Math. Control Signals Systems (2000) 13: 95–124

Mathematics of Control, Signals, and Systems

On Assigning the Derivative of a Disturbance Attenuation Control Lyapunov Function*

Andrew R. Teel† and Laurent Praly‡

Abstract. We consider feedback design for nonlinear, multi-input affine control systems with disturbances and present results on assigning, by choice of feedback, a desirable upper bound to a given control Lyapunov function (clf) candidate's derivative along closed-loop trajectories. Specific choices for the upper bound are motivated by $\mathscr{L}_2$ and $\mathscr{L}_\infty$ disturbance attenuation problems. The main result leads to corollaries on "backstepping" locally Lipschitz disturbance attenuation control laws that are perhaps implicitly defined through a locally Lipschitz equation. The results emphasize that only rough information about the clf is needed to synthesize a suitable controller. A dynamic control strategy for linear systems with bounded controls is discussed in detail.

Key words. Control Lyapunov function (clf), Disturbance attenuation, Iterative design.

1. Introduction

One of the main analysis tools for verifying stability and/or disturbance attenuation properties for closed-loop control systems is the Lyapunov function—a smooth, positive definite, radially unbounded function. If the derivative of the Lyapunov function can be bounded appropriately, then the resulting differential inequality may be integrated to establish desired closed-loop properties, e.g., $\mathscr{L}_2$ or $\mathscr{L}_\infty$ disturbance attenuation. The control synthesis problem can then be seen as the problem of finding a Lyapunov function that can be assigned a desirable derivative by appropriate choice of feedback.

A (global) control Lyapunov function (clf) for a smooth control system of the form $\dot{x} = f(x) + g(x)u$ has been defined in [S1] to be a smooth, positive definite, radially unbounded function whose derivative along the parameterized vector field $f(x) + g(x)u$ can be made negative for each $x \neq 0$ by an appropriate

* Date received: March 27, 1998. Date revised: March 11, 1999. The first author acknowledges the support of the NSF under Grant ECS-9502034 and of the AFOSR under Grants F49620-96-1-0144 and F49620-98-1-0087.

† Department of Electrical and Computer Engineering, University of California, Santa Barbara, California 93106, U.S.A. teel@ece.ucsb.edu.

‡ Centre Automatique et Systèmes, École des Mines de Paris, 35 rue St. Honoré, 77305 Fontainebleau cédex, France. praly@cas.ensmp.fr.

choice of the control parameter u. When a clf for $\dot{x} = f(x) + g(x)u$ is given, a smooth function $k : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}^m$ can be constructed from the clf and its derivatives along $f(x)$ and $g(x)$ so that the derivative of the clf along the vector field $f(x) + g(x)k(x)$ is negative whenever $x \neq 0$. Thus, there is an intimate connection between the existence of this type of clf for $\dot{x} = f(x) + g(x)u$ and the construction of a feedback law that globally asymptotically stabilizes the origin of this system. See [A] and [S1] for more details.

Control Lyapunov functions have also been characterized for control systems with disturbances. In Section 4 of [FK3] a robust control Lyapunov function (rclf) for a system $\dot{x} = f(x,d) + g(x,d)u$ was defined to be a continuously differentiable, positive definite, radially unbounded function whose derivative along $f(x,d) + g(x,d)u$ can, for each $x \neq 0$, be made negative, uniformly in d belonging to a compact set depending on x, by an appropriate choice of u. A similar notion is used in [SW]. In Section 6 of [FK3] it was pointed out how the notion of an rclf encompasses the $\mathscr{L}_\infty$ disturbance attenuation property. The rclf's discussed in [FK3] do not address other disturbance attenuation properties directly but the required modifications to the definition of the rclf are not difficult. Part of the contribution here is in that direction.

In this paper we consider feedback design for nonlinear, multi-input control systems with disturbances of the form

$$\dot{x} = f(x,d) + g(x)u, \qquad x \in \mathbb{R}^n, \quad d \in \mathbb{R}^p, \quad u \in \mathbb{R}^m. \tag{1}$$

We develop a general notion of a clf for these systems in Section 3.1. The definition of a clf is in terms of a desired upper bound, denoted $\tilde{\alpha}(x,d)$, for the clf's derivative and also in terms of a preliminary feedback function $\pi(x)$ which is more general than, but can be thought of as being like, minus the derivative of the clf along the matrix field $g(x)$. (The main assumption on $\pi(x)$ is that the derivative of the clf along the vector field $g(x)\pi(x)$ is nonpositive.) In Section 2 we show how various disturbance attenuation problems motivate specific choices for the upper bound $\tilde{\alpha}(x,d)$. In Section 3 we state and prove our main results and make some connections to other results in the literature. In Section 3.2 we characterize a class of upper bounds $\tilde{\alpha}(x,d)$ that can be assigned to the derivative of the clf based on a bound for the derivative of the clf when the derivative of the clf along the vector field $g(x)\pi(x)$ is zero. In Section 4 we apply our results to $\mathscr{L}_\infty$ and $\mathscr{L}_2$ disturbance attenuation problems, including gain assignment when controlling through perturbed integrators. These problems have also been discussed in [JTP] and Section 9.5 of [I1], for example.

Throughout the paper we emphasize that only rough information about the clf and the system is needed to synthesize a desired controller. A motivating example, discussed in detail in Section 5, is the problem of global nonlinear $\mathscr{L}_2$ disturbance attenuation for the system

$$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} Ax_1 + B\mathrm{sat}(-B^T P(x_2)x_1) + d \\ u \end{pmatrix} = f(x,d) + g(x)u, \tag{2}$$

where $x_1 \in \mathbb{R}^n$, $x_2 \in \mathbb{R}$, and "sat" is a standard saturation function. Here, under appropriate assumptions, a feedback $x_2 = \theta_1(x_1)$ exists, but is only known implicitly through an equation of the form $\pi(x_1, \theta_1(x_1)) = 0$, that gives nonlinear $\mathscr{L}_2$ disturbance attenuation for the x_1 subsystem. (See, for example, [M].) This property is verified via the derivative of a Lyapunov function expressed in terms of $\theta_1(x_1)$. There is a natural choice for a clf $V(x)$ for the full system (2) that has desired properties when the derivative of $V(x)$ along the vector field g is zero. However, this choice is such that $V(x)$ depends explicitly on $\theta_1(x_1)$. Our main results tell us how to choose u to achieve the nonlinear $\mathscr{L}_2$ disturbance attenuation properties of the x_1 subsystem under the feedback $x_2 = \theta_1(x_1)$ without knowing $\theta_1(x_1)$ explicitly, i.e., without knowing the clf explicitly. (In Section 3.2 of [CPT] a general stabilization—disturbances set to zero—problem is considered where the stabilizer for the reduced-order system is known implicitly through an equation while the Lyapunov function for the reduced-order system is known explicitly. It is then shown how to construct an explicit clf for the extended system, at least locally.)

2. Motivation and Preliminaries

A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to *belong to class-$\mathscr{K}$* ($\alpha \in \mathscr{K}$) if it is continuous, zero at zero, and strictly increasing. It is said to *belong to class-$\mathscr{K}_\infty$* if, in addition, it is unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to *belong to class-$\mathscr{K}\mathscr{L}$* if, for each $t \geq 0$, $\beta(\cdot, t)$ is nondecreasing and $\lim_{s \to 0^+} \beta(s, t) = 0$, and, for each $s \geq 0$, $\beta(s, \cdot)$ is nonincreasing and $\lim_{t \to \infty} \beta(s, t) = 0$. A continuous function $\rho : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is said to be *positive definite* if $\rho(x) > 0$ for all $x \neq 0$. A continuous function $\rho : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is said to be *radially unbounded* if $\rho(x) \to \infty$ as $x \to \infty$.

Motivated by certain disturbance attenuation problems discussed below, we develop sufficient conditions for synthesizing a continuous state feedback $u = k(x)$ that assigns a given upper bound $\tilde{\alpha}(x(t), d(t))$ to the time derivative of a locally Lipschitz clf candidate $V(x(t))$ along closed-loop trajectories. In other words, we are looking for a continuous function $k(x)$ so that

$$\dot{V}(x(t)) \leq \tilde{\alpha}(x(t), d(t)) \qquad \text{for almost all } t, \tag{3}$$

where $x(t)$ is any absolutely continuous function satisfying

$$\dot{x}(t) = f(x(t), d(t)) + g(x(t))k(x(t)) \qquad \text{for almost all } t. \tag{4}$$

Throughout the paper when considering solutions of ordinary differential equations, generically denoted $\dot{X} = F(X, t)$, we assume that the Carathéodory conditions are satisfied, i.e., F is continuous in X, measurable in t, and, for each compact set $\mathscr{C}$ of $\mathbb{R}^n$ and each interval $[a, b]$ of $\mathbb{R}_{\geq 0}$, there exists an integrable function $m : [a, b] \to \mathbb{R}_{\geq 0}$ such that

$$|F(X, t)| \leq m(t), \qquad \forall (X, t) \in \mathscr{C} \times [a, b]. \tag{5}$$

This guarantees that, for each initial condition and starting time, at least one absolutely continuous solution of (4) exists locally in time, i.e., on the interval $[t_\circ, T)$ for some $T > t_\circ$ where $t_\circ$ denotes the starting time. (See, for example, Section I.5 of [H].) Then, since V is locally Lipschitz, $V(x(t))$ is absolutely continuous [N, Theorem 2, p. 245] and $\dot{V}(x(t))$ is well-defined for almost all $t \in [t_\circ, T)$ [N, Corollary, p. 246]. Assuming that $\tilde{\alpha}(x(t), d(t))$ is locally integrable, the function φ defined by

$$\varphi(t) := V(x(t)) - V(x(t_\circ)) - \int_{t_\circ}^{t} \tilde{\alpha}(x(s), d(s))\, ds \tag{6}$$

is absolutely continuous [N, Theorem 1, p. 252], its derivative is defined for almost all $t \in [t_\circ, T)$ and, using (3), it satisfies

$$\dot{\varphi}(t) = \dot{V}(x(t)) - \tilde{\alpha}(x(t), d(t)) \le 0. \tag{7}$$

It follows from (7) (see, for example, Theorem 3.1 of [S3]) that $\varphi(t) \le \varphi(t_\circ)$ for all $t \in [t_\circ, T)$, which, from (6), implies that

$$V(x(t)) \le V(x(t_\circ)) + \int_{t_\circ}^{t} \tilde{\alpha}(x(s), d(s))\, ds, \qquad \forall t \in [t_\circ, T). \tag{8}$$

For $\mathscr{L}_2$ disturbance attenuation, one function $\tilde{\alpha}(x, d)$ in (3) that we use is

$$\tilde{\alpha}(x, d) = \gamma^2 |d|^2. \tag{9}$$

If $d \in \mathscr{L}_2$, i.e., d is measurable and $\|d\|_2^2 := \int_0^\infty |d(t)|^2\, dt < \infty$, then it follows from (8) with (9) and $t_\circ = 0$ that

$$V(x(t)) \le V(x(0)) + \gamma^2 \|d\|_2^2, \qquad \forall t \in [0, T). \tag{10}$$

If $V(x)$ is positive definite and radially unbounded in x, then we conclude from (10) that each solution is defined on $[0, \infty)$. If we also have that (3) is satisfied with

$$\tilde{\alpha}(x, d) = -\kappa(x)|h(x)|^2 + \gamma^2 |d|^2, \tag{11}$$

where κ is continuous and positive, then, denoting by $V_{\max}$ the upper bound on $V(x(t))$ from (10), it follows from (8) with (11) and $t_\circ = 0$ that

$$\|h(x)\|_2^2 \le (\gamma^2 \|d\|_2^2 + V(x(0))) \frac{1}{\min\limits_{\{x : V(x) \le V_{\max}\}} \kappa(x)}. \tag{12}$$

When $\kappa(x) = 1/\tilde{\kappa}(V(x))$ with $\tilde{\kappa} : \mathbb{R}_{\ge 0} \to \mathbb{R}_{>0}$ nondecreasing and $V(x(0)) = 0$, we get a nonlinear $\mathscr{L}_2$ gain from d to $y = h(x)$ given by

$$\|h(x)\|_2 \le \sqrt{\tilde{\kappa}(\gamma^2 \|d\|_2^2)}\gamma \|d\|_2. \tag{13}$$

When $\tilde{\kappa}(x) \equiv 1$ this is the standard case of $\mathscr{L}_2$ disturbance attenuation with linear gain $\gamma \cdot s$ ("finite gain" γ).

