PERGAMON

Automatica 36 (2000) 179–187

www.elsevier.com/locate/automatica

Asymptotic tracking of a reference state for systems with a feedforward structure☆

F. Mazenc[a,*], L. Praly[b]

[a]Centre for Process Systems Engineering, Imperial College of Science, Technology and Medicine, London SW7 2BY, UK
[b]Centre Automatique et Systèmes, École des Mines de Paris 35 rue St Honoré, 77305 Fontainebleau cédex, France

Received 30 October 1996; revised 17 December 1997; received in final form 29 March 1999

Abstract

We consider the problem of asymptotic tracking for a system which can be written in a feedforward form. The data of a bounded reference state trajectory is assumed. Our solution relies on a Lyapunov construction. The time-varying state feedbacks obtained are bounded and ensure the global uniform asymptotic stability when the signals are periodic and the local exponential stability of the reference state trajectory of the closed-loop system.

Keywords: Global uniform asymptotic stabilization; Lyapunov design

1. Introduction

Consider the system

$$\dot{x} = h(x, y, u), \qquad \dot{y} = f(y, u). \tag{1}$$

Suppose that a known function $u_r(t)$, defined and bounded on $[0, +\infty[$, and an initial condition $(x_r(0), y_r(0))$ such that the corresponding solution $(x_r(t), y_r(t))$ of (1) is defined and bounded on $[0, +\infty[$. Moreover, suppose that $y_r(t)$ is a globally asymptotically stable (GAS) solution of

$$\dot{y} = f(y, u_r(t)). \tag{2}$$

Problem. When is it possible to design a time-varying feedback law $u(x, y, t)$ such that (x_r, y_r) be a GAS solution of (1)?

In (1), the presence of u in the $\dot{x}$-equation impedes the application of the backstepping technique (Krstic, Kanellakopoulos & Kokotovic, 1995) to solve the problem. The solution we propose here is based on a completely different approach and is appropriate only for the subclass of systems (1) characterized by the assumptions in Section 2.1. This particular class allows us to design a Lyapunov function. This construction utilizes tools and properties analogous to those employed to achieve stabilization of an equilibrium in Mazenc and Praly (1996): Jurdjevic–Quinn approach, higher-order notion, changes of coordinates. Our result is a result of global uniform asymptotic stability (GUAS) and local exponential stability (LES). Moreover, the class of feedback laws obtained contains *bounded* feedbacks.

The problem of asymptotic tracking of trajectories of feedforward systems has been addressed in (Teel, 1992, Corollary 2.1). There, only the case of a chain of integrators is considered and the solution relies heavily on the linear structure. In Liu, Chitour and Sontag (1996), the problem of controlling a linear time-invariant system subject to *input saturation* in order to have its output track (or reject) a family of reference (or disturbance) signals produced by some external generator is examined. Other approaches to the tracking problem for nonlinear systems rely on the properties of linearizability or of partial linearizability (and are not concerned with the issue of the saturation of the feedback laws) by static or dynamic feedback (see Martin, Devasia & Paden, 1996 for an illustrative example). The input–output linearization theory, which leads to the celebrated normal form

☆This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor I. Petersen under the direction of Editor R. Tempo.

* Corresponding author. Tel.: +44-171-594-6604; fax: +44-171-594-6606.

E-mail address: f.mazenc@ic.ac.uk (F. Mazenc)

0005-1098/00/$ - see front matter
PII: S0005-1098(99)00136-3

and to the notion of minimum phase system in the nonlinear context (see Isidori, 1989; Marino & Tomei, 1995), plays a central role in this type of work: for it may provide an efficient help for solving the problem of reproducing a reference output trajectory and next for determining a feedback law which asymptotically stabilizes the solution of exact tracking.

We will proceed regardless of any kind of linearizability property or of minimum phase property. For we center our efforts on the problem of stabilizing a given bounded reference state trajectory. We do not address the problem of designing such a trajectory. The fact that we assume its existence may seem to be a weakness of our work, since, a priori, it does not offer a response to the problem of tracking an *output* trajectory. Nevertheless, as it is pointed out for instance in Martin et al. (1996) and Marino, Kanellakopoulos and Kokotovic (1989), the output tracking problem may be profitably split up into two steps: (1) Determination of a bounded reference state trajectory and of a bounded input which exactly yields the reference output trajectory. (2) Stabilization of the reference state trajectory. Thus, by offering a response to the second step, our technique is complementary to those which offer a response to the first. Moreover, the feedforward structure may also be exploited to construct the reference trajectory.

Organization of the paper. In Section 2 is given a result about the problem of globally asymptotically stabilizing a bounded trajectory for particular systems (1). In Section 3 an improvement of this result is proposed. It is centered on the uniform aspect of the asymptotic stability. Section 4 contains concluding remarks.

1.1. Notations and basic definitions

- Throughout the paper, the symbol c denotes generically a strictly positive real number (i.e. $c + c * c = c$).
- A continuous function $\mathscr{F}(x, y)$ is said to have a zero of order $p \geq 0$ at $y = 0$ if there exists a nonnegative continuous function $\tilde{\mathscr{F}}$ such that, for all (x, y),

$$|\mathscr{F}(x, y)| \leq \tilde{\mathscr{F}}(x, y)|y|^p. \tag{3}$$

- A function $\sigma : \mathbb{R} \to \mathbb{R}$ is said to be a saturation if it is a continuous, bounded, differentiable at 0 and such that

$$\sigma(s)s > 0 \quad \forall s \neq 0, \quad \sigma'(0) > 0, \quad \sigma|_{\mathbb{R}_+} \notin L^1(\mathbb{R}_+),$$
$$\sigma|_{\mathbb{R}_-} \notin L^1(\mathbb{R}_-).$$

  It is said to be a linear saturation if there exists $L > 0$ such that

$$\sigma(s) = s, \quad \forall |s| \leq L.$$

- With Q a positive-definite symmetric matrix, we denote: $|x| = \sqrt{x^\top x}$, $|x|_Q = \sqrt{x^\top Q x}$.
- A function $\alpha : [0, +\infty) \to [0, +\infty)$ is said to be of class $\mathscr{K}$ if it is zero at zero and strictly increasing. If besides it is unbounded, it is said to be of class $\mathscr{K}^\infty$.

