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Ignored input dynamics and a new characterization of control Lyapunov functions☆

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Abstract

Our objective in this paper is to extend as much as possible the dissipativity approach for the study of robustness of stability in the presence of known/unknown but ignored input dynamics. This leads us to:

- give a new characterization of control Lyapunov functions (CLF) where $L_f V$ is upper-bounded by a function of $L_g V$,
- define the dissipativity approach as
  - assuming the ignored dynamics are dissipative with storage function W and (known) supply rate w,
  - analyzing closed-loop stability with the sum of a CLF for the nominal part and the storage function W.

Stability margin are given in terms of an inequality the supply should satisfy. Nevertheless, in spite of this extension, we show that the dissipativity approach cannot cope with ignored dynamics which have nonzero relative degree or are nonminimum phase.

Keywords: Lyapunov methods; Uncertain dynamic systems; Stabilizing controllers; Robustness

1. Introduction

1.1. Problem statement

This last decade has seen some progress made in nonlinear regulation. Lyapunov designs are now available for systems of special kinds such as those exhibiting a dissipativity property or having a peculiar structure in appropriate coordinates. One way to meet such requirements is to work with simplified model obtained for instance by neglecting well or poorly known input dynamics. This leads to the problem of (global) asymptotic stabilization of systems with ignored input dynamics.

The problem is to design a state feedback law[1] $u = k(x)$ which globally asymptotically stabilizes the origin for the system whose dynamics are described by a nominal part

$$\dot{x} = f(x) + g(x)y \tag{1}$$

with state x in $\mathbb{R}^n$. Its input is y in $\mathbb{R}^m$. It may be accessed only through the system

$$\dot{z} = j(z, x, u),$$
$$y = h(z, x, u). \tag{2}$$

In the control law (but not in its design), this system is ignored because it is unknown or it is known but its state z is unavailable or its dynamics are too complicated or irrelevant to the control objective.

In the nonlinear framework, besides the very recent approach via disturbance estimation proposed in Praly and Jiang (1998a,b), two main non overlapping ways of tackling with this problem have been proposed: the dissipativity approach (see Moylan & Anderson, 1973; Glad, 1984; Tsitsiklis & Athans, 1984; Sepulchre, Jankovic, & Kokotovic, 1997 for instance) and the nonlinear small gain approach (see Krstić & Kokotović, 1994; Praly & Wang, 1996; Jiang & Mareels, 1997 for instance). In

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[1] In this paper, we consider only the case of static feedback.

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this paper we concentrate our attention on the former trying to extend it as much as possible.

1.2. Motivation

From the dissipativity approach, we retain:

1. the characterization of the ignored systems (2) as those for which there exists a positive definite, proper and C^1 function W, the storage function, such that
$$\frac{\partial W}{\partial z}(z)j(z,x,u) \leqslant w(u,y) - \alpha(|z|) \quad \forall (z,x,u) \tag{3}$$
with $y = h(z,x,u)$, α a nonnegative continuous function and w a continuous function, called the supply rate. We refer the reader to van der Schaft (1996) for a survey on dissipativity. The supply rate w is the only known data on the ignored input dynamics.
2. the idea of studying the stability of the overall system via a Lyapunov function U which is the sum of W and of a control Lyapunov function (CLF) V for the nominal part. Namely we assume the data of a positive definite, proper and C^1 function V such that[2]
$$\{L_g V(x) = 0,\ x \neq 0\} \quad \Rightarrow \quad L_f V(x) < 0 \tag{4}$$
and we pick
$$U(x,z) = \psi(V(x)) + W(z), \tag{5}$$
where ψ is to be chosen as a positive definite, proper and C^1 function.

From the above, the problem studied in this paper reduces finding ψ and $u = k(x)$ so that the right hand side of
$$\dot U \leqslant L_f \psi(V) + L_g \psi(V) y + w(u,y) \tag{6}$$
is nonpositive for all (z,x). From this we see that if γ is the function defined as (when it makes sense)[3]
$$\gamma(s) = -\inf_u \sup_y \{sy + w(u,y)\} \tag{7}$$
which depends only on w, then ψ should be chosen such that
$$L_f \psi(V(x)) < \gamma(L_g \psi(V(x))) \quad \forall x \neq 0. \tag{8}$$
In Section 2, we shall observe that V is a CLF if and only if for any function γ in an appropriate class, there exists a function ψ so that (8) holds. So there is no loss of generality in considering (8). With such a result, the class of admissible supply rates w is simply the one giving γ in (7) in this appropriate class. This will be stated in Theorem 3.1 in Section 3. Following our arguments, it is the broadest class that we can expect by following the dissipativity approach as defined above. But we shall see in Theorem 3.2 that, at least in the case where u is in $\mathbb{R}$, we must have
$$w(u,y) \leqslant \lambda(u)y \tag{9}$$
for some function λ. This is reminiscent from the input feedback passivity assumption invoked in Sepulchre et al. (1997) where
$$\lambda(u) = u. \tag{10}$$
This necessary property implies that ignored dynamics which are only minimum phase and with zero relative degree can be allowed when the nominal part is not already open loop stable. This exhibits an inherent limitation in the dissipativity approach although as explained above we have tried to extend it as much as possible. In fact, with the help of an example we shall be able to show that this limitation is in part an artifact of this approach and more precisely of the technique of using a Lyapunov function U as the sum (5).

Our paper is organized as follows. In Section 2, we state and prove our necessary and sufficient condition for a Lyapunov function to be a CLF. The problem of robust stabilization in the presence of ignored input dynamics is studied in Section 3.

2. Another characterization of control Lyapunov functions

2.1. Main result

Since Artstein (1983) (see also Sontag, 1989), it is known that the existence of a control Lyapunov function (CLF) for systems of the form
$$\dot x = f(x) + g(x)u \tag{11}$$
with x in $\mathbb{R}^n$ and u in $\mathbb{R}^m$, is equivalent to the existence of a global asymptotic stabilizer $k(x)$, which is C^0 on $\mathbb{R}^n \backslash \{0\}$. Here we propose another way of characterizing such CLF's.

