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Design of Robust Adaptive Controllers for Nonlinear Systems with Dynamic Uncertainties*

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A robustification methodology is described for a broader class of adaptive systems with nonlinear dynamic uncertainties.

Key Words—Uncertain systems; adaptive nonlinear control; input-to-state stability; dynamic uncertainties; robustness; Lyapunov functions; backstepping.

Abstract—In this paper, a modified adaptive backstepping design procedure is proposed for a class of nonlinear systems with three types of uncertainty: (i) unknown parameters; (ii) uncertain nonlinearities and (iii) unmodeled dynamics. Nonlinear damping terms are used to counteract the uncertain nonlinear functions and a dynamic signal is introduced to dominate the dynamic disturbance. The derived adaptive controller guarantees the global boundedness property for all signals and states and at the same time, steers the output to a small neighborhood of the origin. Incidentally an adaptive output-feedback control problem is solved. © 1998 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

Recent years have seen much progress in adaptive control of a class of nonlinear systems, see the recent textbooks by Krstić *et al.* (1995) and Marino and Tomei (1995) and references therein. Most adaptive control algorithms are proposed for nonlinear systems which are linearly parametrized and are state feedback linearizable (see, e.g. Sastry and Isidori, 1989; Kanellakopoulos *et al.*, 1991b; Jiang and Praly, 1991; Krstić *et al.*, 1992; Marino and Tomei, 1993b; Praly, 1992; Seto *et al.*, 1994). However, as pointed out in Krstić *et al.* (1995), robustness issue in adaptive nonlinear control has received less attention and the robust modifications of the proposed adaptive nonlinear controllers are yet to be developed (although see Jiang and Praly, 1992).

A singular perturbation approach was used to take into account the effects of unmodeled dynamics in adaptive control of feedback linearizable systems under a matching condition (Taylor *et al.*, 1989), or an extended matching condition (Kanellakopoulos *et al.*, 1991a). In the work of Marino and Tomei (1993a), the authors studied a class of nonlinear systems satisfying a triangularity condition and assumed the uncertain nonlinearities are unbiased, that is, the zero point is an equilibrium for the system under consideration regardless of the value of the unknown parameters. The results of Marino and Tomei (1993a) were recently extended by Polycarpou and Ioannou (1995) to adaptive systems with biased uncertain nonlinearities. With the additional assumption that the bounds of unknown parameters are known, the adaptive control design procedure of Polycarpou and Ioannou (1995) was modified by Yao and Tomizuka (1995) towards an improved transient performance. However, all these papers (Marino and Tomei, 1993a; Polycarpou and Ioannou, 1995; Yao and Tomizuka, 1995) have not considered the dynamic perturbations. In Jiang and Praly (1992) we generalized to adaptive control of nonlinear systems the idea of using an available dynamic signal to bound some dynamic uncertainty. This idea was further examined in Jiang (1995) via small-gain techniques. More recently, in Jiang and Praly (1996), we proposed a recursive robust adaptive control procedure for a class of non-linear systems in the presence of unmeasured dynamics satisfying an input-to-state stability property. This paper is an extension of Jiang and Praly (1996) from the unbiased case to the biased situation.

The primary goal of this paper is to study the problem of robust adaptive control for uncertain

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nonlinear systems described by

$$\begin{aligned} \dot{z} &= q(z, x, u), \\ \dot{x}_i &= x_{i+1} + \theta^{\mathrm{T}}\phi_i(x_1, \ldots, x_i) + \Delta_i(x, z, u, t), \\ &\quad 1 \le i \le n-1, \\ \dot{x}_n &= u + \theta^{\mathrm{T}}\phi_n(x_1, \ldots, x_n) + \Delta_n(x, z, u, t), \\ y &= x_1, \end{aligned} \tag{1}$$

where $u \in \mathbb{R}$ and $y \in \mathbb{R}$ represent the control input and the output, respectively, $x = (x_1, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$ is comprised of the *measured* states and the $z \in \mathbb{R}^{n_0}$ is the remaining part of the *unmeasured* states, and $\theta \in \mathbb{R}^l$ is a vector of *unknown* constant parameters. Assume that the Δ_i's and q are unknown Lipschitz continuous functions but the ϕ_i's are known smooth functions.

Throughout the paper, the following assumption is made on the system (1).

Assumption 1.1. For each $1 \le i \le n$, there exists an *unknown* positive constant p_i^* such that, for all (z, x, u, t) in $\mathbb{R}^{n_0} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}_{\ge 0}$,

$$|\Delta_i(x, z, u, t)| \le p_i^* \psi_{i1}(|(x_1, \ldots, x_i)|) + p_i^* \psi_{i2}(|z|), \tag{2}$$

where ψ_{i1} and ψ_{i2} are two known nonnegative smooth functions. With no loss of generality, assume that $\psi_{i2}(0) = 0$.

Nonlinear systems meeting Assumption 1.1 will be discussed in Sections 2 and 5 (see also Remark 4.2). The above class of uncertain nonlinear systems (1) is motivated by previous work on the control of triangular systems in the context of both adaptive (see, e.g. Kanellakopoulos *et al.*, 1991b; Jiang and Praly, 1991; Krstić *et al.*, 1992; Marino and Tomei, 1993a, b; Praly, 1992; Seto *et al.*, 1994; Jiang *et al.*, 1996) and nonadaptive control (see, e.g. Byrnes and Isidori, 1989; Tsinias, 1989; Freeman and Kokotovic, 1993; Tsinias, 1995; Teel and Praly, 1993).

In this paper, we present a robustification method for the usual adaptive backstepping design for system (1) in the presence of parametric, static and dynamic uncertainties. The control objective is to find an adaptive controller of the form

$$\dot{\chi} = \varpi(\chi, x), \quad \chi \in \mathbb{R}^{l+1}, \tag{3}$$

$$u = u(x, \chi), \tag{4}$$

in such a way that all the solutions of the closed-loop system (1), (3) and (4) are globally uniformly ultimately bounded* (Khalil, 1996). Furthermore, the output y can be rendered small.

* The solutions of $\dot{x} = f(t, x)$ are said to be globally uniformly ultimately bounded if, for any $a > 0$, there exist two positive constants b and $T = T(a)$ such that

$$|x(t_0)| = a \Rightarrow |x(t)| \le b, \quad \forall t \ge t_0 + T.$$

To achieve this objective, an additional assumption on unmeasured dynamics z is given in Section 4. The contributions of the paper are twofold. Firstly, the uncertain systems under consideration are subject to a general set of uncertainty: parametric uncertainty (which may nonlinearly appear), uncertain nonlinear functions and stable dynamic uncertainties. Secondly, our control design procedure is constructive and incorporates certain scalar dynamic signal $r(t)$ which will be defined in Section 4.

There are several possible avenues to treat our problem. The first possible way is to depart with the idea of exploiting *a priori* information on the system as much as possible. This is exactly what we pursue in this paper since we utilize the fact that there is a linear parameterization and all unmodeled effects are bounded by the above-mentioned dynamic signal $r(t)$. The second way is to go through an intermediate case where we use the fact that there is a linear parameterization but we treat the unmodeled dynamics by a worst-case design on the basis of small-gain arguments (Jiang *et al.*, 1994; Jiang and Mareels, 1997). The third one is the case where everything is treated via a worst-case design by ignoring the system structure, as in Teel and Praly (1993). All these avenues have their own advantages and disadvantages. As far as our approach presented in this paper is concerned, the main advantage is that, under our hypotheses, we obtain a desired asymptotic behavior with a moderate control effort. However, as any other purely adaptive control scheme, this behavior is not robust with respect to perturbations which violate our assumptions. As a result, the typical "bursting" phenomenon may happen. This warns that some precautions must be taken for the application of our algorithm in practice.

The presentation of this paper is as follows: we begin with a motivating problem in Section 2. Section 3 contains some needed definitions and preliminary results. Then, we present in Section 4 a novel robust adaptive backstepping scheme which allows us to solve the above output regulation problem. Illustrative examples are given in Section 5. Section 6 discusses some particular situations where the exact output regulation can be achieved. Our conclusion is in Section 7.

2. A MOTIVATING PROBLEM

The problem of output feedback stabilization of nonlinear systems has recently received attention with renewed interest (see, e.g. Kanellakopoulos *et al.*, 1991c, 1992; Khalil and Saberi, 1987; Marino and Tomei, 1991, 1993b). One common feature in these cited papers is that the zero dynamics of the system to be controlled is linear and is asymptotically stable. In our recent paper (Praly and

Jiang, 1993), we have extended this case to the case where the zero dynamics is nonlinear and is input-to-state stable in the sense of Sontag (see Section 3). An observer-based robust dynamic controller is designed using a nonlinear small gain argument (Jiang *et al.*, 1994). Here, we intend to extend further to the adaptive regulation case, i.e. gain functions are known up to a multiplicative unknown constant.

To this purpose, let us consider the following class of single-input–single-output nonlinear control systems:

$$\begin{aligned} \dot{\zeta} &= q_0(\zeta, y), \\ \dot{\xi}_i &= \xi_{i+1} + \theta^{\mathrm{T}}\phi_i(y) + \omega_i(\xi, \zeta, u, t), \\ &\quad 1 \le i \le n-1, \\ \dot{\xi}_n &= u + \theta^{\mathrm{T}}\phi_n(y) + \omega_n(\xi, \zeta, u, t), \\ y &= \xi_1, \end{aligned} \tag{5}$$

where y represents the system output and the ω_i's are unknown functions satisfying

$$|\omega_i(\xi, \zeta, u, t)| \le \vartheta_i^* \varphi_{i1}(|y|) + \vartheta_i^* \varphi_{i2}(|\zeta|). \tag{6}$$

where the ϑ_i^*'s are unknown positive constants and φ_{i1} and φ_{i2} are known smooth functions.

It is important to note that nonlinear parametrization was considered in Marino and Tomei (1993b, Part II) and Teel and Praly (1993) for classes of uncertain output-feedback systems similar to the form (5). In Marino and Tomei (1993b, Part II), the zero dynamics is linear, asymtotically stable and the coupling terms ω_i $(1 \le i \le n)$ depend linearly on ζ. By taking advantage of these facts, Marino and Tomei (1993b) succeeded in building a globally self-tuning output-feedback stabilizing controller by forcing an appropriate function to be a Lyapunov function. In Teel and Praly (1993), the problem of practical output regulation was solved for a larger class of systems (5) using high-gain control. Here, we intend to propose an alternative solution which is even new in contrast to the existing work on adaptive output-feedback control (Krstić *et al.*, 1995; Marino and Tomei, 1995).

