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Mathematics of Control, Signals, and Systems

Stabilization in Spite of Matched Unmodeled Dynamics and an Equivalent Definition of Input-to-State Stability*

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Abstract. We consider nonlinear systems with input-to-output stable (IOS) unmodeled dynamics which are in the "range" of the input. Assuming the nominal system is globally asymptotically stabilizable and a nonlinear small-gain condition is satisfied, we propose a first control law such that all solutions of the perturbed system are bounded and the state of the nominal system is captured by an arbitrarily small neighborhood of the origin. The design of this controller is based on a gain assignment result which allows us to prove our statement via a Small-Gain Theorem [JTP, Theorem 2.1]. However, this control law exhibits a high-gain feature for all values. Since this may be undesirable, in a second stage we propose another controller with different characteristics in this respect. This controller requires more *a priori* knowledge on the unmodeled dynamics, as it is dynamic and incorporates a signal bounding the unmodeled effects. However, this is only possible by restraining the IOS property into the exp-IOS property. Nevertheless, we show that, in the case of input-to-state stability (ISS)—the output is the state itself—ISS and exp-ISS are in fact equivalent properties.

Key words. Nonlinear systems, Robust control, Uncertain systems, Gain assignment, Input-to-state stability.

1. Introduction

Consider the system

$$\begin{cases} \dot{x} = f(x) + \sum_{i=1}^{p} g_i(x)[u_i + c_i(x, z, u)], \\ \dot{z} = a(x, z, u), \end{cases} \tag{1}$$

where a and f are continuous vector fields, $G = (g_i)$ is a continuous "matrix field," and $c_1, \ldots, c_p$ are continuous functions. The x-subsystem represents, when $c = 0$,

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the nominal system. Its state x, taking values in $\mathbf{R}^n$, is measured and the vector $u = (u_i)$, taking values in $\mathbf{R}^p$, is its input. The z-subsystem represents the unmodeled dynamics, its state z, taking values in $\mathbf{R}^m$, is unmeasured and the functions a and c are unknown.

The problem is to design a feedback law, with x as the only input, guaranteeing boundedness of the solutions of the closed-loop system and regulating x around 0. To solve this problem we assume (see Assumption A1) that the nominal system is globally asymptotically stabilizable and that the z-subsystem has an appropriately "stable" input–output behavior (see Assumptions A2 or A2′).

In the terminology of linear systems, the perturbation introduced via c would be called a stable and proper multiplicative perturbation. Its main characteristics are:

—The relative degree between u and any "generic" output function of x cannot be decreased by the presence of c.
—The so-called matching assumption is met. Namely, if c were measured, we could completely annihilate its effects on the x-subsystem (see Remark 4.6.2 of [I]). Here c is not assumed to be measured. Instead, we impose an amplitude limitation (see Assumption A2 or A2′).
—The state x can be measured, and, consequently, there is no inverse dynamics. This makes it theoretically possible to use "high-gain" controllers. However, we know that if other classes of "real life" unmodeled effects—input saturations, unmeasured noise, unmatched unmodeled dynamics, ...—are present, then "high-gain" controllers may be unsuitable. For this reason, we propose two solutions to the problem stated above with a different high-gain requirement.

The topic of stabilizing (nonlinear) systems with uncertainties has been attracting the attention of many authors for a long time, see, for instance, [BCL], [C], [CL], [G], and [K]. While most of the work in this area focused on unmodeled static (time-varying) uncertainties, less work has been done for systems with dynamic uncertainties. The recent work [KSK] has formulated very properly the problem of stabilizability for nonlinear systems with unmodeled dynamics. There also the authors have proposed a solution for a specific class of systems with linear unmodeled dynamics at the input. Some related work in this area can also be found in [Q1] and [Q2], where the author has investigated the tracking problem for linear systems with unmodeled dynamic uncertainties.

Our problem generalizes the one stated and solved by Krstić *et al.* [KSK, Lemma 3.1] for x in $\mathbf{R}$ and functions c and a linear and not depending on x. The solution proposed by these authors incorporates, in the controller, a signal, called a normalizing signal, which captures the effect of the unmodeled dynamics. This concept of normalizing signal is nowadays widely used in linear adaptive control, and its extension to the nonlinear case has been suggested in [JP1], [JP2], and [J]. Jiang *et al.* have shown in [JMP] that the result of Lemma 3.1 of [KSK] holds also with a static feedback law, without using the normalizing signal. For this, an appropriate change of coordinates of the unmodeled dynamics is made and the technique of propagating the input-to-state stability (ISS) property through integrators proposed in [JTP] is applied. Based on the technique of gain assignment and the Small-Gain Theorem of [JTP], Krstić and Kokotović have obtained another solution in [KK], without a normalizing signal for the system (1), allowing the functions c and a to be nonlinear and to depend on x but still imposing that x be in $\mathbf{R}$.

