}_1, u) + \hat{z}_2 \\ \qquad + k_{n+1} r^{n+1} (y_1 - \hat{y}_1) \\ \quad \vdots \\ \dot{\hat{z}}_m \;=\; h_m(y_1, \hat{y}_2, \ldots, \hat{y}_n, \hat{z}_1, \ldots \hat{z}_m, u) \\ \qquad + k_{n+m} r^{n+m} (y_1 - \hat{y}_1) \\ u \;=\; \alpha_n(r, y_1, \hat{y}_2, \ldots, \hat{y}_n) \;-\; \hat{z}_1. \end{cases} \tag{39}$$

IV. Analysis of the Closed-Loop System

The dynamics of the closed-loop system can be described using the coordinates

$$(r,\, \varepsilon,\, y_1,\, \hat{y}_2,\, \ldots,\, \hat{y}_n,\, z_1,\, \ldots,\, z_m).$$

They satisfy the set of equations shown in (40) at the bottom of the page, where we have used the notation, for $i \in \{1, \ldots, n\}$

$$x_i = \hat{y}_i + r^{i-1+b} \varepsilon_i \tag{41}$$

and, for $i \in \{n+1, \ldots, n+m\}$

$$x_i = z_{i-n}. \tag{42}$$

These dynamics have the following properties.

2) The right-hand side is defined on $(0, +\infty) \times \mathbb{R}^{2(n+m)}$ where it is continuously differentiable. It follows that, to each initial condition in $(0, +\infty) \times \mathbb{R}^{2(n+m)}$, it corresponds a unique solution.

3) The expression of $\dot{r}$ has been chosen such that:

$$r = 1 \quad \Rightarrow \quad \dot{r} > 0. \tag{43}$$

It follows that the set $(1, +\infty) \times \mathbb{R}^{2(n+m)}$ is forward invariant and its boundary $\{1\} \times \mathbb{R}^{2(n+m)}$ is repellent. Hence, any solution initialized in this set remains in it and, if its right maximal interval of definition is bounded, it is unbounded (since its r component cannot go to 1). So, for any such solution, (38) holds.

4) We have an interconnection structure. The $(r, \varepsilon, y_1, \hat{y}_2, \ldots, \hat{y}_n)$ subsystem sends the signals

$$\begin{aligned} &v_0 = -r^{n+b} \varepsilon_{n+1} + \alpha_n, \qquad v_1 = y_1 \\ &v_2 = \hat{y}_2 + r^{1+b} \varepsilon_2, \ldots, v_n = \hat{y}_n + r^{n-1+b} \varepsilon_n \end{aligned} \tag{44}$$

to the z subsystem whose dynamics are then given by (4). Conversely, the z-subsystem sends its state to the $(r, \varepsilon, y_1, \hat{y}_2, \ldots, \hat{y}_n)$ subsystem via the functions g_is. With (the proof of) [18, Cor.] and our minimum phase assumption, it follows that asymptotic stability with domain of attraction $(1, +\infty) \times \mathbb{R}^{2(n+m)}$ holds if the $(r, \varepsilon, y_1, \hat{y}_2, \ldots, \hat{y}_n)$ subsystem has, uniformly in z, an asymptotically stable equilibrium with $(1, +\infty) \times \mathbb{R}^{2n+m}$ as domain of attraction. (See (49) for what we mean by this).

5) With the help of (31), (34), (37), and (38), we see that $(r, \varepsilon, y_1, \hat{y}_2, \ldots, \hat{y}_n)$-subsystem has only one equilibrium at $(r^*, 0, \ldots, 0)$ in the set $(1, +\infty) \times \mathbb{R}^{2n+m}$, with