2. Global asymptotic stability

2.1. Assumptions and result

Consider system (1) rewritten here as

$$\dot{X} = H(X, Y, u), \qquad \dot{Y} = F(Y, u), \tag{4}$$

where $Y \in \mathbb{R}^n$, $X \in \mathbb{R}^m$, $u \in \mathbb{R}^q$ and both H and F are functions of class C^3 which satisfy $H(0,0,0) = 0$ and $F(0, 0) = 0$.

Assumption A1. *The function H can be decomposed as*

$$H(X, Y, u) = MX + H_1(Y) + H_2(Y, u)u. \tag{5}$$

With this assumption and the fact that F is of class C^3, system (4) can be rewritten as

$$\begin{aligned} \dot{X} &= MX + H_1(Y) + H_2(Y, u)u, \\ \dot{Y} &= F_0(Y) + F_2(Y, u)u, \end{aligned} \tag{6}$$

where all the functions are of class C^2.

Assumption A2. *There exists a function* $(X_r(t), Y_r(t), u_r(t))$ *bounded on* $[0, +\infty)$, *of class* C^2, *and verifying*:

$$\begin{aligned} \dot{X}_r(t) &= MX_r(t) + H_1(Y_r(t)) + H_2(Y_r(t), u_r(t))u_r(t), \\ \dot{Y}_r(t) &= F_0(Y_r(t)) + F_2(Y_r(t), u_r(t))u_r(t). \end{aligned} \tag{7}$$

This assumption guarantees that the matrix

$$A(t) = \frac{\partial}{\partial \tilde{Y}}[F_0(\tilde{Y} + Y_r(t)) + F_2(\tilde{Y} + Y_r(t), u_r(t))u_r(t)]|_{\tilde{Y}=0} \tag{8}$$

is well-defined and of class C^1. Let $\Phi_A(t, t_0)$ be the transition matrix associated to this matrix, i.e. the function verifying:

$$\frac{\partial \Phi_A}{\partial t}(t, t_0) = \Phi_A(t, t_0)A(t), \qquad \Phi_A(t_0, t_0) = I. \tag{9}$$

Assumption A3.
(A3.1). *The point* $\tilde{Y} = 0$ *is a GUAS equilibrium point of the system*:

$$\dot{\tilde{Y}} = F_0(\tilde{Y} + Y_r(t)) - F_0(Y_r(t)) + F_2(\tilde{Y} + Y_r(t), u_r(t))u_r(t) - F_2(Y_r(t), u_r(t))u_r(t). \tag{10}$$

(A3.2). *There exist a positive-definite symmetric matrix Q and $c>0$, $\alpha>0$ such that for all $t>0$ and for all $s\in[0,t]$*:

$$|\exp(M(s-t))||\Phi_A(t,s)| \le c\exp(-\alpha(t-s)), \quad (11)$$

$$M^\top Q + QM = -R \le 0. \quad (12)$$

As we shall prove later on, this assumption implies that the equation

$$\dot P(t) = MP(t) - P(t)A(t) - C(t), \quad (13)$$

where

$$C(t) = \frac{\partial}{\partial \tilde Y}[H_1(\tilde Y + Y_r(t)) + H_2(\tilde Y + Y_r(t), u_r(t))u_r(t)]|_{\tilde Y=0}, \quad (14)$$

admits on $[0,+\infty)$ a unique continuous and bounded solution $P(t)$. This allows us to introduce the notation

$$D(t) = \frac{\partial H_2}{\partial u}(Y_r(t),u_r(t))u_r(t) + H_2(Y_r(t),u_r(t)) + P(t)\left[\frac{\partial F_2}{\partial u}(Y_r(t),u_r(t))u_r(t) + F_2(Y_r(t),u_r(t))\right], \quad (15)$$

and to state the assumption:

Assumption A4. *The pair*

$$\left(M, \begin{pmatrix} D(t)^\top Q \\ R^{1/2}\end{pmatrix}\right)$$

is uniformly detectable, i.e. there exist bounded and continuous functions $K_d(t)$ and $K_r(t)$ such that the solution $\chi=0$ of

$$\dot\chi = (M + K_d(t)D(t)^\top Q + K_r(t)R^{1/2})\chi, \quad (16)$$

is ES.

Theorem 1. *If system* (6) *satisfies Assumptions* A1–A4 *then, for all $\tilde u>0$, there exists a continuous feedback law $\bar u(X,Y,t)$ verifying*:

$$|\bar u(X,Y,t) - u_r(t)| < \tilde u \quad (17)$$

and such that the closed-loop system admits (X_r, Y_r) as a GAS solution.