Theorem 2.1. *Let V be a C^1, positive definite and proper function. V is a CLF for* (11) *if and only if and for any $\varepsilon > 0$ and for any function $\gamma \in C^0(\mathbb{R}^p, \mathbb{R}_+)$, such that*:

(i) $\gamma(0) = 0$,
(ii) *for all $s \in \mathbb{R}^p \backslash \{0\}$, $\gamma(\varphi s)/\varphi$ is an increasing function of φ,*

[2] We denote $L_g V(x) = \partial V/\partial x(x) g(x)$.

[3] The worst case definition (7) of γ is the best thing we can use in the design of a robust control law. This follows from the fact that the only quantified knowledge of the ignored input dynamics is the supply rate w. If, for instance, h were also known, the best control would be given by the following min max problem
$$\gamma(s,x) = -\inf_u \sup_z \{sh(z,x,u) + w(u,h(z,x,u))\}.$$
with (8) replaced by $L_f \psi(V(x)) < \gamma(L_g \psi(V(x)), x)$.

(iii) *for all* $s \in \mathbb{R}^p \backslash \{0\}$, $\lim_{\varphi \to +\infty} \gamma(\varphi s)/\varphi = +\infty$,

there exists a positive definite and radially unbounded function $\psi_\varepsilon \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ *such that*:

(a) *the derivative* ψ'_ε *is positive definite on* $\mathbb{R}_+ \backslash \{0\}$,
(b) *we have*:

$$L_f \psi_\varepsilon(V(x)) < \gamma(L_g \psi_\varepsilon(V(x))) \quad \forall |x| \geqslant \varepsilon. \tag{12}$$

Moreover, we can take ψ_ε *independent of* ε *if and only if* γ *is such that*

$$\exists k > 0: \quad \limsup_{\substack{x \to 0 \\ L_g V(x) \neq 0}} \frac{L_f V(x)}{\gamma(k L_g V(x))} < \frac{1}{k}. \tag{13}$$

Proof of Theorem 2.1.
($\Leftarrow$) We want to show

$$\{L_g V(y) = 0,\ y \neq 0\} \ \Rightarrow\ L_f V(y) < 0. \tag{14}$$

For $\varepsilon = |y|/2$ in (12), we know the existence of ψ_ε such that, for all $|x| \geqslant \varepsilon$, we have

$$L_f \psi_\varepsilon(V(x)) < \gamma(L_g \psi_\varepsilon(V(x))) \tag{15}$$

or, since the function ψ'_ε is positive definite,

$$L_f V(x) < \frac{\gamma(\psi'_\varepsilon(V(x)) L_g V(x))}{\psi'_\varepsilon(V(x))}. \tag{16}$$

So in particular for $x = y$, since $L_g V(y) = 0$, this inequality yields

$$L_f V(y) < 0. \tag{17}$$

($\Rightarrow$) When $V(x)$ is a CLF, we want to exhibit a function ψ_ε satisfying (12). For this, let

$$X_n = \{x : 2^n \leqslant V(x) \leqslant 2^{n+1}\}, \quad n = 0, \pm 1, \pm 2, \ldots . \tag{18}$$

There exists a nonnegative real number ψ_n such that, for all x in X_n, we have

$$L_f V(x) - \frac{1}{\psi_n} \gamma(\psi_n L_g V(x)) < 0. \tag{19}$$

Indeed, if not, there exists a sequence x_m such that

$$L_f V(x_m) \geqslant \frac{1}{m} \gamma(m L_g V(x_m)). \tag{20}$$

By compactness of X_n, x_m has a cluster point x^*. We still denote by x_m a converging subsequence. We have $L_g V(x^*) = 0$, otherwise, $L_g V$ being continuous, there would exist $\mu > 0$ and M such that, for all $m \geqslant M$, we would have

$$|L_g V(x_m)| \geqslant \mu. \tag{21}$$

Then for such m's we have, from (20),

$$L_f V(x_m) \frac{m |L_g V(x_m)|}{\gamma(m L_g V(x_m))} \geqslant |L_g V(x_m)|. \tag{22}$$

But, since we have

$$\lim_{|s| \to +\infty} \frac{\gamma(s)}{|s|} = +\infty \tag{23}$$

and since $L_f V(x_m)$ is bounded, we get

$$\lim_{m \to \infty} L_f(V(x_m)) \frac{m |L_g V(x_m)|}{\gamma(m L_g V(x_m))} = 0 \tag{24}$$

which contradicts (21) and establishes $L_g V(x^*) = 0$. Now, since we have $|x^*| \geqslant 2^n$ and V is a CLF, we have also established

$$L_f V(x^*) < 0. \tag{25}$$

On the other hand, from (20) and the continuity of $L_f V(x)$, we have

$$\lim_{m \to \infty} L_f V(x_m) \geqslant 0 \ \Rightarrow\ L_f V(x^*) \geqslant 0. \tag{26}$$

This is a contradiction which establishes the existence of ψ_n satisfying (19).

Now, let η be some fixed real number in $(0, \varepsilon]$. Since V is positive definite and proper, there exists n_0 such that

$$V(x) \leqslant 2^{n_0} \ \Rightarrow\ |x| \leqslant \eta. \tag{27}$$

With all this at hand, we can pick ψ'_ε as any positive definite and continuous function such that

$$\psi'_\varepsilon(v) \geqslant \psi_0 \quad \text{if } v \leqslant 2^{n_0}, \tag{28}$$

$$\psi'_\varepsilon(v) \geqslant \psi_n \quad \text{if } 2^n \leqslant v \leqslant 2^{n+1},\ n \geqslant n_0 \tag{29}$$

with ψ_0 arbitrary. Since $\gamma(\psi\, s)/\psi$ is an increasing function of ψ for all $s \in \mathbb{R}^p$, we have

$$|x| \geqslant \eta \ \Rightarrow\ L_f V(x) - \frac{1}{\psi'_\varepsilon(V(x))} \gamma(\psi'_\varepsilon(V(x)) L_g V(x)) < 0. \tag{30}$$

Then (12) follows by multiplying by $\psi'_\varepsilon(V(x))$.

