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Integrator Backstepping for Bounded Controls and Control Rates

Randy Freeman and Laurent Praly

***Abstract*— We present a backstepping procedure for the design of globally stabilizing state feedback control laws such that the magnitudes of the control signals and their derivatives are bounded by constants which do not depend on the initial conditions. We accomplish this by propagating such boundedness properties through each step of the recursive design.**

I. Introduction

Recursive Lyapunov design procedures developed in recent years have expanded the classes of nonlinear systems for which systematic controller designs are possible. A prime example of such a procedure is integrator backstepping (see [2] and the references therein). The flexibilities of this procedure create opportunities for the improvement of performance and the satisfaction of design constraints.

In this paper we present a new version of the backstepping procedure in which the boundedness of the control signal and its derivative are propagated through each step of the recursive design. We thereby add the powerful backstepping method to the collection of tools available for the global design of control systems with actuator constraints (see [4] and [7] for instance). The achieved bound on the control signal in our design cannot generally be made to satisfy an arbitrary prescribed constraint, unlike the bounds in the designs of [4] and [7]. However, our method applies to a much broader class of nonlinear systems, including those which do not admit controllers satisfying arbitrary constraints.

The key feature of our method is a new choice for the Lyapunov function at each step of the recursive design, a choice based on combining design flexibilities proposed in [1] and [3]. We will give our main result in Section II, followed by its proof in Section III.

II. Backstepping with Actuator Constraints

A. Main Result

Given continuous functions $f, g\colon \mathbb{R}^n \to \mathbb{R}^n$ and $h\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ such that $f(0) = 0$ and $h(0, 0) = 0$, we consider the single-input system

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = F(x, y) + G(x, y)u \tag{1}$$

where $(x, y) \in \mathbb{R}^n \times \mathbb{R}$ is the state variable, $u \in \mathbb{R}$ is the control variable, and F and G are given by

$$F(x, y) := \begin{bmatrix} f(x) + g(x)y \\ h(x, y) \end{bmatrix}, \qquad G(x, y) := \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{2}$$

Our goal in this paper is to present a set of conditions guaranteeing the existence of a stabilizing control law for (1) such that the magnitudes of both the control signal u and its derivative $\dot{u}$ are bounded by a constant which does not depend on initial conditions. Roughly

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speaking, we will show that if the result already holds for the reduced-order system

$$\dot{x} = f(x) + g(x)v \tag{3}$$

with some control law $v = \mu(x)$, then it does so for (1) with a control law $u = \nu(x, y)$. To be precise, we will prove the following.

Theorem 1: Suppose there exist C^1 functions $\mu, r, V, \alpha: \mathbb{R}^n \to \mathbb{R}$ such that

A1) V, α, and r are positive definite, V is proper, $\mu(0) = 0$, and $\inf_{|x| \geq c} r(x) > 0$ for some $c > 0$;
A2) $|y - \mu(x)| \leq r(x)$ implies $L_f V(x) + L_g V(x)\, y \leq -\alpha(x)$;
A3) $\mu, r, L_f\mu, L_g\mu, L_f r, L_g r$, and $L_g V$ are all bounded on $\mathbb{R}^n$;
A4) $L_g V$, $L_f \mu$, and $(L_g\mu \cdot \mu)$ are all $O(\sqrt{\alpha(x)})$ as $x \to 0$;
B1) there exists $r_0 > 0$ such that h is bounded on the set $\{(x, y) \in \mathbb{R}^n \times \mathbb{R}: |y - \mu(x)| \leq r_0\}$;
B2) there exists $h_0 \geq 0$ such that $\operatorname{sign}[y - \mu(x)] \cdot h(x, y) \leq h_0$ on $\mathbb{R}^n \times \mathbb{R}$;
B3) h is $O(\sqrt{\alpha(x) + [y - \mu(x)]^2})$ as $(x, y) \to (0, 0)$.