$$r^* = 1 + \frac{6(p-1)}{a\sqrt{q}}\,\gamma(0). \tag{45}$$

$$\begin{cases} \dot{r} \;=\; -\dfrac{1}{b}\, r\left(\dfrac{a}{3}\,[r-1] - \dfrac{2(p-1)}{\sqrt{q}}\,\gamma(y_1)\right), \qquad r(0) > 1 \\ \dot{\varepsilon}_1 \;=\; r\varepsilon_2 - r k_1 \varepsilon_1 - b\,\dfrac{\dot{r}}{r}\,\varepsilon_1 \\ \quad \vdots \\ \dot{\varepsilon}_i \;=\; \dfrac{g_i(y_1, x_2, \ldots, x_i, u) - g_i(y_1, x_2 - r^{1+b}\varepsilon_2, \ldots, x_i - r^{i-1+b}\varepsilon_i, u)}{r^{i-1+b}} + r\varepsilon_{i+1} - r k_i \varepsilon_1 - (i-1+b)\,\dfrac{\dot{r}}{r}\,\varepsilon_i \\ \quad \vdots \\ \dot{\varepsilon}_p \;=\; \dfrac{g_p(y_1, x_2, \ldots, x_p, u) - g_p(y_1, x_2 - r^{1+b}\varepsilon_2, \ldots, x_p - r^{p-1+b}\varepsilon_p, u)}{r^{p-1+b}} - r k_p \varepsilon_1 - (p-1+b)\,\dfrac{\dot{r}}{r}\,\varepsilon_p \\ \dot{y}_1 \;=\; f_1(y_1) + \hat{y}_2 + r^{1+b}\varepsilon_2 \\ \dot{\hat{y}}_2 \;=\; f_2(y_1, \hat{y}_2) + \hat{y}_3 + k_2 r^{2+b} \varepsilon_1 \\ \quad \vdots \\ \dot{\hat{y}}_n \;=\; f_n(y_1, \hat{y}_2, \ldots, \hat{y}_n) + k_n r^{n+b} \varepsilon_1 + \alpha_n(r, y_1, \hat{y}_2, \ldots, \hat{y}_n) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dot{z}_1 \;=\; h_1\big(y_1, \hat{y}_2 + r^{1+b}\varepsilon_2, \ldots, \hat{y}_n + r^{n-1+b}\varepsilon_n, z_1, -z_1 - r^{n+b}\varepsilon_{n+1} + \alpha_n\big) + z_2 \\ \quad \vdots \\ \dot{z}_m \;=\; h_m\big(y_1, \hat{y}_2 + r^{1+b}\varepsilon_2, \ldots, \hat{y}_n + r^{n-1+b}\varepsilon_n, z_1, \ldots, z_m, -z_1 - r^{n+b}\varepsilon_{n+1} + \alpha_n\big) \end{cases} \tag{40}$$

6) The function $r-r^*-r^*\log(r/r^*)$ is C^1, proper and nonnegative on $(0, +\infty)$. It is zero if and only if $r = r^*$. It satisfies

$$\overbrace{r - r^* - r^* \log(r/r^*)}^{\cdot} = -\frac{a}{3b}\left(r - 1 - \frac{6(p-1)}{a\sqrt{q}}\,\gamma(y_1)\right)(r - r^*) \tag{46}$$

$$= -\frac{a}{3b}\,(r - r^*)^2 - \frac{2(p-1)}{b\sqrt{q}}\,(r - r^*)(\gamma(0) - \gamma(y_1)) \tag{47}$$

$$\leq -\frac{a}{6b}\,(r - r^*)^2 + \frac{6(p-1)^2}{abq}\,(\gamma(0) - \gamma(y_1))^2. \tag{48}$$

We conclude that to show the asymptotic stability of the point $(r^*, 0, \ldots, 0)$ with domain of attraction $(1, +\infty) \times \mathbb{R}^{2(n+m)}$ uniformly in z, it is sufficient to show that for some C^1, unbounded and strictly increasing function Φ: $[0, +\infty) \to [0, +\infty)$, the derivative of

$$V(r, y_1, \hat{y}_2, \ldots, \hat{y}_n, \varepsilon) = [r-r^*-r^*\log(r/r^*)] + \Phi\left(y_1^2 + \sum_{i=2}^{n} \zeta_i^2 + \varepsilon^T Q \varepsilon\right) \tag{49}$$

is negative definite uniformly in z. In view of (38) and (48), we pick φ: $[0, +\infty) \to (0, +\infty)$ as a continuous non decreasing function satisfying, for all y_1

$$\varphi(y_1^2) \geq 2\,\frac{6(p-1)^2}{abq}\left(\frac{\gamma(0) - \gamma(y_1)}{y_1}\right)^2. \tag{50}$$