*Discussion of Assumption A*3

- (11) says that M is marginally stable.
- (11) and (12) imply that

$$P(t) = \left[\int_t^{+\infty} \exp(M(t-s))C(s)\Phi_A(s,t)\,\mathrm{d}s\right], \quad (18)$$

is well-defined, bounded, of class C^1 on $[0,+\infty)$ and solution of (13). Indeed, the functions F_0, F_2, u_r and Y_r being of class C^2, the function $A(t)$ is of class C^1. The functions H_1, H_2, u_r and Y_r being of class C^2, the function $C(t)$ is of class C^1. So $\exp(M(t-s))C(s)\Phi_A(s,t)$ is of class C^1. On the other hand, according to Assumption A2, $Y_r(t)$ is bounded on $[0,+\infty)$. It follows that $|C(t)|$ is bounded by a positive-real number c. With (12), this implies:

$$|P(t)| \le c\int_t^{+\infty}\exp(-\alpha(s-t))\,\mathrm{d}s \le \frac{c}{\alpha} < +\infty. \quad (19)$$

It follows readily that $P(t)$ is well-defined and of class C^1. Next, by simply evaluating the derivative of P, one can check that P satisfies (13), which in turn implies that P is of class C^2.

We will exploit the properties of P mentioned above in the proof of Theorem 1 for designing coordinates which facilitates the construction of a Lyapunov function for (6).

2.2. *Proof of Theorem 1*

2.2.1. *First step*: *Error equation*

We transform the tracking problem into the problem of GAS the origin of a time-varying system. In the coordinates

$$\tilde X = X - X_r(t), \qquad \tilde Y = Y - Y_r(t), \qquad v = u - u_r(t), \quad (20)$$

system (6) rewrites

$$\begin{aligned}\dot{\tilde X} &= M\tilde X + [H_1(\tilde Y + Y_r) - H_1(Y_r)] \\ &\quad + [H_2(\tilde Y + Y_r, u_r + v)(u_r + v) - H_2(Y_r,u_r)u_r],\\ \dot{\tilde Y} &= [F_0(\tilde Y + Y_r) - F_0(Y_r)] \\ &\quad + [F_2(\tilde Y + Y_r, u_r + v)(u_r + v) - F_2(Y_r,u_r)u_r]. \end{aligned} \quad (21)$$

2.2.2. *Second step*: *Appropriate coordinates*

As in Mazenc and Praly (1996), we remark that if we find coordinates (x,y) allowing us to rewrite (21) in the form

$$\begin{aligned}\dot x &= Mx + h_2(y,v,t)v,\\ \dot y &= f_0(y,t) + f_2(y,v,t)v,\end{aligned} \quad (22)$$

then the stabilization problem could possibly be solved by applying a generalization of the Jurdjevic and Quinn's approach (see third step). Under our assumptions, such coordinates exist. More precisely, we have

Lemma 2. *Let $\Phi_Y(t,s,\tilde Y)$ be the solution of the system*:

$$\begin{aligned}\dot{\tilde Y} &= [F_0(\tilde Y + Y_r) - F_0(Y_r)] \\ &\quad + [F_2(\tilde Y + Y_r, u_r) - F_2(Y_r,u_r)]u_r\end{aligned} \quad (23)$$

which satisfies $\Phi_Y(s,s,\tilde{Y}) = \tilde{Y}$. *The change of coordinates*:

$$x = \tilde{X} + \mathscr{P}(t,\tilde{Y}), \qquad y = \tilde{Y}, \tag{24}$$

where $\mathscr{P}(t,\tilde{Y})$ *is the function of class* C^1 *defined by*

$$\mathscr{P}(t,\tilde{Y}) = \int_0^{+\infty} \exp(-Ml)\mathscr{H}(t+l,\Phi_Y(t+l,t,\tilde{Y}))\,\mathrm{d}l \tag{25}$$

with

$$\mathscr{H}(t,\tilde{Y}) = H_1(\tilde{Y}+Y_r(t)) - H_1(Y_r(t)) + H_2(\tilde{Y}+Y_r(t),u_r(t))u_r(t) - H_2(Y_r(t),u_r(t))u_r(t), \tag{26}$$

applied to system (21) *yields*

$$\dot{x} = Mx + h_2(y,v,t)v, \qquad \dot{y} = f_0(y,t) + f_2(y,v,t)v, \tag{27}$$

where h_2, f_0 *and* f_2 *are of class* C^1 *and defined by*

$$\begin{aligned} h_2(y,v,t)v = {} & [H_2(y+Y_r(t),u_r(t)+v) \\ & -H_2(y+Y_r(t),u_r(t))]u_r(t) \\ & +\left[H_2(y+Y_r(t),u_r(t)+v)\right. \\ & \left.+\frac{\partial P}{\partial \tilde{Y}}(t,y)f_2(y,v,t)\right]v, \end{aligned} \tag{28}$$

$$f_0(y,t) = [F_0(y,Y_r(t)) - F_0(Y_r(t))] + [F_2(y+Y_r(t),u_r(t))u_r(t) - F_2(Y_r(t))u_r(t)], \tag{29}$$

$$\begin{aligned} f_2(y,v,t)v = {} & [F_2(y+Y_r(t),u_r(t)+v) \\ & - F_2(y+Y_r(t),u_r(t))]u_r(t) \\ & + F_2(y+Y_r(t),u_r(t)+v)v. \end{aligned} \tag{30}$$

This lemma is proved in Appendix A.

The difficulty encountered with (25) is the complexity of its evaluation. Fortunately, it turns out that, as in Mazenc and Praly (1996), the LES of $\tilde{Y}=0$ when $v=0$ allows us to deal with a system (21) when $\mathscr{H}(t,\tilde{Y})$, defined in (26), is a second-order term in $\tilde{Y}$. This implies that a change of coordinates which just removes the first-order term of $\mathscr{H}(t,\tilde{Y})$ is what we only need. By replacing in (24) the function $\mathscr{P}(t,\tilde{Y})$ by its first variation in $\tilde{Y}$, we get such a change of coordinates. It turns out that the Frechet derivative of $\mathscr{P}(t,\tilde{Y})$ with respect to $\tilde{Y}$ is, when evaluated at $\tilde{Y}=0$, the function $P(t)$ given in (18).

