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automatica

Automatica 37 (2001) 1609–1617

www.elsevier.com/locate/automatica

Brief Paper

Sufficient conditions for the existence of an unbounded solution☆

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Received 23 March 1999; revised 5 April 2000; received in final form 18 January 2001

Abstract

Readily verifiable conditions under which a dynamical system of the form $\dot{x} = f(x)$ possesses an unbounded solution are presented. The results are illustrated by showing they can be used to infer results about lack of global stabilizability for nonlinear control systems. The key observation in the paper is that behaviour at infinity can be studied using local methods applied to an auxiliary system. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Unbounded solutions; Global stabilizability

1. Introduction

Let $\mathbb{R}$ denote the real numbers. Given a dynamical system of the form

$$\dot{x} = f(x) \tag{1}$$

defined on $\mathbb{R}^n$, it is often of interest to know if all possible solutions of the system are bounded or if the system possesses an unbounded solution. Determining this for systems without explicit solutions can be a highly nontrivial task. Presented in this paper are sufficient conditions for a system of the form (1) to possess an unbounded solution.

When considering whether a system possesses an unbounded solution, one is asking how the system behaves arbitrarily far away from the origin, that is, how it behaves near "infinity". The key observation in the paper is that behaviour at infinity can be studied using local methods. It will be shown that the existence of appropriate auxiliary functions and variables allows us to construct a new system of dimension one greater than the original from which the existence of an unbounded solution of the original system can be inferred using local methods.

As mentioned above, the result given in the paper relies on finding appropriate auxiliary functions and variables. In this manner it is similar to Lyapunov's stability theorem which requires one to find an appropriate auxiliary function, namely a Lyapunov function, in order to give a positive result about system stability.

The paper is structured as follows. The main ideas of the paper are further introduced in Section 2. The material in Section 2 is quite concrete and is presented to motivate the slightly more abstract material in the remainder of the paper. The main result of the paper is presented in Section 3. Section 4 contains some examples. Included are examples that show how the results of the paper can be used to infer results about lack of global stabilizability for nonlinear control systems. A partial converse theorem to the main result is presented in Section 5. Section 6 contains some additional comments and the paper ends with some concluding remarks in Section 7. (Two technical lemmas have been placed at the end of the paper in an appendix.)

2. Main ideas

Presented in this section, in a rather nonrigorous manner, are sufficient conditions for a system to possess an unbounded solution. The material presented here is

☆This paper is an extended version of the NOLCOS '98 paper "Sufficient conditions for a dynamical system to possess an unbounded solution". This paper was recommended for publication in revised form by Associate Editor Eugene P. Ryan under the direction of Editor Roberto Tempo.

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PII: S0005-1098(01)00114-5

intended only as an introduction to the main ideas of the paper and rigorous proofs of results based on these ideas are given in the main body of the paper.

Consider an arbitrary system of the form (1) defined on $\mathbb{R}^n$. Let $\lambda \in \mathbb{R}$ be a fixed positive number and let $z \in \mathbb{R}$ be a new variable governed by the dynamics

$$\dot{z} = -\lambda z. \tag{2}$$

Further let $\alpha_1, \ldots, \alpha_n$ be certain real numbers and let us introduce another new variable $y \in \mathbb{R}^n$ and define it via the identity

$$(y_1, \ldots, y_n) = (x_1 z^{\alpha_1}, \ldots, x_n z^{\alpha_n}). \tag{3}$$

Eq. (3) determines the dynamics of y in terms of the dynamics of z and x and differentiating (3) with respect to t and substituting (1)–(3) we arrive at a new system of the form

$$\begin{aligned} \dot{z} &= -\lambda z, \\ \dot{y} &= H(z, y). \end{aligned} \tag{4}$$

The function H may or may not be well defined as a function. Let us suppose that it is and furthermore that it is continuously differentiable. In addition, suppose that $(z, y) = (0, \bar{y})$ is an equilibrium point of (4) and that for some $j \in \{1, \ldots, n\}$, $\alpha_j > 0$ and $\bar{y}_j \neq 0$.

If all the conditions above are met, our original system (1) has an unbounded solution. Indeed the linearization of (4) at $(0, \bar{y})$ has a negative eigenvalue (as $\lambda > 0$) and it can be shown that this implies that (4) has a solution $(z(\cdot), y(\cdot))$, $z(0) > 0$, that converges to $(0, \bar{y})$. This solution in turn defines a solution $x(\cdot)$ of (1) via (3). As $x_j(t) = y_j(t)/z(t)^{\alpha_j}$ and as $z(t) \to 0$, $y_j(t) \to \bar{y}_j \neq 0$ and $\alpha_j > 0$, it follows that $|x_j(t)| \to \infty$ and hence that (1) has an unbounded solution.

3. Main result

This section contains the main result of the paper giving sufficient conditions for a system to possess an unbounded solution. The material in this section is directly motivated by the ideas in Section 2 and the results of that section are presented in rigorous form in Corollary 3.5.

Let $\mathbb{Z}$ denote the integers, $\mathbb{R}_{>0}$ the set $\{z \in \mathbb{R} \mid z > 0\}$ and C^1 the class of continuously differentiable functions. If $\varphi(z, x)$ is a map from $\mathbb{R} \times \mathbb{R}^n$ to $\mathbb{R}^n$ which is differentiable at $(z, x) = (a, b)$, let $D_1 \varphi(a, b)$ denote the partial derivative of the function φ with respect to its first argument z evaluated at the point (a, b). Similarly, let $D_2 \varphi(a, b)$ denote the partial derivative of the function φ with respect to its second argument x evaluated at the point (a, b).