Such a choice is possible since $(\gamma(0) - \gamma(y_1))/y_1$ is a continuous function. Then, in the definition of V, we use

$$\Phi(s) = \int_0^s \varphi(\sigma)\,d\sigma. \tag{51}$$

With (38) and (48), we get, in $(1, +\infty) \times \mathbb{R}^{2n+m}$

$$\dot{V} \leq -\frac{a}{6b}\,(r - r^*)^2 + \frac{6(p-1)^2}{abq}\,(\gamma(0) - \gamma(y_1))^2 - \varphi\left(y_1^2 + \sum_{i=2}^{n} \zeta_i^2 + \varepsilon^T Q\varepsilon\right)\left[y_1^2 + \sum_{i=2}^{n}\zeta_i^2 + \frac{a}{2}\,\varepsilon^T Q\varepsilon\right]. \tag{52}$$

Since φ is nondecreasing and satisfies (50), this yields

$$\dot{V} \leq -\frac{a}{6b}\,(r - r^*)^2 + \frac{6(p-1)^2}{abq}\,(\gamma(0) - \gamma(y_1))^2 - \varphi(y_1^2)y_1^2 - \varphi\left(y_1^2 + \sum_{i=2}^{n}\zeta_i^2 + \varepsilon^T Q\varepsilon\right) \cdot \left[\sum_{i=2}^{n}\zeta_i^2 + \frac{a}{2}\,\varepsilon^T Q\varepsilon\right] \tag{53}$$

$$\leq -\frac{a}{6b}\,(r - r^*)^2 - \frac{1}{2}\,\varphi(y_1^2)y_1^2 - \varphi\left(\sum_{i=2}^{n}\zeta_i^2 + \varepsilon^T Q\varepsilon\right)\left[\sum_{i=2}^{n}\zeta_i^2 + \frac{a}{2}\,\varepsilon^T Q\varepsilon\right]. \tag{54}$$

The right-hand side of (54) is nonpositive and, with (34) and (37), zero if and only if we are at $(r^*, 0, \ldots, 0)$. Hence, we have established the asymptotic stability of this point uniformly in z. Also, this inequality holding everywhere in the set $(1, +\infty) \times \mathbb{R}^{2n+m}$ which is forward invariant, this whole set is the domain of attraction.

To conclude, for (1) satisfying (2), (3), and the minimum phase assumption, the dynamic output feedback we have proposed provides asymptotic stability of the point $(r^*, 0, \ldots, 0)$ with domain of attraction $(1, +\infty) \times \mathbb{R}^{2(n+m)}$.

V. Conclusion

We have shown that, by combining an adapted high gain observer and observer backstepping, we can design globally asymptotically stabilizing output feedbacks for systems admitting the form (1) where the nonlinearities have an incremental rate depending only on the measured output as specified by (2) and (3).

The main contribution here is in the observer gain update law. It is reminiscent from the covariance matrix up date law in the Kalman filter used in [15]. In particular, our update law is not nominally non negative. It follows that, to get asymptotic stability of a compact set, there is no need to add some fix like dead-zone, leakage, or other (see [19] and [9], for instance).

The key to get such an update law is in the coordinate scaling commonly used in the analysis of high gain observer. In our case, without any extra restriction on the observer poles, this scaling

$$\varepsilon_i = \frac{\xi_i}{r^{i-1+b}} \tag{55}$$

depends not only on the rank i in the integrator chain, but also on b, a parameter directly related to the "observer poles" [see (8) and (14)].

Appendix
Construction of the Functions α_is

For the sake of completeness, we reproduce here with some adaptation what can be found in [8, Sec. 7.1.2].