For $\mathscr{L}_\infty$ disturbance attenuation, we are interested in functions $\tilde{\alpha}$ satisfying

$$V(x) \ge \max\{\gamma(|d|), \varepsilon\} \quad \Rightarrow \quad \tilde{\alpha}(x, d) \le -\kappa(V(x)), \tag{14}$$

where $\varepsilon \geq 0$, and γ and κ are functions of class-$\mathcal{K}_\infty$. If $d \in \mathcal{L}_\infty$, i.e., d is measurable and $\|d\|_\infty := \text{ess sup}_{t\geq 0}|d(t)| < \infty$, it follows from (3) with (14) that

$$V(x(t)) \geq \max\{\gamma(\|d\|_\infty), \varepsilon\} \quad \Rightarrow \quad \dot{V} \leq -\kappa(V(x(t))). \tag{15}$$

With the differential inequality $\dot{V} \leq -\kappa(V(x(t)))$, and in other more general situations, we can apply the following result to determine a bound on $V(x(t))$:

Lemma 1 [LL, Theorem 1.10.2]. *Let $\alpha(V, t)$ satisfy the Carathéodory conditions. If $V(t)$ is absolutely continuous, defined on $[t_\circ, T)$ and satisfies*

$$\dot{V} \leq \alpha(V(t), t) \qquad \textit{for almost all} \quad t \in [t_\circ, T) \tag{16}$$

and if $U(t)$ is the maximal solution of

$$\dot{U} = \alpha(U, t), \qquad U(t_\circ) = U_\circ \geq V(t_\circ) \tag{17}$$

with maximal interval of definition $[t_\circ, \bar{T})$, then $V(t) \leq U(t)$ for all $t \in [t_\circ, \min\{\bar{T}, T\})$.

This type of lemma is used in the proof of Theorem 1 of [S2] to conclude for the system (4) that if, in addition to (15), $V(x)$ is radially unbounded and there exists a class-$\mathcal{K}_\infty$ function δ such that $\delta(|h(x)|) \leq V(x)$, then, for each initial condition, the solutions are defined for all $t \geq 0$ and satisfy

$$|h(x(t))| \leq \max\{\beta(V(x_\circ), t), \delta^{-1} \circ \gamma(\|d\|_\infty), \delta^{-1}(\varepsilon)\}, \qquad \forall t \geq 0, \tag{18}$$

where $\beta \in \mathcal{KL}$. The property (18) with $h(x) = x$ and $\varepsilon = 0$ is known as input-to-state stability (ISS). For more general functions $h(x)$, it is known as input-to-output stability (IOS). See [S2].

Regarding the derivative of $V(x(t))$, since $V(x)$ is assumed to be locally Lipschitz, if $\dot{x} = v(t)$ for almost all t, then $\dot{V}$ is defined for almost all t and equals the usual one-sided directional derivative, i.e., for almost all t,

$$\dot{V}(x(t)) = \lim_{h\to 0^+} \frac{V(x(t) + hv(t)) - V(x(t))}{h}. \tag{19}$$

Rather than work directly with the one-sided directional derivative, we work with the generalized directional derivative of Clarke [C] defined as

$$V^\circ(x; v) := \limsup_{h\to 0^+, y\to x} \frac{V(y + hv) - V(y)}{h}. \tag{20}$$

Comparing (19) with (20), we see that the generalized directional directive is an upper bound for the usual directional derivative. Moreover, the generalized directional derivative offers the following convenient properties when $V(x)$ is locally Lipschitz:

1. [C, Proposition 2.2.4] At points where $V(x)$ is continuously differentiable, $V^\circ(x; v) = (\partial V/\partial x)(x)v$.

2. [C, Proposition 2.1.1(a)] If ψ is a nonnegative scalar, then

$$V^\circ(x; v_1 + v_2\psi) \le V^\circ(x; v_1) + V^\circ(x; v_2)\psi. \tag{21}$$

3. [C, Proposition 2.1.1(b)] $V^\circ(x; v)$ is upper semicontinuous, i.e.,

$$\limsup_{(y,w)\to(x,v)} V^\circ(y; w) = V^\circ(x; v). \tag{22}$$

4. [C, Proposition 2.1.1(a)] If $V(x)$ has Lipschitz constant L in an open neighborhood $\mathscr{X}$, then $|V^\circ(x; v)| \le L|v|$ for all $x \in \mathscr{X}$.
5. If $f(x,d)$ and $\tilde\alpha(x,d)$ are continuous and

$$\frac{\partial V}{\partial x}(x) f(x,d) \le \tilde\alpha(x,d), \qquad \forall d, \forall x \notin \Omega, \tag{23}$$

   where Ω is a set of measure zero containing the set where V is not differentiable,[1] then

$$V^\circ(x; f(x,d)) \le \tilde\alpha(x,d), \qquad \forall (x,d). \tag{24}$$

   This follows from the corollary on p. 64 of [C] which says that

$$V^\circ(x; f(x,d)) \le \limsup_{y\to x,\, y\notin\Omega} \frac{\partial V}{\partial x}(y) f(x,d); \tag{25}$$

   rewriting $f(x,d)$ on the right-hand side of the inequality as $f(y,d) + f(x,d) - f(y,d)$, using that $(\partial V/\partial x)(y) f(y,d) \le \tilde\alpha(y,d)$ and then that $\tilde\alpha$ and $f(\cdot,d)$ are continuous and that $(\partial V/\partial x)(y)$ is bounded for $y \notin \Omega$ near x, we are drawn to the conclusion.
6. If $V(x)$, where $x = [x_1^T \;\; x_2^T]^T$, is continuously differentiable in x_2 and $v = [0 \;\; v_2^T]^T$, then $V^\circ(x; v) = (\partial V/\partial x_2)(x) v_2$. This follows from the definition of $V^\circ(x; v)$ and the fact that, from the mean value theorem and the differentiability of V with respect to x_2, $V(y + hv) - V(y) = (\partial V/\partial x_2)(z) h v_2$ for some z on the line segment connecting y to $y + hv$.
7. [C, Proposition 2.3.3] $(V_1 + V_2)^\circ(x, v) \le V_1^\circ(x, v) + V_2^\circ(x, v)$.
8. [C, Proposition 2.3.13] If $V(x) = \mu|x_2 - \theta_1(x_1)|^2$, where $x = [x_1^T \;\; x_2^T]^T$, and θ_1 is locally Lipschitz, then $V^\circ(x; v)_{|x_2 = \theta_1(x_1)} = 0$.

In the rest of the paper, except where precision is necessary, we replace $V^\circ(x, v)$ by the notation $L_v V(x)$, which is usually reserved for continuously differentiable functions where it stands for $(\partial V/\partial x)(x) v$. We do this for the sake of the reader more familiar with results for continuously differentiable Lyapunov functions and because of the similarities between the Lie derivative and the generalized directional derivative described above. As further abuse of notation, when $g(x)\pi(x)$ is a vector field we replace $V^\circ(x; g(x)\pi(x))$ by the notation $L_{g(x)} V(x)\pi(x)$. Also, whenever we write $L_{g(x)} V(x)$ alone, we mean the row vector with ith entry given

[1] Rademacher's theorem says that since V is locally Lipschitz it is differentiable except on a set of measure zero.

by $V^{\circ}(x; g_i(x))$. This last bit of notation will only be used in rigorous statements when $V(x)$ is continuously differentiable.

3. Assignable Upper Bounds for clf's

3.1. Main Result

We have motivated our desire to solve the following problem: given a locally Lipschitz function $V(x)$ and another function $\tilde{\alpha}(x, d)$, find, when possible, a function $k(x)$ so that

$$L_{f(x,d)}V(x) + L_{g(x)}V(x)k(x) \leq \tilde{\alpha}(x, d), \qquad \forall (x, d). \tag{26}$$

In fact, we consider the more specific problem where a function $\pi : \mathbb{R}^n \to \mathbb{R}^m$ is given such that $L_{g(x)}V(x)\pi(x)$ is nonpositive and we must find a locally bounded, i.e., bounded on compact sets, function $\psi : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that

$$k(x) = \pi(x)\psi(x) \tag{27}$$

solves our problem. For the case where there are no disturbances and $V(x)$ is C^1 and $\pi(x) = -L_{g(x)}V(x)^T$, this mirrors Sontag's "universal formula" for stabilization [S1]. We consider more general functions $\pi(x)$ since we will be considering problems where $L_{g(x)}V(x)$ is not known exactly.

Definition 1. The function $\tilde{\alpha}(x, d)$ is said to be an *assignable upper bound* for the derivative of V using π if there exists a locally bounded function $\psi(x)$ so that, with (27), (26) holds.

If $\psi^*(x)$ establishes that $\tilde{\alpha}(x, d)$ is an assignable upper bound using π, then, since $L_{g(x)}V(x)\pi(x)$ is assumed to be nonpositive, any feedback of the form $u = \pi(x)\psi(x)$ where $\psi(x) \geq \psi^*(x)$ also assigns the upper bound $\tilde{\alpha}(x, d)$. As a consequence, we can always take $\psi(x)$ to be locally Lipschitz or smooth.

Two properties will be used to characterize when $\tilde{\alpha}(x, d)$ is an assignable upper bound for the derivative of V using π. Both will be expressed in terms of the function

$$\omega(x) := \sup_d \{L_{f(x,d)}V(x) - \tilde{\alpha}(x, d)\}. \tag{28}$$

The first property, a "clf" property, parallels the "clf" and "rclf" definitions in [S1] and [FK3, Definition 4.1], respectively.

Definition 2. The locally Lipschitz function $V(x)$ is a *control Lyapunov function* (*clf*) *for the pair* $(\pi, \tilde{\alpha})$ if $L_{g(x)}V(x)\pi(x)$ is nonpositive, $\omega(x)$ given in (28) is well-defined, $\max\{0, \omega(x)\}$ is locally bounded, and, for $x \neq 0$,

$$\limsup_{z \to x} L_{g(z)}V(z)\pi(z) = 0 \quad \Rightarrow \quad \limsup_{z \to x} \omega(z) < 0. \tag{29}$$

The next property is related to the "small control property" found in [S1] in the setting of stabilization without disturbances, and in [FK3] for stabilization with disturbances constrained to a state-dependent set. Further connections will be made in Theorem 3 below.

Definition 3. The locally Lipschitz function $V(x)$ satisfies the *bounded control property for the pair* $(\pi, \tilde{\alpha})$ if there exist $\chi > 0$ and $\bar{v} \geq 0$ such that, with $\omega(x)$ defined in (28),

$$\omega(x) + \bar{v} L_{g(x)} V(x)\pi(x) \leq 0, \qquad \forall |x| \leq \chi. \tag{30}$$

The main result of this section is the following:

Theorem 1. *If $V(x)$ is a clf and satisfies the bounded control property for the pair $(\pi, \tilde{\alpha})$, then $\tilde{\alpha}(x,d)$ is an assignable upper bound for the derivative of V using π.*

Proof. Define the function $\psi^* : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ by

$$\psi^*(x) := \begin{cases} \dfrac{\max\{0, \omega(x)\}}{-L_{g(x)} V(x)\pi(x)} & \text{if} \quad L_{g(x)} V(x)\pi(x) \neq 0, \\ 0 & \text{if} \quad L_{g(x)} V(x)\pi(x) = 0. \end{cases} \tag{31}$$

We first establish that $\psi^*(x)$ is locally bounded. From the definition of $\psi^*(x)$, the bounded control property implies that $\psi^*(x) \leq \bar{v}$ for all $|x| \leq \chi$. For $|x| \geq \chi$, since $\max\{0, \omega(x)\}$ is assumed to be locally bounded and since $L_{g(x)} V(x)\pi(x)$ is nonpositive, it is sufficient to have that $\limsup_{z\to x} L_{g(z)} V(z)\pi(z) = 0$ implies $\limsup_{z\to x} \omega(z) < 0$. This follows from (29) in the clf property.

The second fact to establish is that

$$\omega(x) + L_{g(x)} V(x)\pi(x)\psi^*(x) \leq 0, \qquad \forall x \in \mathbb{R}^n, \tag{32}$$

which, from the definition of $\omega(x)$ in (28), is equivalent to

$$L_{f(x,d)} V(x) + L_{g(x)} V(x)\pi(x)\psi^*(x) \leq \tilde{\alpha}(x,d), \qquad \forall (x,d) \in \mathbb{R}^n \times \mathbb{R}^p. \tag{33}$$

If $L_{g(x)} V(x)\pi(x) \neq 0$, then

$$\omega(x) + L_{g(x)} V(x)\pi(x)\psi^*(x) = \omega(x) - \max\{0, \omega(x)\} \leq 0. \tag{34}$$

For the case where $L_{g(x)} V(x)\pi(x) = 0$ we must establish that $\omega(x) \leq 0$. From the bounded control property, when $|x| \leq \chi$ and $L_{g(x)} V(x)\pi(x) = 0$ we have $\omega(x) \leq 0$. For the case where $|x| \geq \chi$, since $L_{g(x)} V(x)\pi(x)$ is nonpositive we have

$$L_{g(x)} V(x)\pi(x) = 0 \quad \Rightarrow \quad \limsup_{z\to x} L_{g(z)} V(z)\pi(z) = 0, \tag{35}$$

and so, from (29) in the clf property, we also have that $L_{g(x)} V(x)\pi(x) = 0$ and $|x| \geq \chi$ imply $\limsup_{z\to x} \omega(z) < 0$. Since $\omega(x) \leq \limsup_{z\to x} \omega(z)$, the result follows. ∎

Remark 3.1. The function $\psi^*(x)$ is the minimum norm value for v, as a function of x, that satisfies

$$\omega(x) + v L_{g(x)} V(x)\pi(x) \leq 0. \tag{36}$$

It is the same choice as in equation (23) of [FK3]. When $\omega(x)$ and $L_{g(x)}V(x)\pi(x)$ are locally Lipschitz on $\mathbb{R}^n\backslash\{0\}$ then it is not difficult to verify that $\psi^*(x)$ defined in (31) is locally Lipschitz on $\mathbb{R}^n\backslash\{0\}$. As pointed out earlier, even when $\psi^*(x)$ is not locally Lipschitz we can always upper bound it by a locally Lipschitz (on $\mathbb{R}^n$) function and assign the same bound $\tilde{\alpha}(x,d)$. If $\pi(x)$ and $\psi^*(x)$ are locally Lipschitz (resp. continuous) on $\mathbb{R}^n\backslash\{0\}$ and $\pi(x)$ is everywhere continuous and zero at zero, then the feedback we are proposing, $u = \pi(x)\psi(x)$, is locally Lipschitz (resp. continuous) on $\mathbb{R}^n\backslash\{0\}$ and everywhere continuous. This observation is useful for guaranteeing the existence of solutions to the closed-loop differential equation.