Here, we extend the work in [KK] to the general case when x is in $\mathbf{R}^n$. Our major assumptions are: (1) the nominal system is stabilizable, and (2) the unmodeled dynamics is input-to-output stable (IOS) with a small enough gain function. In the special case when there is no dynamic uncertainty present in the system, that is, when the functions c_i's do not depend on z, the IOS condition reduces to the usual boundedness condition on the static uncertainties considered, for instance, in [C], [K], and [Q2]. After stating our assumptions in Section 2, we propose, in Section 3, a first control law which solves the problem. It is a static feedback but, as already mentioned, it exhibits a high-gain feature. This feature has been found useful in solving some problems in robust control (see, for instance, [BCL] and [SK]) but it may also be undesirable in other situations. This motivates our propositon of a second controller in Section 4. Our two controllers are compared for a simplistic example in Section 5. In Section 6 we propose a framework allowing us to relax somehow the assumptions made in Section 2. In fact, to prove that our second controller provides the closed-loop system with properties similar to the ones given by the first one, we need to restrain the class of unmodeled dynamics. Nevertheless, in the case $c_i(x, z, u) = z$, i.e., the disturbance is the state of the unmodeled dynamics itself, we prove in Section 7 that there is in fact no restriction.

2. Assumptions

We assume the nominal system is globally asymptotically stabilizable and, more precisely:

A1. We know a C^1 positive definite function V satisfying, for all x,

$$V(x) \geq \alpha_1(|x|), \tag{2}$$

for some function α_1 of class[1] $\mathscr{K}_\infty$, and a C^0 feedback law $u_n(x)$ with $u_n(0) = 0$,[2] such that the function[3]

$$W(x) = -L_{[f+Gu]}V(x) \tag{3}$$

is also positive definite.

According to [S1], if there exists u_n satisfying Assumption A1, then the following

[1] For the definitions of class $\mathscr{K}$, $\mathscr{K}_\infty$, and $\mathscr{KL}$ functions see [H].

[2] Assuming $u(0) = 0$ can be done without loss of generality as far as the nominal system is concerned. Indeed, if $u(0) = u_0 \neq 0$, it is sufficient to replace f by $\hat{f} = f + Gu_0$ and u by $\hat{u} = u - u_0$.

[3] $L_f V$ is the Lie derivative of V along f and $L_G V$ is the row vector $(L_{g_i} V)$.

feedback u_s also globally asymptoically stabilizes the nominal system

$$u_s(x) = \begin{cases} -\dfrac{L_f V(x) + \sqrt{L_f V(x)^2 + \|L_G V(x)\|^4}}{\|L_G V(x)\|^2} L_G V(x), & \text{if} \quad L_G V(x) \neq 0, \\ 0, & \text{if} \quad L_G V(x) = 0, \end{cases} \tag{4}$$

where $\|\cdot\|$ denotes the usual Euclidian norm of $\mathbf{R}^p$. With this control we get the following positive definite function:

$$W_s(x) = -L_{[f+Gu]} V(x). \tag{5}$$

The interest of this particular feedback is that we have, for all x,

$$L_G V(x) \neq 0 \quad \Rightarrow \quad L_G V(x) u_s(x) < 0. \tag{6}$$

A2. The z-subsystem of (1) with input (x, u) and output c is BIBS and IOpS. That is:

BIBS. For each initial condition $z(0)$ and each measurable essentially bounded function (x, u): $\mathbf{R}_{\geq 0} \to \mathbf{R}^n \times \mathbf{R}^p$, the corresponding solution $z(t)$ is defined and bounded on $\mathbf{R}_{\geq 0}$.