Then there exist C^1 functions $\nu, \rho, U, \beta: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ such that

C1) U, β, ρ are positive definite, U is proper, $\nu(0, 0) = 0$, and $\inf\{\rho(x, y): |x| + |y| \geq c\} > 0$;
C2) $|u - \nu(x, y)| \leq \rho(x, y)$ implies $L_F U(x, y) + L_G U(x, y) u \leq -\beta(x, y)$;
C3) $\nu, \rho, L_F\nu, L_G\nu, L_F\rho, L_G\rho$, and $L_G U$ are all bounded on $\mathbb{R}^n \times \mathbb{R}$;
C4) $L_G U$, $L_F \nu$, and $(L_G\nu \cdot \nu)$ are all $O(\sqrt{\beta(x, y)})$ as $(x, y) \to (0, 0)$.

B. Interpreting the Conclusion of Theorem 1: Properties C1)–C4)

If we can satisfy A1)–A4) and B1)–B3), then this theorem generates a control law ν and a Lyapunov function U whose derivative along solutions of (1) with $u = \nu(x, y)$ satisfies, from C2)

$$\dot{U} = L_F U(x, y) + L_G U(x, y)\, \nu(x, y) \;\leq\; -\beta(x, y). \tag{4}$$

Thus this control law globally asymptotically stabilizes the origin of (1), and we see from C3) that the control law $u = \nu(x, y)$ and its derivative

$$\dot{u}(x, y) = L_F \nu(x, y) + L_G \nu(x, y) \nu(x, y) \tag{5}$$

are bounded functions of (x, y) as desired. The control law ν used to prove Theorem 1 is simply

$$\nu(x, y) = -\gamma\sigma(\lambda[y - \mu(x)]) \tag{6}$$

where $\sigma: \mathbb{R} \to \mathbb{R}$ is the C^1 saturation function defined in Section III-A. Following our proof, the constant design parameters γ and λ must be chosen sufficiently large. In general, there is no guarantee that the magnitude limit γ on the control law (6) can be chosen small enough to meet a prescribed constraint. A similar statement holds for the rate limit, which depends on both γ and λ as well as the functions f, g, h, μ, and σ.

Theorem 1 can be applied recursively because properties C1)–C4) are to the complete system (1) as properties A1)–A4) are to the reduced-order system (3). After the first step in a recursive design, one needs only verify properties B1)–B3) at each new step. For example, by applying Theorem 1 twice, one can find constant parameters λ_1, γ_1, λ_2, and γ_2 so that

$$\nu(x, y_1, y_2) = -\gamma_2\, \sigma\left(\lambda_2\left[y_2 + \gamma_1 \sigma\left(\lambda_1\left[y_1 + \frac{2x^2}{1+x^2}\right]\right)\right]\right) \tag{7}$$

is a magnitude- and rate-limited control law which globally asymptotically stabilizes the system

$$\begin{cases} \dot{x} = \dfrac{x^3}{1+x^2} + x y_1 \\ \dot{y}_1 = y_2 - x^2 \max\{0, y_1\} \\ \dot{y}_2 = u + \sin(x^2). \end{cases} \tag{8}$$

Furthermore, we immediately have the following corollary to Theorem 1.

Corollary 2: Let $f, g: \mathbb{R}^n \to \mathbb{R}^n$ be C^0, and suppose there exist C^1 functions $\mu, r, V, \alpha: \mathbb{R}^n \to \mathbb{R}$ satisfying A1)–A4). Then for any $m \geq 1$ there is a C^1 function $\nu: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ such that the system

$$\begin{cases} \dot{x} = f(x) + g(x)\, y_1 \\ \quad\vdots \\ \dot{y}_i = y_{i+1} \qquad 1 \leq i \leq m-1 \\ \quad\vdots \\ \dot{y}_m = \nu(x, y) \end{cases} \tag{9}$$

with $y := [y_1 \ \cdots \ y_m]^T$ is globally asymptotically stable, and furthermore the control law $u = \nu(x, y)$ and its derivative $\dot{u}(x, y)$ are bounded functions of (x, y).