We apply this result to certain disturbance attenuation problems in subsequent sections. For $\mathscr{L}_\infty$ disturbance attenuation, the conclusions that we draw from our main result are very similar to the results in Propositions 4.4 and 4.5 of [FK3] expressed in terms of the notion of an rclf. One of the main aspects we emphasize throughout the paper is that only rough information about the system and $V(x)$ is needed to synthesize our controller. For example, assuming that $V(x)$ is a clf and satisfies the bounded control property for $(\pi, \tilde{\alpha})$, we only need to find an upper bound for the function $\psi^*(x)$ defined in (31) to assign the bound $\tilde{\alpha}$. Moreover, in the case where V is a clf for the pair $(-L_g V^T, \tilde{\alpha})$, we do not need the exact magnitude or direction of $L_{g(x)}V(x)$ to find a smooth function $\pi(x)$ such that V is a clf for the pair $(\pi, \tilde{\alpha})$. In this sense, we do not need precise information about the clf to assign the desired upper bound to its derivative. This observation is related to gain and phase margin properties of $L_g V$ controllers made precise in [SJK].

If $V(x)$ is continuously differentiable and radially unbounded, and if $V(x)$ and $L_{g(x)}V(x)$ are known and $V(x)$ is a clf and satisfies the bounded control property for the pair $(-L_g V^T, \tilde{\alpha})$, where the bounded control property in this case implies that there exist $\chi > 0$ and $\bar{v} \geq 0$ such that, for all $|x| \leq \chi$,

$$\sup_d \{L_{f(x,d)}V(x) - \tilde{\alpha}(x,d)\} \leq \bar{v}|L_{g(x)}V(x)|^2, \tag{37}$$

then the control can be taken as

$$k(x) = -(L_{g(x)}V(x))^T \psi_v(V(x)) = -L_{g(x)}\left(\int_0^{V(x)} \psi_v(s)\, ds\right)^T \tag{38}$$

with ψ_v any continuous function satisfying

$$\psi_v(V(x)) \geq \psi^*(x). \tag{39}$$

So we see that, maybe after reassigning the values of the level sets of V to obtain a new clf

$$\hat{V}(x) = \int_0^{V(x)} \psi_v(s)\, ds, \tag{40}$$

we can assign the upper bound $\psi_v(V(x))\tilde{\alpha}(x,d)$ to the derivative of $\hat{V}$ using a so-called $L_g\hat{V}$ controller (see Section 3.4.3 of [SJK] and see [FP]).

3.2. Sufficient Conditions for a clf

The next two results provide sufficient conditions for V to be a clf for the pair $(\pi, \tilde{\alpha})$. The utility of these results is that they can be used to guarantee that (29) holds without actually having to compute $\omega(x)$ defined in (28). The sufficient conditions are given in terms of a relationship between $\tilde{\alpha}(x,d)$ and a bound on $L_{f(x,d)}V(x)$ when $L_{g(x)}V(x)\pi(x) = 0$ (the function $L_{g(x)}V(x)\pi(x)$ is assumed to be continuous in these results.)

Theorem 2. *Let $f(x,d)$ be continuous, let $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be locally Lipschitz, and let $\pi : \mathbb{R}^n \to \mathbb{R}^m$ be such that $L_{g(x)}V(x)\pi(x)$ is continuous and nonpositive. If $\alpha(x,d)$ is such that*

$$L_{g(x)}V(x)\pi(x) = 0 \quad \Rightarrow \quad L_{f(x,d)}V(x) \leq \alpha(x,d), \tag{41}$$

then $V(x)$ is a clf for the pair $(\pi, \tilde{\alpha})$ for any $\tilde{\alpha}$ satisfying all of the following:

1. $\sup_d \{L_{f(x,d)}V(x) - \tilde{\alpha}(x,d)\}$ *is well-defined and locally bounded,*
2. $\tilde{\alpha}(x,d)$ *is lower semicontinuous on the set $L_{g(x)}V(x)\pi(x) = 0$, i.e.,*

$$L_{g(x)}V(x)\pi(x) = 0 \quad \Rightarrow \quad \liminf_{(y,e)\to(x,d)} \tilde{\alpha}(y,e) = \tilde{\alpha}(x,d), \tag{42}$$

3. *there exist functions $\rho_1(x)$ (continuous and nonnegative) and $\rho_2(x)$ (continuous and positive definite) such that*

$$|d| \geq \rho_1(x) \quad \Rightarrow \quad L_{f(x,d)}V(x) - \tilde{\alpha}(x,d) \leq -\rho_2(x), \tag{43}$$

4. $\tilde{\alpha}(x,d) - \alpha(x,d) \geq \rho(x)$ *for some continuous, positive definite function ρ.*

Proof. Since $L_{g(x)}V(x)\pi(x)$ is continuous, to show that $V(x)$ is a clf for the pair $(\pi, \tilde{\alpha})$ we need to show that, for all $x \neq 0$ such that $L_{g(x)}V(x)\pi(x) = 0$, $\limsup_{z\to x} \omega(z) < 0$ where $\omega(z)$ was defined in (28). To this end we will show that, for each $x \neq 0$ such that $L_{g(x)}V(x)\pi(x) = 0$, there exist $\varepsilon_x > 0$ and a compact neighborhood $\mathscr{V}_x$ of x not containing the origin such that, for all $(z,d) \in \mathscr{V}_x \times \mathbb{R}^p$,

$$L_{f(z,d)}V(z) - \tilde{\alpha}(z,d) \leq -\varepsilon_x. \tag{44}$$

Let $\tilde{\mathscr{V}}_x$ be a compact neighborhood of x not containing the origin, let $D_x = \sup_{z\in\tilde{\mathscr{V}}_x} \rho_1(z)$ and

$$\varepsilon_x = \inf_{z\in\tilde{\mathscr{V}}_x} (\min\{0.5\rho(z), \rho_2(z)\}). \tag{45}$$

Using the compactness of $\tilde{\mathscr{V}}(x)$, $D_x < \infty$ since ρ_1 is continuous, and $\varepsilon_x > 0$ since ρ and ρ_2 are continuous, positive definite, and $0 \notin \tilde{\mathscr{V}}_x$. Then, from condition 3 imposed on $\tilde{\alpha}$, we have

$$\{z \in \tilde{\mathscr{V}}_x, |d| \geq D_x\} \quad \Rightarrow \quad L_{f(z,d)}V(z) - \tilde{\alpha}(z,d) \leq -\varepsilon_x. \tag{46}$$

It remains to show that (44) holds for $|d| \leq D_x$, perhaps by restricting z to a compact neighborhood of x contained in $\tilde{\mathscr{V}}_x$. Assume, for the time being, that there exists $b_x > 0$ such that

$$\{z \in \tilde{\mathcal{V}}_x, |d| \leq D_x, L_{g(z)}V(z)\pi(z) \geq -b_x\} \quad \Rightarrow \quad L_{f(z,d)}V(z) - \tilde{\alpha}(z,d) \leq -\varepsilon_x. \tag{47}$$

We then define

$$\mathcal{V}_x := \tilde{\mathcal{V}}_x \cap \{z : L_{g(z)}V(z)\pi(z) \geq -b_x\}, \tag{48}$$

which is a neighborhood of x from the continuity of $L_{g(x)}V(x)\pi(x)$. It follows from (46) and (47) that (44) holds for all $(x,d) \in \mathcal{V}_x \times \mathbb{R}^p$.

The proof will be complete when we establish the existence of $b_x > 0$ such that (47) holds. If such a b_x does not exist, then there exists a sequence (z_m, d_m) with $z_m \in \tilde{\mathcal{V}}_x$ and $|d_m| \leq D_x$ such that

$$L_{g(z_m)}V(z_m)\pi(z_m) \geq -1/m, \qquad L_{f(z_m,d_m)}V(z_m) - \tilde{\alpha}(z_m,d_m) > -\varepsilon_x. \tag{49}$$

By compactness, there exists a cluster point (x^*, d^*). Still denoting by (z_m, d_m) the converging subsequence, we have, from the continuity and nonpositiveness of $L_g V\pi$,

$$L_{g(x^*)}V(x^*)\pi(x^*) = 0, \qquad \limsup_{m\to\infty}(L_{f(z_m,d_m)}V(z_m) - \tilde{\alpha}(z_m,d_m)) \geq -\varepsilon_x. \tag{50}$$

However, with the continuity of $f(x,d)$ and, in turn, the upper semicontinuity of $L_f V - \tilde{\alpha}$ on the set $L_g V\pi = 0$, which follows from property 3 of the generalized directional derivative given in Section 2 and condition 2 imposed on $\tilde{\alpha}$, this implies

$$L_{g(x^*)}V(x^*)\pi(x^*) = 0, \qquad L_{f(x^*,d^*)}V(x^*) - \tilde{\alpha}(x^*,d^*) \geq -\varepsilon_x \geq -0.5\rho(x^*). \tag{51}$$

On the other hand we know, from (41), the first part of (51) and condition 4 imposed on the function $\tilde{\alpha}$, that

$$L_{f(x^*,d^*)}V(x^*) - \tilde{\alpha}(x^*,d^*) \leq \alpha(x^*,d^*) - \tilde{\alpha}(x^*,d^*) \leq -\rho(x^*). \tag{52}$$

This contradiction of the second part of (51) proves the existence of b_x. ■

A corollary of the previous result is the following:

Corollary 1. *Let $f(x,d)$ be continuous, let $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be locally Lipschitz, and let $\pi : \mathbb{R}^n \to \mathbb{R}^m$ be such that $L_{g(x)}V(x)\pi(x)$ is continuous and nonpositive. If $\alpha(x,d)$ is such that*

$$L_{g(x)}V(x)\pi(x) = 0 \quad \Rightarrow \quad L_{f(x,d)}V(x) \leq \alpha(x,d) \tag{53}$$

and the quantity $\sup_d\{L_{f(x,d)}V(x) - \alpha(x,d)\}$ is well-defined and locally bounded, then $V(x)$ is a clf for the pair $(\pi, \tilde{\alpha})$ for any $\tilde{\alpha}$ satisfying

1. *$\tilde{\alpha}$ is lower semicontinuous on the set $L_{g(x)}V(x)\pi(x) = 0$,*
2. *there exist functions $\rho(x)$ (continuous, positive definite) and $\rho_d(d)$ (continuous, nonnegative, radially unbounded) such that*

$$\tilde{\alpha}(x,d) - \alpha(x,d) \geq \max\{\rho(x), \rho_d(d)\}. \tag{54}$$

Proof. Condition 2 of the corollary implies condition 4 of the previous theorem. Condition 2 of the corollary and the assumption that $\sup_d\{L_{f(x,d)}V(x) - \alpha(x,d)\}$ is well-defined and locally bounded imply condition 1 of the theorem. Condition 1 of the corollary implies condition 2 of the theorem. We will show that condition 2 of the corollary and the assumption that $\sup_d\{L_{f(x,d)}V(x) - \alpha(x,d)\}$ is well-defined and locally bounded imply condition 3 of the theorem.