As in Praly and Jiang (1993), introduce an observer as follows:

$$\begin{aligned} \dot{\hat{\xi}}_i &= \hat{\xi}_{i+1} + L_i(y - \hat{\xi}_1), \quad 1 \le i \le n-1\ , \\ \dot{\hat{\xi}}_n &= u + L_n(y - \hat{\xi}) \end{aligned} \tag{7}$$

where the real numbers L_i $(1 \le i \le n)$ are chosen so that the matrix

$$A = \begin{pmatrix} -L_1 & 1 & 0 & \cdots & 0 \\ -L_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -L_{n-1} & 0 & 0 & \cdots & 1 \\ -L_n & 0 & 0 & \cdots & 0 \end{pmatrix}$$

is asymptotically stable. Letting

$$z_0 = \frac{\hat{\xi} - \xi}{p^*},$$

$$\begin{aligned} \tilde{\omega}(\xi, \zeta, u, t) = (&\theta^{\mathrm{T}}\phi_1(y) + \omega_1(\xi, \zeta, u, t), \ \ldots, \theta^{\mathrm{T}}\phi_n(y) \\ &+ \omega_n(\xi, \zeta, u, t))^{\mathrm{T}} \end{aligned} \tag{8}$$

with $p^* := \max\{|\theta|, \vartheta_1^*, \ \ldots, \vartheta_n^*\}$, we have

$$\dot{z}_0 = Az_0 - \frac{1}{p^*}\tilde{\omega}(\xi, \zeta, u, t). \tag{9}$$

Thanks to the introduction of p^* in equation (8), the z_0-system (9) is input-to-state practically stable (cf. Definition 3.2) with some gain which is available for feedback design. See Section 5.2 below.

Therefore, we get an augmented system for control design:

$$\begin{aligned} \dot{z}_0 &= Az_0 - \frac{1}{p^*}\tilde{\omega}(\xi, \zeta, u, t), \\ \dot{\zeta} &= q_0(\zeta, y), \\ \dot{y} &= \hat{\xi}_2 + \theta^{\mathrm{T}}\phi_1(y) + p^* z_{02} + \omega_1(\xi, \zeta, u, t), \\ \dot{\hat{\xi}}_i &= \hat{\xi}_{i+1} - L_i p^* z_{01}, \quad 2 \le i \le n-1, \\ \dot{\hat{\xi}}_n &= u - L_n p^* z_{01}. \end{aligned} \tag{10}$$

It is easy to see that system (10) belongs to the class of systems (1) with

$$\begin{aligned} &x = (y, \hat{\xi}_2, \ \ldots, \hat{\xi}_n)^{\mathrm{T}}, \qquad z = (z_0^{\mathrm{T}}, \zeta^{\mathrm{T}})^{\mathrm{T}}, \\ &\Delta_1 = p^* z_{02} + \omega_1, \qquad \Delta_i = -L_i p^* z_{01}, \quad 2 \le i \le n. \end{aligned} \tag{11}$$

It is clear that Assumption 1.1 is satisfied for system (10).

Notice that, similar to Praly and Jiang (1993), regulating the output y of (10) using the information of partial-state x achieves the output regulation problem for the original system (5). We will come back to this issue in the Section 5.2.

3. MATHEMATICAL PRELIMINARIES

We begin with definitions of class K, K_∞ and KL functions which are standard in the stability literature, see Khalil (1996).

Definition 3.1. A *K-function* γ, or a *function of class* K, is a function from $\mathbb{R}_{\ge 0}$ into $\mathbb{R}_{\ge 0}$ which is continuous, strictly increasing and is zero at zero. A K_∞*-function* is a function which is of class K and unbounded. A *KL-function* β is a function from $\mathbb{R}_{\ge 0} \times \mathbb{R}_{\ge 0}$ to $\mathbb{R}_{\ge 0}$ with the property that for each fixed t, the function $\beta(\cdot, t)$ is of class K and, for each fixed s, the function $\beta(s, \cdot)$ is decreasing and tends to zero at infinity.

The concepts of input-to-state stability (ISS) and ISS-Lyapunov function due to Sontag (Sontag, 1989, 1990; Sontag and Wang, 1995) have recently been used in various control problems such as nonlinear stabilization, robust control and observer designs (see, e.g. Sontag, 1995; Jiang *et al.*, 1994; Tsinias, 1993; Praly and Jiang, 1993; Praly and Wang, 1996; Jiang and Mareels, 1997)). In the following, we present variants of these notions which are suitable for our application.

Definition 3.2. A control system $\dot{x} = f(x, u)$ is *input-to-state practically stable* (*ISpS*) if there exist a function β of class KL, a function γ of class K and a nonnegative constant d such that, for any initial condition $x(0)$ and each measurable essentially bounded control $u(t)$ defined for all $t \geq 0$, the associated solution $x(t)$ exists for all $t \geq 0$ and satisfies

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma(\|u_t\|) + d, \qquad (12)$$

where u_t is the truncated function of u at t and $\|\cdot\|$ stands for the L^∞ supremum norm.

When $d = 0$ in equation (12), the ISpS property collapses to the *input-to-state stability* (*ISS*) property introduced in Sontag (1990).

Definition 3.3. A C^1 function V is said to be an *ISpS-Lyapunov function* for system $\dot{x} = f(x, u)$ if

- there exist functions α_1, α_2 of class K_∞ such that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|), \quad \forall x \in \mathbb{R}^n, \qquad (13)$$

- there exist class K-functions α_3, χ and a constant $d \geq 0$ such that

$$|x| \geq \chi(|u|) + d \;\Rightarrow\; \frac{\partial V}{\partial x}(x) f(x, u) \leq -\psi_3(|x|). \qquad (14)$$

When equation (14) holds with $d = 0$, V is referred to as an *ISS-Lyapunov function*.

For the purpose of applications studied in this paper and motivated by recent work (Sontag and Wang 1995; Praly and Wang, 1996), we introduce in the sequel a notion of exp-ISpS Lyapunov function.

Definition 3.4. A C^1 function V is said to be an exp-*ISpS Lyapunov function* for system $\dot{x} = f(x, u)$ if

- there exist functions α_1, α_2 of class K_∞ such that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|), \quad \forall x \in \mathbb{R}^n, \qquad (15)$$

- there exist two constants $c > 0$, $d \geq 0$ and a class K_∞-function γ such that

$$\frac{\partial V}{\partial x}(x) f(x, u) \leq -cV(x) + \gamma(|u|) + d. \qquad (16)$$

When equation (16) holds with $d = 0$, the function V is referred to as an exp-*ISS Lyapunov function.*

The three previous definitions are equivalent from Sontag and Wang (1995) and Praly and Wang (1996). Namely

Proposition 3.1. For any control system $\dot{x} = f(x, u)$, the following properties are equivalent:

(i) It is ISpS.
(ii) It has an ISpS-Lyapunov function.
(iii) It has an exp-ISpS Lyapunov function.

As another illustration of this notion of exp-ISpS Lyapunov function, we give the following useful fact.

Lemma 3.1. If V is an exp-*ISpS* Lyapunov function for a control system $\dot{z} = q(z, u)$, i.e. equations (15) and (16) hold, then, for any constants $\bar{c}$ in $(0, c)$, any initial instant $t_0 \geq 0$, any initial condition $z^o = z(t_0)$ and $r^o > 0$, for any function $\bar{\gamma}$ such that $\bar{\gamma}(u) \geq \gamma(|u|)$, there exist a finite $T^o = T^o(\bar{c}, r^o, z^o) \geq 0$, a nonnegative function $D(t_0, t)$ defined for all $t \geq t_0$ and a signal described by

$$\dot{r} = -\bar{c}r + \bar{\gamma}(u(t)) + d, \quad r(t_0) = r^o \qquad (17)$$

such that $D(t_0, t) = 0$ for all $t \geq t_0 + T^o$ and

$$V(z(t)) \leq r(t) + D(t_0, t) \qquad (18)$$

for all $t \geq t_0$ where the solutions are defined.

Proof. By assumptions, we have

$$\dot{V} \leq -cV(z(t)) + \gamma(|u(t)|) + d\,. \qquad (19)$$

From equations (19) and (17) and with the help of Gronwall's lemma, it follows that

$$V(z(t)) \leq r(t) + e^{-c(t-t_0)}V(z^o) - e^{-\bar{c}(t-t_0)}r^o. \qquad (20)$$

Define

$$D(t_0, t) = \max\{0, e^{-c(t-t_0)}V(z^o) - e^{-\bar{c}(t-t_0)}r^o\}. \qquad (21)$$

Since $0 < \bar{c} < c$ and $r^o > 0$, there exists a finite $T^o = T^o(\bar{c}, r^o, z^o) \geq 0$ such that $D(t_0, t) = 0$ for all $t \geq t_0 + T^o$. Finally, from (20) and (21), (18) follows readily. □

Remark 3.1. A memorizing signal similar to the dynamic signal r as in Lemma 3.1 was proposed in Jiang and Praly (1994) in characterizing unmodeled effects for general dynamical systems. The benefits of using such a signal are now well known in adaptive linear control (Ioannou and Sun, 1996). The idea of using it also in the nonlinear context can be found in Jiang and Praly (1992).

We close this section by giving three useful technical lemmas whose proofs are straightforward.

Lemma 3.2. For any x and y in $\mathbb{R}^n$, and for any positive real number ε, we have

$$x^{\mathrm{T}}y = \tfrac{1}{2}|x|^2 + \tfrac{1}{2}|y|^2 - \tfrac{1}{2}|x-y|^2 \le \frac{1}{4\varepsilon}|x|^2 + \varepsilon|y|^2. \quad (22)$$

Lemma 3.3. For any $\varepsilon > 0$, there exists a smooth function g such that $g(0) = 0$ and

$$|x| \le xg(x) + \varepsilon, \quad \forall x \in \mathbb{R}. \quad (23)$$

A simple example of functions satisfying equation (23) follows from Lemma 3.2, i.e. $g(x) = (1/4\varepsilon)x$. Another example verifying equation (23) was given and used in Polycarpou and Ioannou (1995), i.e. $g(x) = \tanh(\delta x/\varepsilon)$ with $\delta > 0$ defined by $\delta = \exp(-\delta - 1)$.

Lemma 3.4. For any $\varepsilon > 0$ and any continuous function f: $\mathbb{R} \to \mathbb{R}$, with $f(0) = 0$, there exists a nonnegative smooth function $\hat{f}$, with $\hat{f}(0) = \partial\hat{f}/\partial x(0) = 0$, such that

$$|f(x)| \le \hat{f}(x) + \varepsilon, \quad \forall x \in \mathbb{R}. \quad (24)$$

An idea of proof for Lemma 3.4 can be found in Praly and Jiang (1993, Lemma A.1).

4. ROBUST ADAPTIVE BACKSTEPPING SCHEME

In order to present the robustification of the adaptive backstepping design with tuning functions in Krstić *et al.* (1992), we need the following assumption on the unmodeled dynamics characterized by the z-equation in equation (1).

Assumption 4.1. The z-system in equation (1) has an exp-ISpS Lyapunov function V_z in the sense of Definition 3.4, i.e. there exist two constants $c_0 > 0$, $d_0 \ge 0$ and three class K_∞-functions α_1, α_2 and γ such that

$$\alpha_1(|z|) \le V_z(z) \le \alpha_2(|z|), \quad \forall z \in \mathbb{R}^{n_0}, \quad (25)$$

$$\frac{\partial V_z}{\partial z}(z)q(z, x, u) \le -c_0 V_z(z) + \gamma(|x_1|) + d_0. \quad (26)$$

Moreover, $\bar{c}_0 \in (0, c_0)$, d_0, γ and α_1 are known.

Without loss of generality, we assume throughout that γ is a smooth function and of the form

$$\gamma(s) = s^2\gamma_0(s^2) \quad (27)$$

with γ_0 a nonnegative smooth function. Otherwise, using Lemma 3.4, it suffices to replace γ in (26) by $x_1^2\gamma_0(x_1^2) + \varepsilon_0$ with $\varepsilon_0 > 0$ being a sufficiently small real number.

