Stabilization of Nonlinear Systems via Forwarding $\mathrm{mod}\{L_gV\}$

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***Abstract*—*Forwarding* is a tool for constructing stabilizers for nonlinear systems. A key step in this design technique is to find an explicit solution to a *partial differential equation* (PDE), which may be hard to find—actually, the PDE may even not be solvable at all. In this brief note we show that it is possible to provide an additional degree of freedom for the solution of the aforementioned PDE, hence effectively extending the realm of application of the forwarding methodology. Our contribution is illustrated with the example of an inverted pendulum with a disk inertia.**

***Index Terms*—Control of mechanical systems, forwarding, nonlinear control, stabilization of NL systems.**

I. Background

In this section, we will briefly review the basic forwarding technique for stabilization of nonlinear systems from a geometric perspective. For further details on this technique the reader is referred to [6], [7]. To motivate the developments, let us first consider a cascade of two systems of the form

$$\begin{aligned}\dot{z} &= h(x)\\ \dot{x} &= f(x)\end{aligned} \tag{1}$$

where z is scalar and f and h are Lipschitz continuous functions and the origin of the x-subsystem is asymptotically stable, namely there exists a positive definite Lyapunov function $V(x)$ such that[1] $L_fV < 0$ for all $x \neq 0$. To study the stability of the cascade we look for the existence of a stable manifold of the origin described by the graph $z = M(x)$. That is, we want to find a C^1 function $M(x)$, with $M(0) = 0$, such that the following implication is true

$$\begin{aligned}(z(0), x(0)) \in & \Omega \triangleq \{(z,x) | z = M(x)\}\\ &\Rightarrow (z(t)x(t)) \in \Omega, \quad \forall\, t \geq 0\end{aligned}$$

where $(z(t), x(t))$ denotes a solution of (1). With the C^1 assumption for M, the existence of Ω is equivalent to the solvability of the PDE

$$L_fM = h \tag{2}$$

with boundary condition $M(0) = 0$. If we can solve the PDE (2) a Lyapunov function for the overall system is given by

$$W(x,z) \triangleq V(x) + \frac{1}{2}[z - M(x)]^2 \tag{3}$$

whose derivative is simply L_fV.

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[1]We use the standard Lie derivative notation $L_fV \triangleq (\partial V/\partial x)f$.

Forwarding builds upon this basic idea to stabilize cascaded systems of the form

$$\Sigma : \begin{cases}\dot{z} = & h(x)\\ \dot{x} = & f(x) + g(x)u.\end{cases} \tag{4}$$

In this case, the derivative of W yields

$$\dot{W} = L_fV + [L_gV - (z - M)L_gM]\,u$$

which clearly suggests the control law

$$u = -[L_gV - (z - M)L_gM]\,. \tag{5}$$

II. Forwarding $\mathrm{mod}\{L_gV\}$

The main stumbling block of the forwarding procedure is therefore the solvability of the PDE (2), a question which is difficult to answer in general. The main objective of this note is to provide an additional degree of freedom for the solution of the PDE, consequently enlarging the class of systems that can be explicitly stabilized with the forwarding procedure. Toward this end we show that we can add to the right-hand side of the PDE a "free" term, and still be able to synthesize a stabilizing controller.

Proposition 1: Consider the system (4) with the following assumptions.

A.1) *(Stability of the x-subsystem)* There exists a positive–definite Lyapunov function $V(x)$ such that $L_fV(x) < 0$ for all $x \neq 0$;

A.2) We know a $\mathcal{C}^0$ function $k(x)$ and a $\mathcal{C}^1$ function $M(x)$, with $M(0) = 0$, such that

i) *(New PDE)* [compare (6) with (2)]

$$L_fM = h + kL_gV. \tag{6}$$

ii) The following implication holds:

$$\{L_gM(x) \neq 0, x \neq 0\} \Rightarrow L_fV(x) - \frac{k(x)L_gV(x)^2}{L_gM(x)} < 0. \tag{7}$$

iii) $L_gM(0) \neq 0$.

Under these conditions, the function

$$W(x,z) \triangleq V(x) + \frac{1}{2}[z - M(x)]^2$$

is a *Control Lyapunov Function* satisfying the *Small Control Property* for the system (4).

A direct consequence of this Proposition is that, under its assumptions, we are guaranteed of the existence of an at least continuous global asymptotic stabilizer. Expressions for such a stabilizer are given by universal formulae as the ones in [4], [10], e.g., as shown in the equation at the bottom of the next page.However, here we can take advantage of the specificity to propose other expressions (see the proof below).