Central to the material in this section and showing that a system has an unbounded solution is the idea of a "stability preserving extension" which we now define.

Definition 3.1. A *stability preserving extension* is defined to be a pair $(\phi, \bar{y})$ consisting of a C^1 function $\phi : \mathbb{R}_{>0} \times \mathbb{R}^n \to \mathbb{R}^n$ and a point $\bar{y} \in \mathbb{R}^n$ that together satisfy the following properties:

(P1) $\phi(z_1, \phi(z_2, x)) = \phi(z_1 z_2, x)$ for all $z_1, z_2 \in \mathbb{R}_{>0}$ and $x \in \mathbb{R}^n$,
(P2) $\phi(1, x) = x$ for all $x \in \mathbb{R}^n$,
(P3) $\psi : \mathbb{R}^n \to \mathbb{R}^n$, $y \mapsto D_1 \phi(1, y)$ is C^1 in a neighbourhood of $\bar{y}$,
(P4) $\lim_{(z,y) \to (0^+, \bar{y})} |\phi(1/z, y)| = \infty$.

An example of a stability preserving extension $(\phi, \bar{y})$ is any pair consisting of a function of the form $\phi(z, x) = (z^{\alpha_1} x_1, \ldots, z^{\alpha_n} x_n)$ ($\alpha_i \in \mathbb{R}$, $i = 1, \ldots, n$) and any point $\bar{y} \in \mathbb{R}^n$ such that for at least one $j \in \{1, \ldots, n\}$, $\bar{y}_j \neq 0$ and $\alpha_j > 0$. Indeed this is exactly the type of transformation that was used in Section 2. It will be considered further in Corollary 3.5.

Stability preserving extensions are "extensions" in the sense that if a system has an unbounded solution and an appropriate stability preserving extension exists, the extension can be used to form the basis for transforming the given system into a certain new system of dimension one greater than the original. (Local methods applied to this new system then infer the existence of an unbounded solution of the original system. The details of this result are given in Theorem 3.2 below.) We have used the term "stability preserving" to indicate that stability preserving extensions do not introduce any unbounded behaviour that was not present in the original system.

Theorem 3.2 below gives sufficient conditions for a given system (5) to possess an unbounded solution. Its proof starts by showing that the requirements of the theorem guarantee that the new system (6) can be formed. Furthermore, the theorem conditions also guarantee that $(z, y) = (0, \bar{y})$ is an equilibrium point of (6) and that the linearization of (6) at this point has a negative eigenvalue. The proof then shows that this implies that there exists a solution of (6) that converges to $(0, \bar{y})$ and finally that this via property (P4) implies that the original system (5) has an unbounded solution.

Theorem 3.2. *Given a function $f : \mathbb{R}^n \to \mathbb{R}^n$ and an arbitrary continuous function $F : \mathbb{R}^n \to \mathbb{R}_{>0}$, define $g = f/F$. If there exists $\lambda \in \mathbb{R}_{>0}$ and a stability preserving extension $(\phi, \bar{y})$ such that*

1. *there exists a function $h : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ which is C^1 in a neighbourhood N of $(0, \bar{y})$ and which equals*

$$\left(D_2 \phi\left(\frac{1}{z}, y\right) \right)^{-1} g\left(\phi\left(\frac{1}{z}, y\right) \right)$$

for all $(z, y) \in N \cap (\mathbb{R}_{>0} \times \mathbb{R}^n)$,

2. $h(0,\bar{y}) - \lambda D_1\phi(1,\bar{y}) = 0$, *then the system*

$$\dot{x} = f(x) \tag{5}$$

has an unbounded solution.

Proof. Consider the system

$$\begin{aligned} \dot{z} &= -\lambda z, \\ \dot{y} &= h(z,y) - \lambda D_1\phi(1,y). \end{aligned} \tag{6}$$

By assumption, h is C^1 in a neighbourhood of $(0,\bar{y})$. In addition, from property (P3), $\psi(y) = D_1\phi(1,y)$ is C^1 in a neighbourhood of $\bar{y}$ and it follows that the vector field of (6) is C^1 in a neighbourhood of $(0,\bar{y})$.

Condition (2) of the theorem statement indicates that the point $(z,y) = (0,\bar{y})$ is an equilibrium point of system (6). Linearization of (6) at this point gives

$$\begin{pmatrix} \dot{\tilde{z}} \\ \dot{\tilde{y}} \end{pmatrix} = A \begin{pmatrix} \tilde{z} \\ \tilde{y} \end{pmatrix},$$

where A has the form

$$A = \begin{pmatrix} -\lambda & 0 \\ A_{21} & A_{22} \end{pmatrix}.$$

The real number $-\lambda$ is an eigenvalue of A and Lemma A.1 implies that associated with this eigenvalue there exists, a real (generalized) eigenvector v whose z component is nonzero. The Center Manifold Theorem (Guckenheimer & Holmes, 1997) now implies that there exists a stable invariant manifold (of dimension at least 1) passing through the equilibrium point $(0,\bar{y})$ and furthermore that this manifold is tangential to the vector v at the point $(0,\bar{y})$. (The manifold is stable and invariant in the sense that all solutions starting on the manifold remain on the manifold and converge to $(0,\bar{y})$.) This implies the existence of a solution of (6) passing through a point (z^0, y^0), $z^0 > 0$, and converging to $(0,\bar{y})$.