Thanks to Assumption 4.1 and Lemma 3.1, we obtain an available signal r defined by

$$\dot{r} = -\bar{c}_0 r + x_1^2\gamma_0(x_1^2) + d_0, \quad r(t_0) = r^o > 0 \quad (28)$$

with the property that

$$V_z(z(t)) \le r(t) + D(t_0, t) \quad (29)$$

for all $t \ge t_0$ where the solutions are defined, with $D(t_0, t)$ defined for each $t \ge t_0 \ge 0$ and $D(t_0, t) = 0$ for all $t \ge t_0 + T^o$ ($T^o \ge 0$ being finite and depending continuously on the initial conditions r^o, z^o). In particular, $T^o \to +\infty$ as $|z^o| \to +\infty$.

Remark 4.1. Upon specialization to linear systems, Assumption 4.1 is checked if the linear system $\dot{z} = q(z, 0, 0)$ in (1) is asymptotically stable with a known stability margin (Ioannou and Sun, 1996).

4.1. *An initial step*

We start with the following subsystem of equation (1):

$$\begin{aligned} \dot{z} &= q(z, x, u), \\ \dot{x}_1 &= x_2 + \theta^{\mathrm{T}}\phi_1(x_1) + \Delta_1(x, z, u, t). \end{aligned} \quad (30)$$

As in Krstić *et al.* (1992) where $\Delta_1 \equiv 0$ and there is no unmeasured dynamics z, consider x_2 as virtual input and let $\hat{\theta}$ be a parameter estimate of θ. We wish to find a stabilizing function w_1 and a tuning function τ_1 such that the system (30) is adaptively stabilized. However, in the present case where there is a dynamic uncertainty Δ_1 containing uncertain nonlinearities and unmeasured dynamics, certain nonlinear damping term and dynamic normalizing signal will be incorporated in w_1 and τ_1.

More precisely, we will design a smooth intermediate control function w_1 and smooth update estimate laws τ_1, ϖ_1 such that, with

$$x_2 = \bar{x}_2 + w_1(x_1, r, \hat{\theta}, \hat{p}) \quad (31)$$

and along solutions of equation (1), the time derivative of the following function:

$$V_1 = \frac{1}{2}x_1^2 + \frac{1}{\lambda_0}r + \frac{1}{2}(\hat{\theta} - \theta)^{\mathrm{T}}\Gamma^{-1}(\hat{\theta} - \theta) + \frac{1}{2\lambda}(\hat{p} - p^*)^2 \quad (32)$$

satisfies

$$\dot{V}_1 \le -c_1 V_1 + x_1\bar{x}_2 + \mu_1(t_0, t) + (\hat{\theta} - \theta)^{\mathrm{T}}\Gamma^{-1}(\dot{\hat{\theta}} - \tau_1) + \frac{1}{\lambda}(\hat{p} - p^*)(\dot{\hat{p}} - \varpi_1), \quad (33)$$

where $p^* \ge \max\{p_1^*, \ldots, p_n^*\}$, $\Gamma > 0$, λ_0, $\lambda > 0$ are design parameters and $c_1 > 0$, $\mu_1(t_0, t)$ is a nonnegative function which is identically equal to some constant whenever t is sufficiently large.

Towards this end, by hypotheses, taking the time derivative of V_1 with respect to solutions of equation (1) yields

$$\dot V_1 \le x_1(x_2 + \theta^T\phi_1(x_1)) + p_1^*|x_1|\psi_{11}(|x_1|) + p_1^*|x_1|\psi_{12}(|z|) - \frac{\bar c_0}{\lambda_0}r + \frac{1}{\lambda_0}(x_1^2\gamma_0(x_1^2) + d_0) + (\hat\theta - \theta)^T\Gamma^{-1}\dot{\hat\theta} + \frac{1}{\lambda}\tilde p\dot{\hat p}, \quad (34)$$

where $\tilde p := \hat p - p^*$.

As in Polycarpou and Ioannou (1995), set

$$\tau_1 := \Gamma(x_1\phi_1(x_1) - \sigma_\theta(\hat\theta - \theta^o)), \quad (35)$$

where $\sigma_\theta > 0$ and $\theta^o \in \mathbb{R}^l$ are design parameters. We have

$$\dot V_1 \le x_1\left(x_2 + \hat\theta^T\phi_1(x_1) + \frac{1}{\lambda_0}x_1\gamma_0\right) - \frac{\bar c_0}{\lambda_0}r + \frac{d_0}{\lambda_0} + (\hat\theta - \theta)^T\Gamma^{-1}(\dot{\hat\theta} - \tau_1) - \sigma_\theta(\hat\theta - \theta)^T(\hat\theta - \theta^o) + p_1^*|x_1|\psi_{11}(|x_1|) + p_1^*|x_1|\psi_{12}(|z|) + \frac{1}{\lambda}\tilde p\dot{\hat p}. \quad (36)$$

Next, we examine the last three terms in equation (36). From Lemmas 3.3 and 3.4, given any $\varepsilon_{11} > 0$, there exists a smooth function $\hat\psi_{11}$, with $\hat\psi_{11}(0) = 0$, such that

$$|x_1|\psi_{11}(|x_1|) \le x_1\hat\psi_{11}(x_1) + \varepsilon_{11}, \quad \forall x_1 \in \mathbb{R}. \quad (37)$$

On the other hand, from equation (29) and Assumption 4.1, using Lemma 3.2, it follows successively that

$$\begin{aligned} p_1^*|x_1|\psi_{12}(|z|) &\le p^*|x_1|\psi_{12}\circ\alpha_1^{-1}(r + D(t_0,t)) \\ &\le p^*|x_1|\psi_{12}\circ\alpha_1^{-1}(2r) + p^*|x_1|\psi_{12}\circ\alpha_1^{-1}(2D(t_0,t)) \\ &\le p^*|x_1|\psi_{12}\circ\alpha_1^{-1}(2r) + \tfrac14 x_1^2 + d_1(t_0,t), \end{aligned} \quad (38)$$

where $d_1(t_0,t)$ is defined by

$$d_1(t_0,t) = (p^*\psi_{12}\circ\alpha_1^{-1}(2D(t_0,t)))^2. \quad (39)$$

Notice that $d_1(t_0,t) = 0$ for all $t \ge t_0 + T^o$.

By application of Lemma 3.4, there exists a smooth function $\hat\psi_{12}$, with $\hat\psi_{12}(0) = 0$, such that

$$\psi_{12}\circ\alpha_1^{-1}(2r) \le \hat\psi_{12}(r) + 1. \quad (40)$$

Then, given any $\varepsilon_{12} > 0$, since $p^* \ge p_1^*$, a repeated use of Lemma 3.3 gives

$$\begin{aligned} p_1^*|x_1|\psi_{12}\circ\alpha_1^{-1}(2r) &\le p_1^*|x_1|\hat\psi_{12}(r) + p_1^*|x_1| \\ &\le p^*x_1\hat\psi_{12}(r)\hat\psi_{13}(x_1,r) + p^*x_1\hat\psi_{14}(x_1) + 2p^*\varepsilon_{12}, \end{aligned} \quad (41)$$

where $\hat\psi_{13}$ and $\hat\psi_{14}$ are two suitable smooth functions which are zero at zero.

Combining equations (37), (38) and (41), since $p^* \ge p_1^*$, equation (36) implies

$$\begin{aligned} \dot V_1 \le\ & x_1\Big(x_2 + \hat\theta^T\phi_1 + \frac{1}{\lambda_0}x_1\gamma_0 + \hat p\hat\psi_{11}(x_1) + \tfrac14 x_1 + \hat p\hat\psi_{12}(r)\hat\psi_{13}(x_1,r) + \hat p\hat\psi_{14}(x_1)\Big) \\ & - \frac{\bar c_0}{\lambda_0}r + \frac{d_0}{\lambda_0} + p^*(\varepsilon_{11} + 2\varepsilon_{12}) + d_1(t_0,t) \\ & + (\hat\theta - \theta)^T\Gamma^{-1}(\dot{\hat\theta} - \tau_1) - \sigma_\theta(\hat\theta - \theta)^T(\hat\theta - \theta^o) \\ & + \frac{1}{\lambda}\tilde p(\dot{\hat p} - \varpi_1) - \sigma_p\tilde p(\hat p - p^o), \end{aligned} \quad (42)$$

where

$$\varpi_1 = \lambda[x_1\hat\psi_{11}(x_1) + x_1\hat\psi_{12}(r)\hat\psi_{13}(x_1,r) + x_1\hat\psi_{14}(x_1) - \sigma_p(\hat p - p^o)] \quad (43)$$

with $\sigma_p > 0$ and $p^o \ge 0$ as design parameters.

Therefore, by choosing the intermediate stabilizing function w_1 as

$$w_1 = -k_1x_1 - \hat\theta^T\phi_1 - \frac{1}{\lambda_0}x_1\gamma_0 - \frac14 x_1 - \hat p(\hat\psi_{11}(x_1) + \hat\psi_{12}(r)\hat\psi_{13}(x_1,r) + \hat\psi_{14}(x_1)) \quad (44)$$

with $k_1 > 0$ as design constant, in view of equations (42) and (31), we obtain

$$\begin{aligned} \dot V_1 \le\ & -k_1x_1^2 + x_1\tilde x_2 - \frac{\bar c_0}{\lambda_0}r + \frac{d_0}{\lambda_0} + p^*(\varepsilon_{11} + 2\varepsilon_{12}) + d_1(t_0,t) \\ & + (\hat\theta - \theta)^T\Gamma^{-1}(\dot{\hat\theta} - \tau_1) + \frac{1}{\lambda}\tilde p(\dot{\hat p} - \varpi_1) \\ & - \sigma_\theta(\hat\theta - \theta)^T(\hat\theta - \theta^o) - \sigma_p\tilde p(\tilde p - p^o). \end{aligned} \quad (45)$$

Finally, equation (33) follows with equation (22) and

$$c_1 = \min\left\{2k_1, \bar c_0, \sigma_p\lambda, \frac{\sigma_\theta}{\lambda_{\max}(\Gamma^{-1})}\right\}, \quad (46)$$

$$\mu_1(t_0,t) = \frac{d_0}{\lambda_0} + p^*(\varepsilon_{11} + 2\varepsilon_{12}) + \tfrac12\sigma_\theta|\theta - \theta^o|^2 + \tfrac12\sigma_p(p^* - p^o)^2 + d_1(t_0,t). \quad (47)$$

Since $d_1(t_0,t) \ge 0$ for all $t \ge t_0$ and $= 0$ if $t \ge t_0 + T^o$, $\mu_1(t_0,t)$ is equal to a constant, denoted as μ_1', if $t \ge t_0 + T^o$. It is important to note that $\mu_1' > 0$ can be made arbitrarily small by choosing appropriately the design parameters λ_0, ε_{11}, ε_{12}, σ_θ, σ_p.

4.2. *Recursive design step*

The main purpose of this section is to show how to extend the property (32)–(33) for the system (30) to the system composed of the z-equation and the $(x_1, \dots, x_i)$-subsystems in (1) when $2 \le i \le n$. Again, the adaptive backstepping idea in Krstić *et al.* (1992) will be employed and extended to the present case of dynamic uncertainties.