Proof of Proposition 1: Evaluating the derivative of the Lyapunov function candidate (3) along the trajectories of (4), and using (6), yields

$$\dot{W} = L_fV + L_gVu - (z - M)[L_gMu + kL_gV].$$

To show that W is a Control Lyapunov Function it is sufficient to find, for each $(z,x) \neq 0$, a control[2] u making $\dot{W}$ strictly negative. We consider several cases.

1) If $k(x) = 0$, we pick:

$$u = -[L_gV(x) - (z - M(x))L_gM(x)]\,.$$

[2]Actually, the fact that W is a Control Lyapunov Function is a direct consequence of A.2.ii) when $L_gM(x) \neq 0$ and cases 1 and 2 below. However, having expressions for u is useful for establishing the Small Control Property.

This yields

$$\dot{W} = L_f V - [L_g V - (z - M) L_g M]^2$$

where $L_f V(x) < 0$ if $x \neq 0$. However, if $L_f V(x) = 0$ then $x = 0$ and $L_g V(x) = M(x) = 0$ and so with A.2.iii), we have $-[L_g V(0) - (z - M(0)) L_g M(0)]^2 < 0$ if $z \neq 0$.

2) If $k(x) \neq 0$ and $L_g M(x) = 0$, then with A.2.iii), we must have $x \neq 0$. Then we pick

$$u = k(x)(z - M(x)).$$

This yields

$$\dot{W} = L_f V < 0.$$

3) If $k(x) L_g M(x) > 0$, we pick

$$u = k(x)(z - M(x)).$$

This yields

$$\dot{W} = L_f V - k L_g M (z - M)^2$$

which is strictly negative for the same reason as in the first case above.

4) If $k(x) L_g M(x) < 0$, we pick

$$u = -\frac{k(x) L_g V(x)}{L_g M(x)} - [L_g V(x) - (z - M(x)) L_g M(x)]. \quad (8)$$

This yields

$$\dot{W} = L_f V - \frac{k (L_g V)^2}{L_g M} - [L_g V(x) - (z - M(x)) L_g M(x)]^2$$

which is strictly negative for the same reason as above.

To show that the Small Control Property holds, it is sufficient to check that the norm of the control u exhibited above can be made as small as we want by picking $|x| + |z|$ small enough. This is true for the cases 1)–3) since we have continuity of the various functions and $L_g V(0) = M(0) = 0$. For the case 4), the result follows with A.2.iii) which implies that, as x goes to zero, $|L_g M(x)|$ is bounded away from 0, i.e., the function $k L_g V / L_g M$ is continuous at the origin and zero at the origin.

Remark 1: To get some further insight into the assumptions of the proposition it is interesting to consider the linear case

$$\begin{aligned} \dot{z} &= c^{\mathrm{T}} x \\ \dot{x} &= Ax + bu. \end{aligned}$$

We then have $V = (1/2) x^{\mathrm{T}} P x$, with $P = P^{\mathrm{T}} > 0$ the solution of the algebraic Lyapunov equation $PA + A^{\mathrm{T}} P = -Q < 0$, and we can take $M(x) = M^{\mathrm{T}} x$, with M defined as $M = A^{-\mathrm{T}}(c + kPb)$. Now, as $L_g M = M^{\mathrm{T}} b$, assumption A.2.iii) reduces to assuming that $M^{\mathrm{T}} b \neq 0$, which is a necessary condition for controllability of the system.[3] Also, assumption A.2.ii) reduces to

$$-\frac{1}{2} Q \leq \frac{k}{b^{\mathrm{T}} A^{-1}(c + kPb)} P b b^{\mathrm{T}} P$$

which is equivalent to

$$-\frac{b^{\mathrm{T}} PQPb}{(b^{\mathrm{T}} P^2 b)^2} \leq \frac{4k}{2c^{\mathrm{T}} A^{-1} b - k b^{\mathrm{T}} A^{-\mathrm{T}} Q A b}.$$

This defines the (nonempty) set of values allowed for k.

Remark 2: Using forwarding $\mathrm{mod}\{L_g V\}$ allows us, in some cases, to relax the assumption A.1) of stability of the x-subsystem to $L_f V \leq 0$. This feature is illustrated in the example below.