IOpS. There exist a function β_c of class $\mathscr{KL}$, two functions γ_u and γ_x of class $\mathscr{K}$, and a positive real number c_0 such that, for each initial condition $z(0)$ and each measurable essentially bounded function (x, u): $\mathbf{R}_{\geq 0} \to \mathbf{R}^n \times \mathbf{R}^p$, the corresponding solution satisfies, for all t in $\mathbf{R}_{\geq 0}$,[4]

$$|c(t)| \leq c_0 + \beta_c(|z(0)|, t) + \gamma_u(U(t)) + \gamma_x\left(\sup_{\tau \in [0,t)} \{|x(\tau)|\}\right), \tag{7}$$

where

$$U(t) = \sup_{\tau \in [0,t)} \{|u(\tau)|\}, \tag{8}$$

and for each vector v in $\mathbf{R}^p$, $|v|$ denotes $\max\{|v_1|, \ldots, |v_p|\}$, and similarly for vectors in $\mathbf{R}^n$.

With Assumption A2, we are able to obtain a control law whose design is based on only the fact that inequality (7) holds. However, Krstić and Kokotović have noticed in [KK] that a better performance can be obtained if more *a priori* knowledge on c is used, namely that the last two terms in the right-hand side of (7) can be evaluated on line and therefore used in the control law. Unfortunately such terms involve U which is the output of an infinite-dimensional system with u as input. To overcome this difficulty, we remark that Assumption A2 could apply to systems with dynamics involving mathematical objects more complex than the

[4] For the sake of simplicity, here and throughout the paper we make the following abuse of notation: sup is to be taken as the essential supremum norm and "for all t" should be "for almost all t with respect to the Lebesgue measure."

system

$$\begin{cases} \dot{z} = a(x, z, u), \\ y = c(x, z, u). \end{cases} \tag{9}$$

In particular, when the initial condition $z(0)$ is fixed, this system provides operators $u \mapsto z$ and $u \mapsto y$ which are finite-dimensional, the former being strictly proper, whereas, in (8), the operator $u \mapsto U$ is only proper and infinite-dimensional. From this, we conjecture that the restriction of Assumption A2 to systems in the particular form (9) should give a stronger property. These arguments lead us to restrain Assumption A2 by replacing the infinite-dimensional operator $\sup_{\tau \in [0,t)} \{\cdot\}$ by a first-order one. This yields:

A2′. The z-subsystem with input u and output c is BIBS and exp-IOpS. That is:

exp-IOpS. For some positive real number μ and some functions γ_{vx}, γ_{vu}, γ_{cx}, and γ_c of class $\mathcal{K}$, there exist a positive real number c_0, a function γ_{cu} of class $\mathcal{K}$, and a function β of class $\mathcal{KL}$, such that, for each initial condition $z(0)$ and for each measurable essentially bounded function (x, u): $\mathbf{R}_{\geq 0} \to \mathbf{R}^n \times \mathbf{R}^p$, the corresponding solution satisfies, for all t in $\mathbf{R}_{\geq 0}$,

$$|c(t)| \leq c_0 + \beta(|z(0)|, t) + \gamma_{cu}(|u(t)|) + \gamma_{cx}(|x(t)|) + \gamma_c(r(t)), \tag{10}$$

where the function $r(t)$ satisifes the following equations:

$$\dot{r} = -\mu r + \gamma_{vu}(|u|) + \gamma_{vx}(|x|), \qquad r(0) = 0. \tag{11}$$

The main difference between IOpS and exp-IOpS is that in (10), through r, an exponentially weighted L^1 norm is used instead of the L^∞ one expressed in U. Clearly, exp-IOpS implies IOpS with the gains of the relations $u \mapsto c$ and $x \mapsto c$ given by

$$\gamma_u = \gamma_{cu} + \gamma_c \circ \frac{1}{\mu}\gamma_{vu}, \qquad \gamma_x = \gamma_{cx} + \gamma_c \circ \frac{1}{\mu}\gamma_{vx}, \tag{12}$$

respectively. However, the previous arguments let us expect that the converse may be true. This is proved in Section 7 for the case when $c = z$.

Assumption A2′ is strongly related with Assumption UEC (73) in [JP1]. From this relation, we note

- —Lemma 1 of [JP1] is a helpful tool for selecting the real number μ and the functions γ_{vx}, γ_{vu}, γ_{cx}, and γ_c.
- —The equations in (11) provide us with r as a pseudostate for the stability analysis. In the proof of Proposition 1 of [JP1], it is shown that the proposition is well suited for the application of Lyapunov's second method.
- —To help the reader get a better understanding of the meaning of this signal r, we refer to Property 1 of [P].