Since $\sup_d\{L_{f(x,d)}V(x) - \alpha(x,d)\}$ is well-defined and locally bounded, we can find a continuous, nonnegative function $\rho_3(x)$ satisfying

$$\rho_3(x) \geq \sup_d\{L_{f(x,d)}V(x) - \alpha(x,d)\}, \qquad \forall x \in \mathbb{R}^n. \tag{55}$$

Let $\rho_2(x)$ be an arbitrary continuous, positive definite function. Since ρ_d is radially unbounded, we can find a continuous, nonnegative function $\rho_1(x)$ such that

$$|d| \geq \rho_1(x) \quad \Rightarrow \quad \rho_d(d) \geq \rho_3(x) + \rho_2(x). \tag{56}$$

Using (54) and (55), we have

$$L_{f(x,d)}V(x) - \tilde{\alpha}(x,d) \leq \rho_3(x) - \tilde{\alpha}(x,d) + \alpha(x,d) \leq \rho_3(x) - \rho_d(d). \tag{57}$$

Then, combining (56) and (57), we have

$$|d| \geq \rho_1(x) \quad \Rightarrow \quad L_{f(x,d)}V(x) - \tilde{\alpha}(x,d) \leq -\rho_2(x). \quad \blacksquare \tag{58}$$

3.3. *Sufficient Conditions for the Bounded Control Property*

Throughout this section we assume that V is continuously differentiable and we address the relationship between the bounded control property (Definition 3) and the small control property used in [S1] and [FK3]. We will show that the bounded control property holds if a small control property holds (see Definition 5) and the function $\pi(x)$ makes $-L_{g(x)}V(x)\pi(x)$ large enough. To characterize the latter, we make the following definition:

Definition 4. Given a C^1 function $V(x)$ and a function $\lambda : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ (resp. $\lambda_\circ : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$), the function π is said to locally dominate λ (resp. $\lambda_\circ$) if there exist $\chi > 0$ and $\mu > 0$ such that, for all $|x| \leq \chi$,

$$-L_{g(x)}V(x)\pi(x) \geq \mu|L_{g(x)}V(x)|\lambda(|L_{g(x)}V(x)|) \tag{59}$$

$$(\text{resp.} -L_{g(x)}V(x)\pi(x) \geq \mu|L_{g(x)}V(x)|\lambda_\circ(|L_{g(x)}V(x)|, |x|)). \tag{60}$$

Remark 3.2. One choice for $\pi(x)$ (locally) so that it locally dominates the identity, i.e., $\lambda(s) = s$, is

$$\pi(x) = -L_{g(x)}V(x)^T. \tag{61}$$

More generally, a choice for $\pi(x)$ (locally) so that it locally dominates λ is

$$\pi(x) = -\frac{\lambda(|L_{g(x)}V(x)|)L_{g(x)}V(x)^T}{|L_{g(x)}V(x)|}. \tag{62}$$

If $L_{g(x)}V(x)$ is smooth and λ is smooth on $(0, \infty)$, continuous everywhere, and zero at zero, then the right-hand side of (62) is smooth on $\mathbb{R}^n \backslash \{L_{g(x)}V(x) = 0\}$ and continuous on $\mathbb{R}^n$. Similarly, one choice for π (locally) so that it locally dominates $\lambda_\circ$ is

$$\pi(x) = -\frac{\lambda_\circ(|L_{g(x)}V(x)|, |x|)L_{g(x)}V(x)^T}{|L_{g(x)}V(x)|}. \tag{63}$$

In what follows we will encounter functions $\lambda_\circ$ that make the right-hand side of (63) smooth on $\mathbb{R}^n \backslash \{0\}$ and continuous on $\mathbb{R}^n$ when $L_{g(x)}V(x)$ is smooth.

Definition 5. The C^1 function V satisfies *the small control property for* $\tilde{\alpha}$ if there exists a continuous, positive definite function $p(x)$ satisfying: for each $\varepsilon > 0$ there exists $v > 0$ such that $|x| \leq v$ implies the existence of u such that $|u| \leq \varepsilon$ and

$$\sup_d \{L_{f(x,d)}V(x) - \tilde{\alpha}(x, d)\} + L_{g(x)}V(x)u + p(x) \leq 0. \tag{64}$$

The following theorem relates the bounded control property and the small control property:

Theorem 3. *If the C^1 function V satisfies the small control property for $\tilde{\alpha}$, then there exists $\lambda : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ smooth on $(0, \infty)$, continuous everywhere, and zero at zero, and there exists $\lambda_\circ$ (with the right-hand side of (63) smooth on $\mathbb{R}^n \backslash \{0\}$ and continuous everywhere when $L_{g(x)}V(x)$ is smooth) such that if π locally dominates λ or π locally dominates $\lambda_\circ$, then $V(x)$ satisfies the bounded control property for $(\pi, \tilde{\alpha})$.*

Proof. *Step 1.* First we establish that the small control property implies that there exist a function $\gamma \in \mathcal{K}$ that is smooth on $(0, \infty)$ and a strictly positive real number χ such that for all $|x| \leq \chi$ we have

$$\sup_d \{L_{f(x,d)}V(x) - \tilde{\alpha}(x, d)\} - |L_{g(x)}V(x)|\gamma(|x|) + p(x) \leq 0. \tag{65}$$

It follows from the small control property that, as long as $\gamma(0) = 0$, the inequality (65) holds for $x = 0$. Without loss of generality, assume that $v(\varepsilon)$ given by the small control property is nondecreasing in ε. Let $\tilde{\chi} > 0$ and $\gamma \in \mathcal{K}_\infty$, smooth on $(0, \infty)$, be such that

$$\gamma^{-1}(s) \leq v(s), \qquad \forall s \in (0, \tilde{\chi}]. \tag{66}$$

Consider x satisfying $0 < |x| \leq \gamma^{-1}(\tilde{\chi}) =: \chi$ and let $\varepsilon = \gamma(|x|) > 0$. Then we have $\varepsilon \leq \tilde{\chi}$ and thus $|x| = \gamma^{-1}(\varepsilon) \leq v(\varepsilon)$. It is sufficient to establish that

$$-|L_{g(x)}V(x)|\gamma(|x|) \leq L_{g(x)}V(x)u, \qquad \forall |u| \leq \varepsilon. \tag{67}$$

The right-hand side is minimized under the constraint by taking

$$u = -\frac{\varepsilon L_{g(x)}V(x)^T}{|L_{g(x)}V(x)|}. \tag{68}$$

This gives

$$L_{g(x)}V(x)u = -|L_{g(x)}V(x)|\varepsilon = -|L_{g(x)}V(x)|\gamma(|x|). \tag{69}$$

Step 2. Let χ and γ come from step 1. Let $\rho \in \mathscr{K}_\infty$ be smooth on $(0, \infty)$ and such that

$$\rho(|x|)\gamma(|x|) \le p(x), \qquad \forall |x| \le \chi. \tag{70}$$

Define $\lambda(s) := \gamma \circ \rho^{-1}(s)$, and note that $\lambda(s)$ is smooth on $(0, \infty)$. Also, define $\lambda_\circ(s, t) := b(s, \rho(t))\gamma(t)$ where $b : \mathbb{R}_{\ge 0} \times \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is smooth on $(0, \infty) \times (0, \infty)$, such that $b(s, \tau) \ge 1$ when $s \ge \tau$ and $b(s, \tau) = 0$ when $s \le \tau/2$ with $b(0, 0) = 0$. (It follows that the right-hand side of (63) is smooth on $\mathbb{R}^n \backslash \{0\}$ and continuous on $\mathbb{R}^n$ when $L_{g(x)}V(x)$ is smooth.) Using (70), we have

$$\{|x| \le \chi, |L_{g(x)}V(x)| \le \rho(|x|)\} \quad \Rightarrow \quad |L_{g(x)}V(x)|\gamma(|x|) \le p(x). \tag{71}$$

On the other hand, when $|L_{g(x)}V(x)| \ge \rho(|x|)$ we have, from the definition of λ and $\lambda_\circ$,

$$\begin{aligned} |L_{g(x)}V(x)|(\gamma(|x|) - \lambda(|L_{g(x)}V(x)|)) &\le |L_{g(x)}V(x)|(\gamma(|x|) - \gamma \circ \rho^{-1}(|L_{g(x)}V(x)|)) \\ &\le 0 \le p(x) \end{aligned} \tag{72}$$

and

$$|L_{g(x)}V(x)|(\gamma(|x|) - \lambda_\circ(|L_{g(x)}V(x)|, |x|)) \le 0 \le p(x). \tag{73}$$

From here we carry out the calculation for λ. The case for $\lambda_\circ$ goes through verbatim.

The inequalities (71) and (72) imply that for all $|x| \le \chi$ we have

$$|L_{g(x)}V(x)|(\gamma(|x|) - \lambda(|L_{g(x)}V(x)|)) \le p(x). \tag{74}$$

Finally, it follows from (65) that, for all (x, d) with $|x| \le \chi$,

$$\begin{aligned} &L_{f(x,d)}V(x) - |L_{g(x)}V(x)|\lambda(|L_{g(x)}V(x)|) \\ &\quad = L_{f(x,d)}V(x) - |L_{g(x)}V(x)|\gamma(|x|) + |L_{g(x)}V(x)|(\gamma(|x|) - \lambda(|L_{g(x)}V(x)|)) \\ &\quad \le \tilde{\alpha}(x, d) - p(x) + |L_{g(x)}V(x)|(\gamma(|x|) - \lambda(|L_{g(x)}V(x)|)) \\ &\quad \le \tilde{\alpha}(x, d). \end{aligned}$$

It follows readily that if π satisfies: there exists $\mu > 0$ such that, for sufficiently small x,

$$-L_{g(x)}V(x)\pi(x) \ge \mu|L_{g(x)}V(x)|\lambda(|L_{g(x)}V(x)|), \tag{75}$$

then the bounded control property is satisfied. ∎

Remark 3.3. The proof of the theorem shows that if there exist strictly positive real numbers k and $\bar{p}$ such that, for sufficiently small x,

$$\sup_d \{L_{f(x,d)}V(x) - \tilde{\alpha}(x, d)\} - |L_{g(x)}V(x)|k|x| + \bar{p}|x|^2 \le 0, \tag{76}$$

then V satisfies the bounded control property for $(\pi, \tilde{\alpha})$ for any π that locally dominates the identity. This amounts to saying that if (76) holds for sufficiently small x, then (37) holds for sufficiently small x and some $\bar{v} > 0$. This also follows directly from completing squares. The more general calculations of the proof lead to this conclusion as well since, in the case of (76), we simply have

$$\gamma(s) = ks, \qquad \rho(s) = \frac{\bar{p}}{k}s, \qquad \lambda(s) = \frac{k^2}{\bar{p}}s. \tag{77}$$

As pointed out in Remark 3.2, one choice for $\pi(x)$ so that it locally dominates the identity is $-L_{g(x)}V(x)^T$.

Remark 3.4. If there exists $\mu > 0$ such that, for sufficiently small x,

$$-L_{g(x)}V(x)\pi(x) \geq \mu|\pi(x)|\,|L_{g(x)}V(x)|, \tag{78}$$

then, with ψ^* given in (31), for sufficiently small x satisfying $L_{g(x)}V(x) \neq 0$, we have

$$|\pi(x)|\psi^*(x) \leq \mu^{-1}\frac{\max\{0, \omega(x)\}}{|L_{g(x)}V(x)|}. \tag{79}$$

When V satisfies the small control property, the right-hand side of (79) converges to zero as x approaches the origin since, according to (65), for x near the origin we have

$$\omega(x) \leq |L_{g(x)}V(x)|\gamma(|x|), \tag{80}$$

where γ is continuous and zero at zero. In this case the magnitude of the feedback $k(x) = \pi(x)\psi^*(x)$ converges to zero even though the bounded control property may not be satisfied for the given $\pi(x)$.

Given a clf V, often there is a tradeoff between smoothness of the control law at the origin and the inherent gain and phase margin robustness of L_gV-type controllers—controllers $u = k(x)$ having the property that $L_{g(x)}V(x)k(x)$ is nonpositive. We illustrate that this is the case even for systems without disturbances. Consider the system

$$\begin{aligned} \dot{x}_1 &= -x_1^3 + x_2, \\ \dot{x}_2 &= u + x_1, \end{aligned} \tag{81}$$

with the clf candidate $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$ for the pair $(-x_2, -x_1^4 - x_2^2)$. The function $V(x)$ is a clf for this pair since $L_{g(x)}V(x)\pi(x) = 0$ implies $x_2 = 0$ and, hence,

$$L_{f(x)}V(x) = -x_1^4 + 2x_1x_2 = -x_1^4 - x_2^2. \tag{82}$$

On the other hand, $V(x)$ does not satisfy the bounded control property for this pair. If it did, then we could smoothly stabilize the origin with a control of the form $-\psi(x)x_2$. However, regardless of the value of $\psi(0)$, the linearization would always have an eigenvalue with positive real part. This is a contradiction to the stability that would be implied by the assigned bound $-x_1^4 - x_2^2$ to the derivative of V. We conclude that, for the given clf, there is no smooth $L_{g(x)}V(x)$-type controller, i.e., one that vanishes when $x_2 = 0$, that assigns the upper bound $-x_1^4 - x_2^2$ globally. On the other hand, the smooth (non-$L_{g(x)}V(x)$-type for the given V)

control $u = -x_2 - 2x_1$ does assign the upper bound $-x_1^4 - x_2^2$ to the closed-loop derivative of V.