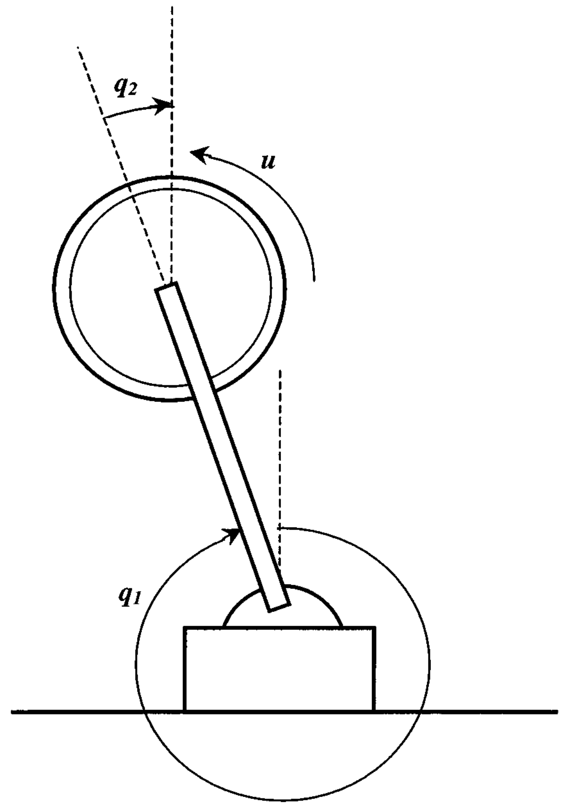

Fig. 1. Schematic representation of the disk inertia pendulum.

III. Stabilization of the Disk Inertia Pendulum

In this section, we apply Proposition 1 to design a global asymptotic stabilizer of the upward position of the pendulum device shown in Fig. 1, which consists of a free pendulum with a rotating mass at the end. The

[3]Controllability being invariant under state feedback, we consider the system

$$\dot{z} = c^{\mathrm{T}} x, \quad \dot{x} = \left[A - \frac{k}{M^{\mathrm{T}} b} b b^{\mathrm{T}} P \right] x + bu.$$

Then, by replacing z by the new coordinates $y = z - M^{\mathrm{T}} x$, we get $\dot{y} = -M^{\mathrm{T}} b u$. The claim follows readily.

$$u = -\frac{\max\left\{0, L_f V - (z - m) k L_g V + (L_g V - (z - m) L_g M)^2\right\}}{L_g V - (z - m) L_g M}.$$

motor torque produces an angular acceleration of the end-mass which generates a coupling torque at the pendulum axis. Unfortunately this motor torque is limited and this input constraint puts a hard bound on our ability to stabilize the upward position. It is worth mentioning that the present study was precisely motivated by this example.

First, we show that the problem is not solvable with "standard" forwarding. In particular we prove that the PDE (2) is not globally solvable. To overcome the problem we apply the forwarding $\mathrm{mod}\{L_gV\}$ technique to get a controller that "almost" globally stabilizes the upward position.[4] In [8], the system is stabilized using passivity-based control, we refer the reader to this reference for further details about this device.

A. Model

The dynamic equations of the device can be written in standard Lagrangian form as

$$\begin{bmatrix} m_1+m_2 & m_1 \\ m_1 & m_1 \end{bmatrix} \ddot{q} = \begin{bmatrix} mgl\sin(q_1) \\ v \end{bmatrix} \tag{9}$$

where in particular the control v is the motor torque. In practice, this torque is limited and this makes the whole difficulty of the stabilization problem. Without such a limit, a backstepping technique or stabilizing a particular planned trajectory would lead easily to other global asymptotic stabilizers. Here, we assume that this limit is still large enough to allow the motor torque for compensating the maximal gravity torque mgl. This assumption implies in particular that we can do the preliminary feedback:

$$v = mgl\sin(q_1) - u.$$

In the following, any limitation on u can be afforded. Unfortunately, the specific device in our lab does not meet the above assumption and therefore does not allow us to test the following proposed controller.

To proceed with our design in order to avoid cluttering the notation, we take $mgl = m_1 = m_2 = 1$.[5] We introduce the new coordinates $x_1 = q_1$, $x_3 = \dot{q}_1$, $z_1 = 2q_1 + q_2$, $z_2 = 2\dot{q}_1 + \dot{q}_2$ and as mentioned above the new control $u = \sin(x_1) - v$. Then the dynamics rewrite in the (block) forwarding form (4) as

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= \sin(x_1) \\ \dot{x}_1 &= x_3 \\ \dot{x}_3 &= u. \end{aligned} \tag{10}$$

The final control objective is to find a stabilizer of the origin of (10) satisfying prescribed limitations.