Let us summarize what we have:

Fact 3. *In the coordinates*

$$x = \tilde{X} + P(t)\tilde{Y}, \qquad y = \tilde{Y}, \tag{31}$$

where P *is the function defined in* (18), *the dynamics* (21) *become*

$$\begin{aligned} \dot{x} &= Mx + h_1(y,t) + h_2(y,v,t)v, \\ \dot{y} &= f_0(y,t) + f_2(y,v,t)v, \end{aligned} \tag{32}$$

where h_1, h_2, f_0 *and* f_2 *are of class* C^1, *bounded with respect to* t, *and defined by*

$$\begin{aligned} h_1(y,t) = {} & H_1(y+Y_r(t)) - H_1(Y_r(t)) \\ & + [H_2(y+Y_r(t),u_r(t)) - H_2(Y_r(t),u_r(t))]u_r(t) \\ & + P(t)[f_0(y,t) - A(t)y] - C(t)y, \end{aligned} \tag{33}$$

$$\begin{aligned} h_2(y,v,t)v = {} & [H_2(y+Y_r(t),u_r(t)+v) \\ & - H_2(y+Y_r(t),u_r(t))]u_r(t) \\ & + [H_2(y+Y_r(t),u_r(t)+v) \\ & + P(t)f_2(y,v,t)]v, \end{aligned} \tag{34}$$

$$f_0(y,t) = [F_0(y,Y_r(t)) - F_0(Y_r(t))] + [F_2(y+Y_r(t),u_r(t))u_r(t) - F_2(Y_r(t))u_r(t)], \tag{35}$$

$$\begin{aligned} f_2(y,v,t)v = {} & [F_2(y+Y_r(t),u_r(t)+v) \\ & - F_2(y+Y_r(t),u_r(t))]u_r(t) \\ & + F_2(y+Y_r(t),u_r(t)+v)v. \end{aligned} \tag{36}$$

Moreover, about h_1 we prove in Appendix B:

Lemma 4. *There exists a continuous positive function* γ *such that*

$$|h_1(y,t)| \le \gamma(|y|)|y|^2. \tag{37}$$

2.2.3. Third step: Assignment of a Lyapunov function

At this point of our proof, we need the following preliminary result:

Lemma 5. *If system* (6) *satisfies Assumptions* A1–A3 *there exist a Lyapunov function* $V(y,t)$ *of class* C^1, *strictly positive-real numbers* p_i, *a function* $W(y,t)$ *and functions* $\alpha_1, \alpha_2, \alpha_3$ *and* α_4 *of class* $\mathscr{K}^\infty$ *such that*

- *for all* $|y| \le p_1$:

$$p_2|y|^2 \le V(y,t) \le p_3|y|^2, \quad \left|\frac{\partial V}{\partial y}(y,t)\right| \le p_4|y|, \tag{38}$$

$$p_5|y|^2 \le \alpha_3(|y|) \le W(y,t) \tag{39}$$

- *for* $v=0$ *and for all* y:

$$\alpha_1(|y|) \le V(y,t) \le \alpha_2(|y|), \tag{40}$$

$$\overbrace{\dot{V(y,t)}}_{(32)} = -W(y,t) \leq -\alpha_3(|y|) < 0 \quad \forall y \neq 0, \tag{41}$$

$$\left|\frac{\partial V}{\partial y}(y,t)\right| \leq \alpha_4(|y|). \tag{42}$$

Proof. This result is obtained by a convex combination of the Lyapunov functions given by the converse Lyapunov theorems for equilibrium points which are LES or GUAS. Details can be found in (Mazenc, 1996, Annexe G). □

We focus our attention on the family of candidate Lyapunov functions:[1]

$$U(x,y,t) = \kappa(V(y,t)) + \int_0^{|x|_Q} \sigma(s)\,\mathrm{d}s, \tag{43}$$

where κ is a smooth function of class $\mathscr{K}^\infty$ with a derivative strictly larger than 1, where σ is a smooth saturation and where Q is the matrix given by Assumption A3.2. Any function U thus defined is smooth, proper, positive definite and lower bounded by a positive-definite quadratic form in a neighborhood of the origin. Its derivative along (32) is

$$\begin{aligned}\overbrace{\dot{U(x,y,t)}}_{(32)} = &-\kappa'(V(y,t))W(y,t)\\ &+\kappa'(V(y,t))\frac{\partial V}{\partial y}(y,t)f_2(y,v,t)v\\ &+\sigma(|x|_Q)\frac{x^\top Q}{|x|_Q}[Mx + h_1(y,t) + h_2(y,v,t)v].\end{aligned} \tag{44}$$

According to Mazenc and Praly (1996, Lemma B.2), we deduce from (38) and (39) that there exists a smooth function κ such that

$$\frac{1}{2}\,\kappa'(\alpha_1(|y|)) \geq \left(\sup_{s\in\mathbb{R}} |\sigma(s)|\right)\frac{|y|^2}{\alpha_3(|y|)}\gamma(|y|), \tag{45}$$

$$\kappa'(s) \geq 1, \quad \forall s \geq 0. \tag{46}$$

Such a choice for κ yields, with (37) and (11),

$$\begin{aligned}\overbrace{\dot{U(x,y,t)}}_{(32)} \leq &-\frac{1}{2}\kappa'(V(y,t))W(y,t) - \frac{1}{2}\sigma(|x|_Q)\frac{x^\top Rx}{|x|_Q}\\ &+ a(x,y,v,t)v\end{aligned} \tag{47}$$

with

$$\begin{aligned}a(x,y,v,t) = &\kappa'(V(y,t))\frac{\partial V}{\partial y}(y,t)f_2(y,v,t)\\ &+\sigma(|x|_Q)\frac{x^\top Q}{|x|_Q}h_2(y,v,t).\end{aligned} \tag{48}$$