Let $(z(\cdot), y(\cdot))$ denotes the solution of (6) with initial point (z^0, y^0) and let $x:[0,\infty) \to \mathbb{R}^n$ be the function defined by

$$x(t) = \phi\left(\frac{1}{z(t)}, y(t)\right). \tag{7}$$

Note that $z(t) > 0$ for all $t \in [0,\infty)$. Differentiating (7) with respect to t gives

$$\dot{x} = D_1\phi\left(\frac{1}{z}, y\right)\left(-\frac{\dot{z}}{z^2}\right) + D_2\phi\left(\frac{1}{z}, y\right)\dot{y}. \tag{8}$$

Substituting (6) into (8) and then using condition (1) of the theorem statement and simplifying gives

$$\begin{aligned} \dot{x} &= \frac{\lambda}{z} D_1\phi\left(\frac{1}{z}, y\right) + g\left(\phi\left(\frac{1}{z}, y\right)\right) \\ &\quad - \lambda D_2\phi\left(\frac{1}{z}, y\right) D_1\phi(1,y). \end{aligned}$$

It now follows from (A.1) and (7) that $x(\cdot)$ is a solution of

$$\dot{x} = g(x). \tag{9}$$

As $(z(t), y(t)) \to (0^+, \bar{y})$, property (P4) implies that $x(\cdot)$ is an unbounded solution of (9).

Define $\tau(t) = \int_0^t F(x(s))^{-1}\,\mathrm{d}s$. As $F(x(s))$ depends continuously on s and is strictly greater than zero for all $s \in [0,\infty)$, it follows that $\mathrm{d}\tau(t)/\mathrm{d}t = F(x(t))^{-1}$. As τ is a strictly monotonically increasing function of t, it follows that t can be considered as a function of τ and that $\mathrm{d}t(\tau)/\mathrm{d}\tau = F(x(t(\tau)))$. Now τ converges to some value $\bar{\tau} \in (0,\infty]$ as $t \to \infty$. Let $\xi:[0,\bar{\tau}) \to \mathbb{R}^n$ be the function defined by

$$\xi(\tau) = x(t(\tau)). \tag{10}$$

Then

$$\begin{aligned} \frac{\mathrm{d}\xi(\tau)}{\mathrm{d}\tau} &= \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} \\ &= \frac{f(x(t(\tau)))}{F(x(t(\tau)))} F(x(t(\tau))) \\ &= f(\xi(\tau)). \end{aligned}$$

From the relationship between τ and t, t as a function of τ converges to infinity as $\tau \to \bar{\tau}$ and it follows that (10) is an unbounded solution of (5). □

Remark 3.3. A priori no continuity/smoothness conditions are imposed on the function f itself. Such conditions only appear indirectly through the auxiliary system (6) and the requirement that h be continuously differentiable in an appropriate neighbourhood N. In this way the unboundedness established in the theorem is also an existence result for the solutions in the first place. (Note however that f is required to be continuous if (5) is to have a classical solution through each point in $\mathbb{R}^n$.)

Remark 3.4. Note that the net result of dividing the vector field f by the positive continuous function F is just a nonlinear scaling of time. The orientation of the vector field and the trajectories of the system remain unchanged.

As to the usefulness of actually having the freedom to choose such an F, this will become apparent when we consider some examples. We now give a corollary to Theorem 3.2 that presents the ideas of Section 2 in rigorous form. The result presented below in Corollary 3.5 is more general than the result of Section 2 in that, like Theorem 3.2, it allows normalization of the vector field.

Corollary 3.5. *Given a function* $f:\mathbb{R}^n \to \mathbb{R}^n$ *and an arbitrary continuous function* $F:\mathbb{R}^n \to \mathbb{R}_{>0}$, *define* $g = f/F$ *and* *let* $g_i: \mathbb{R}^n \to \mathbb{R}$, $i = 1, \dots, n$, *denote the components of* g. *If there exists* $\lambda \in \mathbb{R}_{>0}$, $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ *and* $(\bar{y}_1, \dots, \bar{y}_n) \in \mathbb{R}^n$ *such that*

1. *for each* $i = 1, \dots, n$, *there exists a function* $h_i(z, y)$ *from* $\mathbb{R} \times \mathbb{R}^n$ *into* $\mathbb{R}^n$ *which is* C^1 *in a neighbourhood* N *of* $(0, \bar{y}_1, \dots, \bar{y}_n)$ *and which equals*

$$\tilde{h}_i := z^{\alpha_i} g_i\left(\frac{y_1}{z^{\alpha_1}}, \dots, \frac{y_n}{z^{\alpha_n}}\right) \tag{11}$$

for all $(z, y_1, \dots, y_n) \in N \cap (\mathbb{R}_{>0} \times \mathbb{R}^n)$,

2. $h_i(0, \bar{y}_1, \dots, \bar{y}_n) - \lambda \alpha_i \bar{y}_i = 0$, *for all* $i = 1, \dots, n$,
3. $\bar{y}_j \neq 0$, $\alpha_j > 0$, *for some* $j \in \{1, \dots, n\}$,

then the system $\dot{x} = f(x)$ *has an unbounded solution.*

Proof. The result follows in a straightforward manner from Theorem 3.2 by choosing the stability preserving extension consisting of $\phi(z, x) = (z^{\alpha_1} x_1, \dots, z^{\alpha_n} x_n)$ and $\bar{y} = (\bar{y}_1, \dots, \bar{y}_n)$. □

4. Examples

In this section the results of the previous section are illustrated with some examples. In particular, some of the examples in this section show how the results of Section 3 can be used to infer results about lack of global stabilizability for nonlinear control systems.