Now let us prove that we can take ψ_ε independent of ε, when (13) is satisfied. To do so it is sufficient to show that η can be taken independent of ε. But, if $V(x)$ satisfies (13) (which is a strict inequality), there exists $\eta > 0$ such that, for all $\psi_0 > 0$, we have

$$|x| \leqslant \eta \Rightarrow \frac{L_f V(x)}{\gamma(k L_g V(x))} \leqslant \frac{1}{k}, \tag{31}$$

$$\Rightarrow \psi_0 L_f V(x) - \gamma(\psi_0 L_g V(x)) \leqslant \psi_0 \left[\frac{\gamma(k L_g V(x))}{k} - \frac{\gamma(\psi_0 L_g V(x))}{\psi_0} \right]. \tag{32}$$

Since V is a CLF and $\gamma(\varphi\ s)/\varphi$ is an increasing function of φ for $s \neq 0$, it is sufficient to choose ψ_0 as satisfying

$$\psi_0 > k. \tag{33}$$

This way, we have obtained, for all $x \neq 0$,

$$L_f V(x) - \frac{1}{\psi_\varepsilon'(V(x))}\gamma(\psi_\varepsilon'(V(x))L_g V(x)) < 0. \quad (34)$$

Let us conclude the proof by showing that (13) is necessary for having ψ independent of ε. Indeed, according to (12), we have for $L_g V(x) \neq 0$:

$$\frac{\psi'(V(x))L_f V(x)}{\gamma(\psi'(V(x))L_g V(x))} < 1. \quad (35)$$

So, let

$$k = \sup_{0 \leqslant |x| \leqslant \eta} \psi'(V(x)). \quad (36)$$

Since $\gamma(\varphi s)/\varphi$ is an increasing function of φ for $|s| \neq 0$, we have

$$0 < \frac{k}{\gamma(ks)} \leqslant \frac{\psi'(V(x))}{\gamma(\psi'(V(x))s)}. \quad (37)$$

Then, for all x such that $|x| \leqslant \eta$,

- if $L_g V(x) \neq 0$ and $L_f V(x) > 0$, we have

$$\frac{kL_f V(x)}{\gamma(kL_g V(x))} \leqslant \frac{\psi'(V(x))L_f V(x)}{\gamma(\psi'(V(x))L_g V(x))} < 1. \quad (38)$$

- if $L_f V(x) \leqslant 0$ then we have trivially

$$\frac{kL_f V(x)}{\gamma(kL_g V(x))} < 1. \quad (39)$$

We have proved

$$\{L_g V(x) \neq 0,\ |x| \leqslant \eta\} \ \Rightarrow\ \frac{L_f V(x)}{\gamma(kL_g V(x))} \leqslant \frac{1}{k}. \qquad \square \quad (40)$$

Remark. When $\gamma(s) = |s|^{(1+c)}$, (i)–(iii) are satisfied and, in this case, ψ_n can be obtained by the following maximization problem:

$$\psi_n = \sup_{x \in X_n} \left\{ \frac{L_f V(x)}{|L_g V(x)|^{(1+c)}} \right\}^{1/c}. \quad (41)$$

2.2. Related results

2.2.1. Case $\gamma(s) = r|s|^2$

The case $\gamma(s) = r|s|^2$ was already known. It was established indirectly invoking the relation between CLF and optimal value functions. Indeed in Sepulchre et al. (1997) and Krstić and Li (1997), the authors prove that if V is a CLF and satisfies a local condition, discussed below, at the origin then there exists a C^1 function ψ such that $\psi(V(x))$ is the optimal value function associated to the cost functional

$$J(x) = \int_0^\infty \left[\ell(X(x,t)) + \frac{1}{4r}|u(t)|^2 \right] \mathrm{d}t \quad (42)$$

with ℓ being positive definite. More precisely, $\psi(V(x))$ is a solution of the following Hamilton–Jacobi–Bellman (HJB) equation

$$\ell(x) + L_f \psi(V(x)) - r|L_g \psi(V(x))|^2 = 0. \quad (43)$$

Our result with $\gamma(s) = r|s|^2$ follows since ℓ being positive definite, we get readily

$$L_f \psi(V(x)) < r|L_g \psi(V(x))|^2 \quad \forall x \neq 0. \quad (44)$$

The local condition, mentioned above, has been stated in Krstić & Li (1997) (see also Praly (1997)) as

$$\limsup_{x \to 0} \frac{L_f V(x)}{|L_g V(x)|^2} < +\infty. \quad (45)$$

This is nothing but (13) for the case $\gamma(s) = r|s|^2$.