The control law ν in this corollary is given by

$$\begin{aligned} \nu(x, y) = -\gamma_m\sigma(\lambda_m[y_m + \cdots \\ + \gamma_2\sigma(\lambda_2[y_2 + \gamma_1\sigma(\lambda_1[y_1 - \mu(x)])])\cdots]) \end{aligned} \tag{10}$$

where the constants γ_i and λ_i are positive design parameters. It is reminiscent of the nested saturation control laws proposed in [5] and [6].

C. Interpreting the Assumptions of Theorem 1: Properties A1)–A4) and B1)–B3)

Assumptions A1)–A4) concern only the reduced-order system (3). Essentially, we require knowledge of a bounded function $\mu(x)$ such that with $v = \mu(x)$, this reduced-order system is globally asymptotically stable with Lyapunov function $V(x)$. We require further that the functions $L_f\mu$ and $L_g\mu$ be bounded, which is tantamount to requiring that the control law μ, and its rate $\dot{\mu}$ be bounded along solutions to (3). The function r is a measure of the stability robustness to errors in the implementation of μ for the reduced-order system (3). Because some amount of robustness will always exist, the only assumption concerning r is that it not vanish outside a neighborhood of $x = 0$; the requirements that r, $L_f r$, and $L_g r$ be bounded can be satisfied by taking r to be constant outside a compact set.

In Condition A3), we require that the function $L_g V$ be bounded. This requirement is an important part of Theorem 1. Indeed, let us consider the $n = 1$ system

$$\dot{x} = -x^3 + x^3 y, \qquad \dot{y} = u. \tag{11}$$

The feedback $v = \mu(x) \equiv 0$ is bounded with bounded rate and globally asymptotically stabilizes

$$\dot{x} = -x^3 + x^3 v. \tag{12}$$

For $r(x) \leq \frac{1}{2}$, we see that conditions A1)–A4) hold, except that there is no proper C^1 function $V(x)$ such that $L_g V(x) = V'(x) \cdot x^3$ is bounded. Therefore Theorem 1 does not apply, which is consistent with the observation that no bounded control law $u = \nu(x, y)$ for (11) can prevent finite escape times from all initial conditions.

The final requirement A4) on (3) is a mild condition on the local behavior of the functions V and μ in a neighborhood of $x = 0$. This condition allows us to conclude the existence of a C^1 control law ν for (1) given a C^1 control law μ for (3). This is in contrast to standard backstepping results in which one degree of differentiability is lost, namely in which a C^1 control law μ yields a merely continuous (C^0) control law ν.

Assumptions B1)–B3) concern only the function h in the y-subsystem of (1). Conditions B1) and B2) will always be satisfied when h is bounded, but they also allow h to be unbounded in certain directions. Condition B3) is a mild condition on the local behavior of the function h in a neighborhood of the point $(x, y) = (0, 0)$.

III. Proof of Theorem 1

A. Definitions and Technical Preliminaries

- We will use the function K defined in [3, eq. (11)] as

$$K(p, q) = \int_q^p [a(s-q) + b(s|s| - q|q|)]\, ds \tag{13}$$

$$= \tfrac{1}{2}a(p-q)^2 + b(\tfrac{1}{3}|p|^3 - pq|q| + \tfrac{2}{3}|q|^3) \tag{14}$$

where $a > 0$ and $b > 0$ are design parameters. One can verify that $K(p, q) \geq 0$ for all $p, q \in \mathbb{R}$, and furthermore $K(p, q) = 0$ if and only if $p = q$. Partial derivatives of the function K are

$$K_1(p, q) := \frac{\partial K}{\partial p}(p, q) = (p-q)\, M(p, q) \tag{15}$$

$$K_2(p, q) := \frac{\partial K}{\partial q}(p, q) = -(p-q)(a + 2b|q|) \tag{16}$$

where M is the continuous function given by

$$M(p, q) := a + b \cdot \begin{cases} |p| + |q|, & \text{when } pq \geq 0 \\ \dfrac{p^2 + q^2}{|p| + |q|}, & \text{when } pq < 0. \end{cases} \tag{17}$$