We remark that Assumptions A1 and A2 or A2′ are not sufficient for guaranteeing the existence of a feedback law solving our problem. Consider the system

$$\begin{cases} \dot{x} = x^2 - (u - \gamma(z))x, \\ \dot{z} = (u - z)z^2, \end{cases} \tag{13}$$

where γ is a smooth odd function satisfying

$$\operatorname{sign}(r)[r - \gamma(r)] \leq M, \qquad \forall r \in \mathbf{R}, \tag{14}$$

for some positive real number M. This condition says roughly that the function γ grows at least as much as the identity function. Assumption A1 holds, with

$$V(x) = \tfrac{1}{2}x^2, \qquad u_n(x) = x + x^2, \tag{15}$$

and Assumption A2 also holds since the z-subsystem with input u and output $\gamma(z)$ is IOS with $\gamma_x \equiv 0$ and γ_u, any gain function of class $\mathscr{K}$ strictly greater than γ. However, system (13) is not asymptotically controllable. We prove in Appendix A that there is no control law $u(t)$ that can drive to zero the x-component of any solution starting from $(x_0, 1)$ with $x_0 > M \exp(1)$.

This example shows that it is in general impossible to solve the problem stated in Section 1 if the function $\mathrm{Id} - \gamma_u$ is bounded.

3. First Solution with a Static Feedback

Proposition 1. *Assume Assumptions* A1 *and* A2 *hold. Under this condition, for any functions κ_u and κ_x of class $\mathscr{K}_\infty$, there exists a continuous feedback law $\omega(x)$ such that all the solutions of the closed-loop system* (1) *are bounded provided that we have*

$$(\mathrm{Id} + \rho_2) \circ [\gamma_u \circ (\mathrm{Id} + \rho_1) \circ (\mathrm{Id} + \kappa_u) + \gamma_x \circ (\mathrm{Id} + \rho_1) \circ \kappa_x] \leq \mathrm{Id} \tag{16}$$

for some functions ρ_1 and ρ_2 of class $\mathscr{K}_\infty$. Moreover, for each closed-loop solution, we have

$$\limsup_{t \to +\infty} |x(t)| \leq \kappa_x \circ (\mathrm{Id} + \rho_2^{-1})(c_0). \tag{17}$$

Remark 2. If, in (7), $c_0 = 0$, that is, when the z-subsystem of (1) is IOS with (x, u) as input and c as output, then it follows immediately from (17) that

$$\lim_{t \to +\infty} |x(t)| = 0. \tag{18}$$

If $c_0 \neq 0$, since κ_x can be chosen as an arbitrarily small function of class $\mathscr{K}_\infty$, (16) is mainly a condition on γ_u and (17) gives a practical convergence result. In fact, it can be shown that, for any given functions γ_u and γ_x of class $\mathscr{K}$, satisfying

$$\mathrm{Id} - \gamma_u > \rho_0, \tag{19}$$

for some function ρ_0 of class $\mathscr{K}_\infty$, we can find functions ρ_1, ρ_2, κ_u, and κ_x of class $\mathscr{K}_\infty$ such that (16) holds. Note also that we do not claim stability in the proposition.

Proposition 1 is established by showing first the existence of a continuous feedback law $\omega(x)$ assigning appropriate gains to the system

$$\dot{x} = f(x) + G(x)[\omega(x) + c] \tag{20}$$

with c as input and $(x, \omega(x))$ as output. The conclusion then follows from the Small-Gain Theorem [JTP, Theorem 2.1].

Lemma 3 (Strong Gain Assignment Theorem). *Assume Assumption* A1 *holds. Then, for any functions κ_u and κ_x of class $\mathcal{K}_\infty$, there exists a continuous feedback law $\omega(x)$ and functions β_u and β_x of class $\mathcal{KL}$, such that, for each initial condition $x(0)$ and for each measurable essentially bounded function c: $\mathbf{R}_{\geq 0} \to \mathbf{R}^p$, the corresponding solutions of*

$$\dot{x} = f(x) + G(x)[\omega(x) + c(t)] \tag{21}$$

satisfy, for all $0 \leq s \leq t$,

$$|\omega(x(t))| \leq \beta_u(|x(s)|, t - s) + (\mathrm{Id} + \kappa_u)\left(\sup_{\tau \in [s,t)} \{|c(\tau)|\}\right), \tag{22}$$

$$|x(t)| \leq \beta_x(|x(s)|, t - s) + \kappa_x\left(\sup_{\tau \in [s,t)} \{|c(\tau)|\}\right). \tag{23}$$

This result is to be compared with Theorem 2.2 of [JTP]. We have here a stronger statement since not only can any gain be assigned to the relation $c \mapsto x$ but we can also limit the gain of the relation $c \mapsto \omega$.