Given a clf $V(x)$, it is possible to get smoothness near the origin while still retaining $L_g V$-type controller properties away from the origin. Suppose we know a (smooth) feedback $k_s(x)$ and a strictly positive real number χ such that, for all $|x| \le \chi$,

$$\sup_d \{L_{f(x,d)} V(x) - \tilde{\alpha}(x,d)\} + L_{g(x)} V(x) k_s(x) \le 0. \tag{83}$$

Let $\ell : \mathbb{R}_{\ge 0} \to [0,1]$ be smooth such that $\ell(s) = 1$ for $s \le \chi/2$ and $\ell(s) = 0$ for $s \ge \chi$. Next, note that if $V(x)$ is a clf for the pair $(\pi, \tilde{\alpha})$ and

$$\limsup_{z \to x} L_{g(z)} V(z) \pi(z) = 0 \quad \Rightarrow \quad \limsup_{z \to x} L_{g(z)} V(z) \ell(|z|) k_s(z) \le 0 \tag{84}$$

(which is the case when V is continuously differentiable and $\pi(x) = -L_{g(x)} V(x)^T$), then $V(x)$ is also a clf for the pair $(\pi, \tilde{\alpha} - L_{g(x)} V(x) \ell(|x|) k_s(x))$. Moreover, from the definition of ℓ and (83), $V(x)$ satisfies the bounded control property for this new pair. It follows that we can find a smooth function $\psi(x)$ such that the control $u = \ell(|x|) k_s(x) + \pi(x) \psi(x)$ assigns the upper bound $\tilde{\alpha}(x,d)$ to the closed-loop derivative of $V(x)$.

4. Disturbance Attenuation clf's

In this section we provide conditions which guarantee that we can assign to the derivative of $V(x)$ an upper bound that is useful for establishing either $\mathscr{L}_\infty$ or $\mathscr{L}_2$ disturbance attenuation properties. Of course, the resulting control law must guarantee the existence of solutions for the disturbance attenuation properties actually to hold. This section includes a study of assigning $\mathscr{L}_2$ or $\mathscr{L}_\infty$ input–output gain when controlling through perturbed integrators. In particular, we study the system

$$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} q(x,d) \\ u + \varphi(x,d) \end{pmatrix} = f(x,d) + g(x)u, \qquad y = h(x) \tag{85}$$

and we discuss what can be said about gain assignment for the full system based on what can be said for the x_1 subsystem with x_2 thought of as a control that is allowed to be only locally Lipschitz in x_1, perhaps defined implicitly through a locally Lipschitz equation. For other results on "backstepping" locally Lipschitz control laws, see Section 5.4 of [FK2] and see [FK1].

Recall that we say $d \in \mathscr{L}_\infty$ if d is essentially bounded, i.e., $\|d\|_\infty := \operatorname{ess\,sup}_{t \ge 0} |d(t)| < \infty$. We say that $d \in \mathscr{L}_2$ if d is square integrable, i.e., $\|d\|_2^2 := \int_0^\infty |d(t)|^2 \, dt < \infty$.

4.1. ISS clf's

For the $\mathscr{L}_\infty$ (ISS/IOS) case, we have motivated in Section 2 that we want to assign to the derivative of V a bound $\tilde{\alpha}(x,d)$ that satisfies

$$V(x) \ge \max\{\gamma(|d|), \varepsilon\} \quad \Rightarrow \quad \tilde{\alpha}(x,d) \le -\kappa(V(x)) \tag{86}$$

for some $\varepsilon \geq 0$ and functions γ and κ of class-$\mathscr{K}_\infty$. We will see that the following assumption on V and π will make V a clf for a pair $(\pi, \tilde{\alpha})$ with $\tilde{\alpha}$ satisfying (86).

Assumption 1. The function $f(x,d)$ is continuous and the locally Lipschitz, positive definite, radially unbounded function $V(x)$, the function $\pi(x)$, and the class-$\mathscr{K}_\infty$ functions δ, γ, and κ satisfy

1. $\delta(|h(x)|) \leq V(x)$,
2. $L_{g(x)}V(x)\pi(x)$ is continuous and nonpositive,
3. $\{L_{g(x)}V(x)\pi(x) = 0, V(x) \geq \gamma(|d|)\} \Rightarrow L_{f(x,d)}V(x) \leq -\kappa(V(x))$.

Remark 4.1. According to Definition 4.1 of [FK3], if V is C^1 and if $L_{g(x)}V(x)\pi(x) = 0$ only when $L_{g(x)}V(x) = 0$, then Assumption 1 makes V an rclf and so the results of [FK3] apply.

Remark 4.2. In [SW] the authors give a sufficient condition for the third condition of the assumption to hold for some γ when f is affine in d, i.e., $f(x,d) = f_0(x) + f_1(x)d$.

Corollary 2. *If Assumption* 1 *holds, then there exists* $\tilde{\alpha}(x,d)$ *satisfying*

$$V(x) \geq \gamma(|d|) \quad \Rightarrow \quad \tilde{\alpha}(x,d) \leq -0.5\kappa(V(x)) \tag{87}$$

such that $V(x)$ *is a clf for the pair* $(\pi, \tilde{\alpha})$.

Proof. The proof is based on Corollary 1. First we define

$$\alpha(x,d) := \begin{cases} -\kappa(V(x)) & \text{if} \quad V(x) \geq \gamma(|d|), \\ \rho(|d|) & \text{if} \quad V(x) < \gamma(|d|), \end{cases} \tag{88}$$

where $\rho(\cdot)$ is a continuous function satisfying

$$\sup_{\{x: V(x) \leq \gamma(|d|)\}} L_{f(x,d)}V(x) \leq \rho(|d|). \tag{89}$$

We claim that

$$L_{g(x)}V(x)\pi(x) = 0 \quad \Rightarrow \quad L_{f(x,d)}V(x) \leq \alpha(x,d). \tag{90}$$

This follows by considering the two cases $V(x) \geq \gamma(|d|)$ and $V(x) < \gamma(|d|)$. For the former case, (90) follows from condition 3 of Assumption 1 and the definition of $\alpha(x,d)$ in (88). For the latter case, (90) follows from (89) and the definition of $\alpha(x,d)$ in (88).

To see that the quantity $\sup_d\{L_{f(x,d)}V(x) - \alpha(x,d)\}$ is well-defined and locally bounded, we note that, from the choice for ρ in (89), we have

$$\sup_d\{L_{f(x,d)}V(x) - \alpha(x,d)\} \leq \max\left\{0, \sup_{\{d: V(x) \geq \gamma(|d|)\}} L_{f(x,d)}V(x) + \kappa(V(x))\right\}. \tag{91}$$

Since γ is radially unbounded and V is continuous, the right-hand side is locally bounded.

We also remark that the function α is lower semicontinuous, i.e.,

$$\alpha(x,d) = \liminf_{(y,e)\to(x,d)} \alpha(y,e), \tag{92}$$

and

$$V(x) \geq \gamma(|d|) \quad \Rightarrow \quad \alpha(x,d) = -\kappa(V(x)). \tag{93}$$

We now take

$$\tilde{\alpha}(x,d) := \alpha(x,d) + 0.5\kappa(\max\{V(x), \gamma(|d|)\}) \tag{94}$$

so that

$$V(x) \geq \gamma(|d|) \quad \Rightarrow \quad \tilde{\alpha}(x,d) = -0.5\kappa(V(x)). \tag{95}$$

Moreover, we have that $\tilde{\alpha}$ is lower semicontinuous and condition 2 of Corollary 1 holds with $\rho(x) = 0.5\kappa(V(x))$ and $\rho_d(d) = 0.5\kappa \circ \gamma(|d|)$. It follows from Corollary 1 that $V(x)$ is a clf for the pair $(\pi, \tilde{\alpha})$. ■

The next result, which applies to the system (85) where a perturbed integrator is added, is similar to what is reported in [JTP] and [PJ].

Proposition 1. *For the system* (85), *if there exist a locally Lipschitz function* $\theta_1(x_1)$, *a locally Lipschitz function* $\pi(x)$, *a positive definite, radially unbounded, locally Lipschitz function* $V_1(x_1)$, *and three class-*$\mathcal{K}_\infty$ *functions* δ, γ, *and* κ *satisfying*

1. $\delta(|h(x)|) \leq V_1(x_1)$,
2. $\{V_1(x_1) \geq \gamma(|d|), x_2 = \theta_1(x_1)\} \Rightarrow L_{q(x,d)} V_1(x_1) \leq -\kappa(V_1(x_1))$,
3. $(x_2 - \theta_1(x_1))^T \pi(x)$ *is nonpositive and zero only when* $x_2 = \theta_1(x_1)$,

then, for each $\mu > 0$, *the functions*

$$V(x) = V_1(x_1) + \mu|x_2 - \theta_1(x_1)|^2, \tag{96}$$

$\pi(x)$, δ, γ, *and* κ *satisfy Assumption* 1.

Proof. Using property 6 of the generalized directional derivative given in Section 2, we have that

$$L_{g(x)} V(x)\pi(x) = 2\mu(x_2 - \theta_1(x_1))^T \pi(x). \tag{97}$$

Since $\pi(x)$ and $\theta_1(x_1)$ are assumed to be locally Lipschitz, so is $L_{g(x)} V(x)\pi(x)$. By assumption, $L_{g(x)} V(x)\pi(x)$ is nonpositive and zero only when $x_2 = \theta_1(x_1)$. Also, using property 7 of the generalized directional derivative given in Section 2, we have

$$L_{f(x,d)} V(x) \leq L_{q(x,d)} V_1(x_1) + L_{f(x,d)}(\mu|x_2 - \theta_1(x_1)|^2). \tag{98}$$

By assumption, when $L_{g(x)}V(x)\pi(x) = 0$ we have $x_2 = \theta_1(x_1)$ and thus, using property 8 of the generalized directional derivative given in Section 2, we have

$$L_{g(x)}V(x)\pi(x)=0 \quad \Rightarrow \quad \{x_2=\theta_1(x_1), L_{f(x,d)}V(x)\le L_{q(x,d)}V_1(x_1), V(x)=V_1(x_1)\}. \tag{99}$$

We conclude that

$$\{V(x) \ge \gamma(|d|), L_{g(x)}V(x)\pi(x) = 0\} \quad \Rightarrow \quad L_{f(x,d)}V(x) \le -\kappa(V(x)). \quad \blacksquare \tag{100}$$

It follows from Corollary 2 that Proposition 1 provides conditions under which $V(x)$ defined in (96) is a clf for the pair $(\pi, \tilde{\alpha})$ for the system (85) with $\tilde{\alpha}$ satisfying (87). If $V(x)$ also satisfies the bounded control property for the pair $(\pi, \tilde{\alpha})$, then, combining Corollary 2 with Theorem 1, we have a new locally Lipschitz feedback $\theta_2(x) = \pi(x)\psi(x)$ and a new positive definite, radially unbounded, locally Lipschitz function $V_2(x) = V(x)$ that can be used for another application of Proposition 1 if another perturbed integrator is added. In the process, γ, which characterizes the IOS gain, remains unchanged, and κ becomes 0.5κ.

By iterating this process for a chain of n perturbed integrators (which is possible if, at each step, a bounded control property holds), we get a control law of the form

$$u = \psi_{n-1}(x)\pi_{n-1}(x), \tag{101}$$

where

$$\pi_{i-1}(x) = -(x_i - \pi_{i-2}(x)\psi_{i-2}(x)) \tag{102}$$

and $\psi_i(x)$ comes from the ith application of Corollary 2 with Theorem 1. The form of this control law is very similar to what is used in [TP] for semiglobal stabilization with partial state feedback. The only difference is that, there, the functions $\psi_i(x)$ are (sufficiently large) constants.

Regarding whether, under Assumption 1, V satisfies the bounded control property for the pair $(\pi, \tilde{\alpha})$ where $\tilde{\alpha}$ was constructed in the proof of Corollary 2, it follows from (91) and the definition of $\tilde{\alpha}$ in (94) that

$$\sup_d\{L_{f(x,d)}V(x) - \tilde{\alpha}(x,d)\}$$
$$\le \max\left\{0, \sup_{\{d: V(x)\ge\gamma(|d|)\}} L_{f(x,d)}V(x) + \kappa(V(x))\right\} - 0.5\kappa(V(x)). \tag{103}$$

So, V satisfies the bounded control property for $(\pi, \tilde{\alpha})$ if there exist $\chi > 0$ and $\bar{v} \ge 0$ such that

$$\sup_{\{d: V(x)\ge\gamma(|d|)\}} L_{f(x,d)}V(x) + 0.5\kappa(V(x)) \le -\bar{v}L_{g(x)}V(x)\pi(x), \qquad \forall |x| \le \chi. \tag{104}$$

When the condition (104) is not satisfied, it still may be possible to modify the function $\tilde{\alpha}$ near $(x,d) = (0,0)$, thereby only changing the IOS gain locally, to induce the bounded control property. Two examples of this are described in the next corollaries which are proved in the Appendix.