2.2.2. A link with "$L_g V$ controllers"

Another related result is given in Teel and Praly (1998). There, it is proved that if a given function $\Pi: \mathbb{R}^n \to \mathbb{R}^m$, is continuous on $\mathbb{R}^m \backslash \{L_g V(x) = 0\}$, locally bounded on $\mathbb{R}^n$, and such that $L_g V(x)\Pi(x)$ is nonpositive and

$$\{L_g V(x)\Pi(x) = 0, x \neq 0\} \ \Rightarrow\ L_f V(x) < 0, \quad (46)$$

then with

$$\Psi(x) = \frac{\max\{0, L_f V(x) + |L_g V(x)|^2\}}{-L_g V(x)\Pi(x)}, \quad (47)$$

the control law

$$u(x) = \Psi(x)\Pi(x), \quad (48)$$

called $L_g V$-controller, is continuous on $\mathbb{R}^n \backslash \{0\}$ and gives

$$L_f V(x) + L_g V(x)u(x) < 0 \quad \forall x \neq 0. \quad (49)$$

To meet (12), we choose the function Π as

$$\Pi(x) = -\frac{\gamma(L_g V(x))L_g V^{\mathrm{T}}(x)}{|L_g V(x)|^2}. \quad (50)$$

It satisfies the required properties and the function Ψ given by (47) is such that

$$L_f V(x) < \Psi(x)\gamma(L_g V(x)). \quad (51)$$

And, from the properties of V, we can find a C^1 function ψ on $\mathbb{R}^n \backslash \{0\}$ such that

$$\psi'(V(x)) \geqslant \Psi(x). \quad (52)$$

This yields

$$L_f V(x) < \psi_\varepsilon(V(x))\gamma(L_g V(x)). \tag{53}$$

But (53) is not (12) yet. However (12) can be recovered for instance with $\psi_\varepsilon = \Psi^{1/c}$ if

$$\gamma(s) = |s|^{(1+c)}. \tag{54}$$

The connection with Teel and Praly (1998) is also interesting since, by mimicking what is done there, we can relate the condition (13) in Theorem 2.1 to the small control property of Sontag (1989).

3. Robustness to input dynamics

In this section we use the CLF characterization given in Section 2 to solve the stabilization design problem stated in Introduction. More specifically, we consider the class of systems of the following form:

$$\dot{x} = f(x) + g(x)y,$$

$$\dot{z} = j(z,x,u),$$

$$y = h(z,x,u), \tag{55}$$

where $x \in \mathbb{R}^n$ represents the state of the system to be controlled, $z \in \mathbb{R}^p$ represents the state of the ignored part and is not available for feedback design, $u \in \mathbb{R}^m$ is the control input, $y \in \mathbb{R}^m$ is the output of the uncertain z-subsystem and the input of the x-subsystem.

For nonlinear systems and within the dissipativity approach, the study of the margin of stability of systems in the presence of input uncertainties began by exploiting the properties of optimal controllers. Precisely, it has been established that if $u = k(x)$ is a minimizer of the cost functional:

$$J(x) = \int_0^\infty [\ell(X(x,t)) + r(u(t))]\,\mathrm{d}t \tag{56}$$

with r and ℓ being positive definite functions, then this control law guarantees global asymptotic stability in presence of ignored input dynamics for which the supply rate w in (3) satisfies

$$w(u,y) \leqslant (y-u)r'(u) + (1-c)r(u), \tag{57}$$

where c is a strictly positive real number. This is established, for instance, in Moylan and Anderson (1973) where r is quadratic and in Glad (1984) and Tsitsiklis and Athans (1984) for general r. Since it is sufficient to have an optimal control to get such a property, this leads to the question of when a control law is optimal. Such a question is addressed and solved in the nonlinear context in Moylan and Anderson (1973) and Sepulchre et al. (1997) under the constraint of a quadratic r, i.e. (57) takes the form

$$w(u,y) \leqslant uy - ru^2 \tag{58}$$

and the corresponding ignored dynamics are said input feedforward passive (IFP). In particular in Sepulchre et al. (1997), it is established that the knowledge of a CLF satisfying (45) is sufficient to derive an optimal control law (see Section 2.2.1). This proves that optimal synthesis is not necessary to design a robust control law.

In this section we consider the case of a general supply rate and propose a controller design adapted to it and providing global asymptotic stability for system (55).

3.1. Main results

To design a control law for the system (55), we assume:

1. We know a CLF V for (11).
2. The z-subsystem satisfies the following dissipativity inequality:

$$\frac{\partial W}{\partial z}(z)j(z,x,u) \leqslant w(u,y) - \alpha(|z|) \quad \forall (z,x,u) \tag{59}$$

with $y = h(z,x,u)$, W a positive definite, proper and C^1 function, α a nonnegative continuous function and w a continuous function which is known for the design.

Theorem 3.1. *Assume the supply rate w is such that there exists a continuous function π such that a function γ satisfying*

$$\gamma(s) \leqslant -\sup_y \{w(\pi(s),y) + sy\} \tag{60}$$

meets the properties (i)–(iii) *in Theorem* 2.1. *Under these conditions, if condition* (13) *in Theorem* 2.1 *holds, there exists*[4] *a function ψ and a controller*:

$$u = \pi(L_g\psi(V(x))) \tag{61}$$

which guarantees global stability of the origin for (55) *and*

$$\lim_{t\to+\infty} x(t) = 0. \tag{62}$$

Moreover this controller is globally asymptotically stabilizing if α is positive definite.

Proof of Theorem 3.1. By assumption, the functions V and γ satisfy the conditions of Theorem 2.1. So we know the existence of a function ψ so that, for all $x \neq 0$, we have

$$L_f\psi(V(x)) < \gamma(L_g\psi(V(x))). \tag{63}$$

[4] The procedure for getting u in (61) is:

- to find γ and π satisfying (60). They depend only on w.
- to find ψ satisfying (12) with the above γ. It depends only on f, g, V and γ.