Let $\bar{x}_1 = x_1$ and assume that we have designed smooth intermediate stabilizing functions w_j and smooth update estimate functions τ_j and ϖ_j $(1 \le j \le i-1)$, such that, with

$$x_{j+1} = \bar{x}_{j+1} + w_j(x_1, \dots, x_j, r, \hat{\theta}, \hat{p}) \tag{48}$$

$\forall 1 \le j \le i-1$, the time derivative of the following function w.r.t. solutions of equation (1)

$$V_{i-1} = \sum_{j=1}^{i-1} \frac{1}{2} x_j^2 + \frac{1}{\lambda_0} r + \tfrac{1}{2}(\hat{\theta} - \theta)^{\mathrm{T}} \Gamma^{-1} (\hat{\theta} - \theta) + \frac{1}{2\lambda}(\hat{p} - p^*)^2 \tag{49}$$

satisfies

$$\dot{V}_{i-1} \le -c_{i-1} V_{i-1} + \bar{x}_{i-1}\bar{x}_i + \mu_{i-1}(t_0, t) + \left((\hat{\theta} - \theta)^{\mathrm{T}}\Gamma^{-1} - \sum_{j=1}^{i-2} \bar{x}_{j+1} \frac{\partial w_j}{\partial \hat{\theta}}\right)(\dot{\hat{\theta}} - \tau_{i-1}) + \left(\frac{1}{\lambda}(\hat{p} - p^*) - \sum_{j=1}^{i-2} \bar{x}_{j+1} \frac{\partial w_j}{\partial \hat{p}}\right)(\dot{\hat{p}} - \varpi_{i-1}) \tag{50}$$

where c_{i-1} is a positive design parameter and $\mu_{i-1}(t_0, t)$ is a nonnegative function which identically equals to some constant $\mu'_{i-1} > 0$ whenever $t \ge t_0 + T^o$.

We prove in the sequel that a similar property holds for the system composed of the z-equation and the $(x_1, \dots, x_i)$-subsystems in equation (1).

Consider the function V_i defined by

$$V_i = V_{i-1} + \tfrac{1}{2}\bar{x}_i^2 \tag{51}$$

Notice that the variable $\bar{x}_i$ satisfies

$$\dot{\bar{x}}_i = x_{i+1} - \kappa_i(x_1, \dots, x_i, r, \hat{\theta}, \hat{p}) + \theta^{\mathrm{T}}\left(\phi_i - \sum_{j=1}^{i-1} \frac{\partial w_{i-1}}{\partial x_j}\phi_j\right) - \frac{\partial w_{i-1}}{\partial \hat{\theta}}\dot{\hat{\theta}} - \frac{\partial w_{i-1}}{\partial \hat{p}}\dot{\hat{p}} + \bar{\Delta}_i \tag{52}$$

where κ_i and $\bar{\Delta}_i$ are defined by

$$\kappa_i := \sum_{j=1}^{i-1} \frac{\partial w_{i-1}}{\partial x_j} x_{j+1} + \frac{\partial w_{i-1}}{\partial r}(-\bar{c}_0 r + x_1^2 \gamma_0 + d_0), \tag{53}$$

$$\bar{\Delta}_i := \Delta_i - \sum_{j=1}^{i-1} \frac{\partial w_{i-1}}{\partial x_j}\Delta_j. \tag{54}$$

Then, by equation (50), the time derivative of V_i along the solutions of equation (1) satisfies

$$\dot{V}_i \le -c_{i-1}V_{i-1} + \mu_{i-1}(t_0, t) + \left((\hat{\theta} - \theta)^{\mathrm{T}}\Gamma^{-1} - \sum_{j=1}^{i-2} \bar{x}_{j+1}\frac{\partial w_j}{\partial \hat{\theta}}\right)(\dot{\hat{\theta}} - \tau_{i-1}) + \left(\frac{1}{\lambda}(\hat{p} - p^*) - \sum_{j=1}^{i-2} \bar{x}_{j+1}\frac{\partial w_j}{\partial \hat{p}}\right)(\dot{\hat{p}} - \varpi_{i-1}) + \bar{x}_i\left[x_{i+1} + \bar{x}_{i-1} - \kappa_i + \theta^{\mathrm{T}}\left(\phi_i - \sum_{j=1}^{i-1}\frac{\partial w_{i-1}}{\partial x_j}\phi_j\right) - \frac{\partial w_{i-1}}{\partial \hat{\theta}}\dot{\hat{\theta}} - \frac{\partial w_{i-1}}{\partial \hat{p}}\dot{\hat{p}} + \bar{\Delta}_i\right] \tag{55}$$

with $x_{i+1} := u$ if $i = n$.

Using Assumption 1.1 since $p^* \ge \max\{p_1^*, \dots, p_n^*\}$, the following holds:

$$\bar{x}_i\bar{\Delta}_i \le p^*|\bar{x}_i|\left(\psi_{i1} + \sum_{j=1}^{i-1}\left|\frac{\partial w_{i-1}}{\partial x_j}\right|\psi_{j1}\right) + p^*|\bar{x}_i|\left(\psi_{i2}(|z|) + \sum_{j=1}^{i-1}\left|\frac{\partial w_{i-1}}{\partial x_j}\right|\psi_{j2}(|z|)\right). \tag{56}$$

By pursuing the same arguments as in the Section 4.1, it follows that, given any $\varepsilon_{i1}, \varepsilon_{i2} > 0$, there are smooth functions $\hat{\psi}_{i1}$ and $\hat{\psi}_{i2}$ of variables $(x_1, \dots, x_i, r, \hat{\theta}, \hat{p})$ such that

$$p^*|\bar{x}_i|\left(\psi_{i1} + \sum_{j=1}^{i-1}\left|\frac{\partial w_{i-1}}{\partial x_j}\right|\psi_{j1}\right) \le p^*\bar{x}_i\hat{\psi}_{i1} + p^*\varepsilon_{i1}, \tag{57}$$

$$p^*|\bar{x}_i|\left(\psi_{i2}(|z|) + \sum_{j=1}^{i-1}\left|\frac{\partial w_{i-1}}{\partial x_j}\right|\psi_{j2}(|z|)\right) \le p^*\bar{x}_i\hat{\psi}_{i2} + \frac{1}{4}\bar{x}_i^2\left[1 + \sum_{j=1}^{i-1}\left(\frac{\partial w_{i-1}}{\partial x_j}\right)^2\right] + 2ip^*\varepsilon_{i2} + d_i(t_0, t). \tag{58}$$

where

$$d_i(t_0, t) = \sum_{j=1}^{i} (p^*\psi_{j2} \circ \alpha_1^{-1}(2D(t_0, t)))^2. \tag{59}$$

Notice that $d_i(t_0, t) \ge 0$ for all $t \ge t_0$ and $= 0$ if $t \ge t_0 + T^o$.

Define

$$\tau_i = \tau_{i-1} + \Gamma\left(\phi_i - \sum_{j=1}^{i-1}\frac{\partial w_{i-1}}{\partial x_j}\phi_j\right)\bar{x}_i, \tag{60}$$

$$\varpi_i = \varpi_{i-1} + \lambda\bar{x}_i(\hat{\psi}_{i1} + \hat{\psi}_{i2}), \tag{61}$$

$$\mu_i(t_0, t) = \mu_{i-1}(t_0, t) + p^*(\varepsilon_{i1} + 2i\varepsilon_{i2}) + d_i(t_0, t). \tag{62}$$

Consequently, as in Krstić *et al.* (1992), with the observation that

$$\dot{\hat{\theta}} - \tau_{i-1} = \dot{\hat{\theta}} - \tau_i + \Gamma\left(\phi_i - \sum_{j=1}^{i-1}\frac{\partial w_{i-1}}{\partial x_j}\phi_j\right)\bar{x}_i, \tag{63}$$

$$\dot{\hat{p}} - \varpi_{i-1} = \dot{\hat{p}} - \varpi_i + \lambda(\hat{\psi}_{i1} + \hat{\psi}_{i2})\bar{x}_i. \tag{64}$$

equation (55) implies

$$\dot V_i \le -c_{i-1}V_{i-1} + \mu_i(t_0,t) + \bar x_i\left[x_{i+1}+\bar x_{i-1} - \kappa_i + \frac{1}{4}\bar x_i\left(1+\sum_{j=1}^{i-1}\left(\frac{\partial w_{i-1}}{\partial x_j}\right)^2\right) - \frac{\partial w_{i-1}}{\partial \hat\theta}\tau_i - \frac{\partial w_{i-1}}{\partial \hat p}\varpi_i + \left(\hat\theta^{\mathrm T} - \sum_{j=1}^{i-2}\bar x_{j+1}\frac{\partial w_j}{\partial \hat\theta}\Gamma\right)\left(\phi_i - \sum_{j=1}^{i-1}\frac{\partial w_{i-1}}{\partial x_j}\phi_j\right) + \left(\hat p - \lambda\sum_{j=1}^{i-2}\bar x_{j+1}\frac{\partial w_j}{\partial \hat p}\right)(\hat\psi_{i1}+\hat\psi_{i2})\right] + \left((\hat\theta-\theta)^{\mathrm T}\Gamma^{-1} - \sum_{j=1}^{i-1}\bar x_{j+1}\frac{\partial w_j}{\partial \hat\theta}\right)(\dot{\hat\theta} - \tau_i) + \left(\frac{1}{\lambda}(\hat p - p^*) - \sum_{j=1}^{i-1}\bar x_{j+1}\frac{\partial w_j}{\partial \hat p}\right)(\dot{\hat p} - \varpi_i). \quad (65)$$

By choosing the intermediate stabilizing function w_i as

$$w_i = -k_i\bar x_i - x_{i-1} + \kappa_i - \frac{1}{4}\bar x_i\left(1+\sum_{j=1}^{i-1}\left(\frac{\partial w_{i-1}}{\partial x_j}\right)^2\right) + \frac{\partial w_{i-1}}{\partial \hat\theta}\tau_i + \frac{\partial w_{i-1}}{\partial \hat p}\varpi_i - \left(\hat\theta^{\mathrm T} - \sum_{j=1}^{i-2}\bar x_{j+1}\frac{\partial w_j}{\partial \hat\theta}\Gamma\right)\left(\phi_i - \sum_{j=1}^{i-1}\frac{\partial w_{i-1}}{\partial x_j}\phi_j\right) - \left(\hat p - \lambda\sum_{j=1}^{i-2}x_{j+1}\frac{\partial w_j}{\partial \hat p}\right)(\hat\psi_{i1}+\hat\psi_{i2}) \quad (66)$$

with $k_i > 0$, a direct substitution of $x_{i+1} := \bar x_{i+1} + w_i$ in equation (65) yields the desired inequality

$$\dot V_i \le -c_iV_i + \bar x_i\bar x_{i+1} + \mu_i(t_0,t) + \left((\hat\theta-\theta)^{\mathrm T}\Gamma^{-1} - \sum_{j=1}^{i-1}\bar x_{j+1}\frac{\partial w_j}{\partial \hat\theta}\right)(\dot{\hat\theta} - \tau_i) + \left(\frac{1}{\lambda}(\hat p - p^*) - \sum_{j=1}^{i-1}\bar x_{j+1}\frac{\partial w_j}{\partial \hat p}\right)(\dot{\hat p} - \varpi_i), \quad (67)$$

where

$$c_i = \min\{c_{i-1}, 2k_i\}. \quad (68)$$

Note that, by assumption, $\mu_i(t_0,t) \ge 0$ for all $t \ge t_0$ and identically equals to a constant denoted as $\mu_i' > 0$ if $t \ge t_0 + T^o$.