Since f_2 and h_2 are of class C^1 and $|\sigma(|x|_Q)x^\top Q/|x|_Q|$ is a bounded function, this function a satisfies the condition (C.1) of Appendix C. It follows from Lemma 7 that there exists a C^1 function λ satisfying:

$$\forall c_1 \geq 0, \quad \exists c_2 > 0\colon \{|y| \leq c_1 \Rightarrow \lambda(y) \geq c_2\}, \tag{49}$$

and such that the time-varying feedback law:

$$\begin{aligned}v(y,x,t) = -\lambda(y)\Big[&\kappa'(V(y,t))\frac{\partial V}{\partial y}(y,t)f_2(y,0,t)\\ &+\sigma(|x|_Q)\frac{x^\top Q}{|x|_Q}h_2(y,0,t)\Big]^\top,\end{aligned} \tag{50}$$

is bounded by $\tilde{u}$ and verifies

$$\begin{aligned}\Big[&\kappa'(V(y,t))\frac{\partial V}{\partial y}(y,t)f_2(y,v,t)\\ &+\sigma(|x|_Q)\frac{x^\top Q}{|x|_Q}h_2(y,v,t)\Big]v(y,x,t)\\ &\leq -\frac{1}{2}\lambda(y)\Big|\kappa'(V(y,t))\frac{\partial V}{\partial y}(y,t)f_2(y,0,t)\\ &+\sigma(|x|_Q)\frac{x^\top Q}{|x|_Q}h_2(y,0,t)\Big|^2.\end{aligned} \tag{51}$$

Such a feedback law yields:

$$\begin{aligned}\overbrace{\dot{U(x,y,t)}}_{(32)} \leq &-\frac{1}{2}\kappa'(V(y,t))W(y,t) - \frac{1}{2}\sigma(|x|_Q)\frac{x^\top Rx}{|x|_Q}\\ &-\frac{1}{2}\lambda(y)\Big|\kappa'(V(y,t))\frac{\partial V}{\partial y}(y,t)f_2(y,0,t)\\ &+\sigma(|x|_Q)\frac{x^\top Q}{|x|_Q}h_2(y,0,t)\Big|^2.\end{aligned} \tag{52}$$

This inequality implies the GUS of the trajectory $(X_r(t), Y_r(t))$. Unfortunately, the right-hand side of (52) is a priori not smaller than a negative-definite function independent of time. So the GAS does not follow from a standard Lyapunov theorem. Nevertheless, this result may be inferred by a study of the consequences of (52). This is established in the next paragraph.

[1] The motivation for choosing $\int_0^{|x|_Q}\sigma(s)\,\mathrm{d}s$ is that the partial derivatives with respect to x are bounded. Other choices are possible.

2.2.4. Fourth step: GAS analysis

By integrating (52) between 0 and $+\infty$, we deduce that

$$\int_0^{+\infty} \kappa'(V(y(s),s))W(y(s),s)\,\mathrm{d}s + \int_0^{+\infty} \sigma(|x(s)|_Q)\frac{x(s)^\top Rx(s)}{|x(s)|_Q}\,\mathrm{d}s + \int_0^{+\infty}\frac{|v(y(s),x(s),s)|^2}{\lambda(y(s))}\,\mathrm{d}s < +\infty. \quad (53)$$

Let us analyze the consequences of this inequality:

(1) Using (46) and (39), the fact that W is positive definite and the boundedness of $y(t)$ and of the control, we have, for some $c > 0$:

$$c|y(s)|^2 \le \kappa'(V(y(s),s))W(y(s),s) \quad \forall s\in[0,+\infty). \quad (54)$$

With (54), inequality (53) implies that $y(t)\in L^2([0,+\infty))$. On the other hand (32), and the boundedness of $y(t)$ implies that $\dot{\overline{y^2}}(t)$ is bounded as well. Then, by applying (Khalil, 1992, Lemma 4.4), we get

$$\lim_{t\to+\infty} y(t) = 0. \quad (55)$$

(2) Since σ is a linear saturation and $x(t)$ is bounded, there exists $c_\sigma > 0$ such that

$$0 < c_\sigma \le \frac{\sigma(|x(t)|_Q)}{|x(t)|_Q} \quad \forall t \ge 0, \quad (56)$$

which, according to (53), implies that $x(t)^\top Rx(t)\in L^1([0,+\infty))$.

(3) Since $y(t)$ is bounded, $\lambda(y(t))$ is bounded as well. So with this property and (53), we have that $v(y(t),x(t),t)$ is in $L^2([0,+\infty))$. It follows readily from (37) and the fact that $y(t)\in L^2([0,+\infty))$ that the function:

$$\varphi_1(t) = h_1(y(t),t) + h_2(y(t),v(x(t),y(t),t),t)\, v(x(t),y(t),t) \quad (57)$$

belongs to $L^2([0,+\infty))$.