This function M satisfies the inequalities

$$a + \tfrac{1}{2}b[|p| + |q|] \leq M(p, q) \leq a + b[|p| + |q|] \tag{18}$$

for all $p, q \in \mathbb{R}$. Also, given any compact set $\mathcal{Q} \subset \mathbb{R}$, there exists $\omega \geq 0$ such that

$$\lim_{|p|\to\infty} \frac{K(p,q)}{|p|^3} = \frac{b}{3}$$
$$\lim_{|p|\to\infty} \frac{|K_1(p,q)|}{p^2} = b, \qquad \left|\frac{K_1(p,q)}{1 + K(p,q)}\right| \leq \omega \tag{19}$$

for all $p \in \mathbb{R}$ and all $q \in \mathcal{Q}$.

- We define a saturation function σ as follows. Given $\sigma_0 > 1$, let $\sigma\colon \mathbb{R} \to [-1, 1]$ be C^1, odd, nondecreasing, and such that with σ' denoting the derivative of σ

$$\sigma(s) = \operatorname{sign}(s), \qquad \text{when } |s| \geq 1 \tag{20}$$

$$s^2 \leq s\sigma(s) \leq \sigma_0 s^2, \qquad \text{when } |s| \leq 1 \tag{21}$$

$$0 \leq \sigma'(s) \leq \sigma_0, \qquad \forall\, s \in \mathbb{R}. \tag{22}$$

- We define C^1 functions $\mu^+, \mu^-\colon \mathbb{R}^n \to \mathbb{R}$ by

$$\mu^+(x) := \mu(x) + r(x), \qquad \mu^-(x) := \mu(x) - r(x). \tag{23}$$

We use these functions to define the following sets in $\mathbb{R}^n \times \mathbb{R}$:

$$A^+ := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}\colon y > \mu^+(x)\} \tag{24}$$

$$A^0 := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}\colon \mu^-(x) \leq y \leq \mu^+(x)\} \tag{25}$$

$$A^- := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}\colon y < \mu^-(x)\} \tag{26}$$

$$A^\pm := A^+ \cup A^-. \tag{27}$$

Note that $\mathbb{R}^n \times \mathbb{R} = A^+ \cup A^0 \cup A^-$ and that A^+, A^0, and A^- are disjoint. In the following we shall typically write:

$$0 \leq I(x, y, \mu^\pm), \qquad \forall\, (x, y) \in A^\pm \tag{28}$$

where I is some function. This must be understood as

$$0 \leq I(x, y, \mu^+(x)), \qquad \forall\, (x, y) \in A^+$$

and

$$0 \leq I(x, y, \mu^-(x)), \qquad \forall\, (x, y) \in A^-. \tag{29}$$

Using this notation and letting $R := \sup_x r(x)$, we have, for all $(x, y) \in A^\pm$

$$0 < (y - \mu^\pm(x))(y - \mu(x)) \leq |y - \mu(x)|^2 \tag{30}$$

$$r(x) \leq \min\{R, |y - \mu(x)|\}. \tag{31}$$

- We note that the function

$$(x, y) \mapsto \begin{cases} |\sigma(\lambda[y - \mu(x)])|, & \text{when } (x, y) \in A^\pm \\ \sigma(\lambda r(x)), & \text{when } (x, y) \in A^0 \end{cases} \tag{32}$$

is continuous, positive definite, and bounded away from zero outside a compact neighborhood of $(x, y) = (0, 0)$.

- We assume, without loss of generality, that

$$r,\ L_f r, \text{ and } (L_g r \cdot \mu) \text{ are all } o(\sqrt{\alpha(x)}) \text{ as } x \to 0. \tag{33}$$

Indeed, if the given function r violates this condition, we can always flatten it near $x = 0$ while preserving A1)–A4) and B1)–B3) so that this condition is satisfied.