Example 4.1. For arbitrary (but fixed) real numbers a, b, c and d, with c and d nonzero, the system

$$\begin{aligned} \dot{x}_1 &= a x_2 + x_1 x_3 + b x_3^2, \\ \dot{x}_2 &= c x_1, \\ \dot{x}_3 &= d x_2 x_3 \end{aligned} \tag{12}$$

has an unbounded solution. This can be verified by checking that

$$F(x) = \sqrt{1 + x_3^2},\ \lambda = 0.5,\ \alpha_1 = 2,\ \alpha_2 = 1,\ \alpha_3 = 1,$$

$$\bar{y} = (c/d, |c|/d, 2|c|)$$

and

$$\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = \frac{1}{\sqrt{z^2 + y_3^2}} \begin{pmatrix} a y_2 z^2 + y_1 y_3 + b y_3^2 z \\ c y_1 \\ d y_2 y_3 \end{pmatrix} \tag{13}$$

satisfy the requirements of Corollary 3.5. We will now attempt to give some insight into how these functions and variables were found.

Let us first attempt to prove the result using $F = 1$. With $F = 1$, the $\tilde{h}_i$ equations from (11) are

$$\tilde{h}_1 = z^{\alpha_1}\left(a \frac{y_2}{z^{\alpha_2}} + \frac{y_1}{z^{\alpha_1}} \frac{y_3}{z^{\alpha_3}} + b \frac{y_3^2}{z^{2\alpha_3}}\right),$$

$$\tilde{h}_2 = z^{\alpha_2} c \frac{y_1}{z^{\alpha_1}},$$

$$\tilde{h}_3 = z^{\alpha_3} d \frac{y_2}{z^{\alpha_2}} \frac{y_3}{z^{\alpha_3}}$$

and simplifying gives

$$\begin{aligned} \tilde{h}_1 &= a y_2 z^{\alpha_1 - \alpha_2} + y_1 y_3 z^{-\alpha_3} + b y_3^2 z^{\alpha_1 - 2\alpha_3}, \\ \tilde{h}_2 &= c y_1 z^{-\alpha_1 + \alpha_2}, \\ \tilde{h}_3 &= d y_2 y_3 z^{-\alpha_2}. \end{aligned} \tag{14}$$

Each $\tilde{h}_i$ is only defined for $z > 0$ and in order to satisfy condition (1) of Corollary 3.5 we would like to find a choice of α_i's that enables us to extend each $\tilde{h}_i$ to a function h_i which is C^1 in a neighbourhood of $(0, \bar{y}_1, \dots, \bar{y}_n)$ (a point which we are free to choose). Hence, we would like to find α_i's such that for each term in (14) of the form z^β, $\beta \geqslant 0$. This imposes the requirement that $\alpha_1 - \alpha_2 \geqslant 0$, $-\alpha_3 \geqslant 0$, $\alpha_1 - 2\alpha_3 \geqslant 0$, $-\alpha_1 + \alpha_2 \geqslant 0$ and $-\alpha_2 \geqslant 0$. These inequalities imply $\alpha_1 = \alpha_2 \leqslant 0$ and $\alpha_3 \leqslant 0$ and as none of the α_i's can be positive, condition (3) of Corollary 3.5 cannot be satisfied. This implies that using our initial choice for F no appropriate h_i's can be found. In order to try to demonstrate the existence of an unbounded solution of (12) there is no choice but to try a different F.

Let γ be a fixed but as yet unspecified real number. Multiplying each of the $\tilde{h}_i$'s in (14) by z^γ gives

$$\begin{aligned} z^\gamma \tilde{h}_1 &= a y_2 z^{\alpha_1 - \alpha_2 + \gamma} + y_1 y_3 z^{-\alpha_3 + \gamma} + b y_3^2 z^{\alpha_1 - 2\alpha_3 + \gamma}, \\ z^\gamma \tilde{h}_2 &= c y_1 z^{-\alpha_1 + \alpha_2 + \gamma}, \\ z^\gamma \tilde{h}_3 &= d y_2 y_3 z^{-\alpha_2 + \gamma}. \end{aligned} \tag{15}$$

The z exponents on the right hand sides of (15) are all nonnegative if for example $\alpha_1 = 2$, $\alpha_2 = 1$, $\alpha_3 = 1$ and $\gamma = 1$. Hence in order to satisfy condition (1) of Corollary 3.5, it is sufficient to find a F that introduces a factor z^γ ($\gamma = 1$). Leaving $\alpha_1 = 2$, $\alpha_2 = 1$ and $\alpha_3 = 1$, a suitable choice is $F = \sqrt{1 + x_3^2}$ which upon substitution of $x_3 = y_3 / z^{\alpha_3}$ becomes $F = \sqrt{z^2 + y_3^2}/z$. With this new F, the $\tilde{h}_i$'s in (14) become

$$\begin{pmatrix} \tilde{h}_1 \\ \tilde{h}_2 \\ \tilde{h}_3 \end{pmatrix} = \frac{1}{\sqrt{z^2 + y_3^2}} \begin{pmatrix} a y_2 z^2 + y_1 y_3 + b y_3^2 z \\ c y_1 \\ d y_2 y_3 \end{pmatrix}. \tag{16}$$

Note that the right hand side of (16) is the same as that of (13). With this choice of h_i's and α_i's, condition (1) of

Corollary 3.5 is satisfied if $\bar{y}_3$ is nonzero. If this is the case, condition (3) of Corollary 3.5 will also be satisfied as $\alpha_3 > 0$.