On the other hand, thanks to (49) and (53), we deduce that $[v(y(t),x(t),t)\lambda(y(t))]\in L^2([0,+\infty))$. Moreover, by taking advantage of (38), (40),(55), (50) and the boundedness of $y(t)$, there exist $c > 0$ and $t_c > 0$ such that for all $t\in[t_c,+\infty)$, we have

$$\begin{aligned} c_\sigma|x(t)^\top Qh_2(y(t),0,t)| &\le \left|\sigma(|x(t)|_Q)\frac{x(t)^\top Q}{|x(t)|_Q}h_2(y(t),0,t)\right|, \\ &\le \left|\frac{v(y(t),x(t),t)}{\lambda(y(t))}\right| + \left|\kappa'(V(y,t))\frac{\partial V}{\partial y}(y,t)f_2(y,0,t)\right|, \\ &\le \left|\frac{v(y(t),x(t),t)}{\lambda(y(t))}\right| + c|y(t)|. \end{aligned} \quad (58)$$

Since h_2 is of class C^1, this inequality implies that $x(t)^\top Qh_2(0,0,t)$ belongs to $L^2([0,+\infty))$.

Let us sum up our previous results:

- The function $y(t)$ belongs to $L^2([0,+\infty))$ and

$$\lim_{t\to+\infty} y(t) = 0. \quad (59)$$

- The function $x(t)$ satisfies

$$\dot{x}(t) = Mx(t) + \varphi_1(t) \quad (60)$$

with the functions φ_1 and $x(t)^\top Qh_2(0,0,t)$ in $L^2([0,+\infty))$ and $x(t)^\top Rx(t)$ in $L^1([0,+\infty))$.

Since our notation is $D(t) = h_2(0,0,t)$, this last equation can be rewritten as follows with K_d and K_r given by Assumption A4:

$$\dot{x}(t) = (M + K_d(t)D(t)^\top Q + K_r(t)R^{1/2})x(t) + [\varphi_1(t) - K_d(t)D(t)^\top Qx(t) - K_r(t)R^{1/2}x(t)]. \quad (61)$$

But with Assumption A4, this can be seen as an ES linear system driven by inputs in L^p spaces. This implies

$$\lim_{t\to+\infty} x(t) = 0. \quad (62)$$

Finally, noticing that $P(t)$ is bounded, and returning to the initial coordinates, we get

$$\lim_{t\to+\infty}(X(t) - X_r(t)) = 0, \qquad \lim_{t\to+\infty}(Y(t) - Y_r(t)) = 0. \quad (63)$$

3. Global uniform asymptotic stability

Theorem 1 is not entirely satisfactory: while it requires that Y_r be a GUAS solution of the Y-subsystem with $u = u_r$, it only ensures the existence of a feedback law which GAS the solution (X_r, Y_r) of (6). Since the property of uniformity is missing in this stabilizability result, there is a priori no way to apply repeatedly Theorem 1. Fortunately, we show in this section that GUAS may be achieved when the reference signal is a periodic function.

3.1. Periodic case

First, consider the case where $(X_r(t), Y_r(t))$ and $u_r(t)$ are periodic functions.

Corollary 6. *Assume that Assumptions* A1–A4 *are satisfied and that* $X_r(t)$, $Y_r(t)$, $u_r(t)$ *are periodic functions of period* T. *Then* (X_r, Y_r) *is a GUAS and a LES solution of system* (6).

Proof. *Step* 1: GAS follows from Theorem 1.

Step 2: *Periodic closed-loop system.* The closed-loop system can be described by (32) with the functions defined in (33)–(36) and the feedback v given in (50). From these various definitions, this closed-loop system is periodic if $P(t)$ and $V(y,t)$ are periodic in t.

About $P(t)$, we have (18) with $C(t)$ and Φ_A defined in (14), (9), (8). So if T is the period for the reference trajectory, we have: $C(t+T) = C(t)$, $A(t+T) = A(t)$,$\Phi_A(s+T, t+T) = \Phi_A(s,t)$ which implies that P is of period T.

About $V(y,t)$, we remark that a converse Lyapunov Theorem which can be used in the proof of Lemma 5 is Hahn (1967, Theorem 49.4, p. 237). It guarantees that $V(y,t)$ is periodic whenever $f_0(y,t)$ is periodic.

Step 3: *GUAS.* For periodic systems GAS implies GUAS (see Yoshizawa, 1966, Theorem 11.3 for instance). So from Steps 1 and 2, it follows that we can design a feedback which GUAS the system (6).

Step 4: *LES.* By applying to the linear approximation of (32) the strategy of design of the proof of Theorem 1, we obtain a family of asymptotically stabilizing feedbacks which contains the linear approximation of (50). On the other hand, the linear approximations of (32) and (50) are periodic. It follows that the corresponding closed-loop system is uniformly asymptotically stable. Since we know that if a time-varying linear system is UAS, then it is ES (Khalil, 1992, Chapter 3, Section 3.5), this last property is proved for the linear system we are interested in. Next, we conclude by mentioning that the conditions of Khalil (1992, Theorem 3.11) which guarantee that if the linear approximation at the origin of a time-varying system is ES, then this system admits the origin as a LES point are met by (32) with (50) as control. □

4. Concluding remarks

In this paper, we have constructed a family of feedback laws (which contains arbitrarily small bounded functions) which GUAS periodic reference state trajectories of nonlinear systems obtained after adding one integration. Such systems are in general not feedback linearizable which makes the tracking problem even more difficult. A recursive application of the design we have proposed is possible and provides us with a new technique for dealing with tracking problems for nonlinear systems, in feedforward form. In future works, we will prove, under slightly more restrictive assumption, the GUAS of signals which are not periodic functions of the time and illustrate our design of control law on the cart–pendulum system.