The first result says that if an $\mathscr{L}_\infty$ gain with an arbitrarily small offset at the origin is allowed, i.e., $\varepsilon > 0$ in (86), then the bounded control property will hold as long as $x = 0$ is an equilibrium point of the system when $d \equiv 0$.

The second result says that if, when $d \equiv 0$, V satisfies the bounded control property for (π, α) for some continuous, negative definite function $\alpha(x)$, then the function γ in (86) can be modified near the origin so that $\tilde{\alpha}(x, d)$ satisfies (86) with $\varepsilon = 0$ and $V(x)$ satisfies the bounded control property for $(\pi, \tilde{\alpha})$.

Corollary 3. *If Assumption* 1 *holds and* $f(0,0) = 0$, *then, for each* $\varepsilon > 0$, *there exists* $\tilde{\alpha}_\varepsilon(x, d)$ *satisfying*

$$V(x) \geq \max\{\gamma(|d|), \varepsilon\} \quad \Rightarrow \quad \tilde{\alpha}_\varepsilon(x, d) \leq -0.5\kappa(V(x)) \tag{105}$$

such that $V(x)$ *is a clf and satisfies the bounded control property for the pair* $(\pi, \tilde{\alpha})$.

Corollary 4. *If Assumption* 1 *holds,* $V(0) = 0$, *and, for the system* $\dot{x} = f(x, 0) + g(x)u$, *there exists* $\tilde{\kappa} \in \mathscr{K}_\infty$ *such that* $V(x)$ *satisfies the bounded control property for the pair* $(\pi, -\tilde{\kappa}(V(x)))$, *then, for each* $v > 0$, *there exist class-*$\mathscr{K}_\infty$ *functions* γ_v *and* κ_v *and a function* $\tilde{\alpha}_v(x, d)$ *satisfying*

$$\begin{aligned} s \geq v \quad &\Rightarrow \quad \gamma_v(s) = \gamma(s), \qquad \kappa_v(s) = \kappa(s), \\ V(x) \geq \gamma_v(|d|) \quad &\Rightarrow \quad \tilde{\alpha}_v(x, d) \leq -0.5\kappa_v(V(x)) \end{aligned} \tag{106}$$

such that $V(x)$ *is a clf and satisfies the bounded control property for the pair* $(\pi, \tilde{\alpha}_v)$.

4.2. $\mathscr{L}_2$ *clf's*

For $\mathscr{L}_2$ disturbance attenuation problems, we have motivated in Section 2 that we want to assign to the derivative of V a bound $\tilde{\alpha}(x, d)$ of the form

$$\tilde{\alpha}(x, d) = -\kappa(x)|h(x)|^2 + \gamma^2|d|^2, \tag{107}$$

where κ is a continuous, positive-valued function. We will see that the following assumption on V and π will make V a clf for the pair $(\pi, \tilde{\alpha})$ with $\tilde{\alpha}$ of the form given in (107).

Assumption 2. The functions $f(x, d)$ and $h(x)$ are continuous and the locally Lipschitz function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, the continuous, positive definite function $\alpha : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, the strictly positive real number γ, the continuous function $\kappa : \mathbb{R}^n \to \mathbb{R}_{>0}$, and the function π satisfy

1. $L_{g(x)}V(x)\pi(x)$ is continuous and nonpositive,
2. $L_{g(x)}V(x)\pi(x) = 0 \Rightarrow L_{f(x,d)}V(x) \leq -\alpha(x) - \kappa(x)|h(x)|^2 + \gamma^2|d|^2$,
3. there exists $\rho_1(x)$ (continuous, nonnegative) and $\rho_2(x)$ (continuous, positive definite) such that

$$|d| \geq \rho_1(x) \quad \Rightarrow \quad L_{f(x,d)}V(x) + 0.5\alpha(x) + \kappa(x)|h(x)|^2 - \gamma^2|d|^2 \leq -\rho_2(x). \tag{108}$$

Corollary 5. *If Assumption* 2 *is satisfied, then* $V(x)$ *is a clf for the pair* $(\pi, \tilde{\alpha})$ *where*

$$\tilde{\alpha}(x, d) = -0.5\alpha(x) - \kappa(x)|h(x)|^2 + \gamma^2|d|^2. \tag{109}$$

Proof. Follows from Theorem 2. ■

If $f(x, d)$ is affine in d, i.e., $f(x, d) = f_\circ(x) + f_1(x)d$, then the third condition of Assumption 2 is automatically satisfied. This follows by noting that, with $d \in \mathbb{R}^p$ and $f_{1i}(x)$ representing the ith column of $f_1(x)$, we have from property 2 of the generalized directional derivative given in Section 2 that

$$L_{f_1(x)d} V(x) \leq \sum_{i=1}^{p} L_{f_{1i}(x)d_i} V(x) \leq \sum_{i=1}^{p} |d_i| L_{f_{1i}(x)\,\mathrm{sgn}(d_i)} V(x). \tag{110}$$

For this affine case and when V is C^1 and $L_{g(x)} V(x)\pi(x) = 0$ only when $L_{g(x)} V(x) = 0$ and $\kappa(x) \equiv 1$, related results have been established in Lemmas 9.5.2 and 9.5.3 of [I1]. The control laws in Lemmas 9.5.2 and 9.5.3 of [I1] are everywhere smooth but not of the $L_g V$-type—the controllers $u = k(x)$ are not such that $L_{g(x)} V(x)k(x)$ is nonpositive. (The use of $L_g V$-type controllers for achieving the result we have presented is discussed at the end of Section 9.5 of [I1].) Thus, for the affine case, if the $V(x)$ does not satisfy the bounded control property for the pair $(\pi, \tilde{\alpha})$, where $\tilde{\alpha}$ is defined in (109) with $\kappa(x) \equiv 1$, the results of [I1] and the discussion at the end of Section 3 may be used to get a controller that is smooth at the origin but with the $L_{g(x)} V(x)$ structure away from the origin.

We again consider the system (85) with the added assumption that d enters in an affine manner. The result here is essentially the same as those in Theorem 9.5.4 and Corollary 9.5.6 of [I1] (see also I2).

Proposition 2. *For the system* (85) *under the assumption that d enters in an affine manner, if there exist a locally Lipschitz function* $\theta_1(x_1)$, *a locally Lipschitz function* $\pi(x)$, *a nonnegative, locally Lipschitz function* $V_1(x_1)$, *a continuous, positive definite function* α_1, *a continuous, positive-valued function* $\kappa_1(x_1)$, *and a strictly positive real number* γ *such that*

1. $x_2 = \theta_1(x_1) \Rightarrow L_{q(x,d)} V_1(x_1) \leq -\alpha_1(x_1) - \kappa_1(x_1)|h(x)|^2 + \gamma^2|d|^2$,
2. $(x_2 - \theta_1(x_1))^T \pi(x)$ *is nonpositive and zero only when* $x_2 = \theta_1(x_1)$,

then, for each $\mu > 0$ *and each continuous, positive definite function* $\alpha(x)$ *satisfying*

$$x_2 = \theta_1(x_1) \quad \Rightarrow \quad \alpha(x) \leq \alpha_1(x_1), \tag{111}$$

the functions

$$V(x) = V_1(x_1) + \mu|x_2 - \theta_1(x_1)|^2, \tag{112}$$

$\pi(x)$, $\alpha(x)$, $\kappa(x) = \kappa_1(x_1)$, *and* γ *satisfy Assumption* 2.

Proof. The proof is the same as that for Proposition 1 except that from (98) we conclude

$$L_{g(x)} V(x)\pi(x) = 0 \quad \Rightarrow \quad L_{f(x,d)} V(x) \leq -\alpha(x) - \kappa(x)|h(x)|^2 + \gamma^2|d|^2. \quad ■ \tag{113}$$

Combining Proposition 2 with Corollary 5, we have conditions under which $V(x)$ defined in (112) is a clf for the pair $(\pi, \tilde{\alpha})$ for the system (85) with $\tilde{\alpha}$ satisfying (109). If $V(x)$ also satisfies the bounded control property for this pair, then, with Theorem 1, we have a new locally Lipschitz feedback $\theta_2(x) = \pi(x)\psi(x)$ and a new nonnegative locally Lipschitz function $V_2(x) = V(x)$ that can be used for another application of Proposition 2 if another perturbed integrator is added. In the process, γ and κ, which characterize the $\mathscr{L}_2$ gain, remain unchanged. This procedure can be repeated for a chain of perturbed integrators of length n as long as at each step a bounded control property is satisfied.

5. $\mathscr{H}_\infty$-Based Control of Linear Systems with Input Saturation

Much progress has been made on the problem of global stabilization and disturbance attenuation for linear systems with bounded controls in recent years. One set of results in this area has been to show that algebraic Riccati equations can be used to construct control laws that yield global, or semiglobal, stabilization and disturbance attenuation for all stabilizable linear systems having no exponentially unstable open-loop modes. For semiglobal results, see [SLT], [L], and the references therein. For global results, see [WB], [M], [T], [SF], and [SAS] among others. For most of the global results, a family of algebraic Riccati equations is employed and a selection from the family is made by solving an implicit equation. In this section we apply the results of this paper to show that the implicit control law can be made explicit and dynamic without paying a price in the level of disturbance attenuation.

We consider the control system

$$\dot{x}_1 = Ax_1 + B\phi(v) + d, \tag{114}$$

where (A, B) is stabilizable and A has no eigenvalues in the open right half-plane. The function ϕ is continuous and such that

$$v^T\phi(v) \geq \min\{|v|^2, |v|\}. \tag{115}$$

Letting $\mathscr{M}_n$ denote the set of positive definite, symmetric $n \times n$ matrices, we assume we have constructed a function $P : \mathbb{R} \to \mathscr{M}_n$ such that $P(r)$ is locally Lipschitz, $P(r) \to 0$ as $r \to \infty$, for each $r \in \mathbb{R}_{\geq 0}$ we have

$$A^T P(r) + P(r)A + P(r)\left(\frac{1}{\gamma^2}I - BB^T\right)P(r) =: Q(r) < 0, \tag{116}$$

and, where $P(\cdot)$ is differentiable, we have

$$-p_2(r) \cdot \mathrm{Id} \leq \frac{dP}{dr} \leq -p_1(r) \cdot \mathrm{Id} \tag{117}$$

for some continuous, strictly positive functions p_1 and p_2. (Note that $p_1(r) \to 0$ as $r \to \infty$.) Since P is supposed to be only locally Lipschitz, all subsequent derivatives should be interpreted as holding almost everywhere. For example, such a function could come from the positive definite, stabilizing solution to the algebraic Riccati equation

$$A^T P + PA + P\left(\frac{1}{\gamma^2} I - BB^T\right) P + \frac{1}{1 + \chi(r)} I = 0, \tag{118}$$

where $\chi(r)$ is smooth, monotonically strictly increasing with $\lim_{r\to\infty} \chi(r) = \infty$ and $\lim_{r\to-\infty} \chi(r) = -0.5$, $\chi(0) = 0$, and γ is sufficiently large. See [T] and [M], for example, for a discussion of the properties of such an equation. Alternatively (see [M]), if we have an infinite sequence $\{P_i\}_{i=-\infty}^{+\infty}$ of positive definite matrices such that $P_{i+1} - P_i > 0$, $\lim_{i\to\infty} P_i = 0$, and, for each $i \geq 0$, the inequality (116) is satisfied, then we can synthesize $P(r)$ as

$$P(r) = [(1 - q(r))P_{i(r)}^{-1} + q(r)P_{i(r)+1}^{-1}]^{-1}, \tag{119}$$

where $i(r)$ is the largest integer i such that $i \leq r$ and $q(r) = r - i(r)$.

Now we let ε be some small positive real number and define

$$\eta(x_1, x_2) := [x_1^T P(x_2) x_1 + \varepsilon] \cdot \operatorname{tr}(B^T P(x_2) B). \tag{120}$$

The value of ε should be small enough so that $\varepsilon \cdot \operatorname{tr}(B^T P(0) B) < 1$, so that we have $\eta(x_1, 0) < 1$ for x_1 sufficiently small. The function $\eta(x_1, x_2)$ is always positive and, since $dP/dr \leq -p_1(r)I$, there exists a continuous, positive definite function $p_3(x_2)$ such that $(\partial\eta/\partial x_2)(x_1, x_2) \leq -p_3(x_2)$. It follows from the mean-value theorem for Lipschitz functions [C, Proposition 2.6.5] that $\eta(x_1, x_2)$ is strictly decreasing in x_2. Next, we implicitly define a function $\theta_1(x_1)$ as the unique solution of the equation $C(x_1, x_2) = 0$ where

$$C(x_1, x_2) := \eta(x_1, x_2) - \min\{\eta(x_1, 0), 1\}. \tag{121}$$

We are guaranteed that the equation $C(x_1, x_2) = 0$ has a unique solution for each $x_1 \in \mathbb{R}^n$ since $\eta(x_1, x_2)$ is strictly decreasing in x_2 and $\lim_{x_2\to\infty} \eta(x_1, x_2) = 0$. In particular, $\theta_1(x_1) = 0$ for $\eta(x_1, 0) < 1$ which is the case for x_1 sufficiently small. It follows from $(\partial\eta/\partial x_2)(x_1, x_2) \leq -p_3(x_2)$ and the corollary on p. 256 of [C] that the function $\theta_1(x_1)$ is locally Lipschitz. Note that $\theta_1(x_1)$ takes values in $[0, \infty)$. Also, $\theta_1(x_1) \to \infty$ and $P(\theta_1(x_1)) \to 0$ as $|x_1| \to \infty$ so that

$$|x_1| \to \infty \quad \Rightarrow \quad x_1^T P(\theta_1(x_1)) x_1 \cdot \operatorname{tr}(B^T P(\theta_1(x_1)) B) = 1 - \varepsilon \cdot \operatorname{tr}(B^T P(\theta_1(x_1)) B) \to 1. \tag{122}$$

It follows that the function

$$V_1(x_1) = x_1^T P(\theta_1(x_1)) x_1 \tag{123}$$

is locally Lipschitz, positive definite, and radially unbounded.