Before presenting a solution to the constrained problem, we solve the nonconstrained problem with the new forwarding $\mathrm{mod}\{L_gV\}$.

B. Standard Forwarding

Step 1) We consider the subsystem

$$\begin{aligned} \dot{x}_1 &= x_3 \\ \dot{x}_3 &= u \end{aligned}$$

and design a controller invoking passivity, i.e., with the Lyapunov function

$$V_1(x_1,x_3) = (1-\cos(x_1)) + \frac{\alpha}{2}x_3^2$$

where $\alpha > 0$ is a tuning parameter. This yields

$$\dot{V}_1 = x_3\left[\sin(x_1) + \alpha u\right]$$

Hence the first control loop is

$$u = -\frac{1}{\alpha}\sin(x_1) - \beta x_3 + v_1$$

where v_1 is a new control to be defined in the next step and $\beta > 0$.

Step 2) We add an integration to the subsystem as

$$\begin{aligned} \dot{z}_2 &= \sin(x_1) \\ \dot{x}_1 &= x_3 \\ \dot{x}_3 &= -\frac{1}{\alpha}\sin(x_1) - \beta x_3 + v_1 \end{aligned} \tag{11}$$

and, following [7], look for a function $M_1(x_1,x_3)$ such that, with $v_1 = 0$, we have $\dot{M}_1 = \sin(x_1)$. This is tantamount to solving the PDE (2) which, in this case, takes the form

$$\frac{\partial M_1}{\partial x_1}x_3 - \frac{\partial M_1}{\partial x_3}\left[\frac{1}{\alpha}\sin(x_1) + \beta x_3\right] = \sin(x_1). \tag{12}$$

A solution is given as $M_1 = -\alpha(x_3 + \beta x_1)$. The second controller is defined by the formula (5) and takes the form

$$v_1 = \underbrace{-\alpha x_3}_{-L_gV_1} + \underbrace{[z_2 + \alpha(x_3+\beta x_1)]}_{z_2 - M_1}\underbrace{(-\alpha)}_{L_gM_1} + v_2.$$

Step 3) We add the last integration to the subsystem as

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= \sin(x_1) \\ \dot{x}_1 &= x_3 \\ \dot{x}_3 &= -\frac{1}{\alpha}\sin(x_1) - \left(\beta+\alpha+\alpha^2\right)x_3 - \alpha z_2 - \beta\alpha^2 x_1 + v_2. \end{aligned}$$

Now, we look for a function $M_2(z_2,x_1,x_3)$ such that, with $v_2 = 0$, we have $\dot{M}_2 = z_2$. Unfortunately, the associated PDE *does not* have a global solution. Indeed, it is clear that a *necessary* condition for the existence of a global solution of the PDE (2) is that the function $h(x)$ is equal to zero at the equilibria of the subsystem $\dot{x} = f(x)$. That is, $f(\bar{x}) = 0 \Rightarrow h(\bar{x}) = 0$. In our case, the equilibria of the "f subsystem" are given by $(\bar{z}_2, \bar{x}_1, \bar{x}_3) = (\alpha\beta j\pi, j\pi, 0)$, with $j \in \mathbb{Z}$, hence the "h function", which is equal to z_2, is nonzero at some of the equilibria and we cannot complete our design.

We should underscore that the procedure was stymied by the presence of the term $\beta\alpha^2 x_1$ in the "f subsystem". We will show below that this term can be removed with the new forwarding technique.

C. Forwarding $\mathrm{mod}\{L_gV\}$

Step 1) Is the same as above.

Step 2) Proceeding from the subsystem (11) we look now for a solution of the new PDE (6), which in this case takes the form

$$\frac{\partial M_1}{\partial x_1}x_3 - \frac{\partial M_1}{\partial x_3}\left[\frac{1}{\alpha}\sin(x_1) + \beta x_3\right] = \sin(x_1) + k(x_1,x_3)\underbrace{\alpha x_3}_{L_gV_1}.$$

Compare with (12). To remove the term dependent on x_1 we propose a solution $M_1 = -\alpha x_3$ and $k = \beta$.

[4]The qualifier "almost" is needed because there is a set of initial conditions which do not converge to the upright position, but it has zero Lebesgue measure. This stems from the well known fact that if a system with cylindrical configuration space has an asymptotically stable equilibrium, then it has at least another equilibrium which is unstable [11].

[5]In Section III-E, where we present simulation results, we give the expression of the control law for the general case.