It remains to show that condition (2) of Corollary 3.5 can be satisfied. That is, it remains to show that there exists $\bar{y}_1$, $\bar{y}_2$, $\bar{y}_3$ and λ, with $\bar{y}_3 \neq 0$ and $\lambda > 0$, such that

$$\frac{\bar{y}_1 \bar{y}_3}{|\bar{y}_3|} - 2\lambda \bar{y}_1 = 0, \tag{17}$$

$$\frac{c\bar{y}_1}{|\bar{y}_3|} - \lambda \bar{y}_2 = 0, \tag{18}$$

$$\frac{d\bar{y}_2 \bar{y}_3}{|\bar{y}_3|} - \lambda \bar{y}_3 = 0. \tag{19}$$

We leave it to the reader to check that the values for $\bar{y}_1$, $\bar{y}_2$, $\bar{y}_3$ and λ given at the start of this example satisfy the above requirements. (Note that other choices for the $\bar{y}_i$'s also satisfy the requirements.) □

Remark 4.2. While each function $\tilde{h}_i$ may not technically be defined for $z \leqslant 0$, requirement (1) of Corollary 3.5 ensures that each of these functions can be extended to a function h_i which is well defined in a neighbourhood of $(0,\bar{y})$. The particular values taken by the extension for $z \leqslant 0$ are not used in the proof of Theorem 3.2 and hence while it is required that a (continuously differentiable) extension exists, the specifics of the extension are unimportant. In the rest of this paper when using Corollary 3.5, no further distinction will be made between the functions $\tilde{h}_i$ and h_i. For convenience we will simply refer to the function h_i.

The following result in Example 4.3 was originally proved in Sepulchre, Janković, and Kokotović (1997) using quite different methods.

Example 4.3. Consider the control system

$$\dot{x}_1 = -x_1 + x_2 x_3, \dot{x}_2 = -x_2 + x_1^2 x_2, \dot{x}_3 = u, \tag{20}$$

where $(x_1, x_2, x_3) \in \mathbb{R}^3$ denotes the state and u the input. It is now shown that there exists no C^1 partial-state feedback $u = k(x_3)$ that globally stabilizes (20).

Let $u = k(x_3)$ be an arbitrary C^1 function. As in Example 4.1, the proof of this result is first attempted using $F = 1$. Using this choice for F, condition (1) of Corollary 3.5 requires that the functions

$$\begin{aligned} h_1 &= -y_1 + z^{\alpha_1 - \alpha_2 - \alpha_3} y_2 y_3, \\ h_2 &= -y_2 + z^{-2\alpha_1} y_1^2 y_2, \\ h_3 &= z^{\alpha_3} k\left(\frac{y_3}{z^{\alpha_3}}\right) \end{aligned} \tag{21}$$

be C^1 in a neighbourhood of some point $(0, \bar{y}_1, \bar{y}_2, \bar{y}_3)$ (which is still to be determined). A sufficient condition that this be true for h_3 is that $\alpha_3 = 0$. The equations for h_1 and h_2 then imply that $\alpha_1 - \alpha_2 \geqslant 0$ and $-2\alpha_1 \geqslant 0$, that is, $\alpha_2 \leqslant \alpha_1 \leqslant 0$. This however violates condition (3) of Corollary 3.5 that requires at least one of the exponents α_1, α_2 or α_3 to be positive.

To try to overcome this problem we will try a different function F. Leaving $\alpha_3 = 0$, let α_1 be positive. F is now chosen so that h_2 is locally C^1. A suitable choice is $F = 1 + x_1^2$, which upon substitution of $x_1 = y_1/z^{\alpha_1}$ gives $F = (z^{2\alpha_1} + y_1^2)/z^{2\alpha_1}$. With this new F,

$$h_1 = \frac{-z^{2\alpha_1} y_1 + z^{3\alpha_1 - \alpha_2} y_2 y_3}{z^{2\alpha_1} + y_1^2},$$

$$h_2 = \frac{-z^{2\alpha_1} y_2 + y_1^2 y_2}{z^{2\alpha_1} + y_1^2},$$

$$h_3 = \frac{z^{2\alpha_1} k(y_3)}{z^{2\alpha_1} + y_1^2}.$$

Conditions (1) and (3) of Corollary 3.5 are now satisfied if $\alpha_1 > 0 \in \mathbb{Z}$, $3\alpha_1 - \alpha_2 \geqslant 0 \in \mathbb{Z}$ and $\bar{y}_1 \neq 0$.

It must now only be ensured that $\lambda > 0$ and that condition (2) of Corollary 3.5 is satisfied. Noting that $h_3 = 0$ when $z = 0$ and that $\alpha_3 = 0$, the only additional requirements needed to satisfy condition (2) of Corollary 3.5 are that

$$\frac{z^{3\alpha_1 - \alpha_2} \bar{y}_2 \bar{y}_3}{\bar{y}_1^2} - \lambda \alpha_1 \bar{y}_1 = 0$$

and

$$\bar{y}_2 - \lambda \alpha_2 \bar{y}_2 = 0$$

when $z = 0$. In order that $\bar{y}_1 \neq 0$ it is required that $3\alpha_1 = \alpha_2$, $\bar{y}_2 \neq 0$, $\bar{y}_3 \neq 0$ and $\lambda = 1/\alpha_2$. Hence all conditions of Corollary 3.5 are satisfied if for example $\alpha_1 = 1, \alpha_2 = 3, \alpha_3 = 0,$ $\lambda = 1/3$ and $(\bar{y}_1, \bar{y}_2, \bar{y}_3) = (3, 3, 3)$. □

Example 4.4. It is now shown that system (20) cannot be globally stabilized via linear full state feedback. That is, it will be shown that there does not exist a feedback of the form

$$u = b_1 x_1 + b_2 x_2 + b_3 x_3,$$

$(b_1, b_2, b_3) \in \mathbb{R}^3$, which globally stabilizes (20).