Appendix A

A.1. Proof of Lemma 2

First, let us notice that by using the same arguments as those employed in Mazenc and Praly (1996, Appendix C), we can prove that the function $\mathscr{P}(t,\tilde{Y})$ defined in (25) is well-defined and of class C^1.

Next, let us show that this function satisfies, for all $\tilde{Y}, t \geq 0$,:

$$\frac{\partial \mathscr{P}}{\partial t}(t,\tilde{Y}) + \frac{\partial \mathscr{P}}{\partial \tilde{Y}}(t,\tilde{Y})[F_0(\tilde{Y} + Y_r(t)) - F_0(Y_r(t))] = M\mathscr{P}(t,\tilde{Y}) - \mathscr{H}(t,\tilde{Y}). \tag{A.1}$$

For all $\tilde{Y}_0$ and $t \geq 0$, we have, using (23):

$$\begin{aligned}\mathscr{P}(t,\Phi_Y(t,0,\tilde{Y}_0)) &= \int_0^{+\infty} \exp(-Ml)\mathscr{H}(t+l,\Phi_Y(t+l,t,\Phi_Y(t,0,\tilde{Y}_0)))\,\mathrm{d}l,\\ &= \int_0^{+\infty} \exp(-Ml)\mathscr{H}(t+l,\Phi_Y(t+l,0,\tilde{Y}_0))\,\mathrm{d}l,\\ &= \exp(Mt)\int_t^{+\infty} \exp(-Mr)\mathscr{H}(r,\Phi_Y(r,0,\tilde{Y}_0))\,\mathrm{d}r.\end{aligned} \tag{A.2}$$

It follows

$$\overbrace{\mathscr{P}(t,\Phi_Y(t,0,\tilde{Y}_0))}^{\cdot}{}_{(23)} = M\mathscr{P}(t,\Phi_Y(t,0,\tilde{Y}_0)) - \mathscr{H}(t,\Phi_Y(t,0,\tilde{Y}_0)). \tag{A.3}$$

On the other hand, a direct computation gives

$$\begin{aligned}\overbrace{\mathscr{P}(t,\Phi_Y(t,0,\tilde{Y}_0))}^{\cdot}{}_{(23)} &= \frac{\partial \mathscr{P}}{\partial t}(t,\Phi_Y(t,0,\tilde{Y}_0))\\ &\quad + \frac{\partial \mathscr{P}}{\partial \tilde{Y}}(t,\Phi_Y(t,0,\tilde{Y}_0))[F_0(\Phi_Y(t,0,\tilde{Y}_0) + Y_r) - F_0(Y_r)\\ &\quad + (F_2(\Phi_Y(t,0,\tilde{Y}_0) + Y_r,u_r) - F_2(Y_r,u_r))u_r].\end{aligned} \tag{A.4}$$

Combining (A.4) and (A.3) and choosing $\tilde{Y}_0 = \Phi_Y(0,t,\tilde{Y})$, we obtain:

$$\begin{aligned}&\frac{\partial P}{\partial t}(t,\tilde{Y}) + \frac{\partial P}{\partial \tilde{Y}}(t,\tilde{Y})\\ &\quad \times [F_0(\tilde{Y} + Y_r) - F_0(Y_r) + (F_2(\tilde{Y} + Y_r,u_r)\\ &\quad - F_2(Y_r,u_r))u_r]\\ &= M\mathscr{P}(t,\tilde{Y}) - \mathscr{H}(t,\tilde{Y}).\end{aligned} \tag{A.5}$$

To conclude, we notice that equality (A.5) implies that the change of coordinates (24) transforms system (21) into system (27). □

Appendix B

B.1. Proof of Lemma 4

First, observe that

$$h_1(0,t) = 0 \quad \forall t \geq 0. \tag{B.1}$$

Next, let us take the differential of h_1 with respect to y at $y=0$:

$$\frac{\partial h_1}{\partial y}(0,t) = \frac{\partial H_1}{\partial y}(Y_r(t)) + \frac{\partial H_2}{\partial y}(Y_r(t), u_r(t))u_r(t) + P(t)\left[\frac{\partial f_0}{\partial y}(0,t) - A(t)\right] - C(t). \tag{B.2}$$

Using the definitions of $A(t)$ in (8) and of $C(t)$ in (14), we get:

$$\frac{\partial h_1}{\partial y}(0,t) = P(t)\left[\frac{\partial f_0}{\partial y}(0,t) - \frac{\partial F_0}{\partial Y}(Y_r(t)) - \frac{\partial F_2}{\partial Y}(Y_r(t), u_r(t))u_r(t)\right]. \tag{B.3}$$

From the definition of f_0, it follows readily that

$$\frac{\partial h_1}{\partial y}(0,t) = 0 \quad \forall t \geq 0. \tag{B.4}$$

With (B.1) and (B.4) and the fact that H_1, H_2, F_0 and F_2 are of class C^2, we deduce that

$$|h_1(y,t)| \leq \left[\int_0^1 \left|\frac{\partial h_1}{\partial y}(sy,t) - \frac{\partial h_1}{\partial y}(0,t)\right| \mathrm{d}s\right]|y|, \leq \left[\int_0^1 \left(\int_0^1 \left|\frac{\partial^2 h_1}{\partial^2 y}(usy,t)\right| \mathrm{d}u\right) s\,\mathrm{d}s\right]|y|^2. \tag{B.5}$$