Noting that $kL_gM_1 = -\beta\alpha < 0$ we choose the control law (see (8))

$$v_1 = -\frac{1}{\alpha}[z_2 + (1-\alpha\beta+\alpha)x_3] + v_2.$$

Remark that we have succeeded in eliminating the troublesome term.

The Lyapunov function corresponding to this step is

$$V_2(x_1, x_3, z_2) = (1-\cos(x_1)) + \frac{\alpha}{2}x_3^2 + \frac{1}{2}(z_2+\alpha x_3)^2. \tag{13}$$

It is positive definite and proper on $\mathbb{S}^1 \times \mathbb{R}^2$ and, with the control v_1 above with $v_2 = 0$, we get

$$\dot{V}_2 = -[z_2 + (1+\alpha)x_3]^2.$$

This derivative is not negative definite and therefore does not satisfy the assumption of Proposition 1. Nevertheless, we go on with the design. We will check only at the end that we do get global asymptotic stabilization.

Step 3) The last step is classical forwarding similar to Step 3 above, with the fundamental difference that the PDE is now

$$\begin{aligned}\frac{\partial M_2}{\partial z_2}\sin(x_1) &+ \frac{\partial M_2}{\partial x_1}x_3 \\ &- \frac{\partial M_2}{\partial x_3}\frac{1}{\alpha}[\sin(x_1) + z_2 + (1+\alpha)x_3] = z_2\end{aligned}$$

for which we have a solution $M_2 = -z_2 - \alpha x_3 - (1+\alpha)x_1$. We then compute the final outer-loop control

$$v_2 = -\alpha[z_1 + 2z_2 + (1+\alpha)x_1 + (1+2\alpha)x_3].$$

We have the following.

Proposition 2: The system (10) in closed-loop with the static state feedback control

$$\begin{aligned}u = &-\frac{1}{\alpha}\sin(x_1) - \alpha z_1 - \left(\frac{1}{\alpha}+2\alpha\right)z_2 - \alpha(1+\alpha)x_1 \\ &- \left(\frac{1}{\alpha}+\alpha+1+2\alpha^2\right)x_3\end{aligned} \tag{14}$$

with $\alpha > 0$, has an asymptotically stable equilibrium at zero with Lyapunov function

$$\begin{aligned}W(x_1, x_3, z_1, z_2) = (1-\cos(x_1)) &+ \frac{\alpha}{2}x_3^2 + \frac{1}{2}(z_2+\alpha x_3)^2 \\ &+ \frac{1}{2}(z_1 + z_2 + \alpha x_3 + (1+\alpha)x_1)^2.\end{aligned} \tag{15}$$

Its domain of attraction is the whole space minus a set of Lebesgue measure zero.

Remark 3: It is interesting to note that the controller derived in [8] using passivity considerations is of the form of the controller above, that is

$$u = -c_1\sin(x_1) - c_2z_1 - c_3z_2 - c_4x_1 - c_5x_3 \tag{16}$$

with c_i, $i = 1\ldots,5$ some suitably defined positive constants. Also, the Lyapunov function in that paper is the sum of a quadratic function of the state and the potential energy term exactly as W is in (15).

Remark 4: The commissioning of nonlinear controllers on actual physical devices is far from obvious. Hence, it is interesting to know what are the available degrees of freedom in the tuning parameters. In Proposition 2 above we have restricted, for the sake of clarity of the presentation, to the single parameter α. A natural question is then what is the largest range of the constants c_i, $i = 1\ldots,5$ in (16), so as to globally stabilize the pendulum *with* a Lyapunov function consisting of the sum of a quadratic function of the state and the potential energy term as W in (15). The answer is provided by the following choices:

$$\begin{aligned}c_1 &= \frac{c}{b} \quad c_2 = \frac{ae}{d}, \quad c_3 = \left[\left(d+\frac{b}{c}\right)a + \frac{ce}{bd^2}\right] \\ c_4 &= \left[\left(d+\frac{b}{c}\right)\left(c+\frac{ab}{c}\right) + \frac{e}{d^2}\right] \quad c_5 = \frac{e}{d}\left(c+\frac{ab}{c}\right)\end{aligned}$$

where $a, b, c, d, e > 0$. One final remark is that it is possible to prove that there do not exist gains c_i, $i = 1\ldots,5$, such that W in (15) is a *strict* Lyapunov function.