4.3. *Main result*

According to the recursive control design procedure in the above subsections, at the last step (i.e. $i = n$), picking the adaptive controller $u = w_n$ (so $\bar x_{n+1} = 0$) and the adaptive laws $\dot{\hat\theta} = \tau_n$ and $\dot{\hat p} = \varpi_n$, we arrive at

$$\dot V_n \le -c_nV_n + \mu_n(t_0,t). \quad (69)$$

By construction, the constant $c_n > 0$ and the nonnegative function μ_n are given by

$$c_n := \min\left\{\bar c_0, 2k_i, \sigma_p\lambda, \frac{\sigma_\theta}{\lambda_{\max}(\Gamma^{-1})}; i = 1,\ldots,n\right\}, \quad (70)$$

$$\mu_n(t_0,t) := \frac{d_0}{\lambda_0} + \sum_{i=1}^{n} p^*(\varepsilon_{i1} + 2i\varepsilon_{i2}) + \frac{\sigma_p}{2}(p^* - p^o)^2 + \frac{\sigma_\theta}{2}|\theta - \theta^o|^2 + \sum_{i=1}^{n} d_i(t_0,t) := \mu_n' + \sum_{i=1}^{n} d_i(t_0,t). \quad (71)$$

We are now in a position to state our main result on global adaptive regulation.

Theorem 4.1. Under Assumptions 1.1 and 4.1, all the solutions $(x(t), z(t), \hat\theta(t), \hat p)$ of the derived closed-loop system (1) are globally uniformly ultimately bounded. Furthermore, given any $\mu^* > 0$ and bounds on θ and the p_i^*'s, we can tune our controller parameters such that the output $y(t)$ satisfies

$$\limsup_{t\to+\infty} |y(t)| \le \mu^*. \quad (72)$$

Proof. By construction, $\sum_{i=1}^{n} d_i(t_0,t)$ is nonnegative for all $t \ge t_0$ and is equal to zero for all $t \ge t_0 + T^o$. As a consequence,

$$\int_{t_0}^{\infty}\sum_{i=1}^{n} d_i(t_0,t)\,\mathrm dt < +\infty.$$

From equation (69), we have

$$V_n(t) \le \frac{\mu_n'}{c_n} + \left(V_n(t_0) - \frac{\mu_n'}{c_n}\right)\mathrm e^{-c_n(t-t_0)} + \int_{t_0}^{t} d_n(t_0,s)\,\mathrm ds, \quad \forall t \ge t_0 \ge 0. \quad (73)$$

It results that the signals $\bar x_i(t)$, $r(t)$, $\hat\theta(t)$, $\hat p(t)$ and then $x_i(t)$ are globally uniformly ultimately bounded. Due to equations (29) and (25), the trajectory $z(t)$ is also globally uniformly ultimately bounded. In particular, for all initial conditions whose norms are less than $0 < a < +\infty$, there exists a $T^o < +\infty$ such that the $d_i(t_0,t)$ are zero for all $t \ge t_0 + T^o$. This fact together with equations (73), (32) and (51) implies that, for any $\mu > \sqrt{2\mu_n'/c_n}$, there exists a $T < +\infty$ so that $|y(t)| \le \mu$ for all $t \ge T$. The last statement follows readily since $\sqrt{2\mu_n'/c_n}$ can be made arbitrarily small if the design parameters λ_0, σ_θ, Γ, σ_p, λ, k_i, ε_{i1} and ε_{i2} $(1 \le i \le n)$ are chosen appropriately. □

Remark 4.2. Assumption 1.1 (A.1.1 for short) in Theorem 4.1 implies the following condition, which was used in Teel and Praly (1993).

(A.1.1′) For each $1 \le i \le n$, there exist an unknown nonnegative constant s_i and two known smooth functions ψ_{ix}, ψ_{iz} such that

$$|\Delta_i(x, z, u, t)| \le \psi_{ix}(|(x_1, \dots, x_i)|) + \psi_{iz}(|z|) \quad (74)$$

for all $t \ge 0$ and all (x, z, u) satisfying $|(x_1, \dots, x_i)| + |z| \ge s_i$.

Indeed, under (A.1.1), (A.1.1)′) holds with

$$s_i = \tfrac{1}{2} p_i^{*2}, \quad \psi_{ix}(s)^2 = s + \psi_{i1}(s)^2, \quad \psi_{iz}(s) = s + \psi_{i2}(s)^2.$$

Conversely, if Δ_i is locally Lipschitz with respect to $(x_1, \dots, x_i, z)$, uniformly in $(x_{i+1}, \dots, x_n, u)$ and t, and if there exists a (unknown) constant p_{i0} such that

$$|\Delta_i(0, \dots, 0, x_{i+1}, \dots, x_n, 0, u, t)| \le p_{i0} \quad (75)$$

for all $t \ge 0$ and all $(x_{i+1}, \dots, x_n, u)$, then (A.1.1′) also implies (A.1.1). Indeed, with equation (74), for all $t \ge 0$ and all (x, z, u) satisfying $|(x_1, \dots, x_i)| + |z| \le s_i$, there exists an (unknown) nonnegative constant p_i so that

$$|\Delta_i| \le p_{i0} + p_i(|(x_1, \dots, x_i)| + |z|)\,. \quad (76)$$

Combining equations (74) and (76), it follows that, for all (z, x, u, t) in $\mathbb{R}^{n_0} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}_{\ge 0}$,

$$|\Delta_i| \le p_{i0} + p_i(|(x_1, \dots, x_i)| + |z|) + \psi_{ix}(|(x_1, \dots, x_i)|) + \psi_{iz}(|z|), \quad (77)$$

which implies (A.1.1).

Although (A.1.1) and (A.1.1′) are equivalent theoretically, (A.1.1) is more desirable in practice. In fact, we used a linear function to bound the nonlinear function Δ_i in equation (76). This results in a sufficiently large p_i [or, p_i^* in (A.1.1)] and therefore, a high-gain adaptive controller is required in order to guarantee the smallness of the output of equation (1).

5. EXAMPLES AND DISCUSSIONS

The class of nonlinear systems (1) includes the class of parametric strict-feedback systems introduced in Kanellakopoulos *et al.* (1991b) where the Δ_i's are identically zero and there is no dynamic perturbation characterized by the z-equation in equation (1). It also extends the class of nonlinear systems studied recently in Polycarpou and Ioannou (1995) in which uncertain nonlinearities were considered but the information of full state is required and an overparameterization occurs.

In the sequel, we give several examples to illustrate our robust adaptive backstepping procedure.

5.1. *Three third-order nonlinear systems*

Example 1. Consider the following control system:

$$\begin{aligned} \dot{x}_1 &= x_2 + \theta_1 x_1^2 + \theta_2 \frac{x_1^2 x_2}{1 + x_2^2}, \\ \dot{x}_2 &= x_3, \\ \dot{x}_3 &= u, \\ y &= x_1, \end{aligned} \quad (78)$$

which is in the form of equation (1) with $\Delta_1 = \theta_2 x_1^2 x_2/(1 + x_2^2)$ and $\Delta_2 = \Delta_3 = 0$. Since the dynamic uncertainty does not occur in system (78), we do not need the dynamic signal r as introduced in Section 4. A variant of our proposed recursive control procedure in Section 6 is applicable and leads to a global adaptive controller which achieves the output regulation while keeping boundedness of the states. In fact, for system (78), a detailed analysis proves that the derived adaptive controller achieves the global adaptive stabilization, i.e. $x(t) \to 0$. It is worth noting that this system is in a *parametric pure-feedback* form by borrowing the terminology from Kanellakopoulos *et al.* (1991b). However, the systematic adaptive algorithm presented in Kanellakopoulos *et al.* (1991b) is applicable to system (78) only in some *non-global* feasibility region.

The second elementary example is a control system with nonlinearly appearing unknown parameter and input/state stable dynamic uncertainty.

Example 2. Consider the nonlinear system:

$$\begin{aligned} \dot{z} &= -z + x_1^2, \\ \dot{x}_1 &= x_2 + \theta_1 e^{x_1} + \theta_1(x_1 e^{\theta_2 x_1} + \theta_3 \sin(t^2 u)) + \theta_4 z, \\ \dot{x}_2 &= u + \theta_5 x_2^2 + \theta_6 z^2 x_1, \\ y &= x_1, \end{aligned} \quad (79)$$

where θ_i $(1 \le i \le 6)$ are unknown constant parameters and z is unmeasured.

Obviously, system (79) is in the form equation (1) with:

$$\begin{aligned} &\theta = (\theta_1, \theta_5)^T, \quad \phi_1 = (e^{x_1}, 0)^T, \quad \phi_2 = (0, x_2^2)^T, \\ &\Delta_1 = \theta_1(x_1 e^{\theta_2 x_1} + \theta_3 \sin(t^2 u)) + \theta_4 z, \quad \Delta_2 = \theta_6 z^2 x_1. \end{aligned} \quad (80)$$

Then, Assumption 1.1 holds with:

$$\begin{aligned} p_1^* &= \max\{0.5|\theta_1| e^{0.5\theta_2^2}, |\theta_1\theta_3|, |\theta_4|\}, \\ &\psi_{11}(s) = s\,e^{0.5s^2}, \quad \psi_{12}(s) = s, \end{aligned} \quad (81)$$

$$p_2^* = 0.5|\theta_6|, \quad \psi_{21}(s) = s^2, \quad \psi_{22}(s) = s^4. \quad (82)$$

It is direct to verify that Assumption 4.1 holds for the z-subsystem in equation (79) with

$$V_z(z) = z^2, \qquad c = 1.2, \qquad \gamma(s) = 1.25s^4. \quad (83)$$

Therefore, our recursive control design procedure and Theorem 4.1 are applicable to system (79).

In order to compare the proposed robustification method with previous adaptive backstepping design algorithms in the presence of dynamic uncertainty, we consider the following simple example whose nominal system was used in, e.g. Krstić *et al.* (1995), Pomet and Praly (1992); see also Polycarpou and Ioannou (1995).

Example 3. For the three-dimensional system,

$$\begin{aligned} \dot{z} &= -z + x_1^2 + \delta_0, \\ \dot{x}_1 &= x_2 + \theta x_1^2 + \delta_1(t) + 2z, \\ \dot{x}_2 &= u, \\ y &= x_1, \end{aligned} \tag{84}$$

where θ is an unknown constant parameter, δ_0 and $\delta_1(t)$ are two unknown bounded disturbances and z is the unmeasured state component. For comparison purposes, we take exactly the same simulation values as in Polycarpou and Ioannou (1995) for θ and $\delta_1(t)$, i.e. $\theta = 0.1$ and $\delta_1(t) = 0.6\sin(2t)$. Also, $\delta_0 = 0.5$.

It is directly checked that the z-equation in equation (84) fulfills the Assumption 4.1 with

$$V_z(z) = z^2, \qquad \gamma(s) = 2.5s^4, \qquad c_0 = 1.2, \qquad d_0 = 0.625. \tag{85}$$

So an available dynamic signal r is defined as follows:

$$\dot{r} = -r + 2.5x_1^4 + 0.625, \qquad r(0) = r^o > 0. \tag{86}$$

Here, for simulation use, we take $r^o = 1$.