Proof of Lemma 3. Let V and α_1 be the functions as in Assumption A1, and let $\dot{V}_{(21)}$ denote the function

$$\dot{V}_{(21)}(x, t) = \frac{\partial V}{\partial x}(x)(f(x) + G(x)[\omega(x) + c(t)]). \tag{24}$$

With Assumption A1, we have

$$\dot{V}_{(21)}(x, t) \leq -W(x) + L_G V(x)\omega(x) - L_G V(x)u_n(x) + L_G V(x)c(t). \tag{25}$$

We restrict our attention to feedback laws ω of the form

$$\omega_i(x) = -\mathrm{sign}(L_{g_i}V(x))\hat{\omega}_i(x), \qquad \hat{\omega}_i(x) \geq 0, \quad i = 1, 2, \ldots, p, \tag{26}$$

where the functions $\hat{\omega}_i$ are defined below. This yields

$$\dot{V}_{(21)}(x, t) \leq -W(x) - \sum_{i=1}^{p} |L_{g_i}V(x)|(\hat{\omega}_i(x) - |u_{n_i}(x)| - |c(t)|). \tag{27}$$

To define $\hat{\omega}_i$, we let $\mathcal{S}$ be a function of class $\mathcal{K}_\infty$ such that, for all s and x, we have

$$\kappa_u^{-1}(|u_n(x)|) \leq \mathcal{S}(V(x)), \qquad \kappa_x^{-1} \circ \alpha_1^{-1}(s) \leq \mathcal{S}(s). \tag{28}$$

Such a function exists since V is positive definite and proper. Then we choose $\hat{\omega}_i$ as

$$\hat{\omega}_i(x) = \theta_i(x)\bar{b}_i(x), \tag{29}$$

where

$$\bar{b}_i(x) = |u_{n_i}(x)| + \mathcal{S}(V(x)) \tag{30}$$

and θ_i is a function introduced to enforce continuity and defined as follows: For each i, let

$$\mathcal{B}_{0i} = \{x\colon L_{g_i}V(x) = 0, x \neq 0\} \tag{31}$$

and

$$\mathscr{B}_{1i} = \left\{x\colon |L_{g_i}V(x)|(\mathscr{S}(V(x)) + |u_{n_i}(x)|) \geq \frac{W(x)}{2p},\ x \neq 0\right\}. \tag{32}$$

Since W is positive definite, $\mathscr{B}_{0i}$ and $\mathscr{B}_{1i}$ are closed and disjoint subsets of $\mathbf{R}^n\backslash\{0\}$. It follows that we can define this function $\theta_i\colon \mathbf{R}^n\backslash\{0\} \to [0, 1]$ as a continuous function satisfying (see Appendix B for an explicit expression of such a function)

$$\theta_i(x) = \begin{cases} 1, & \text{if} \quad x \in \mathscr{B}_{1i}, \\ 0, & \text{if} \quad x \in \mathscr{B}_{0i}. \end{cases} \tag{33}$$

To obtain a definition of θ_i on $\mathbf{R}^n$ we simply add $\theta_i(0) = 0$. Then, though θ_i may fail to be continuous at zero, the function $\hat{\omega}$ is continuous on $\mathbf{R}^n$ since $\bar{b}_i(0) = 0$. Hence, from (27), we get

$$\begin{aligned} \dot{V}_{(21)}(x, t) &\leq -W(x) - \sum_{i=1}^{p} |L_{g_i}V(x)|[\theta_i(x)(\mathscr{S}(V(x)) + |u_{n_i}(x)|) - |u_{n_i}(x)| - |c(t)|] \\ &\leq -W(x) + \sum_{i=1}^{p} |L_{g_i}V(x)|(1 - \theta_i(x))(\mathscr{S}(V(x)) + |u_{n_i}(x)|) \\ &\quad - \left(\sum_{i=1}^{p} |L_{g_i}V(x)|\right)(\mathscr{S}(V(x)) - |c(t)|) \\ &\leq -\frac{W(x)}{2} - \left(\sum_{i=1}^{p} |L_{g_i}V(x)|\right)(\mathscr{S}(V(x)) - |c(t)|). \end{aligned} \tag{34}$$