Proof of Proposition 2: An important preliminary remark is that the introduction of a non periodic function of x_1 in the control (14) forces us to consider the closed-loop system not on $\mathbb{S}^1 \times \mathbb{R}^3$ but on $\mathbb{R}^4$. And then W is not a proper function for the coordinates (x_1, x_3, z_1, z_2). Nevertheless, with the new coordinate

$$y_1 = z_1 + z_2 + \alpha x_3 + (1+\alpha)x_1$$

the system (10) with the control (14) rewrites

$$\begin{aligned}\dot{y}_1 &= -\alpha^2 y_1 - \alpha^2[z_2 + (1+\alpha)x_3] \\ \dot{z}_2 &= \sin(x_1) \\ \dot{x}_1 &= x_3 \\ \dot{x}_3 &= -\frac{1}{\alpha}\sin(x_1) - \alpha y_1 - \left(\frac{1}{\alpha}+\alpha\right)z_2 \\ &\quad - \left(\frac{1}{\alpha}+1+\alpha+\alpha^2\right)x_3\end{aligned} \tag{17}$$

which does live in $\mathbb{S}^1 \times \mathbb{R}^3$. The function W becomes

$$\begin{aligned}\overline{W}(x_1, x_3, y_1, z_2) &= (1-\cos(x_1)) \\ &+ \frac{\alpha}{2}x_3^2 + \frac{1}{2}(z_2+\alpha x_3)^2 + \frac{1}{2}(y_1)^2\end{aligned}$$

which is positive–definite and proper on $\mathbb{S}^1 \times \mathbb{R}^3$. Then, since we get

$$\dot{\overline{W}} = -\alpha^2\left[y_1 + [z_2 + (1+\alpha)x_3]\right]^2 - [z_2 + (1+\alpha)x_3]^2$$

the solutions of (17) are bounded on $\mathbb{S}^1 \times \mathbb{R}^4$. Also, by invoking LaSalle's invariance principle it is easy to check that all the solutions converge to

$$(x_3, y_1, z_2) = 0 \quad \cos(x_1) = \pm 1.$$

Now the linearization at the equilibrium corresponding to $\cos(x_1) = 1$ has its four eigenvalues with strictly negative real part.[6] Whereas the one at the equilibrium corresponding to $\cos(x_1) = -1$ has eigenvalues with strictly positive real part but also at least one eigenvalue with strictly negative real part.[7] We conclude with [5] that $(x_3, y_1, z_2) = 0$, $\cos(x_1) = 1$ is attractive with domain of attraction in $\mathbb{S}^1 \times \mathbb{R}^3$ the whole set minus a set of zero Lebesgue measure. Also, for the converging solutions, x_3 converges exponentially and therefore its integral x_1 is bounded. Hence the solutions of (17) are also bounded on $\mathbb{R}^4$. The statement of the Proposition follows in particular with the fact that the countable union of sets of zero Lebesgue measure is a set of zero Lebesgue measure.

D. Saturated Control

We will consider in this subsection the practically important case when the control signal must satisfy a bound $|u| \le u_M$. To obtain a saturated control we take-off from the second step above and evaluate the derivative of V_2 (13) (along the trajectories of (10)) as

$$\dot{V}_2 = [z_2 + (1+\alpha)x_3][\alpha u + \sin(x_1)]$$

[6] This follows from the linear counterpart of the above arguments.

[7] This follows from the fact that the matrix of the Jacobian linearization of (17) has a negative trace and the determinant has the sign of $\cos(x_1)$.