- With A1), we see that outside a compact neighborhood of $x = 0$, $r(x)$ can be used to bound any bounded function. From (31) the same holds on $A^\pm$ with $|y - \mu(x)|$. Consequently, from A3), A4), (31), and (33) there exist nonnegative constants c_0 and c_1 such that for all $(x, y) \in A^\pm$

$$\begin{aligned}&|L_g V(x) - (a + 2b|\mu^\pm(x)|)(L_f \mu^\pm(x) + L_g \mu^\pm(x) \mu^\pm(x))| \\ &\quad \leq c_1(c_0 + a + b)[\sqrt{\tfrac{2}{3}\alpha(x)} + |y - \mu(x)|].\end{aligned} \tag{34}$$

Similarly, it follows from (18), (31), B2), and B3) that there exists $c_2 \geq 0$ such that for all $(x, y) \in A^\pm$:

$$\begin{aligned}&M(y, \mu^\pm(x)) \operatorname{sign}(y - \mu^\pm(x)) h(x, y) \\ &\quad \leq c_2(a + b)[\sqrt{\tfrac{2}{3}\alpha(x)} + |y - \mu(x)|] \\ &\qquad + c_2 M(y, \mu^\pm(x)) \min\{R, |y - \mu(x)|\}.\end{aligned} \tag{35}$$

- By using (21) of σ, (18) on M, and by imposing

$$\frac{1}{\sup_x\{r(x)\}} = \frac{1}{R} \leq \lambda \tag{36}$$

we get the following inequalities:

$$\lambda|y-\mu(x)| \leq 1 \Rightarrow$$

$$|y-\mu(x)| \leq \left|\frac{1}{a\gamma\lambda}\right| a|\gamma\sigma(\lambda[y-\mu(x)])| \tag{37}$$

$$1 \leq \lambda|y-\mu(x)| \Rightarrow$$

$$\begin{aligned}|y-\mu(x)| &\leq (|y| + |\mu^{\pm}(x)| + r(x)) \\ &\quad\cdot \frac{1}{\gamma}\,|\gamma\sigma(\lambda[y-\mu(x)])| \qquad (38)\\ &\leq \frac{\max\left\{\dfrac{R}{a},\, \dfrac{2}{b}\right\}}{\gamma}\left[a + \frac{b(|y| + |\mu^{\pm}(x)|)}{2}\right] \\ &\quad\cdot |\gamma\sigma(\lambda[y-\mu(x)])|. \qquad (39)\end{aligned}$$

This yields, for all $(x,\, y)$

$$\begin{aligned}|y-\mu(x)| \leq \frac{1}{\gamma}\,\max\left\{\frac{R}{a},\, \frac{2}{b}\right\} M(y,\, \mu^{\pm}(x)) \\ \cdot |\gamma\sigma(\lambda[y-\mu(x)])|.\end{aligned} \tag{40}$$

Also, we have

$$\begin{aligned}\min\{R, |y-\mu(x)|\} &\leq R\,\min\{1,\, \lambda|y-\mu(x)|\} \\ &\leq \frac{R}{\gamma}\,|\gamma\sigma(\lambda[y-\mu(x)])|.\end{aligned} \tag{41}$$

B. Proof of Theorem 1

1) Proof of Global Stability: We propose a Lyapunov function $W(x,\, y)$ which belongs to the family of Lyapunov functions described in [3] and is "flattened" inside the set A^0, as proposed in [1]

$$W(x,\, y) := V(x) + \begin{cases} K(y,\, \mu^{\pm}(x)), & \text{when } (x,\, y) \in A^{\pm} \\ 0, & \text{when } (x,\, y) \in A^0 \end{cases} \tag{42}$$

where K is given by (14). One can verify that W is C^1, positive definite, and proper. We next compute $\dot{W}$ in each of the two sets $A^{\pm}$ and A^0.