Applying the robust adaptive backstepping design procedure in Section 4 to system (84), we get a stabilizing control function w_1 defined by

$$\begin{aligned} w_1 = &-(k_1+1)x_1 - \hat{\theta}x_1^2 - \frac{2.5}{\lambda_0}x_1^3 \\ &- \hat{p}\left[\tanh\left(\frac{\delta x_1}{\varepsilon_{11}}\right) + 2\tanh\left(\frac{2\delta x_1}{\varepsilon_{12}}\right)\right. \\ &\left. + r\tanh\left(\frac{\delta x_1 r}{\varepsilon_{12}}\right)\right] \end{aligned} \tag{87}$$

with $\delta = 0.2785$. Notice that for system (84) the bounding functions ψ_{ij} as given in Assumption 1.1 are: $\psi_{11} \equiv 1$, $\psi_{12}(s) = s$, $\psi_{21} = \psi_{22} \equiv 0$. Letting $x_2 = x_2 - w_1$, we obtain the following adaptive laws and adaptive controller for $\hat{\theta}$, $\hat{p}_1$, $\hat{p}_2$ and u, respectively,

$$\dot{\hat{\theta}} = \Gamma\left(x_1^3 - \bar{x}_2 x_1^2\frac{\partial w_1}{\partial x_1} - \sigma_\theta(\hat{\theta} - \theta^o)\right), \tag{88}$$

$$\begin{aligned} \dot{\hat{p}} = \lambda\bigg[&-\sigma_p(\hat{p} - p^o) + x_1\tanh\frac{\delta x_1}{\epsilon_{11}} \\ &+ 2x_1\tanh\frac{2\delta x_1}{\varepsilon_{12}} + x_1 r\tanh\frac{\delta x_1 r}{\varepsilon_{12}} \\ &+ \bar{x}_2\frac{\partial w_1}{\partial x_1}\tanh\left(\frac{\delta}{\varepsilon_{21}}\bar{x}_2\frac{\partial w_1}{\partial x_1}\right) + \bar{x}_2 r\tanh\left(\frac{\delta}{\varepsilon_{22}}\bar{x}_2 r\right) \\ &+ 2\bar{x}_2\tanh\left(\frac{2\delta}{\varepsilon_{22}}\bar{x}_2\right)\bigg], \end{aligned} \tag{89}$$

and

$$\begin{aligned} u = &-(k_2+1)\bar{x}_2 - x_1 + \frac{\partial w_1}{\partial x_1}(x_2 + \hat{\theta}x_1^2) \\ &+ \frac{\partial w_1}{\partial \hat{\theta}}\dot{\hat{\theta}} + \frac{\partial w_1}{\partial r}(-r + 2.5x_1^4 + 0.625) \\ &+ \frac{\partial w_1}{\partial \hat{p}}\dot{\hat{p}} - \hat{p}\frac{\partial w_1}{\partial x_1}\tanh\left(\frac{\delta}{\varepsilon_{21}}\bar{x}_2\frac{\partial w_1}{\partial x_1}\right) \\ &- 2\hat{p}r\tanh\left(\frac{\delta}{\varepsilon_{22}}\bar{x}_2 r\right) - 2\hat{p}\tanh\left(\frac{2\delta}{\varepsilon_{22}}x_2\right). \end{aligned} \tag{90}$$

The simulations in Figs 1 and 2 were performed using MATLAB with the following choice of the initial conditions and design parameters:

$$\begin{aligned} &z(0) = x_1(0) = x_2(0) = 1, \quad \hat{\theta}(0) = 0.5, \quad \hat{p}(0) = 0, \\ &\varepsilon_{11} = \varepsilon_{12} = \varepsilon_{21} = \varepsilon_{22} = 0.5\ k_1 = k_2 = 1, \\ &\lambda_0 = 10, \quad \lambda = \Gamma = 1, \quad \sigma_p = \sigma_\theta = 1, \\ &\theta^o = 0, \quad p^o = 1. \end{aligned}$$

We observe that our robustified adaptive controller yields better performance than previous adaptive controllers in Krstić *et al.* (1992) and Polycarpou and Ioannou (1995).

5.2. *Output-feedback form systems*

We return to the uncertain output-feedback-form system (5). As shown in Section 2, the output-feedback control of equation (5) may be translated into the problem of partial-state feedback control of a new system (10). With the help of Proposition 3.1, if the ζ-system in equation (5) has an exp-ISpS Lyapunov function V_ζ, then it is ISpS. Since A is an asymptotically stable matrix, from equation (6), it follows that the z_0-system in equation (10) is ISpS with (ζ, y) as input (note that we do not assume that the functions φ_{i1} and φ_{i2} are zero at zero). Therefore, the cascaded (z_0, ζ)-system is ISpS (Jiang *et al.*, 1994). So, using again Proposition 3.1, this system has an exp-ISpS Lyapunov function.

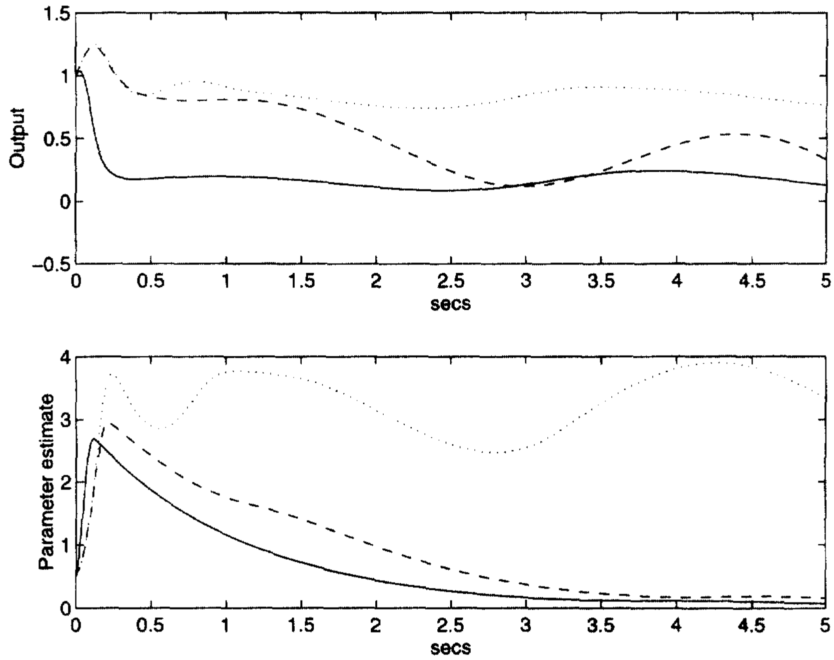

Fig. 1. Output y and parameter estimate $\hat{\theta}$ for three different algorithms: the dotted line is given by Krstić *et al.* (1992), the dashed line by Polycarpou and Ioannou (1995) and the solid line by our control laws (88) and (90).

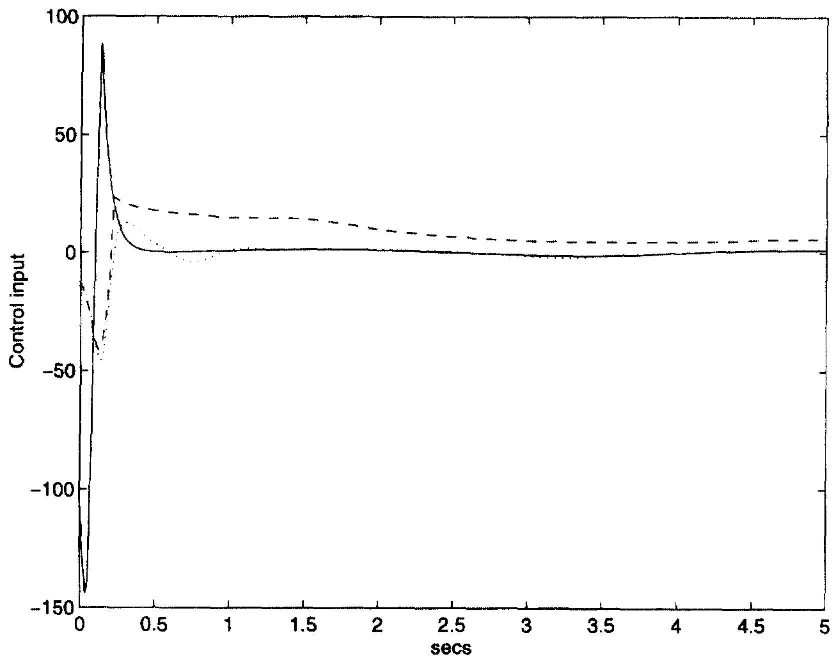

Fig. 2. Control input u for three different algorithms as in Fig. 1.

In the sequel, it is shown how such an ISpS Lyapunov function can be found in some interesting situations. For the sake of simplicity, we consider the following particular case where the ζ-system in equation (5) is described by

$$\dot{\zeta} = Q_0\zeta + q_{01}(\zeta, y), \tag{91}$$

where Q_0 is a stable constant matrix and there exist a constant $\sigma_0 > 0$ and a smooth nonnegative function $\hat{q}$ such that

$$\begin{aligned} &\mathrm{Re}\,\sigma(Q_0) \le -\sigma_0, \\ &|q_{01}(\zeta, y)| \le \hat{q}(|y|) + \nu|\zeta|, \\ &|\omega_i(\xi, \zeta, u, t)| \le \vartheta_i^* \varphi_{i1}(|y|) + \vartheta_i^*|\zeta|, \end{aligned} \tag{92}$$

where ν is a small nonnegative constant.

We will show that an exp-ISpS Lyapunov for the composite (z_0, ζ)-system in equation (10) can be easily obtained in this case.

Let $P_1 > 0$ and $P_2 > 0$ be the solutions of Lyapunov matrix equations

$$\begin{aligned} P_1 Q_0 + Q_0^{\mathrm{T}} P_1 &= -2I, \\ P_2 A + A^{\mathrm{T}} P_2 &= -2I \end{aligned} \tag{93}$$

Introduce the functions:

$$V_\zeta(\zeta) = \zeta^{\mathrm{T}} P_1 \zeta, \qquad V_0(z_0) = z_0^{\mathrm{T}} P_2 z_0. \tag{94}$$

As it can be directly checked, V_ζ and V_0 are exp-ISpS Lyapunov functions for the ζ-subsystem and the z_0-subsystem of equation (10), respectively. More precisely, we have

$$\begin{aligned} \lambda_{\min}(P_1)|\zeta|^2 &\le V_\zeta(\zeta) \le \lambda_{\max}(P_1)|\zeta|^2, \\ \lambda_{\min}(P_2)|z_0|^2 &\le V_0(z_0) \le \lambda_{\max}(P_2)|z_0|^2 \end{aligned} \tag{95}$$

and

$$\frac{\partial V_\zeta}{\partial \zeta}(Q_0\zeta + q_{01}(\zeta, y)) \le -\frac{1 - 2\nu|P_1|}{\lambda_{\max}(P_1)} V_\zeta(\zeta) + |P_1|^2 \hat{q}(|y|)^2, \tag{96}$$

$$\begin{aligned} \frac{\partial V_0}{\partial z_0}\left(Az_0 - \frac{\tilde{\omega}}{p^*}\right) \le &-\frac{1}{\lambda_{\max}(P_2)} V_0(z_0) + \frac{2n|P_2|^2}{\lambda_{\min}(P_1)} V_\zeta(\zeta) \\ &+ 2|P_2|^2 \sum_{i=1}^{n} (\varphi_{i1}(|y|) + \phi_i(y))^2. \end{aligned} \tag{97}$$