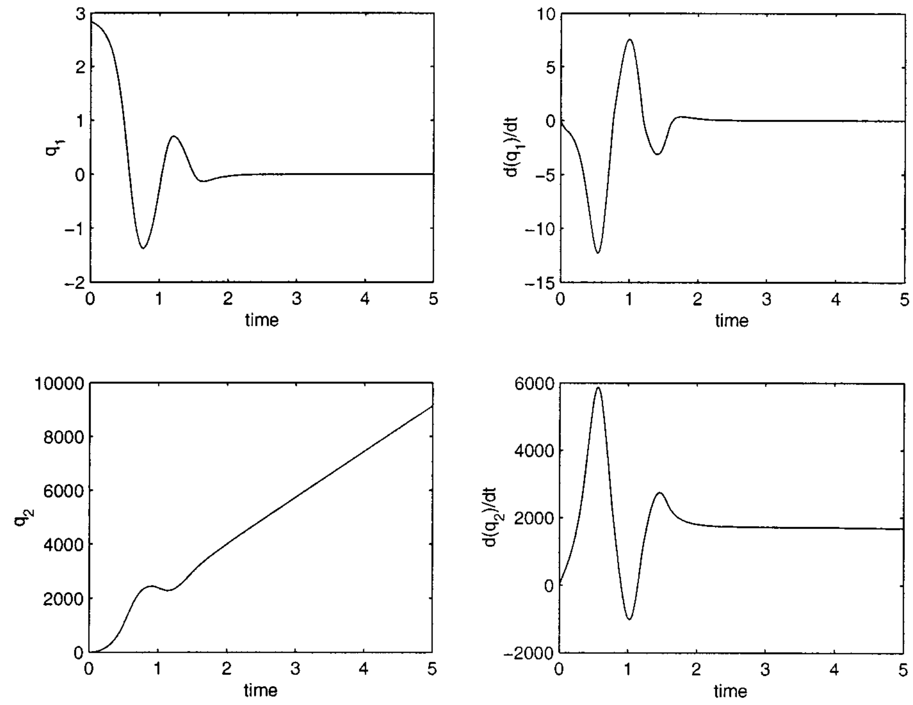

Fig. 2. Swing up response of the pendulum.

Let us define

$$y_1 \triangleq z_1 - M_2 = z_1 + z_2 + \alpha x_3 + (1+\alpha)x_1 \tag{18}$$

whose derivative yields

$$\dot{y}_1 = [z_2 + (1+\alpha)x_3] + [\alpha u + \sin(x_1)]$$

Instead of proceeding with the third forwarding step, we propose a Lyapunov function candidate which is suggested by the computations above

$$V_3(x_1, x_3, z_1, z_2) = V_2(x_1, x_3, z_2) + \int_0^{y_1} a(s)ds$$

where a is any continuous bounded odd function. Indeed, this yields

$$\begin{aligned}\dot{V}_3 &= a(y_1)\,[(1+\alpha)x_3 + z_2] \\ &\quad + [(1+\alpha)x_3 + z_2 + a(y_1)]\,[\sin(x_1) + \alpha u]\end{aligned}$$

which leads to the control

$$u = -\frac{1}{\alpha}\sin(x_1) - b(y_0) \tag{19}$$

where we have defined

$$y_0 \triangleq (1+\alpha)x_3 + z_2 + a(y_1) \tag{20}$$

where b is, again, any continuous bounded odd function. Notice that the control is bounded by any prescribed bound u_M by choosing appropriately $\alpha > 1/u_M$, and the function b to be bounded in norm by $u_M - 1/\alpha$.

Working with the coordinates (x_1, x_3, y_1, z_2) as in the proof of Proposition 2, we see that V_3 is positive definite and proper on $\mathbb{S}^1 \times \mathbb{R}^3$. Its derivative can be expressed in the form

$$\dot{V}_3 = -a(y_1)^2 + y_0 a(y_1) - \alpha y_0 b(y_0)$$

which is a quadratic form in a. Because of the negative sign in front of the maximum order term, we can ensure negativity of $\dot{V}_3$, by imposing some—not very restrictive—constraints on the functions b and a. The asymptotic stability can then be established, by invoking invariant set arguments. We omit these calculations here, for lack of space.

Proposition 3: Consider the system (10) with bounded input $|u| \leq u_M$ in closed-loop with the *saturated* static state feedback control (18), (19) and (20), where a, b are odd bounded functions. Fix $\alpha > 0$ such that $(1/u_M) \geq \alpha$, and choose a, b such that $|a| \leq a_M$ and $|b| \leq u_M - (1/\alpha)$. Further, select the function b such that

$$\begin{aligned}\frac{a_M}{\alpha}\left(1 - \frac{a_M}{|y_0|}\right) &\leq |b(y_0)| \text{ if } 2a_M \leq |y_0| \\ \frac{|y_0|}{4\alpha} &< |b(y_0)| \text{ if } |y_0| < 2a_M\end{aligned}$$

Under these conditions, zero is an asymptotically stable equilibrium of the closed loop, with domain of attraction the whole space minus a set of Lebesgue measure zero.

E. Simulations

Repeating the derivations leading to Proposition 3 for the general model (9) we obtain the controller

$$\begin{aligned}u &= -\frac{1}{\alpha m_2}\sin\left(\frac{x_1}{m_2}\right) - b(y_0) \\ y_0 &\triangleq \frac{1}{\alpha^2 \gamma m_2 mgl} a(y_1) + m_2 mgl\, z_2 + \left(1 + \alpha m_2^2 (mgl)^2\right) x_3 \\ y_1 &\triangleq z_1 + \frac{1}{\alpha^2 \gamma m_2^2 (mgl)^2} z_2 + \frac{1 + \alpha m_2^2 (mgl)^2}{m_2 mgl} x_1 \\ &\quad + \frac{1}{\alpha \gamma m_2 mgl} x_3\end{aligned}$$