Then, from (59), we get

$$\overbrace{\psi(V(x)) + W(z)}^{\cdot} \leqslant L_f\psi(V(x)) + L_g\psi(V(x))y + w(u,y) - \alpha(|z|). \quad (64)$$

So when the control u is given by (61), (60) gives, for all $x \neq 0$,

$$\overbrace{\psi(V(x)) + W(z)}^{\cdot} \leqslant L_f\psi(V(x)) - \gamma(L_g\psi(V(x))) - \alpha(|z|), \quad (65)$$

$$< -\alpha(|z|). \quad (66)$$

This implies global stability of the origin. And, with LaSalle's invariance Theorem, we are guaranteed of the convergence of $x(t)$ to 0 and of global asymptotic stability when α is positive definite. □

Example. Consider the following system, which is not input feedforward passive in the sense of Sepulchre et al. (1997):

$$\dot{x} = x + y,$$

$$\dot{z} = -z^3 + u^3,$$

$$y = u + \tfrac{1}{2}(u^3 + z^3)^{1/3}, \quad (67)$$

$V(x) = \frac{1}{2}x^2$ is a CLF for the nominal system $\dot{x} = x + y$ and the z-subsystem is dissipative with

$$\overbrace{\tfrac{1}{2}z^2}^{\cdot} = -z^4 + u^3(8[y-u]^3 - u^3)^{1/3}. \quad (68)$$

So, the appropriate supply rate

$$w(u,y) = u^3(8[y-u]^3 - u^3)^{1/3} \quad (69)$$

is not in the form $(y-u)r'(u) + r(u)$ as in (57). Nevertheless, for such a function w, we have

$$\sup_y (sy + w(u(s),y)) = \begin{cases} -2^{1/3}s^{4/3} & \text{if } s + 2u(s)^3 = 0, \\ +\infty & \text{if } s + 2u(s)^3 \neq 0. \end{cases} \quad (70)$$

So according to the statement of Theorem 3.1, we let

$$\pi(s) = -\left(\frac{s}{2}\right)^{1/3}, \quad (71)$$

$$\gamma(s) = 2^{1/3}\,|s|^{4/3}. \quad (72)$$

Then, since we have

$$L_f V(x) = x^2 = 2V, \quad (73)$$

$$L_g V(x) = x = \operatorname{sign}(x)\sqrt{2V}, \quad (74)$$

we look for a function ψ so that, for $V \neq 0$,

$$L_f\psi(V) = 2V\psi'(V) < \gamma(L_g\psi(V)) = 2^{1/3}\psi'(V)^{4/3}(2V)^{2/3} \quad (75)$$

This yields, for $V \neq 0$,

$$\psi'(V) > V. \quad (76)$$

So for instance, we choose

$$\psi(V) = V^2. \quad (77)$$

Then, according to (61), a control law is

$$u(x) = -[\tfrac{1}{2}]^{1/3}x. \quad (78)$$

It provides global asymptotic stability for the system (67) but not for the nominal system

$$\dot{x} = x + u. \quad (79)$$

Note that the small gain design of Praly and Wang (1996) applies also for system (67).

Example. As in Sepulchre et al. (1997), consider the system

$$\dot{x} = x^2 + u. \quad (80)$$

Following the design suggested in Tsitsiklis and Athans (1984), we introduce the cost functional:

$$J(x) = \int_0^\infty (X(t,x)^2 + u(t)^2)\,\mathrm{d}t. \quad (81)$$

To find a minimizer, it is sufficient to find a positive solution V for the following HJB equation:

$$x^2 + \frac{\partial V(x)}{\partial x}x^2 - \frac{1}{2}\left(\frac{\partial V(x)}{\partial x}\right)^2 = 0, \qquad V(0) = 0. \quad (82)$$

Solving this equation in $\partial V(x)/\partial x$ and integrating with respect to x, we get (see Sepulchre et al., 1997):

$$V(x) = \tfrac{2}{3}(x^3 + (x^2+1)^{3/2} - 1). \quad (83)$$

The minimizer is then given by

$$u = -\tfrac{1}{2}L_g V(x) = -x^2 - x\sqrt{x^2+1}. \quad (84)$$

From Tsitsiklis and Athans (1984), we know it provides robust stability for dissipative systems with supply rate

$$w(u,y) = 2uy - (1+c)u^2 \quad (85)$$

with c a strictly positive real number.

Following our design, let us show that the same controller provides actually a broader stability margin. Since V given in (83) is a CLF for (80) and according to (61), we look for π and ψ such that

$$-x^2 - x\sqrt{x^2+1} = \pi(L_g\psi(V(x))). \quad (86)$$

Restricting ourselves with ψ = identity, we get, with (83),

$$\pi(s) = -\tfrac{1}{2}s. \quad (87)$$

Then the allowed supply rate is related to π via (60) with γ satisfying (from (12)):

$$L_f V(x) - \gamma(L_g(V(x)) = (2x^2 + 2x\sqrt{x^2+1})x^2 - \gamma(2x^2 + 2x\sqrt{x^2+1}) < 0. \quad (88)$$

A function γ satisfying this constraint is[5]

$$\gamma(s) = (1+c)\,\frac{|s|^3}{4(1+|s|)}. \quad (89)$$

It follows that allowed supply rates w are such that

$$(1+c)\,\frac{|s|^3}{4(1+|s|)} \leqslant -\sup_y \{w(-s/2, y) + sy\} \quad (90)$$

or

$$w(u,y) \leqslant 2uy - 2(1+c)\,\frac{|u|^3}{(1+|2u|)}. \quad (91)$$

Compared to (85), this gives a larger class of supply rates w. For instance, consider the ignored dynamics

$$\dot{z} = 2u,$$

$$y = z + (1+c)\,\frac{u|u|}{(1+2|u|)}. \quad (92)$$

This system is dissipative with supply rate satisfying (91). It is therefore allowed as a disturbing system of (80) when the control is given by (84). Remark that the relative degree of the linearization of this system at the origin is 1.

3.2. Discussion

3.2.1. Known results with specific $w(u,y)$ in (59)

As already mentioned the result of Theorem 3.1 is not new at least for the following two specific expressions of the supply rate w.