Letting

$$V_z = V_\zeta(\zeta) + \frac{(1 - 2\nu|P_1|)\lambda_{\min}(P_1)}{4n|P_2|^2 \lambda_{\max}(P_1)} V_0(z_0), \tag{98}$$

from equations (96) and (97), we have

$$\begin{aligned} \dot{V}_z \le &- cV_z + |P_1|^2 \hat{q}(|y|)^2 + \frac{(1 - 2\nu|P_1|)\lambda_{\min}(P_1)}{2n\lambda_{\max}(P_1)} \\ &\times \sum_{i=1}^{n} (\varphi_{i1}(|y|) + \phi_i(y))^2 \end{aligned} \tag{99}$$

where c is a positive real number given by

$$c = \min\left\{\frac{1 - 2\nu|P_1|}{2\lambda_{\min}(P_1)}, \frac{1}{\lambda_{\max}(P_2)}\right\} \tag{100}$$

From equations (98) and (99), V_z is an exp-ISpS Lyapunov function for the z-subsystem of equation (10) with

$$\begin{aligned} \gamma(s) = &\sup_{|y| \le s} \left\{|P_1|^2 \hat{q}(|y|)^2 + \frac{(1 - 2\nu|P_1|)\lambda_{\min}(P_1)}{2n\lambda_{\max}(P_1)} \sum_{i=1}^{n} (\varphi_{i1}(|y|) + \phi_i(y))^2\right\} \\ &- |P_1|^2 \hat{q}(0)^2 - \frac{(1 - 2\nu|P_1|)\lambda_{\min}(P_1)}{2n\lambda_{\max}(P_1)} \times \sum_{i=1}^{n} (\varphi_{i1}(0) + \phi_i(0))^2, \end{aligned} \tag{101}$$

$$d = |P_1|^2 \hat{q}(0)^2 + \frac{(1 - 2\nu|P_1|)\lambda_{\min}(P_1)}{2n\lambda_{\max}(P_1)} \sum_{i=1}^{n} (\varphi_{i1}(0) + \phi_i(0))^2. \tag{102}$$

Therefore, Assumption 4.1 is checked for the system (10).

A direct application of the proposed control design procedure in Section 4 and Theorem 4.1 yields a globally regulating dynamic output-feedback controller for system (5). Namely:

Proposition 5.1. Under the conditions (91) and (92), we can find an adaptive output-feedback dynamic controller such that, for sufficiently small $\nu > 0$, the solutions of the closed-loop system (5) are globally uniformly ultimately bounded. Furthermore, the output $y(t)$ can be steered to the origin with any prescribed accuracy.

6. FURTHER RESULTS

Up to now, we have studied the general case where the system (1) does not necessarily have an equilibrium point and the practical output regulation has been obtained via robust adaptive control. In this section, we concentrate on a subclass of systems (1) which have an equilibrium and propose sufficient conditions which result in a solution for *exact* adaptive output regulation.

To simplify the presentation, we assume $n = 2$ in equation (1), that is, consider the following class of uncertain systems:

$$\begin{aligned} \dot{z} &= q(z, x, u), \\ \dot{x}_1 &= x_2 + \theta^{\mathrm{T}}\phi_1(x_1) + \Delta_1(x, z, u, t), \\ \dot{x}_2 &= u + \theta^{\mathrm{T}}\phi_2(x_1, x_2) + \Delta_2(x, z, u, t), \\ y &= x_1. \end{aligned} \tag{103}$$

We make the following assumptions on the system (103).

Assumption 6.1. For each $i = 1, 2$, Δ_i satisfies the property (2) in Assumption 1.1 with ψ_{i1} and

ψ_{i2} both vanishing at zero and ψ_{i1} depending only on $|x_1|$. Namely,

$$|\Delta_i(x, z, u, t)| \le p_i^* \psi_{i1}(|x_1|) + p_i^* \psi_{i2}(|z|). \quad (104)$$

Assumption 6.2. The z-system in (103) fulfills Assumption 4.1 with $d_0 = 0$ and γ smooth and satisfying (27). Namely,

$$\alpha_1(|z|) \le V_z(z) \le \alpha_2(|z|)\,, \quad \forall z \in \mathbb{R}^{n_0}, \quad (105)$$

$$\frac{\partial V_z}{\partial z}(z) q(z, x, u) \le -c_0 V_z(z) + x_1^2 \gamma_0(x_1^2). \quad (106)$$

Under these assumptions, we see that $\Delta_1 = \Delta_2 \equiv 0$ and $q \equiv 0$ as long as $x_1 = 0$ and $z = 0$. Thus, $(x_1^e, x_2^e, z) = (0, -\theta^T\phi_1(0), 0)$ is an equilibrium point of the system (103).

In order to regulate y to zero and x to $x^e = (x_1^e, x_2^e)$, the following additional conditions are needed.

Assumption 6.3. The functions ψ_{12}, ψ_{22} and α_1 satisfy

$$\limsup_{s \to 0+} \frac{\psi_{i2} \circ \alpha_1^{-1}(s^2)}{s} < +\infty, \quad i = 1, 2. \quad (107)$$

Assumption 6.4. Denote $X := (z^T, x^T, u)^T$. For any $i = 1, 2$ and any compact set S in R^{n_0+3}, the functions $\partial\Delta_i/\partial t(X, t)$ and $\partial\Delta_i/\partial X(X, t)$ are bounded on $S \times \mathbb{R}_{\ge 0}$.

Notice that Assumption 6.3 implies that the input/output gain of $x_1 \mapsto \Delta_i$, $i = 1, 2$, is linearly bounded near zero. This is a common condition in asymptotic analysis based on nonlinear small-gain results, see Jiang *et al.* (1994).

As in Section 4, we construct our adaptive controller and adaptive laws through a stepwise control design procedure.

Step 1: Using the same notations as in the Section 4.1 but with different expressions. For instance, consider the function V_1 defined by, (see equation (32))

$$V_1 = \tfrac{1}{2}x_1^2 + \frac{1}{\lambda_0} r + \tfrac{1}{2}(\hat\theta - \theta)^T \Gamma^{-1}(\hat\theta - \theta) + \frac{1}{2\lambda}(\hat p - \bar p^*)^2 \quad (108)$$

with $\bar p^* \ge \max\{p_1^{*2}, p_1^{*2}, p_2^*, p_2^{*2}\}$.

We will prove that there exist suitable functions τ_1, w_1 and π_1 so that the time derivative of V_1 satisfies, instead of equation (33),

$$\dot V_1 \le -k_1 x_1^2 + x_1 \bar x_2 - b_1 r + (\hat\theta - \theta)^T \Gamma^{-1}(\dot{\hat\theta} - \tau_1) + \frac{1}{\lambda}(\hat p - \bar p^*)(\dot{\hat p} - \varpi_1) + d_1(t_0, t) \quad (109)$$

with $k_1 > 1$, $b_1 > 0$ and $d_1(t_0, t)$ as defined in (39).

Indeed, by Assumption 6.2, equation (34) holds with $d_0 = 0$ and $\tilde p := \hat p - \bar p^*$. Now look at the terms $p_1^*|x_1|\psi_{11}(|x_1|)$ and $p_1^*|x_1|\psi_{12}(|z|)$ in equation (34).

Since ψ_{11} is smooth and is zero at zero, there exists a smooth function $\hat\psi_{11}$ such that

$$p_1^*|x_1|\psi_{11}(|x_1|) \le \bar p^* x_1^2 \hat\psi_{11}(x_1), \quad \forall x_1 \in \mathbb{R}. \quad (110)$$

According to equation (38),

$$p_1^*|x_1|\psi_{12}(|z|) \le p^*{}_1|x_1|\psi_{12}\circ\alpha_1^{-1}(2r) + \tfrac{1}{4}x_1^2 + d_1(t_0, t). \quad (111)$$

Then, by Assumption 6.3, there exists a smooth nonnegative function $\hat\psi_{12}$ such that

$$p_1^*|x_1|\psi_{12}\circ\alpha_1^{-1}(2r) \le p_1^*|x_1|\sqrt{r}\hat\psi_{12}(r) \quad (112)$$

$$\le p_1^{*2}\frac{\lambda_0}{\bar c_0} x_1^2 \hat\psi_{12}(r)^2 + \frac{\bar c_0}{4\lambda_0} r. \quad (113)$$

Consequently, in view of equations (110), (111) and (113) equation (34) implies

$$\dot V_1 \le x_1\left(x_2 + \hat\theta\phi_1(x_1) + \frac{1}{\lambda_0}x_1\gamma_0 + \tfrac{1}{4}x_1 + \hat p x_1(\hat\psi_{11} + \hat\psi_{12}^2)\right) - \frac{3\bar c_0}{4\lambda_0} r + (\hat\theta - \theta)^T\Gamma^{-1}(\dot{\hat\theta} - \Gamma x_1\phi_1) + \frac{1}{\lambda}(\hat p - \bar p^*)(\dot{\hat p} - \lambda x_1^2\hat\psi_{11}(x_1) - \lambda x_1^2 \hat\psi_{12}(r)^2) + d_1(t_0, t). \quad (114)$$

Setting

$$\tau_1 = \Gamma x_1 \phi_1(x_1), \quad (115)$$

$$\varpi_1 = \lambda(x_1^2\hat\psi_{11}(x_1) + x_1^2\hat\psi_{12}(r)^2), \quad (116)$$

$$w_1 = -k_1 x_1 - \hat\theta\phi_1(x_1) - \frac{1}{\lambda_0}x_1\gamma_0 - \tfrac{1}{4}x_1 - \hat p x_1(\hat\psi_{11}(x_1) + \hat\psi_{12}(r)^2), \quad (117)$$

where $k_1 > 1$, equation (109) follows with $b_1 = 3\bar c_0/(4\lambda_0)$.