We also are guaranteed, from the definition of $\theta_1(x_1)$, that

$$x_1^T P(\theta_1(x_1)) x_1 \cdot \operatorname{tr}(B^T P(\theta_1(x_1)) B) < 1. \tag{124}$$

Also, from the properties of the trace,

$$|B^T P(\theta_1(x_1)) x_1|^2 \leq x_1^T P(\theta_1(x_1)) x_1 \cdot \operatorname{tr}(B^T P(\theta_1(x_1)) B) < 1 \tag{125}$$

and, in particular,

$$|B^T P(\theta_1(x_1))x_1|^2 < |B^T P(\theta_1(x_1))x_1|. \tag{126}$$

We let $\tau : \mathbb{R}^n \to [0.5, \infty)$ be locally Lipschitz but otherwise arbitrary, define

$$\begin{aligned} q(x,d) &= q((x_1,x_2),d) = Ax_1 + B\varphi(-\tau(x_1)B^T P(x_2)x_1) + d, \\ \bar{q}(x_1,d) &= q((x_1,\theta_1(x_1)),d), \end{aligned} \tag{127}$$

and consider the Lie derivative of $V_1(x_1)$ along the vector field $q(x,d)$ under the constraint $x_2 = \theta_1(x_1)$, i.e., along the vector field $\bar{q}$. (The Lie derivative calculations that follow should be interpreted as holding almost everywhere since $V_1(x_1)$ is only locally Lipschitz.) We get, with (115), (116), and (126),

$$\begin{aligned} L_{\bar{q}}V_1(x_1) &\le x_1^T(A^T P + PA)x_1 + 2x_1^T Pd \\ &\quad + 2x_1^T PB\varphi(-\tau(x_1)B^T P(\theta_1(x_1))x_1) + x_1^T \frac{dP}{dr} x_1 \cdot L_{\bar{q}}\theta_1(x_1) \\ &\le -x_1^T Qx_1 + |B^T Px_1|^2 + \gamma^2|d|^2 \\ &\quad - 2\min\{\tau(x_1)|B^T Px_1|^2, |B^T Px_1|\} + x_1^T \frac{dP}{dr} x_1 \cdot L_{\bar{q}}\theta_1(x_1) \\ &\le -x_1^T Qx_1 + \gamma^2|d|^2 + x_1^T \frac{dP}{dr} x_1 \cdot L_{\bar{q}}\theta_1(x_1). \end{aligned} \tag{128}$$

Next we relate the last term of the right-hand side to the left-hand side. First note that for all x_1 such that $\eta(x_1,0) < 1$, the last term of the right-hand side is zero. The function $\theta_1(x_1)$ is not differentiable on the measure zero set $\{x_1 : \eta(x_1,0) = 1\}$. When $\eta(x_1,0) > 1$, we have that

$$\eta(x_1,\theta_1(x_1)) = 1. \tag{129}$$

Differentiating this equation, we get

$$L_{\bar{q}}V_1(x_1) \cdot \mathrm{tr}(B^T PB) = -(V_1(x_1) + \varepsilon) \cdot \mathrm{tr}\left(B^T \frac{dP}{dr} B\right) \cdot L_{\bar{q}}\theta_1(x_1). \tag{130}$$

We then have

$$x_1^T \frac{dP}{dr} x_1 \cdot L_{\bar{q}}\theta_1(x_1) = -L_{\bar{q}}V_1(x_1) \frac{\mathrm{tr}(B^T PB)x_1^T (dP/dr)x_1}{(V_1(x_1) + \varepsilon) \cdot \mathrm{tr}(B^T (dP/dr)B)}. \tag{131}$$

We let $\vartheta(x_1)$ be a continuous, nonnegative function satisfying

$$\vartheta(x_1) \ge \frac{\mathrm{tr}(B^T PB)x_1^T (dP/dr)x_1}{(V_1(x_1) + \varepsilon) \cdot \mathrm{tr}(B^T (dP/dr)B)}. \tag{132}$$

Here we have used the bounds on dP/dr in (117). Then, considering the two cases $L_{\bar{q}}V_1(x_1) \ge 0$ and $L_{\bar{q}}V_1(x_1) \le 0$, we have, for all x_1,

$$x_1^T \frac{dP}{dr} x_1 \cdot L_{\bar{q}}\theta_1(x_1) \le \max\{0, -\vartheta(x_1) \cdot L_{\bar{q}}V_1(x_1)\}. \tag{133}$$

Combining this with (128) we have

$$L_{\bar{q}}V_1(x_1) \leq -\frac{1}{1+\vartheta(x_1)}x_1^T Q(\theta_1(x_1))x_1 + \gamma^2|d|^2. \tag{134}$$

We let $v \in (0,1)$ and let

$$\kappa_1(x_1) = v\frac{1}{(1+\vartheta(x_1))|x_1|^2}x_1^T Q(\theta_1(x_1))x_1 \tag{135}$$

and

$$\alpha_1(x_1) = (1-v)\frac{1}{1+\vartheta(x_1)}x_1^T Q(\theta_1(x_1))x_1. \tag{136}$$

Since ϑ, Q, and θ_1 are continuous and ϑ is nonnegative and $Q(r) > 0$ for all r, it follows that κ_1 and α_1 are continuous and $\kappa_1(x_1) > 0$ for all x_1 while $\alpha_1(x_1)$ is positive definite. We then have

$$L_{\bar{q}}V_1(x_1) \leq -\alpha_1(x_1) - \kappa(x_1)|x_1|^2 + \gamma^2|d|^2. \tag{137}$$

The right-hand side and the function $\bar{q}(x_1, d)$ are continuous and so, according to property 5 of the generalized directional derivative given in Section 2, the bound holds everywhere when the left-hand side is interpreted as the Clarke generalized directional derivative. Henceforth, we make this interpretation. Now we use the results of Section 4.2 to produce a dynamic controller that calculates $\theta_1(x_1)$ on-line and provides the same level of $\mathscr{L}_2$ disturbance attenuation guaranteed by (137). Consider the extended control system

$$\begin{aligned} \dot{x}_1 &= Ax_1 + B\varphi(-\tau(x_1)B^T P(x_2)x_1) + d = q(x,d), \\ \dot{x}_2 &= u \end{aligned} \tag{138}$$

which is in the form (85). Define $\pi(x) = -C(x_1, x_2)$ and note that the quantity $(x_2 - \theta_1(x_1))\pi(x)$ is nonpositive and zero only if $x_2 = \theta_1(x_1)$. We now use Proposition 2 and Corollary 5 to conclude that, for each $\mu > 0$,

$$V(x) = V_1(x_1) + \mu(x_2 - \theta_1(x_1))^2 \tag{139}$$

is a clf for the pair $(\pi, \tilde{\alpha})$ with

$$\tilde{\alpha}(x,d) = -0.5[\alpha_1(x_1) + (x_2 - \theta_1(x_1))^2] - \kappa_1(x_1)|x_1|^2 + \gamma^2|d|^2. \tag{140}$$

We now verify that V satisfies the bounded control property for the pair $(\pi, \tilde{\alpha})$ as well. We start by establishing that there exists a neighborhood $\mathscr{U}_2$ of the origin and a constant L such that

$$|(x_2 - \theta(x_1))\pi(x)| \geq L|x_2 - \theta(x_1)|^2 \tag{141}$$

for all $x \in \mathscr{U}_2$. This would establish, in the terminology of Definition 4, that π locally dominates the identity. We know from the definition of $\theta(x_1)$ that $\theta(x_1) \equiv 0$ on a neighborhood of the origin. So we need to establish that, for sufficiently small x_1 and x_2,

$$|\pi(x)| \geq L|x_2|. \tag{142}$$

Now, for x_1 sufficiently small,

$$\pi(x) = \eta(x_1, x_2) - \eta(x_1, 0). \tag{143}$$

Then since $(\partial\eta/\partial x_2)(x_1, x_2) \leq -p_3(x_2)$ the bound (142) holds according to the result of Section 7.1 (Lemma 2) of [C]. Note that $p_3(x_2)$ depends on ε and decreases to zero as $\varepsilon \to 0$ so L decreases to zero as $\varepsilon \to 0$. Again using that $\theta(x_1) \equiv 0$ on a neighborhood of the origin, we have

$$L_{f(x,d)}V(x) = L_{\bar{q}}V_1(x_1) + L_{q-\bar{q}}V_1(x_1), \tag{144}$$

where $\bar{q}$ and q were defined in (127). Then using that $q(x,d)$ is locally Lipschitz and $V_1(x_1)$ is smooth in a neighborhood of the origin, and using that (137) holds and the definition of $\alpha_1(x_1)$ in (136), we have that there exist strictly positive real numbers q_1 and q_2 such that, for all x sufficiently small,

$$\begin{aligned} L_{f(x,d)}V(x) - \tilde{\alpha}(x,d) &\leq -q_1|x_1|^2 + 0.5|x_2 - \theta(x_1)|^2 + q_2|x_1||x_2 - \theta(x_1)| \\ &\leq \left(\frac{q_2^2}{4q_1} + 0.5\right)|x_2 - \theta(x_1)|^2 \\ &\leq \frac{-1}{L}\left(\frac{q_2^2}{4q_1} + 0.5\right)(x_2 - \theta(x_1))\pi(x). \end{aligned} \tag{145}$$

Thus the bounded control property is satisfied and the proof of Theorem 1 generates a locally Lipschitz function $\psi(x_1, x_2)$ so that the following is true: the locally Lipschitz dynamic controller

$$\begin{aligned} v &= -\tau(x_1)B^T P(x_2)x_1, \\ \dot{x}_2 &= -C(x_1, x_2)\psi(x_1, x_2), \end{aligned} \tag{146}$$

with the functions τ, P, C, and ψ described above, assigns the bound $\tilde{\alpha}(x,d)$ given in (140) to the derivative of the clf candidate $V(x_1, x_2)$ given in (139) along the trajectories of (114), (146). We emphasize that ψ is a positive-valued function that just needs to be sufficiently large and the function P (which also appears in the function C) can be given by (119) where the elements of the sequence $\{P_i\}$ are computed on-line.

6. Conclusion

In this paper we presented results on using the (multi-input) control variable u to assign a desirable upper bound $\tilde{\alpha}(x,d)$ to the derivative of a clf $V(x)$ along the vector field $f(x,d) + g(x)u$. In particular, we have given a relationship between the class of assignable upper bounds, using a control $u = \pi(x)\psi(x)$, where $L_{g(x)}V(x)\pi(x)$ is continuous and nonpositive, and a bound on $L_{f(x,d)}V(x)$ that is satisfied when $L_{g(x)}V(x)\pi(x) = 0$ (see Theorem 2 and Corollary 1). We have emphasized that the control law can be synthesized with only rough information about $g(x)$, $f(x,d)$, and $V(x)$.

Our results on assigning an upper bound to the derivative of V were applied to nonlinear $\mathscr{L}_\infty$ and $\mathscr{L}_2$ disturbance attenuation problems, including various "backstepping locally Lipschitz, disturbance attenuation control law" problems. One particular application was to nonlinear $\mathscr{L}_2$ disturbance attenuation for linear systems with bounded controls.