Let $F = 1 + x_1^2$. Then it is straightforward to verify that

$$h_1 = \frac{-z^{2\alpha_1} y_1 + z^{3\alpha_1 - \alpha_2 - \alpha_3} y_2 y_3}{z^{2\alpha_1} + y_1^2},$$

$$h_2 = \frac{-z^{2\alpha_1} y_2 + y_1^2 y_2}{z^{2\alpha_1} + y_1^2},$$

$$h_3 = \frac{z^{\alpha_1 + \alpha_3} b_1 y_1 + z^{2\alpha_1 - \alpha_2 + \alpha_3} b_2 y_2 + z^{2\alpha_1} b_3 y_3}{z^{2\alpha_1} + y_1^2}.$$

Three sub-cases are considered.

Case 1: $b_2 \neq 0$. Let $\alpha_1 = 2$, $\alpha_2 = 5$, $\alpha_3 = 1$, $\lambda = 1/5$ and $(\bar{y}_1, \bar{y}_2, \bar{y}_3) = ((25b_2/2)^{\frac{1}{5}}, 1, 2\bar{y}_1^3/5)$. It is left to the reader to verify that all the conditions of Corollary 3.5 are satisfied. Note that $b_2 \neq 0$ ensures $\bar{y}_1 \neq 0$ which is required in order to satisfy condition (1) of Corollary 3.5.

Case 2: $b_1 \neq 0$, $b_2 = 0$. Let $\alpha_1 = 1$, $\alpha_2 = 4$, $\alpha_3 = -1$, $\lambda = 1/4$ and $(\bar{y}_1, \bar{y}_2, \bar{y}_3) = (2|b_1|^{1/4}, -b_1/|b_1|, -4b_1/\bar{y}_1)$. Again it is left to the reader to verify that all the conditions of Corollary 3.5 are satisfied.

Case 3: $b_1 = 0$, $b_2 = 0$. In this case $u = b_3 x_3$ and the result follows from Example 4.3.

This proves the desired result. □

5. A partial converse theorem

In this section a partial converse theorem to Theorem 3.2 is presented.

Theorem 5.1. *Suppose* $f: \mathbb{R}^n \to \mathbb{R}^n$ *is* C^1 *and that the system*

$$\dot{x} = f(x) \tag{22}$$

has an unbounded solution. Then there exists a C^1 *function* $\phi: \mathbb{R}_{>0} \times \mathbb{R}^n \to \mathbb{R}^n$ *and a point* $\bar{y} \in \mathbb{R}^n$ *that together satisfy properties* (P1)–(P3), *and*

(P4′) $\limsup_{z \to 0^+} |\phi(1/z, \bar{y})| = \infty$.

Furthermore, there exists a continuous function $F: \mathbb{R}^n \to \mathbb{R}_{>0}$ *and a scalar* $\lambda \in \mathbb{R}_{>0}$ *that together with* ϕ *and* $\bar{y}$ *satisfy conditions* (1) *and* (2) *of Theorem* 3.2.

Proof. Define $F = 1 + |f|^2$ and $g = f/F$. For each $x \in \mathbb{R}^n$, let $\xi(\cdot, x)$ denote the solution of

$$\dot{w} = g(w), \quad w(0) = x. \tag{23}$$

As g is C^1 and $|g(w)| < 1$ for all $w \in \mathbb{R}^n$, it follows that $\xi(t, x)$ is uniquely defined for all $(t, x) \in \mathbb{R} \times \mathbb{R}^n$ and furthermore that $\xi(t, x)$ is C^1 in (t, x) (Hale, 1980). Define

$$\phi(z, x) = \xi(\log(z), x). \tag{24}$$

Note that ϕ is a C^1 function on $\mathbb{R}_{>0} \times \mathbb{R}^n$. That ϕ satisfies properties (P1) and (P2) follows from the fact that, for each x, $\xi(\cdot, x)$ is the solution of (23).

Differentiating (24) with respect to z gives

$$D_1 \phi(z, x) = \frac{1}{z} D_1 \xi(\log(z), x). \tag{25}$$

Substituting $z = 1$ into (25) gives

$$\begin{aligned} D_1 \phi(1, x) &= D_1 \xi(0, x) \\ &= g(\xi(0, x)) \\ &= g(x), \end{aligned} \tag{26}$$

where the last two equalities follow from the fact that $\xi(\cdot, x)$ satisfies (23). Property (P3) now follows as g is a C^1 function.

As (22) has an unbounded solution, it follows from Remark 3.4 that the system $\dot{w} = g(w)$ has an unbounded solution and hence that there exists $\bar{x} \in \mathbb{R}^n$ such that $\limsup_{t \to \infty} |\xi(t, \bar{x})| = \infty$. This implies

$$\limsup_{z \to 0^+} \left| \phi\left(\frac{1}{z}, \bar{y}\right) \right| = \infty,$$

where $\bar{y} = \bar{x}$ and hence property (P4′) is satisfied.