Since $Y_r(t)$ and $u_r(t)$ are bounded functions and $\partial^2 h_1/\partial^2 y(y,t)$ is continuous, there exists a continuous positive function $\gamma(|y|)$ such that

$$\int_0^1 \left(\int_0^1 \left|\frac{\partial^2 h_1}{\partial^2 y}(usy,t)\right| \mathrm{d}u\right) s\,\mathrm{d}s \leq \gamma(|y|). \tag{B.6}$$

This concludes the proof. □

Appendix C

C.1. Lemma 7

Lemma 7. *Let $\tilde{u}$ be any strictly positive-real number. Let $a(x,y,v,t)$ be a function which satisfies*:

$$|a(x,y,0,t)| + \frac{|a(x,y,v,t) - a(x,y,0,t)|}{|v|} \leq \Omega(|y|), \quad \forall v\colon |v| \leq \tilde{u} \tag{C.1}$$

for some continuous and positive function Ω. Let φ be a positive and decreasing function of class C^1 such that

$$\varphi(0) = 1, \quad \varphi(s) > 0, \quad \forall s \in [0,1[, \quad \varphi(0) = 0, \quad \forall s > 1. \tag{C.2}$$

Then, there exists $\delta > 0$ such that

$$a(x,y,v_s,t)v_s \leq -\frac{\delta}{2}\varphi(|y|)|a(x,y,0,t)|^2 \tag{C.3}$$

with

$$v_s = -\delta\varphi(|y|)a(x,y,0,t)^{\top} \tag{C.4}$$

such that $|v_s| \leq \tilde{u}$.

Proof. Simple calculations yield:

$$\begin{aligned} a(x,y,v_s,t)v_s &\leq a(x,y,0,t)v_s + \frac{|a(x,y,v_s,t) - a(x,y,0,t)|}{|v_s|}|v_s|^2, \\ &\leq a(x,y,0,t)v_s + \Omega(|y|)|v_s|^2, \\ &\leq -\delta\varphi(|y|)|a(x,y,0,t)|^2 + \Omega(|y|)|\delta\varphi(|y|)a(x,y,0,t)|^2, \\ &\leq -\delta\varphi(|y|)|a(x,y,0,t)|^2 + \delta^2\varphi(|y|)|a(x,y,0,t)|^2 \sup_{0\leq s\leq 1}\{\Omega(s)\}. \end{aligned} \tag{C.5}$$

The result follows with $0 < \delta \leq \min\{1,\tilde{u}\}/2\sup_{0\leq s\leq 1}\{\Omega(s)\}$. □

References

Hahn, W. (1967). *Stability of Motion.* New York: Springer.

Isidori, A. (1989). *Nonlinear control system.* Berlin: Springer.

Krstic, M., Kanellakopoulos, I., & Kokotovic, P. V. (1995). *Nonlinear and adaptive control design.* New York: Wiley.

Khalil, H. (1992). *Nonlinear systems.* New York: Macmillan. Canada Toronto: Maxwell Macmillan.

Liu, W., Chitour, Y., & Sontag, E. (1996). Remarks on finite gain stabilizability of linear systems subject to input saturation. *SIAM Journal of Control and Optimization*, *34*, 1190–1219.

Marino, R., Kanellakopoulos, I., & Kokotovic, P. (1989). Adaptative tracking for feedback linearizable SISO Systems. *Proceedings of the 28th conference on decision and control.* Tampa, FL.

Marino, R., & Tomei, P. (1995). *Nonlinear control design.* London: Prentice-Hall.

Martin, P., Devasia, S., & Paden, B. (1996). A different look at output tracking: control of a VTOL aircraft. *Automatica*, *32*(1), 101–107.

Mazenc, F. (1996). *Stabilisation de trajectoires, Ajout d'intégration, Commande saturées.* Thèse en Mathématiques et Automatique. École des Mines de Paris.

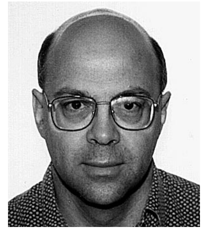
Mazenc, F., & Praly, L. (1996). Adding an integration and Global asymptotic stabilization of feedforward systems. *IEEE Transactions on Automatic Control*, *41*(11), 1559–1578.

Teel, A. (1992). *Feedback stabilization: nonlinear solutions to inherently nonlinear problems.* Memorandum No. UCB/ERL M92/65.

Yoshizawa, T. (1966). *Stability theory by Lyapunov's second method.* The mathematical Society of Japan.

Dr. Frédéric Mazenc was born in 1969. He received the D.E.A in 1993 from the University of Orsay and the Ph.D. degree in 1996, in automatic control and mathematics, from l'Ecole des Mines de Paris. He has been a Postdoctoral Fellow at the Catholic University of Louvain, in the CESAME.

He is currently working as a Postdoctoral Fellow at the Imperial College, in the Centre for Process Systems Engineering. His research interests include nonlinear control theory.



Dr. Laurent Praly graduated form École Nationale Supérieure des Mines de Paris in 1976. After working in industry for three years, in 1980 he joined the Centre Automatique et Systémes at École des Mines de Paris. From July 1984 to June 1985, he spent a sabbatical year as a visiting assistant professor in the Department of Electrical and Computer Engineering at the University of Illinois at Urbana-Champaign. Since 1985 he has continued at the Centre Automatique et Systèmes where he served as director for two years. In 1993, he spent a quarter at the Insitute for Mathematics and its Applications at the University of Minnesota where he was an invited researcher. He served as organizer and author in a Mini-course at ECC'95 conference.

His main interest is in feedback stabilization of controlled dynamical systems under various aspects — linear and nonlinear, dynamic, output, with parametric or dynamic uncertainty —. On this topic he is contributing both on the theoritical aspect with many academic publications and the practical aspect with several industrial applications.