Appendix

A.1. Proof of Corollary 3

Taking α as in (88) and defining (see (94))

$$\tilde{\alpha}_\varepsilon(x,d) := \alpha(x,d) + 0.5\kappa(\max\{V(x), \gamma(|d|), \varepsilon\}) \tag{147}$$

with $\varepsilon > 0$ it can be verified, as in the proof of Corollary 2, that $V(x)$ is a clf for the pair $(\pi, \tilde{\alpha}_\varepsilon)$ and

$$V(x) \geq \max\{\gamma(|d|), \varepsilon\} \quad \Rightarrow \quad \tilde{\alpha}_\varepsilon(x,d) = -0.5\kappa(V(x)). \tag{148}$$

Moreover, using (91) and (147), we get

$$\begin{aligned} &\sup_d \{L_{f(x,d)}V(x) - \tilde{\alpha}_\varepsilon(x,d)\} \\ &\qquad \leq \max\left\{0, \sup_{\{d: V(x) \geq \gamma(|d|)\}} L_{f(x,d)}V(x) + \kappa(V(x))\right\} \\ &\qquad\quad - 0.5\kappa(\max\{V(x), \varepsilon\}). \end{aligned} \tag{149}$$

So, the bounded control property for $(\pi, \tilde{\alpha}_\varepsilon)$ is satisfied if there exist $\chi > 0$ and $\bar{v} \geq 0$ such that

$$\begin{aligned} &\sup_{\{d: V(x) \geq \gamma(|d|)\}} L_{f(x,d)}V(x) + \kappa(V(x)) - 0.5\kappa(\max\{V(x), \varepsilon\}) \\ &\qquad \leq -\bar{v} L_{g(x)}V(x)\pi(x), \qquad \forall |x| \leq \chi. \end{aligned} \tag{150}$$

We are assuming that $f(0,0) = 0$ and $\varepsilon > 0$. In the case where $V(0) = 0$, it follows from property 4 of the generalized directional derivative given in Section 2, the continuity of f and V, and the fact that γ^{-1} and κ are continuous and zero at zero that there is a neighborhood of the origin in which the left-hand side of (150) is negative. Since the right-hand side of (150) is nonnegative, it follows that (150) holds for some $\chi > 0$ and all $\bar{v} \geq 0$. In the case where $V(0) > 0$, it follows from condition 3 of Assumption 1 with $(x,d) = (0,0)$, the assumption that $f(0,0) = 0$, $\gamma(0) = 0$, and property 4 of the generalized directional derivative given in Section 2 that $L_{g(0)}V(0)\pi(0) \neq 0$. From the continuity and nonpositiveness of $L_{g(x)}V(x)\pi(x)$ there exist $\chi > 0$ and $b > 0$ such that

$$-L_{g(x)}V(x)\pi(x) \geq b, \qquad \forall |x| \leq \chi. \tag{151}$$

Again, from the continuity of f and V and $\gamma \in \mathcal{K}_\infty$, it follows that (150) holds for this $\chi > 0$ and for some $\bar{v} > 0$. It follows that $V(x)$ satisfies the bounded control property for $(\pi, \tilde{\alpha}_\varepsilon)$. ■

A.2. Proof of Corollary 4

Without loss of generality assume $\tilde{\kappa}(s) \le \kappa(s)$ for all $s \ge 0$ and then let $\tilde{\gamma} \in \mathcal{K}$ be such that, for x sufficiently small,

$$V(x) \ge \tilde{\gamma}(|d|) \quad \Rightarrow \quad L_{f(x,d)-f(x,0)} V(x) \le 0.5\tilde{\kappa}(V(x)). \tag{152}$$

For example, let $\chi > 0$, let L be a Lipschitz constant for $V(x)$ on the set $|x| \le 2\chi$, and let $\gamma_f \in \mathcal{K}_\infty$ satisfy

$$|f(x,d) - f(x,0)| \le \gamma_f(|d|), \qquad \forall |x| \le \chi, \quad \forall d. \tag{153}$$

Such a function γ_f exists since f is continuous. Then define

$$\tilde{\gamma}(s) := \max\{\gamma(s), \tilde{\kappa}^{-1}(2L \cdot \gamma_f(s))\}. \tag{154}$$

According to property 4 of the generalized directional derivative given in Section 2 and (153), with this choice of $\tilde{\gamma}$, (152) is satisfied. With this construction we see that, without loss of generality, we can assume that $\tilde{\gamma}(s) \ge \gamma(s)$ for all $s \ge 0$.

Let γ_v, κ_v be of class-$\mathcal{K}_\infty$ and match $\tilde{\gamma}$ and $\tilde{\kappa}$, respectively, for small s and match γ and κ for $s \ge v$ and satisfy $\gamma_v(s) \ge \gamma(s)$ and $\kappa_v(s) \le \kappa(s)$ for all s. For example, let $\bar{v}_1 \in (0, v)$ be such that $\tilde{\gamma}(\bar{v}_1) < \gamma(v)$ and then pick

$$\gamma_v(s) = \begin{cases} \tilde{\gamma}(s), & s \in [0, \bar{v}_1], \\ \max\left\{\gamma(s), \dfrac{v-s}{v-\bar{v}_1}\tilde{\gamma}(\bar{v}_1) + \dfrac{s-\bar{v}_1}{v-\bar{v}_1}\gamma(v)\right\}, & s \in [\bar{v}_1, v], \\ \gamma(s), & s \in [v, \infty). \end{cases} \tag{155}$$

Similarly for κ_v, let $\bar{v}_2 \in (0, v)$ be such that $\tilde{\kappa}(\bar{v}_2) < \kappa(v)$ and then pick

$$\kappa_v(s) = \begin{cases} \tilde{\kappa}(s), & s \in [0, \bar{v}_2], \\ \min\left\{\kappa(s), \dfrac{v-s}{v-\bar{v}_2}\tilde{\kappa}(\bar{v}_2) + \dfrac{s-\bar{v}_2}{v-\bar{v}_2}\kappa(v)\right\}, & s \in [\bar{v}_2, v], \\ \kappa(s), & s \in [v, \infty). \end{cases} \tag{156}$$

Similarly for κ_v, let $\bar{v}_2 \in (0, v)$ be such that $\tilde{\kappa}(\bar{v}_2) < \kappa(v)$ and then pick

$$\kappa_v(s) = \begin{cases} \tilde{\kappa}(s), & s \in [0, \bar{v}_2], \\ \min\left\{\kappa(s), \dfrac{v-s}{v-\bar{v}_2}\tilde{\kappa}(\bar{v}_2) + \dfrac{s-\bar{v}_2}{v-\bar{v}_2}\kappa(v)\right\}, & s \in [\bar{v}_2, v], \\ \kappa(s), & s \in [v, \infty). \end{cases} \tag{157}$$

With these choices, point 3 of Assumption 1 holds with γ_v replacing γ and κ_v replacing κ. Define $\tilde{\alpha}_v(x,d)$ using (88), (89), and (94) but with γ_v in place of γ and κ_v in place of κ. It can be verified, as in the proof of Corollary 2, that $V(x)$ is a clf for the pair $(\pi, \tilde{\alpha}_v)$ and

$$V(x) \geq \gamma_v(|d|) \quad \Rightarrow \quad \tilde{\alpha}_v(x,d) = -0.5\kappa_v(V(x)). \tag{158}$$

Now, V satisfies the bounded control property for $(\pi, \tilde{\alpha}_v)$ if there exist $\chi > 0$ and $\bar{v} \geq 0$ such that

$$\sup_{\{d: V(x) \geq \gamma_v(|d|)\}} L_{f(x,d)}V(x) + 0.5\kappa_v(V(x)) \leq -\bar{v}L_{g(x)}V(x)\pi(x), \qquad \forall |x| \leq \chi. \tag{159}$$

We use property 2 of the generalized directional derivative given in Section 2, the fact that $V(0) = 0$ and V is continuous, the fact that $\gamma_v(s) = \tilde{\gamma}(s)$ for small s, and (152) to see that, for sufficiently small x,

$$\begin{aligned} &\sup_{\{d: V(x) \geq \gamma_v(|d|)\}} L_{f(x,d)}V(x) + 0.5\kappa_v(V(x)) \\ &\qquad \leq \sup_{\{d: V(x) \geq \gamma_v(|d|)\}} \{L_{f(x,d)-f(x,0)}V(x)\} + L_{f(x,0)}V(x) + 0.5\tilde{\kappa}(V(x)) \\ &\qquad \leq L_{f(x,0)}V(x) + \tilde{\kappa}(V(x)). \end{aligned} \tag{160}$$

By assumption, for the system $\dot{x} = f(x,0) + g(x)u$, V satisfies the bounded control property for $(\pi, -\tilde{\kappa}(V))$. It follows from (160) that, for the system $\dot{x} = f(x,d) + g(x)u$, V satisfies the bounded control property for $(\pi, \tilde{\alpha}_v)$. ■

References

[A] Z. Artstein. Stabilization with relaxed controls. *Nonlinear Anal.*, 7 (1983), 1163–1173.

[C] F. H. Clarke. *Optimization and Nonsmooth Analysis*. SIAM, Philadelphia, PA, 1990.

[CPT] J.-M. Coron, L. Praly, and A. R. Teel. Feedback stabilization of nonlinear systems: sufficient conditions and Lyapunov and input–output techniques. In *Trends in Control*, A. Isidori, editor, Springer-Verlag, New York, 1995, pp. 293–348.

[FK1] R. A. Freeman and P. V. Kokotovic. Backstepping design with nonsmooth nonlinearities. In *Proceedings of the IFAC Nonlinear Control Systems Design Symposium*, Tahoe City, CA, 1995, pp. 483–488.

[FK2] R. A. Freeman and P. V. Kokotovic. *Robust Nonlinear Control Design*. Birkhäuser, Boston, 1996.

[FK3] R. A. Freeman and P. V. Kokotovic. Inverse optimality in robust stabilization. *SIAM Journal on Control and Optimization*, 34 (1996), 1365–1391.

[FP] R. A. Freeman and J. A. Primbs. Control Lyapunov functions: new ideas from an old source. In *Proceedings of the 35th IEEE Conference on Decision and Control*, Kobe, Japan, 1996, pp. 3926–3931.

[H] J. K. Hale. *Ordinary Differential Equations*. Krieger, Malabar, FL, 1980.

[I1] A. Isidori. *Nonlinear Control Systems*, third edition. Springer-Verlag, New York, 1995.

[I2] A. Isidori. A note on almost disturbance decoupling for nonlinear minimum phase systems. *Systems & Control Letters*, 27 (1996), 191–194.

[JTP] Z. P. Jiang, A. R. Teel, and L. Praly. Small gain theorem for ISS systems and applications. *Mathematics of Control, Signals, and Systems*, 7 (1994), 95–120.

[K] H. K. Khalil. *Nonlinear Systems*. Prentice-Hall, Englewood Cliffs, NJ, 1996.

[LL] V. Lakshmikantham and S. Leela. *Differential and Integral Inequalities*, Vol. 1. Academic Press, New York, 1969.

[L] Z. Lin, A. Saberi, and A. R. Teel. The almost disturbance decoupling problem with internal stability for linear systems subject to input saturation—state feedback case. *Automatica*, 32 (1996), 619–624.

[M] A. Megretski. $\mathscr{L}_2$ BIBO output feedback stabilization with saturated control. In *Proceedings of the 13th World Congress of IFAC*, vol. D, 1996, pp. 435–440.

[N] I. P. Natanson. *Theory of Functions of a Real Variable* (translated from Russian). Ungar, New York, 1974.

[PJ] L. Praly and Z. P. Jiang. Stabilization by output feedback for systems with ISS inverse dynamics. *Systems & Control Letters*, 21 (1993), 19–33.

[SLT] A. Saberi, Z. Lin, and A. R. Teel. Control of linear systems with saturating actuators. *IEEE Transactions on Automatic Control*, 41 (1996), 368–378.

[SAS] R. Saurez, J. Alvarez-Ramirez, and Mario Sznaier. Global stabilization and $\mathscr{L}_2$ disturbance attenuation for linear systems with bounded controls. In *Proceedings of the 1997 European Control Conference*, CD-ROM.

[SJK] R. Sepulchre, M. Jankovic, and P. Kokotovic. *Constructive Nonlinear Control*. Springer-Verlag, New York, 1997.

[SF] J. M. Shewchun and E. Feron. High performance bounded control. In *Proceedings of the American Control Conference*, 1997, pp. 2350–2354.

[S1] E. D. Sontag. A "universal" construction of Artstein's theorem on nonlinear stabilization. *Systems & Control Letters*, 13 (1989), 435–443.

[S2] E. D. Sontag. Smooth stabilization implies coprime factorization. *IEEE Transactions on Automatic Control*, 34 (1989), 435–443.

[SW] E. D. Sontag and Y. Wang. On characterizations of input-to-state stability with respect to compact sets. In *Proceedings of the Nonlinear Control Systems Design Symposium (NOLCOS) 1995*, Lake Tahoe, CA, June 1995, pp. 226–231.

[S3] J. Szarski. *Differential Inequalities*. Polish Scientific, Warsaw, 1965.

[T] A. R. Teel. Linear systems with input nonlinearities: global stabilization by scheduling a family of $\mathscr{H}_\infty$ type controllers. *International Journal of Robust and Nonlinear Control*, 5 (1995), 399–411.

[TP] A. R. Teel and L. Praly. Tools for semiglobal stabilization by partial state and output feedback. *SIAM Journal on Control and Optimization*, 33 (1995), 1443–1488.

[WB] G. F. Wredenhagen and P. R. Belanger. Piecewise-linear LQ control for systems with input constraints. *Automatica*, 30 (1994), 403–416.