Substituting $x = \phi(1/z, y)$ into (26) gives

$$g\left(\phi\left(\frac{1}{z}, y\right)\right) = D_1 \phi\left(1, \phi\left(\frac{1}{z}, y\right)\right). \tag{27}$$

Combining (27) and (A.1) now gives

$$g\left(\phi\left(\frac{1}{z}, y\right)\right) = D_2 \phi\left(\frac{1}{z}, y\right) D_1 \phi(1, y). \tag{28}$$

Define the C^1 function $h: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $(z, y) \mapsto D_1 \phi(1, y)$. Lemma A.2 shows that $(D_2 \phi(1/z, y))^{-1}$ exists and hence (28) implies

$$h(z, y) = \left(D_2 \phi\left(\frac{1}{z}, y\right)\right)^{-1} g\left(\phi\left(\frac{1}{z}, y\right)\right) \tag{29}$$

for all $(z, y) \in \mathbb{R}_{>0} \times \mathbb{R}^n$. Hence condition (1) of Theorem 3.2 is satisfied and so is condition (2) by letting $\lambda = 1$. □

If property (P4) rather than (P4′) was satisfied in Theorem 5.1, Theorem 5.1 would be a complete converse for Theorem 3.2. So does there exist a system with an unbounded solution for which no ϕ, $\bar{y}$, F and λ exists that simultaneously satisfies properties (P1)–(P4), and conditions (1) and (2) of Theorem 3.2? The answer to this question is not known to us. There does however exist examples for which using the ϕ, $\bar{y}$, F and λ combination in the proof of Theorem 5.1 leads to an inability to satisfy property (P4). Indeed, one such example is the following system:

$$\begin{aligned} \dot{x}_1 &= -x_1^3 x_2, \\ \dot{x}_2 &= -1 + x_1^2. \end{aligned} \tag{30}$$

(We do not prove this fact here due to space limitations.)

It is interesting to note that Corollary 3.5 does detect the existence of an unbounded solution of (30) (using $\bar{y} = (0, -1)$, $\lambda = 0.5$, $\alpha_1 = -2$, $\alpha_2 = 2$ and $F = 1/\sqrt{1 + x_2^2}$). Hence for this example there does exist a ϕ, $\bar{y}$, F and λ combination that also satisfies (P4).

Note that the construction of ϕ in the proof of Theorem 5.1 presumes knowledge of the solutions of (23). Let A be a real $n \times n$ matrix with eigenvalue $\tilde{\lambda}$, $\mathrm{Re}(\tilde{\lambda}) > 0$, and consider the linear system

$$\dot{x} = Ax. \tag{31}$$

Presuming knowledge of the solutions of (31), one can easily find appropriate $(\phi, \bar{y})$, F and λ to satisfy Theorem 3.2. What can be said in this case without using knowledge of the solutions? Taking $\alpha_i = 1$, $i = 1, \dots, n$, and $F = 1$,the conditions of Corollary 3.5 are satisfied if and only if there exists $\bar{y} \neq 0 \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}_{>0}$ such that $A\bar{y} = \lambda\bar{y}$. Hence, if $\tilde{\lambda}$ is real and positive, Corollary 3.5 implies (31) has an unbounded solution. If $\tilde{\lambda}$ is complex, the choice of α_i's and F used above fails to conclude that the system has an unbounded solution.

This and the gap that exists between properties (P4) and (P4′) suggests that there may be value in extending the results of this paper to the complex domain and/or using auxiliary systems similar to (6) but with $z \in \mathbb{R}^m$ rather than $z \in \mathbb{R}$. Such extensions would in all likelihood enable one to capture a wider range of unbounded behaviour.

6. Additional comments

Given a system $\dot{x} = f(x)$ and a continuous function $F: \mathbb{R}^n \to \mathbb{R}_{>0}$, let $g = f/F$ and consider the normalized system $\dot{x} = g(x)$. If the requirements of Theorem 3.2 are met, it follows from the proof of Theorem 3.2 that there exists a solution to (6) passing through a point (z^0, y^0), $z^0 > 0$, and converging to $(0,\bar{y})$. If $(z(\cdot), y(\cdot))$ denotes such a solution then it was shown that $x(t) = \phi(1/z(t), y(t))$ is a solution of $\dot{x} = g(x)$. Suppose now that $\phi(z,x) = (z^{\alpha_1}x_1, \dots, z^{\alpha_n}x_n)$ and that for some $j \in \{1, \dots, n\}$, $\bar{y}_j \neq 0$ and $\alpha_j > 0$ as in Corollary 3.5. As $z(t) \to 0$ exponentially (see (6)) and $y_j(t) \to \bar{y}_j \neq 0$, it follows that $x(\cdot)$ is an exponentially unbounded solution of the system $\dot{x} = g(x)$. Indeed this fact shows that the only unbounded solutions that can be detected using Corollary 3.5 are solutions of the normalized systems $\dot{x} = g(x)$ that become unbounded exponentially. Hence being able to normalize by F, and furthermore choosing an appropriate F, is quite important.

Lastly, consider the system

$$\begin{aligned} \dot{x}_1 &= -1 + x_1 x_2, \\ \dot{x}_2 &= -x_2^3. \end{aligned} \tag{32}$$

While it is clear that $x_1(t) = x_1(0) - t$, $x_2(t) = 0$ is an unbounded solution of (32), what is perhaps not quite as easy to see is that (32) also has unbounded solutions for which $x_1(t) \to +\infty$. Indeed, consider the closed region in the positive x_1 half-plane bounded below by the curve $x_1 x_2 = 2$ and bounded above by the line $x_2 = 0.5$. Let R denote this region and let $w = x_1 x_2 - 2$. Differentiating w with respect to t and substituting (32) gives $\dot{w} = (-1 + x_1 x_2)x_2 - x_1 x_2^3$. If $w = 0$ (i.e., if $x_1 x_2 = 2$) then $\dot{w} = x_2(1 - 2x_2)$ and $\dot{w} > 0$ if $0 < x_2 < 0.5$. Hence, except at the point (4,0.5), the vector field along the lower boundary of R points into the interior of R. It is also easily verified that the vector field along the upper boundary of R also points into the interior of R and hence it follows that R is an invariant set. As system (32) does not possess any fixed points nor any periodic orbits ($x_2(0) \neq 0$ implies $x_2(t) \to 0$) it follows from the Poincaré–Bendixson Theorem (Hale, 1980) that all solutions of (32) are unbounded and hence that the x_1 component of any solution starting in R converges to $+\infty$.