- $w(u,y) = uy - ru^2$, $r > 0$: This is the case of an (IFP(r)) uncertainty (see Sepulchre et al., 1997). For such a supply rate, $\sup_y\{w(\pi(s), y) + sy\}$ is finite if and only if $\pi(s) = -s$, and γ can be chosen as $\gamma(s) = rs^2$.
- $w(u,y) = (y-u)r'(u) + r(u)$: with r defining the cost functional (56) (see Tsitsiklis & Athans, 1984). For such a supply rate, we get $\pi(s)$ as the solution of

$$r'(\pi(s)) + s = 0. \quad (93)$$

Then (60) defines γ as

$$\gamma(s) = -r(\pi(s)) + \pi(s)r'(\pi(s)). \quad (94)$$

[5] By letting $z = 2x^2 + 2x\sqrt{x^2+1}$, we get $x^2 = f(z) = z^2/4(1+z)$. So (88) can be written as $zf(z) - \gamma(z) < 0$.

We observe that if $r(u) - ur'(u)$ is not positive, such a function γ is not appropriate for Theorem 3.1. This restriction on r is not present in Tsitsiklis and Athans (1984). It follows in our case from the fact that we adopt a worst case approach and we do not consider the case where the CLF for the nominal system may be such that $L_f V(x)$ is negative for some x.

3.2.2. Stabilization of the nominal system (11)

In general, control (61) does not stabilize the nominal system. This is definitely not a drawback in particular for the case where the ignored dynamics are well known but we do not want to take them into account in the control law.

If we insist on having (61) to stabilize the nominal system it is sufficient to have, for $x \neq 0$,

$$(\dot{V} =)L_f V(x) + L_g V(x)\pi(L_g\psi(V(x))) < 0. \quad (95)$$

Since, according to Theorem 2.1, we have

$$L_f V(x) + L_g V(x)\pi(L_g\psi(V(x))) < \frac{\gamma(\psi' L_g V(x))}{\psi'(V(x))} + L_g V(x)\pi(L_g\psi(V(x))), \quad (96)$$

a sufficient condition for the stability of the nominal system is that γ and π satisfy

$$\gamma(s) \leqslant -s\pi(s). \quad (97)$$

3.2.3. Zero relative degree and minimum phase

Following our approach, (60) characterizes the class of allowed supply rates for the ignored dynamics. A supply rate satisfying (60) has the property:

Property P. *A continuous supply rate w is said to have property* P *if there exists a continuous function π such that*

$$\sup_y (w(\pi(s), y) + sy) < 0, \quad \forall s \in \mathbb{R}^m \backslash \{O\}. \quad (98)$$

Let us show that, at least in the single input case, a system admitting w as a supply rate satisfying property P, is contained in the class of systems with a zero relative degree (in a sense to be made precise) and minimum phase.

Theorem 3.2. *Given w satisfying property* P, *then*:

- *the systems which admits w as supply rate are such that, for all nonzero (s,x), there exist u such that $sh(0,x,u) < 0$ (a zero relative degree property),*
- *when $m = 1$, there exists a function λ such that, for all $(u,y) \in \mathbb{R}^2$, we have*

$$w(u,y) \leqslant \lambda(u)y. \quad (99)$$

Corollary 3.3. *For $m=1$, the systems which admits w, satisfying property* P, *as supply rate, have globally stable zero dynamics* (*when they exist*).

Proof of Theorem 3.2. *Zero relative degree*: We first remark that (59) implies

$$\frac{\partial W}{\partial z}(z)j(z,x,u) \leqslant w(u,h(z,x,u)). \tag{100}$$

Then, since W is positive definite, we have

$$\frac{\partial W}{\partial z}(0)=0. \tag{101}$$

This yields, for all $(s,x)\in\mathbb{R}^m\times\mathbb{R}^n$,

$$w(u,h(0,x,u)) \geqslant 0. \tag{102}$$

But, on the other hand, the fact that w satisfies property P gives, for all $(s,x)\in(\mathbb{R}^m\backslash\{0\})\times\mathbb{R}^n$,

$$w(\pi(s),h(0,x,\pi(s))) + sh(0,x,\pi(s)) \leqslant \sup_y\{sy+w(\pi(s),y)\}<0. \tag{103}$$

We conclude that for any $s\neq 0$, we have

$$sh(0,x,\pi(s))<0. \tag{104}$$

This implies that, for all x and for all directions s, we can find u such that the vector $h(0,x,u)$ is in an half space defined by this direction. So $h(0,x,u)$ must definitely depend on u. This is a weak notion of zero relative degree.

Existence of λ: We need the following Lemma:

Lemma 3.4. *Assume* $m=1$. *Then any continuous function* π *satisfying* (98), *is neither lower nor upper bounded on* $\mathbb{R}$.

Proof.[6] We prove this result by contradiction. Assume the existence of a continuous function π and a real number κ so that, for all $s\in\mathbb{R}$, we have

$$\pi(s)>\kappa \quad (\pi(s)<\kappa \text{ resp.}). \tag{105}$$

We have from (98)

$$sy+w(\pi(s),y)<0 \quad \forall(s,y)\in(\mathbb{R}\backslash\{0\})\times\mathbb{R}. \tag{106}$$

In particular for $y=1$, we get, for all $s\neq 0$:

$$w(\pi(s),1)<-s. \tag{107}$$

[6] The main idea of this proof was suggested to the authors by Jean-Michel Coron.