- In the set A^0 we obtain, using A2)

$$\dot{W}(x,\, y) = L_f V(x) + L_g V(x)\, y \leq -\alpha(x). \tag{43}$$

Therefore, $\dot{W}(x,\, y)$ is negative definite on A^0, regardless of the value of the control variable u.

- In the set $A^{\pm}$ we obtain, with $u = \nu(x,\, y)$ given by (6)

$$\begin{aligned}\dot{W}(x,\, y) \leq\; &-\alpha(x) + T(x,\, y) - [y - \mu^{\pm}(x)] \\ &\cdot M(y,\, \mu^{\pm}(x))\gamma\sigma(\lambda[y-\mu(x)])\end{aligned} \tag{44}$$

where M is from (17) and

$$\begin{aligned}T(x,\, y) = [y-\mu^{\pm}(x)][L_g V(x) + M(y,\, \mu^{\pm}(x))h(x,\, y) \\ - (a + 2b|\mu^{\pm}(x)|)(L_f\mu^{\pm}(x) + L_g\mu^{\pm}(x)y)].\end{aligned}$$

It remains to determine the negativeness of $\dot{W}(x,\, y)$ on the set $A^{\pm}$. For this we observe that by completing the squares and using (30), (34), and (35), we get, for all $(x,\, y) \in A^{\pm}$

$$\begin{aligned}T(x,\, y) \leq\; &\tfrac{1}{3}\,\alpha(x) + |y-\mu^{\pm}(x)| \\ &\cdot [(c_1^2(c_0 + a + b)^2 + c_0c_1 + c_2^2(a+b)^2 \\ &+ (c_1 + c_2 + c_3)(a+b))|y-\mu(x)| \\ &+ c_2 M(y,\, \mu^{\pm}(x))\,\min\{R,\, |y-\mu(x)|\}]\end{aligned} \tag{45}$$

where c_3 is given by the boundedness of $L_g\mu^{\pm}$ and $\mu^{\pm}$. Then, with (40) and (41), we get more simply

$$\begin{aligned}T(x,\, y) \leq\; &\tfrac{1}{3}\alpha(x) + |y-\mu^{\pm}(x)|M(y,\, \mu^{\pm}(x)) \cdot |\gamma\sigma(\lambda[y-\mu(x)])| \\ &\cdot \frac{1}{\gamma}\Bigg([c_1^2(c_0 + a + b)^2 + c_0c_1 \\ &\quad + c_2^2(a+b)^2 + (c_1 + c_2 + c_3)(a+b)] \\ &\quad \cdot \max\left\{\frac{R}{a},\, \frac{2}{b}\right\} + c_2 R\Bigg).\end{aligned} \tag{46}$$

So by imposing that γ be large enough, we finally arrive at

$$\begin{aligned}\dot{W}(x,\, y) \leq -\tfrac{2}{3}\alpha(x) - \tfrac{1}{2}|y-\mu^{\pm}(x)| \\ \cdot M(y,\, \mu^{\pm}(x))|\gamma\sigma(\lambda[y-\mu(x)])|\end{aligned} \tag{47}$$

for all $(x,\, y) \in A^{\pm}$. This proves the negative definiteness of $\dot{W}(x,\, y)$ on $A^{\pm}$.

2) Construction of the Function ρ: With the properties of the function defined in (32), we can construct a C^1 positive definite function $\rho(x,\, y)$ such that

$$\rho(x,\, y) \leq \begin{cases} \frac{1}{4}|\gamma\sigma(\lambda[y-\mu(x)])|, & \text{when } (x,\, y) \in A^{\pm} \\ \frac{1}{4}\gamma\sigma(\lambda r(x)), & \text{when } (x,\, y) \in A^0 \end{cases} \tag{48}$$

and furthermore ρ is constant outside some compact set. With this choice for ρ we obtain