where, to provide more tuning flexibility, we have included some additional gains that were set to one in the procedure described above. Note also that the change of coordinates is now $x_1 = m_2 q_1$, $x_3 = m_2 \dot{q}_1$, $z_1 = (m_1 + m_2)q_1 + m_1 q_2$, $z_2 = (m_1 + m_2)\dot{q}_1 + m_1 \dot{q}_2$ and $v = mgl \sin(x_1/m_2) - u$. We simulated the response of the pendulum using the system parameters of an existing hardware setup , namely: $m_1 = 32 \times 10^{-6}$, $m_2 = 0.0048$, and $ml = 38.7 \times 10^{-3}$ [kg]. The controller gains were set at $\alpha = 700$, $\gamma = 10$. Fig. 2 shows the swing up response of the pendulum starting at nearly the vertically downward position, with the remaining initial conditions zero. Notice that the response is very fast without any initial swinging of the pendulum.

The following remarks are in order.

- It is clear from the simulations that the stabilization mechanism of our controller consists of spinning-up the disk inertia to lift the pendulum, which might impose some unrealistic values to the disk speed. This should be contrasted with the alternative method of [9]—also studied in [1], [3]—where the energy is first pumped-up through a balancing motion before lifting the pendulum. Two drawbacks of the latter approach are the slow convergence and the need to switch the controller close to the upward position. From the theoretical viewpoint both methods also differ, our controller (as well as the one reported in [8]) stabilizes the *equilibrium point*, while the energy-pumping methods stabilizes the homoclinic orbit, hence the need for the switching.
- Although we have solved the stabilization problem of the system (10) with any prescribed saturation of the control, when we come back to the original disk inertia pendulum (9), we have to add $\sin(x_1)$ to the above control. So the above procedure does not give an answer to the problem where the maximal torque that the motor can deliver is smaller than the maximal gravity torque. Simulations and experiments have shown that stability cannot be guaranteed if we impose this saturation limit.

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Per-Queue Stability Analysis of a Random Access System

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***Abstract*—In this note, we have extended previous studies of the system stability of buffered ALOHA systems to study an individual queue's stability, i.e., per-queue stability. The main result obtained in this work is a necessary and sufficient per-queue stability condition, which can be computed analytically only for several cases. For other noncomputable cases, we have evaluated several inner and outer bounds. They are generally quite tight for not-so-asymmetric systems.**

***Index Terms*—ALOHA, multiaccess systems, per-queue stability, queue stability ordering, system stability.**

I. Introduction

Stability analyses of single-resource–multiple-queue systems, such as random access protocols, polling schemes, and token-passing rings, have been studied quite extensively in the past. By stability, we mean that the queue length process of a queue with unlimited buffer space possesses a limiting distribution. Almost all previous studies in this area, however, concern stability of the whole system (system stability). Study of an individual queue's stability (per-queue stability), on the other hand, has hardly received any attention. The per-queue stability problem is more general than the system stability problem, because some queues may remain stable in an unstable system. Therefore, system stability, being a special case of per-queue stability, is inadequate to address the entire stability region of an individual queue. In this note, we consider per-queue stability of a buffered ALOHA system. Our goal is to obtain a necessary and sufficient per-queue stability condition as well as other related results.

So far, only system stability has been studied for the buffered ALOHA system. Computable system stability conditions are well known for two-queue systems and symmetric systems (e.g., see [1], [2]). Szpankowski employed Loynes' theorem and an induction approach to obtain necessary and sufficient system stability conditions for more than two queues, but the conditions are generally noncomputable [2]. Rao and Ephremides, on the other hand, obtained lower bounds for the system stability region using a simple concept of dominance [1]. Luo and Ephremides revisited the same problem and obtained a tighter bound [3]. Their main approach was based on an *instability rank* that helped construct appropriate dominant systems to obtain sufficient conditions. An instability rank, or a stability order, specifies the sequence of queues to become unstable when the system traffic increases according to a certain pattern.

Unlike previous work, our focus in this note is on per-queue stability. Besides describing the system, we present, in Section II, two preliminary results that are essential to obtaining a stability condition, say, for a target queue q_t. We first obtain in Lemma 1 q_t's necessary and sufficient stability condition for a path with a *known* stability order. We then use this result to obtain a criterion for comparing stabilities of any two queues in the system, and the criterion is essentially the same as the one obtained recently by Luo and Ephremides [3]. By combining

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