Consider the system

$$\begin{cases} \dot{r} = -\frac{1}{b}\,r\left(\frac{a}{3}\,[r-1] - \frac{2(p-1)}{\sqrt{q}}\,\gamma(y_1)\right) \\ \dot{y}_1 = f_1(y_1) + \hat{y}_2 + r^{1+b}\varepsilon_2 \\ \dot{\hat{y}}_2 = f_2(y_1, \hat{y}_2) + \hat{y}_3 + k_2 r^{2+b}\varepsilon_1 \\ \quad\vdots \\ \dot{\hat{y}}_i = f_i(y_1, \hat{y}_2, \ldots, \hat{y}_i) + v_i + k_i r^{i+b}\varepsilon_1 \end{cases} \tag{56}$$

where the ε_js are components of a vector ε. Aiming at establishing a result by recurrence, we assume the existence of functions α_j which are $n+1-j$ times continuously differentiable, respectively, satisfy

$$\alpha_j(r, 0, 0, \ldots, 0) = 0 \tag{57}$$

and are such that, by letting

$$\zeta_{j+1} = \hat{y}_{j+1} - \alpha_j(r, y_1, \hat{y}_2, \ldots, \hat{y}_j) \tag{58}$$

we have

$$\overbrace{y_1^2 + \sum_{j=2}^{i}\zeta_j^2 + \varepsilon^T Q\varepsilon}^{\cdot} \leq -y_1^2 - \sum_{j=2}^{i}\zeta_j^2 - \frac{a(2n-i)}{2n}\,\varepsilon^T Q\varepsilon + 2\zeta_i(v_i - \alpha_i). \tag{59}$$

Now, we consider the system

$$\begin{cases} \dot{r} = -\frac{1}{b}\,r\left(\frac{a}{3}\,[r-1] - \frac{2(p-1)}{\sqrt{q}}\,\gamma(y_1)\right) \\ \dot{y}_1 = f_1(y_1) + \hat{y}_2 + r^{1+b}\varepsilon_2 \\ \dot{\hat{y}}_2 = f_2(y_1, \hat{y}_2) + \hat{y}_3 + k_2 r^{2+b}\varepsilon_1 \\ \quad\vdots \\ \dot{\hat{y}}_i = f_i(y_1, \hat{y}_2, \ldots, \hat{y}_i) + \hat{y}_{i+1} + k_i r^{i+b}\varepsilon_1 \\ \dot{\hat{y}}_{i+1} = f_{i+1}(y_1, \hat{y}_2, \ldots, \hat{y}_{i+1}) + v_{i+1} + k_{i+1} r^{i+1+b}\varepsilon_1 \end{cases} \tag{60}$$

and we let

$$\zeta_{i+1} = \hat{y}_{i+1} - \alpha_i(r, y_1, \hat{y}_2, \ldots, \hat{y}_i). \quad (61)$$

In this case, (59) gives

$$\overbrace{y_1^2 + \sum_{j=2}^{i} \zeta_j^2 + \varepsilon^T Q \varepsilon}^{\cdot} \leq -y_1^2 - \sum_{j=2}^{i} \zeta_j^2 - \frac{a(2n-i)}{2n} \varepsilon^T Q \varepsilon + 2\zeta_i \zeta_{i+1}. \quad (62)$$

This yields

$$\overbrace{y_1^2 + \sum_{j=2}^{i+1} \zeta_j^2 + \varepsilon^T Q \varepsilon}^{\cdot} \leq -y_1^2 - \sum_{j=2}^{i} \zeta_j^2 - \frac{a(2n-i)}{2n} \varepsilon^T Q \varepsilon + 2\zeta_{i+1}\left(\zeta_i + \dot{\hat{y}}_{i+1} - \dot{\alpha}_i\right) \quad (63)$$

where, in particular, we have

$$\begin{aligned} \dot{\alpha}_i = &\frac{\partial \alpha_i}{\partial r}\left[-\frac{1}{b} r \left(\frac{a}{3}[r-1] - \frac{2(p-1)}{\sqrt{q}} \gamma(y_1)\right)\right] \\ &+ \frac{\partial \alpha_i}{\partial y_1}\left[f_1(y_1) + \hat{y}_2 + r^{1+b}\varepsilon_2\right] \\ &\vdots \\ &+ \frac{\partial \alpha_i}{\partial \hat{y}_i}\left[f_i(y_1, \hat{y}_2, \ldots, \hat{y}_i) + \hat{y}_{i+1} + k_i r^{i+b} \varepsilon_1\right]. \end{aligned} \quad (64)$$