$$|u - \nu(x,\, y)| \leq \rho(x,\, y) \quad \Longrightarrow \quad \dot{W} \leq -S(x,\, y) \tag{49}$$

where

$$S(x,\, y) = \begin{cases} \dfrac{2}{3}\alpha(x) + \dfrac{a}{4}\,|y-\mu^{\pm}(x)| & \\ \quad\cdot|\gamma\sigma(\lambda[y-\mu(x)])|, & \text{when } (x,\, y) \in A^{\pm} \\ \alpha(x), & \text{when } (x,\, y) \in A^0. \end{cases} \tag{50}$$

3) Construction of the Functions U and β: We define U as

$$U(x,\, y) := \ln[1 + W(x,\, y)] \tag{51}$$

where W is given by (42). We now show that there exists a function $\beta(x,\, y)$ such that

- β is C^1 and positive definite;
- for all $(x,\, y) \in \mathbb{R}^n \times \mathbb{R}$

$$\beta(x,\, y) \leq \frac{S(x,y)}{1 + W(x,y)}; \tag{52}$$

- there exist $\mathcal{C} \subset \mathbb{R}^n \times \mathbb{R}$, a compact neighborhood of the origin, and a constant $\delta > 0$ such that for all $(x,\, y) \in \mathcal{C}$

$$\beta(x,\, y) \;\geq\; \delta[\alpha(x) + (y-\mu(x))^2]. \tag{53}$$

First, because $s^2 \leq s\sigma(s)$ when $|s| \leq 1$, and (33) holds, there is some compact neighborhood $\mathcal{C}$ of the origin such that for all $(x,\, y) \in \mathcal{C}$

$$|\gamma\sigma(\lambda[y-\mu(x)])| \geq \gamma\lambda|y-\mu(x)|, \qquad \frac{1}{8}a\gamma\lambda r(x)^2 \leq \frac{1}{3}\alpha(x). \tag{54}$$

Let us now bound $S(x,\, y)$ from below. We begin by observing from (54) that on $A^{\pm} \cap \mathcal{C}$

$$\begin{aligned}&\frac{a}{4}\,|y-\mu^{\pm}(x)|\,|\gamma\sigma(\lambda[y-\mu(x)])| \\ &\quad\geq \tfrac{1}{4}a\gamma\lambda[(y-\mu(x))^2 - r(x)|y-\mu(x)|] \qquad (55)\\ &\quad\geq \tfrac{1}{4}a\gamma\lambda[\tfrac{1}{2}(y-\mu(x))^2 - \tfrac{1}{2}r(x)^2] \qquad (56)\\ &\quad\geq \tfrac{1}{8}a\gamma\lambda(y-\mu(x))^2 - \tfrac{1}{3}\alpha(x). \qquad (57)\end{aligned}$$

Furthermore, because $|y-\mu(x)| \leq r(x)$ on A^0, we have from (54) that

$$\tfrac{1}{3}\alpha(x) \geq \tfrac{1}{8}a\gamma\lambda(y-\mu(x))^2 \tag{58}$$

on $A^0 \cap \mathcal{C}$. We conclude from (57) and (58) that for all $(x, y) \in \mathcal{C}$

$$S(x, y) \geq \tfrac{1}{3}\alpha(x) + \tfrac{1}{8}a\gamma\lambda(y - \mu(x))^2. \tag{59}$$

Therefore, the function β having the properties listed above must indeed exist.