Unfortunately, using the methods of this paper, we have had no success showing that system (32) has an unbounded solution for which $x_1(t) \to +\infty$. On the other hand, Theorem 5.1 indicates that there may well be an appropriate stability preserving extension that can be used to demonstrate this fact. In some cases, as in the example above, finding an appropriate stability preserving extension can be difficult. We would suggest that such systems are inherently difficult to analyse and the reader should keep in mind that arguments similar to the ones used for (32) can rarely be found, especially for higher dimensional systems.

7. Concluding remarks

In this paper a start was made at exploring the use of local methods to analyse behaviour at infinity. Presented were sufficient conditions for a dynamical system to possess an unbounded solution and it was shown that these results can be used to infer results about lack of global stabilizability for nonlinear control systems.

Appendix

The following lemma is used in the proof of Theorem 3.2.

Let $\mathbb{C}$ denote the complex numbers. If A is a real (or complex) square matrix, let $\sigma(A) \subset \mathbb{C}$ denote the set of eigenvalues of A.

Lemma A.1. *Suppose $\gamma \in \mathbb{R}$ and that A is a real $n \times n$ matrix of the form*

$$A = \begin{pmatrix} \gamma & 0 \\ B & C \end{pmatrix}.$$

Then associated with the eigenvalue γ, A has a real (generalized) eigenvector whose first component is nonzero.

Proof. We will consider three separate cases.

Case 1: $\gamma \notin \sigma(C)$. Then if I denotes the identity matrix, the matrix $C - \gamma I$ is invertible and it is easily verified that

$$v = \begin{pmatrix} 1 \\ -(C - \gamma I)^{-1} B \end{pmatrix}$$

satisfies $Av = \gamma v$.

Case 2: $\sigma(C)=\{\gamma\}$. The result is clear in this case as the only eigenvalue of the real matrix A is the real eigenvalue γ.

Case 3: $\gamma\in\sigma(C)$ but $\sigma(C)\neq\{\gamma\}$. Then there exist real matrices T, C_1 and C_2 such that T is invertible, $\sigma(C_1)=\{\gamma\}$, $\gamma\notin\sigma(C_2)$, and

$$C=T^{-1}\begin{pmatrix}C_1 & 0\\ 0 & C_2\end{pmatrix}T.$$

This implies

$$A=\begin{pmatrix}1 & 0\\ 0 & T^{-1}\end{pmatrix}\begin{pmatrix}\gamma & 0\\ TB & \begin{pmatrix}C_1 & 0\\ 0 & C_2\end{pmatrix}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & T\end{pmatrix}.$$

If C_1 is $m\times m$, let $(TB)_1$ denote the vector consisting of the first m entries in TB and let $(TB)_2$ denote the vector consisting of the remaining entries of TB. Let

$$S=\begin{pmatrix}1 & 0 & 0\\ 0 & I & 0\\ (C_2-\gamma I)^{-1}(TB)_2 & 0 & I\end{pmatrix},$$

$$U=S\begin{pmatrix}1 & 0\\ 0 & T\end{pmatrix} \quad\text{and}\quad D=\begin{pmatrix}\gamma & 0\\ (TB)_1 & C_1\end{pmatrix}.$$

Then it can be verified that

$$A=U^{-1}\begin{pmatrix}D & 0\\ 0 & C_2\end{pmatrix}U.$$

As D is a real matrix whose only eigenvalue is the real eigenvalue γ, it follow that D has a real generalized eigenvector u whose first entry is nonzero.

This now implies that associated with the eigenvalue γ,

$$v=U^{-1}\begin{pmatrix}u\\ 0\end{pmatrix}$$

is a generalized eigenvector of A. Noting that U applied to a vector leaves the first entry of the vector unchanged (and hence that the same is true for U^{-1}), it follows that v satisfies the requirements of the lemma. □

The next lemma contains results used in the proofs of Theorems 3.2 and 5.1.

Lemma A.2. *If* $\phi:\mathbb{R}_{>0}\times\mathbb{R}^n\to\mathbb{R}^n$ *is a* C^1 *function that satisfies properties* (P1) *and* (P2) *then*

$$D_2\phi\left(\frac{1}{z},y\right)D_1\phi(1,y)=\frac{1}{z}D_1\phi\left(\frac{1}{z},y\right)=D_1\phi\left(1,\phi\left(\frac{1}{z},y\right)\right) \quad \text{(A.1)}$$

and

$$\left(D_2\phi\left(\frac{1}{z},y\right)\right)^{-1}=D_2\phi\left(z,\phi\left(\frac{1}{z},y\right)\right) \quad \text{(A.2)}$$

for all $z\in\mathbb{R}_{>0}$ *and* $y\in\mathbb{R}^n$.

Proof. Differentiating the identity given in (P1) with respect to z_2 gives $D_2\phi(z_1,\phi(z_2,x))D_1\phi(z_2,x)=z_1D_1\phi(z_1z_2,x)$. The first equality of (A.1) now follows by choosing $z_1=1/z$, $z_2=1$, $x=y$ and applying (P2).

Differentiating the identity given in (P1) with respect to z_1 gives $D_1\phi(z_1,\phi(z_2,x))=z_2D_1\phi(z_1z_2,x)$. Setting $z_1=1$, $z_2=1/z$ and $x=y$ gives the second equality of (A.1).

From properties (P1) and (P2), $\phi(z,\phi(1/z,y))=y$ and differentiating with respect to y gives $D_2\phi(z,\phi(1/z,y))D_2\phi(1/z,y)=I$. This proves (A.2). □

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