Step 2: The time derivative of $\bar x_2 = x_2 - w_1$ satisfies

$$\dot{\bar x}_2 = u + \theta^T\phi_2(x) + \Delta_2 - \frac{\partial w_1}{\partial x_1}(x_2 + \theta^T\phi_1 + \Delta_1) - \frac{\partial w_1}{\partial \hat\theta}\dot{\hat\theta} - \frac{\partial w_1}{\partial r}(-\bar c_0 r + x_1^2\gamma_0) - \frac{\partial w_1}{\partial \hat p}\dot{\hat p}$$

$$:= u + \theta^T\left(\phi_2 - \frac{\partial w_1}{\partial x_1}\phi_1\right) + \left(\Delta_2 - \frac{\partial w_1}{\partial x_1}\Delta_1\right) - \kappa_2 \quad (118)$$

Consider the function V_2 defined by

$$V_2 = V_1 + \tfrac{1}{2}\bar x_2^2. \quad (119)$$

With equations (109) and (118), differentiating V_2 along solutions of (90) yields:

$$\dot{V}_2 \le -k_1 x_1^2 + x_1\bar{x}_2 - b_1 r + (\hat{\theta}-\theta)^{\mathrm{T}}\Gamma^{-1}(\dot{\hat{\theta}}-\tau_1) + \frac{1}{\lambda}(\hat{p}-\bar{p}^*)(\dot{\hat{p}}-\varpi_1) + d_1(t_0,t) + \bar{x}_2\left[u + \theta^{\mathrm{T}}\left(\phi_2 - \frac{\partial w_1}{\partial x_1}\phi_1\right) + \left(\Delta_2 - \frac{\partial w_1}{\partial x_1}\Delta_1\right) - \kappa_2\right]. \quad (120)$$

By Assumptions 6.1 and 6.2, it holds

$$\bar{x}_2\left(\Delta_2 - \frac{\partial w_1}{\partial x_1}\Delta_1\right) \le |\bar{x}_2|(p_2^*\psi_{21}(|x_1|) + p_2^*\psi_{22}(|z|)) + \left|\bar{x}_2\frac{\partial w_1}{\partial x_1}\right|(p_1^*\psi_{11}(|x_1|) + p_1^*\psi_{12}(|z|)). \quad (121)$$

Using the similar arguments as in Step 1, we get

$$\left|\bar{x}_2\frac{\partial w_1}{\partial x_1}\right|(p_1^*\psi_{11}(|x_1|) + p_1^*\psi_{12}(|z|)) \le \tfrac{1}{4}x_1^2 + p_1^{*2}\bar{x}_2^2\left(\frac{\partial w_1}{\partial x_1}\right)^2\hat{\psi}_{11} + p_1^*\left|\bar{x}_2\frac{\partial w_1}{\partial x_1}\right|\psi_{12}\circ\alpha_1^{-1}(2r) + \tfrac{1}{4}\bar{x}_2^2\left(\frac{\partial w_1}{\partial x_1}\right)^2 + d_1(t_0,t). \quad (122)$$

In addition, with equation (113), we have

$$p_1^*\left|\bar{x}_2\frac{\partial w_1}{\partial x_1}\right|\psi_{12}\circ\alpha_1^{-1}(2r) \le \frac{b_1}{4}r + p_1^{*2}\frac{1}{b_1}x_2^2\left(\frac{\partial w_1}{\partial x_1}\right)^2\hat{\psi}_{12}(r)^2 \quad (123)$$

As for the first term in the right-hand side of equation (121), since ψ_{21} is smooth and is zero at zero, there exists a smooth nonnegative function $\hat{\psi}_{21}$ so that

$$p_2^*|x_2|\psi_{21}(|x_1|) \le \tfrac{1}{4}x_1^2 + p_2^{*2}\bar{x}_2^2\hat{\psi}_{21}(x_1). \quad (124)$$

Similar to equation (113), letting $d_{21}(t_0,t) = [p_2^*\psi_{22}\circ\alpha_1^{-1}(2D(t_0,t))]^2$, there exists a smooth nonnegative function $\hat{\psi}_{22}$ so that

$$p_2^*|x_2|\psi_{22}(|z|)) \le p_2^*|x_2|\psi_{22}\circ\alpha_1^{-1}(2r) + \tfrac{1}{4}\bar{x}_2^2 + d_{21}(t_0,t) \le p_2^*|x_2|\sqrt{r}\hat{\psi}_{22}(r) + \tfrac{1}{4}\bar{x}_2^2 + d_{21}(t_0,t) \le p_2^{*2}\frac{1}{b_1}\bar{x}_2^2\hat{\psi}_{22}(r)^2 + \frac{b_1}{4}r + \tfrac{1}{4}\bar{x}_2^2 + d_{21}(t_0,t). \quad (125)$$

Therefore, by the definition of $\bar{p}^*$,

$$\bar{x}_2\left(\Delta_2 - \frac{\partial w_1}{\partial x_1}\Delta_1\right) \le \tfrac{1}{2}x_1^2 + \frac{b_1}{2}r + d_1(t_0,t) + d_{21}(t_0,t) + \bar{x}_2\left[\tfrac{1}{4}\bar{x}_2 + \bar{p}^*\bar{x}_2\left(\frac{\partial w_1}{\partial x_1}\right)^2 \times\left(\hat{\psi}_{11} + \frac{1}{b_1}\hat{\psi}_{12}^2 + \tfrac{1}{4}\right) + \bar{p}^*\bar{x}_2\left(\hat{\psi}_{21} + \frac{1}{b_1}\hat{\psi}_{22}^2\right)\right]. \quad (126)$$

By choosing the adaptive controller w_2 and adaptive laws τ_2 and π_2 as

$$u = w_2 = -k_2\bar{x}_2 - x_1 - \hat{\theta}^{\mathrm{T}}\left(\phi_2 - \frac{\partial w_1}{\partial x_1}\phi_1\right) + \kappa_2 - \tfrac{1}{4}\bar{x}_2 - \bar{p}^*\bar{x}_2\left(\hat{\psi}_{21} + \frac{1}{b_1}\hat{\psi}_{22}^2\right) - \hat{p}x_2\left(\frac{\partial w_1}{\partial x_1}\right)^2\left(\hat{\psi}_{11} + \frac{1}{b_1}\hat{\psi}_{12}^2 + \tfrac{1}{4}\right), \quad (127)$$

$$\dot{\hat{\theta}} = \tau_2 = \tau_1 + \Gamma\bar{x}_2\left(\phi_2 - \frac{\partial w_1}{\partial x_1}\phi_1\right), \quad (128)$$

$$\dot{\hat{p}} = \varpi_2 = \varpi_1 + \lambda\bar{x}_2^2\left(\frac{\partial w_1}{\partial x_1}\right)^2\left(\hat{\psi}_{11} + \frac{1}{b_1}\hat{\psi}_{12}^2 + \frac{1}{4}\right) + \lambda\bar{x}_2^2\left(\hat{\psi}_{21} + \frac{1}{b_1}\hat{\psi}_{22}^2\right) \quad (129)$$

where $k_2 > 0$, from equations (120) and (126), it follows that

$$\dot{V}_2 \le -(k_1 - 0.5)x_1^2 - k_2x_2^2 - \frac{b_1}{2}r + 2d_1(t_0,t) + d_{21}(t_0,t). \quad (130)$$

Notice that $d_2(t_0,t) := 2d_1(t_0,t) + d_{21}(t_0,t) \ge 0$ for all $t \ge t_0$ and $=0$ if $t \ge t_0 + T^o$.

Finally, we establish the following.

Proposition 6.1. Under Assumptions 6.1–6.4, all the signals and states of the closed-loop system (103), (127)–(129) and (28) are globally uniformly ultimately bounded. Furthermore,

$$\lim_{t\to\infty}(|y(t)| + |x(t) - x^e| + |z(t)|) = 0. \quad (131)$$

In particular, if $\operatorname{rank}[\phi_1(0)\ \phi_2(0, x_2^e)] = l$, then

$$\lim_{t\to\infty}|\hat{\theta}(t) - \theta| = 0. \quad (132)$$

Remark 6.1. Proposition 6.1 can be extended to higher-dimensional uncertain nonlinear systems of the form (1) by induction. Also consult Jiang and Praly (1996) for robust adaptive regulation of a larger class of systems having the origin as an equilibrium via a worst-case design.

Proof. For each initial instant t_0 and each initial condition $(x(t_0), \hat{\theta}(t_0), \hat{p}(t_0), z(t_0))$ in $\mathbb{R}^{l+n_0+3}$ and $r(t_0) = r^o$ in $\mathbb{R}_{>0}$, let $(x(t), \hat{\theta}(t), \hat{p}(t), z(t), r(t))$ be the corresponding solution defined on the maximal interval $[t_0, T_s)$, with $T_s > t_0$. Since $d_2(t_0, t)$ is non-negative and integrable over $[t_0, \infty)$, this together with the definition of V_2 in equation (119) implies that $x(t)$, $\hat{\theta}(t)$, $\hat{p}(t)$, and $r(t)$ are bounded on $[t_0, T_s)$. With equation (29), $z(t)$ is bounded. Therefore, $T_s = \infty$.

From equation (130), it follows that $x_1(t)^2$, $x_2(t)^2$ and $r(t)$ are integrable. Further, the derivatives of these signals are bounded and therefore they are uniformly continuous. A straightforward application of Barbalat's lemma (Khalil, 1996) yields that $x_1(t)$, $x_2(t)$ and $r(t)$ go to zero as t goes to ∞. Then, equation (29) ensures that $z(t)$ tends to zero as $t \to \infty$. Further, equation (104) implies that Δ_1 and Δ_2 tend to zero. By definition of $\bar{x}_2$, it follows

$$\lim_{t\to\infty} [x_2(t) - \hat{\theta}(t)\phi_1(x_1(t))] = 0. \quad (133)$$

On the other hand, by means of Assumption 6.4 and Barbalat's lemma, it results that $\dot{x}_1(t)$ and $\dot{x}_2(t)$ converge to zero as t goes to ∞. Back to the equations (103), (104) and (127), we have

$$\lim_{t\to\infty} [x_2(t) + \theta\phi_1(x_1(t))] = 0, \quad (134)$$

$$\lim_{t\to\infty} (\hat{\theta}(t) - \theta)^{\mathrm{T}} \left[\phi_2(x(t)) - \frac{\partial w_1}{\partial x_1} \phi_1(x_1(t)) \right] = 0. \quad (135)$$

Therefore, equation (134) gives that $x_2(t)$ converges to $-\theta\phi_1(0) = x_2^e$ and equation (131) follows readily. Moreover, equations (133) and (134) imply

$$\lim_{t\to\infty} (\hat{\theta}(t) - \theta)^{\mathrm{T}} \phi_1(0) = 0. \quad (136)$$

So, with equation (135), we obtain

$$\lim_{t\to\infty} (\hat{\theta}(t) - \theta)^{\mathrm{T}} \phi_2(0, x_2^e) = 0. \quad (137)$$

If $\mathrm{rank}[\phi_1(0) \;\; \phi_2(0, x_2^e)] = \dim\theta = l$, from equations (136) and (137) and by contradiction, we conclude the property (132). □

7. CONCLUSIONS

We have considered in this paper a wide class of uncertain nonlinear systems with unknown parameters, static and dynamic uncertainties. A modified robust adaptive backstepping design procedure is proposed, which demonstrates a robustification method for previous backstepping-based adaptive nonlinear controllers (see, e.g. Krstić *et al.*, 1992, 1995; Polycarpou and Ioannou, 1995). This method is nothing but the translation to the nonlinear case of standard fixes proposed in linear adaptive control (Ioannou and Sun, 1996), i.e.

- the introduction of a mechanism to keep the estimated parameter bounded, here a σ-modification,
- the introduction of a dynamic dominating signal to inform about the size of dynamic uncertainties.

The robust adaptive controllers obtained in this paper drive the output to a small neighborhood of the origin while guaranteeing the internal Lagrange stability for all signals. For a subclass of perturbed systems which possess an equilibrium point, sufficient conditions are proposed under which the state converges to the desired equilibrium and the parameter estimate converges to the true parameter. The proposed algorithm can be extended to handle a larger class of block-strict-feedback systems (Krstić *et al.*, 1995, Section 4.5.2) in the presence of nonlinear disturbances and unmodeled dynamics. Extension to the adaptive tracking problem follows readily.

By allowing perturbations of the nominal case, the recursive control design procedure proposed in this paper allows us to deal with a broader class of systems in the presence of nonlinear parametrization, uncertain nonlinearities and unmodeled dynamics. Simulation results on a particular example showed that our robustified adaptive controllers lead to better performance than previous adaptive controllers (Krstić *et al.*, 1992; Polycarpou and Ioannou, 1995).

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