4) Properties C1)–C4) are Satisfied:

- Property C1) follows from the construction of ν, ρ, U, and β.
- Property C2) follows from (49), (51), and (52).
- The functions ρ, $L_F\rho$, and $L_G\rho$ are bounded because ρ is constant outside a bounded set. By definition, ν is bounded on $\mathbb{R}^n \times \mathbb{R}$. We calculate $L_G\nu$ and $L_F\nu$ as follows:

$$L_G\nu(x, y) = -\gamma\lambda\sigma'(\lambda[y - \mu(x)]) \tag{60}$$

$$\begin{aligned} L_F\nu(x, y) &= \gamma\lambda\sigma'(\lambda[y - \mu(x)]) \\ &\quad \cdot [L_f\mu(x) + L_g\mu(x)y - h(x, y)]. \end{aligned} \tag{61}$$

Recall that $|\sigma'(s)| \leq \sigma_0$ for all $s \in \mathbb{R}$; from this we conclude that $L_G\nu$ is bounded on $\mathbb{R}^n \times \mathbb{R}$. On the other hand, if we require λ to be large enough to satisfy both (36) and

$$\frac{1}{r_0} \leq \lambda \tag{62}$$

then, with B1), A3), and the fact that $\sigma'(s) = 0$ for $|s| \geq 1$, we see that the function

$$\sigma'(\lambda[y - \mu(x)])[L_g\mu(x)\, y - h(x, y)]$$

is bounded. We conclude from this and A3) that $L_F\nu$ is bounded.

We next verify that L_GU is bounded on $\mathbb{R}^n \times \mathbb{R}$. It follows from (42) and (51) that

$$\begin{aligned} L_GU(x, y) &= \frac{1}{1 + V(x) + K(y, \mu^{\pm}(x))} \\ &\quad \cdot \begin{cases} K_1(y, \mu^{\pm}(x)), & \text{when } (x, y) \in A^{\pm} \\ 0, & \text{when } (x, y) \in A^0. \end{cases} \end{aligned} \tag{63}$$

Since $\mu^{\pm}$ is bounded, we conclude from (19) that L_GU is bounded; thus C3) holds.

- We have left to verify C4).

1) From (6), (15), (30), and (63) we see that both ν and L_GU are $O([y - \mu(x)])$ as $(x, y) \to (0, 0)$, and it follows from (53) that L_GU and $(L_G\nu \cdot \nu)$ are $O(\sqrt{\beta(x, y)})$ as $(x, y) \to (0, 0)$.

2) From (61) we have

$$\begin{aligned} |L_F\nu(x, y)| &\leq \gamma\lambda\sigma_0[|L_f\mu(x)| + |(L_g\mu \cdot \mu)(x)| \\ &\quad + |L_g\mu(x)|\,|y - \mu(x)| + |h(x, y)|] \end{aligned}$$

and it follows from A4), B3), and (53) that $L_F\nu$ is $O(\sqrt{\beta(x, y)})$ as $(x, y) \to (0, 0)$.

IV. Concluding Remarks

We have presented a new backstepping procedure for the design of state feedback control laws which are bounded both in magnitude and rate. Although the proposed Lyapunov function is necessarily more complicated than the standard Lyapunov function for backstepping, the resulting control law has a simple form.

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On the Computation of the Induced $\mathcal{L}_2$ Norm of Single-Input Linear Systems with Saturation

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***Abstract*—In this paper, a means of determining an upper bound of the induced $\mathcal{L}_2$ norm for a class of single-input linear systems with saturation is given in terms of the existence of a candidate function which satisfies three differential inequalities. A technique to calculate such a function for systems with linear controllers is also developed.**

***Index Terms*—Finite gain stability, nonlinear $\mathcal{H}_\infty$ control, saturating systems.**

I. Introduction

The extension of H_∞ control methodologies to the robust control problem for nonlinear systems is a research topic which has recently attracted attention. One of the core analysis problems which needs to be addressed is the induced-norm computation problem, which must be solved before the synthesis problem can be seriously examined.

Using the concept of dissipativity introduced by Willems in [12], there has been some effort on this topic for affine nonlinear systems, some recent papers on which are [6] and [11]. The class of systems examined in this paper are those with input constraints; recent related work includes [7] and [8].

The development undertaken in this paper does not use any norm-bound assumptions to estimate away the effect of the memoryless nonlinearity, hence it is possible to undertake nonconservative analysis. This is also done in the paper [4], although for another problem

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