We observe that the term $\zeta_i + \dot{\hat{y}}_{i+1} - \dot{\alpha}_i$ admits the following decomposition

$$\zeta_i + \dot{\hat{y}}_{i+1} - \dot{\alpha}_i = v_{i+1} + \mu_i(r, y_1, \hat{y}_2, \ldots, \hat{y}_{i+1}) + \nu_i(r, y_1, \hat{y}_2, \ldots, \hat{y}_i)\varepsilon_1 + \frac{\partial \alpha_i}{\partial y_1} r^{1+b} \varepsilon_2 \quad (65)$$

with a straightforward identification of the function μ_i and ν_i. Also, note that (57) implies

$$\frac{\partial \alpha_i}{\partial r}(r, 0, 0, \ldots, 0) = 0. \quad (66)$$

Since the f_j's are zero at the origin, with (57) and (66), it follows that

$$\mu_i(r, 0, \ldots, 0) = 0. \quad (67)$$

Finally, by completing the squares, we get

$$\begin{aligned} 2\zeta_{i+1}&\left(\nu_i \varepsilon_1 + \frac{\partial \alpha_i}{\partial y_1} r^{1+b} \varepsilon_2\right) \\ &\leq \frac{2nq}{a} \zeta_{i+1}^2 \left(\nu_i^2 + \frac{\partial \alpha_i}{\partial y_1}^2 r^{2+2b}\right) + \frac{a}{2nq}(\varepsilon_1^2 + \varepsilon_2^2) \quad (68) \\ &\leq \frac{2nq}{a} \zeta_{i+1}^2 \left(\nu_i^2 + \frac{\partial \alpha_i}{\partial y_1}^2 r^{2+2b}\right) + \frac{a}{2n} \varepsilon^T Q \varepsilon. \quad (69) \end{aligned}$$

Using this inequality in (63), we obtain

$$\begin{aligned} &\overbrace{y_1^2 + \sum_{j=2}^{i+1} \zeta_j^2 + \varepsilon^T Q \varepsilon}^{\cdot} \\ &\leq -y_1^2 - \sum_{j=2}^{i} \zeta_j^2 - \frac{a(2n-(i+1))}{2n} \varepsilon^T Q \varepsilon \\ &\quad + 2\zeta_{i+1}\left[v_{i+1} + \mu_i(r, y_1, \hat{y}_2, \ldots, \hat{y}_{i+1}) + \frac{nq}{a} \zeta_{i+1}\left(\nu_i^2 + \frac{\partial \alpha_i}{\partial y_1}^2 r^{2+2b}\right)\right]. \end{aligned} \quad (70)$$

So, by defining α_{i+1} as

$$\alpha_{i+1}(r, y_1, \hat{y}_2, \ldots, \hat{y}_{i+1}) = -\left[\mu_i(r, y_1, \hat{y}_2, \ldots, \hat{y}_{i+1}) + \frac{nq}{a} \zeta_{i+1}\left(\nu_i^2 + \frac{\partial \alpha_i}{\partial y_1}^2 r^{2+2b}\right)\right] - \frac{1}{2}\zeta_{i+1} \quad (71)$$

we get [compare with (59)]

$$\overbrace{y_1^2 + \sum_{j=2}^{i+1} \zeta_j^2 + \varepsilon^T Q \varepsilon}^{\cdot} \leq -y_1^2 - \sum_{j=2}^{i+1} \zeta_j^2 - \frac{a(2n-(i+1))}{2n} \varepsilon^T Q \varepsilon + 2\zeta_{i+1}(v_{i+1} - \alpha_{i+1}). \quad (72)$$

Note also that α_{i+1} is $n-i$ times continuously differentiable and satisfies

$$\alpha_{i+1}(r, 0, \ldots, 0) = 0. \quad (73)$$

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