Since w is continuous, this yields

$$\lim_{s\to+\infty}|\pi(s)|=+\infty \tag{108}$$

and, with (105)

$$\lim_{s\to+\infty}\pi(s)=+\infty. \tag{109}$$

Similarly, by taking $y=-1$, we get

$$\lim_{s\to-\infty}\pi(s)=+\infty. \tag{110}$$

But (109), (110) and the continuity of π imply the existence of non zero real numbers s_1, s_2 and κ_0 such that

$$s_1s_2<0, \quad \pi(s_1)=\pi(s_2)=\kappa_0. \tag{111}$$

So, with (104), we get

$$s_1h(0,x,\pi(s_1))=s_1h(0,x,\kappa_0)<0,$$
$$s_2h(0,x,\pi(s_2))=s_2h(0,x,\kappa_0)<0 \tag{112}$$

or

$$s_1s_2h(0,x,\kappa_0)^2>0. \tag{113}$$

This contradicts (111). □

With Lemma 3.4, we know that, when $m=1$, any continuous function π satisfying (98) is neither upper-bounded nor lower-bounded. Since, π being continuous, $\pi(\mathbb{R})$ is a connected set, we have

$$\pi(\mathbb{R})=\mathbb{R}. \tag{114}$$

This means that π is surjective. It follows that we can find a function λ such that, for all $v\in\mathbb{R}$, we have

$$\pi(-\lambda(v))=v. \tag{115}$$

Then, since from (98) and the continuity of w and π, we have, for all $(s,y)\in\mathbb{R}^2$,

$$sy+w(\pi(s),y)\leqslant 0, \tag{116}$$

we have also, for all $(v,y)\in\mathbb{R}^2$,

$$w(v,y)\leqslant\lambda(v)y. \quad \square \tag{117}$$

Proof of Corollary 3.3. From (59), (117) implies that the zero dynamics of the ignored input dynamics satisfy:

$$\frac{\partial W}{\partial z}(z)j(z,x,u)\leqslant-\alpha(|z|) \quad \forall(z,x,u): y=h(z,x,u)=0. \quad \square \tag{118}$$

We know now that, even by extending as much as possible the dissipativity approach (see the Introduction), we cannot relax the zero relative degree and minimum phase property requirement. This leads us to the question:

Is this requirement intrinsic to the problem of global asymptotic stability in spite of ignored input dynamics or is it coming (at least in part) from the proof technique?

We show below that the fact of using the sum of Lyapunov functions for each subsystem involved in the dissipativity approach is an obstruction to relax this requirement.

Consider the linear system

$$\dot{x} = Ax + By,$$
$$\dot{z} = -cz + u, \quad c > 0, \tag{119}$$
$$y = dz + eu$$

with $x \in \mathbb{R}^n$ is the state of the nominal part, $z \in \mathbb{R}$ is the state of the ignored dynamics. Let $x^T Px$ be a CLF for the nominal system, i.e.

$$\{x^T PB = 0,\ x \neq 0\} \ \Rightarrow \ x^T PAx < 0. \tag{120}$$

May be after rescaling P, let us try the following function to design a globally asymptotically stabilizing control law

$$V(x,z) = \tfrac{1}{2}(x^T Px + z^2). \tag{121}$$

It is the sum of Lyapunov functions for each component of the system. Its time derivative is

$$\overset{\cdot}{\overbrace{V(x,z)}} = x^T PAx + x^T PB(\mathrm{d}z + eu) - cz^2 + uz. \tag{122}$$

Since we look for a control law not depending on z, the best static control that we can choose to make this derivative negative is given by the following min max problem:

$$\min_u \max_z \overset{\cdot}{\overbrace{V(x,z)}}.$$

It is solved by taking

$$u = -x^T PB(2ec + d) \tag{123}$$

which yields

$$\min_u \max_z \overset{\cdot}{\overbrace{V(x,z)}} = x^T(PA - e(ec + d)PBB^T P)x. \tag{124}$$

A sufficient condition to get global stability is that the right hand side of this expression is a negative definite matrix. But when $e(ec + d)$ is nonpositive, this is possible only if $PA + A^T P$ is negative definite, i.e. the nominal system is open loop asymptotically stable. Without such a restriction on the nominal system, we must have:

$$e(ec + d) > 0. \tag{125}$$

Now we observe that the ignored dynamics are

$$\dot{z} = -cz + u,$$
$$y = dz + eu. \tag{126}$$

Their relative degree is zero if $e \neq 0$ and its zero is

$$\text{zer} = -\left(c + \frac{d}{e}\right). \tag{127}$$

It follows that (125) implies that the ignored dynamics have zero relative degree and are minimum phase.

Let us now try another Lyapunov function for (119) which is no more only a sum of Lyapunov functions. For ease of computation, we restrict our attention to the case $n = 1$. Let

$$V(x,z) = \tfrac{1}{2}\theta x^2 + \tau xz + \tfrac{1}{2}z^2 \tag{128}$$

with

$$\theta > \tau^2. \tag{129}$$

We get

$$\overset{\cdot}{\overbrace{V(x,z)}} = \theta Ax^2 + (\theta Bd + \tau(A - c))xz + (\theta Be + \tau)xu + (1 + \tau Be)uz + (\tau Bd - c)z^2. \tag{130}$$

For this expression to be upper-bounded in z, we have to choose τ satisfying

$$\tau Bd < c. \tag{131}$$

Then $\max_z \overset{\cdot}{\overbrace{V(x,z)}}$ is given by

$$z = \frac{(\theta Bd + \tau(A - c))x + (1 + \tau Be)u}{2(c - \tau Bd)} \tag{132}$$

and $\min_u \max_z \overset{\cdot}{\overbrace{V(x,z)}}$ is given by

$$u = -\frac{2(c - \tau Bd)(\theta Be + \tau) + (\theta Bd + \tau(A - c))(1 + \tau Be)}{(1 + \tau Be)^2}x. \tag{133}$$

This yields

$$\min_u \max_z \overset{\cdot}{\overbrace{V(x,z)}} = \frac{[\theta - \tau^2][A - \tau B(d + e(c - A)) - \theta B^2 e(d + ec)]}{(1 + \tau Be)^2}x^2 \tag{134}$$

and shows that θ and τ should be chosen such that the right hand side of this expression is negative definite and (129) and (131) hold. So, in particular, we observe that

- If the relative degree of the ignored dynamics is 1, i.e. $e = 0$, stability is achievable when

$$A < c. \tag{135}$$

- If the ignored dynamics are non minimum phase, i.e. $e(ec+d)<0$, and the open loop nominal system is unstable, i.e. $A>0$, stability is achievable when[7]

$$c^2e^2+de(c-A)<0 \tag{136}$$

or[8]

$$\frac{1}{\text{zer}}+\frac{1}{c}<\frac{1}{A}. \tag{137}$$

This shows that indeed nonzero relative degree and non minimum phase are possible.

3.2.4. About the condition (13)

In the statement of Theorem 3.1, we impose that γ also satisfies condition (13) concerning its behavior around 0. Let us illustrate why we need this restriction by considering the following nominal system studied in Jankovic, Sepulchre, and Kokotović (1998)

$$\dot{x}=x^3+x^2u. \tag{138}$$

It admits a CLF but if we pick $\gamma(s)=s^2$, there is no C^1 function V such that

$$\frac{L_fV(x)}{\gamma(L_gV(x))}=\frac{L_fV(x)}{(L_gV(x))^2}=\frac{1}{V'(x)x} \tag{139}$$

is bounded on a neighborhood of zero, i.e. (13) cannot hold. However when we take

$$V(x)=2\ln(\varepsilon^2+x^2), \tag{140}$$

(12) is satisfied. This shows that indeed, the only problem for having (12) to hold globally is only when x is small. In fact this opens the possibility of getting unbounded solutions with ignored dynamics satisfying (59). Indeed, let

$$\begin{aligned}\dot{z}&=-zf(z)+u,\\ y&=z+u\end{aligned} \tag{141}$$

with the function $f(z)$ positive definite. This system satisfies (59) with the supply rate

$$w(u,y)=uy-u^2. \tag{142}$$

Specifically, we have

$$\overbrace{\tfrac{1}{2}z^2}^{\cdot}=-z^2f(z)+uy-u^2. \tag{143}$$

Also, we have

$$\inf_u\sup_y\{w(u,y)+sy\}=-s^2=-\gamma(s), \tag{144}$$

where $\inf_u$ is given by:

$$u=\pi(s)=-s. \tag{145}$$

This establishes that condition (60) of Theorem 3.1 holds.

So all the assumptions of Theorem 3.1 and (97) are satisfied, except (13). But, we can show (see Hamzi and Praly, 1999) that for instance when

$$f(z)=\exp(z^3), \tag{146}$$

there is no static feedback depending only on x which guarantees both global asymptotic stability for the nominal system and boundedness of the z-components of the solution of the overall system

$$\begin{aligned}\dot{x}&=x^3+x^2y,\\ \dot{z}&=-zf(z)+u,\\ y&=z+u.\end{aligned} \tag{147}$$

It is interesting to observe however that when

$$\gamma(s)=|s|^{(1+c)} \tag{148}$$

with $c>0$, then for a C^1 function V we have

$$\frac{L_fV(x)}{\gamma(L_gV(x))}=\frac{L_fV(x)}{|L_gV(x)|^{(1+c)}}=|V'(x)|^{-c}|x|^{1-2c}. \tag{149}$$

There exists a C^1 function V making this ratio bounded on a neighborhood of 0 iff $c\in(0,\frac{1}{2})$. For such c's, i.e. such γ's, Theorem 3.1 applies and gives a stability margin but which is not for supply rates in the form (142).

We end this section by noting that, when condition (13) is not satisfied, boundedness of all the solutions is achievable when the ignored dynamics have a stronger stability property, i.e. α is a class $\mathscr{K}^\infty$ function.

Theorem 3.5. *Under the conditions of Theorem* 3.1, *if α is a class $\mathscr{K}^\infty$ function, then there exists a controller which guarantees boundedness of all the solutions.*

Proof of Theorem 3.5. By assumption, the functions V and γ satisfy the conditions of Theorem 2.1. This allows us to conclude the existence, for any $\varepsilon>0$, of a function ψ_ε so that, for all $|x|\geqslant\varepsilon$, we have

$$L_f\psi_\varepsilon(V(x))<\gamma(L_g\psi_\varepsilon(V(x))). \tag{150}$$

We take

$$u=\pi(L_g\psi_\varepsilon(V(x))) \tag{151}$$

[7] To "meet" (129), we choose $\tau=\sqrt{\theta}\,\text{sign}(B)\text{sign}(d+e(c-A))=-\sqrt{\theta}\,\text{sign}(B)\text{sign}(e)$. Then (134) is negative if $|B|\sqrt{\theta}$ is between $1/|e|$ and $A/|d+ec|$. (136) is a necessary and sufficient condition for at least one value on this interval to meet (131).

[8] When $A>0$, $c>0$ and zer >0, (137) is also a necessary and sufficient condition for the asymptotic stabilizability of (119) by a partial static state feedback of the form $u=kx$.

with π given by the conditions of the Theorem. Then, from (59) and (60), we get

$$\overbrace{\psi_\varepsilon(V(x)) + W(z)}^{\cdot} \leqslant L_f\psi_\varepsilon(V(x)) - \gamma(L_g\psi_\varepsilon(V(x))) - \alpha(|z|). \tag{152}$$

So, for $|x| \geqslant \varepsilon$, we have

$$\overbrace{\psi_\varepsilon(V(x)) + W(z)}^{\cdot} < -\alpha(|z|). \tag{153}$$

and for $|x| < \varepsilon$, we let

$$A = \sup_{|x|<\varepsilon} L_f\psi_\varepsilon(V(x)) < +\infty. \tag{154}$$

Since α is a class $\mathscr{K}^\infty$ function and W is proper there exists B such that

$$W(z) \geqslant B \;\Rightarrow\; \alpha(|z|) \geqslant A + 1. \tag{155}$$

It follows that, for $|x| < \varepsilon$ and $W(z) \geqslant B$, we have

$$\overbrace{\psi_\varepsilon(V(x)) + W(z)}^{\cdot} \leqslant -1. \tag{156}$$

(153) and (156) allow us to conclude that all the solutions are bounded. □

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