From this latter inequality, by using the fact that W is positive definite, V and $\mathscr{S}$ are positive definite and proper and following the same lines as in the Claims on p. 441 in [S2], we can show the existence of a function β_v of class $\mathscr{KL}$ such that, for all $0 \leq s \leq t$, we have

$$V(x(t)) \leq \max\left\{\beta_v(V(x(s)), t - s),\ \mathscr{S}^{-1}\left(\sup_{\tau \in [s,t)} \{|c(\tau)\}|\right)\right\}. \tag{35}$$

Inequality (23) follows readily with (28) and (2). Then, since we have

$$|\omega(x)| \leq (\mathrm{Id} + \kappa_u) \circ \mathscr{S}(V(x)), \tag{36}$$

the conclusion follows. ■

Remark 4. If instead of using u_n, we use u_s satisfying (6), the control law ω can be made simpler by modifying (29) and (30) so that

$$\omega_i(x) = u_{s_i}(x) - \theta_i(x)\,\mathrm{sign}(L_{g_i}V(x))\mathscr{S}(V(x)), \tag{37}$$

and $\mathscr{B}_{1i}$ in (32) into

$$\mathscr{B}_{1i} = \left\{x\colon |L_{g_i}V(x)|\,\mathscr{S}(V(x)) \geq \frac{W_s(x)}{2p},\ x \neq 0\right\}. \tag{38}$$

Remark 5. The control law ω can be made smooth if the addition of arbitrarily small positive numbers to the right-hand side of (22) and (23) are allowed (see

[JTP]). More specifically, for any $\varepsilon_0 > 0$, each $\omega_i(x)$ can always be approximated by a smooth function $\tilde{\omega}_i(x)$ so that, for all $x \in \mathbf{R}^n$, we have

$$|\omega_i(x) - \tilde{\omega}_i(x)| < \varepsilon_0. \tag{39}$$

However, with such a choice of $\tilde{\omega}_i$,

$$L_{g_i} V(x)\tilde{\omega}_i(x) \leq 0, \qquad \forall x \in \mathbf{R}^n, \tag{40}$$

may fail to hold.

To obtain a smooth feedback $\tilde{\omega}_i$ satisfying restriction (40), we proceed as follows: For each $m \in \{1, \ldots, m\}$, we let $\mathscr{B}_{2i}$ denote the open subset of $\mathbf{R}^n$ where $\omega_i(x) \neq 0$. We define

$$\sigma_i(x) = \min\left\{\frac{|\omega_i(x)|}{2}, \frac{\varepsilon_0}{2}\right\}, \tag{41}$$

so that $\sigma_i(x) > 0$ for all $x \in \mathscr{B}_{2i}$. Hence, there exists a function $\overline{\omega}_i(x)$ that is smooth on $\mathscr{B}_{2i}$ and such that

$$|\overline{\omega}_i(x) - \omega_i(x)| < \sigma_i(x) \tag{42}$$

for all $x \in \mathscr{B}_{2i}$ (see Theorem 4.8, p. 197, of [B]). The domain of $\overline{\omega}_i$ can then be extended to $\mathbf{R}^n$ by letting $\overline{\omega}_i(x) = 0$ for $x \notin \mathscr{B}_{2i}$. Note then that $\overline{\omega}_i(x)$ is continuous everywhere, and, for all $x \in \mathbf{R}^n$,

$$\overline{\omega}_i(x)\omega_i(x) \geq 0. \tag{43}$$

Now we let $\bar{\theta}_i(x)\colon \mathbf{R}^n \to [0, 1]$ be a smooth function satisfying the following:

$$\bar{\theta}_i(x) = \begin{cases} 0, & \text{if } x \in \mathscr{B}_{3i}, \\ 1, & \text{if } x \in \mathscr{B}_{4i}, \end{cases} \tag{44}$$

where the two sets $\mathscr{B}_{3i}$ and $\mathscr{B}_{4i}$ are defined by

$$\mathscr{B}_{3i} = \left\{x \in \mathbf{R}^n\colon |\overline{\omega}_i(x)| \leq \frac{\varepsilon_0}{4}\right\}, \qquad \mathscr{B}_{4i} = \left\{x \in \mathbf{R}^n\colon |\overline{\omega}_i(x)| \geq \frac{\varepsilon_0}{2}\right\}. \tag{45}$$

As before, such a smooth function exists because $\mathscr{B}_{3i}$ and $\mathscr{B}_{4i}$ are two disjoint closed subsets of $\mathbf{R}^n$. Finally we let

$$\tilde{\omega}_i(x) = \bar{\theta}_i(x)\overline{\omega}_i(x). \tag{46}$$

Then $\tilde{\omega}_i$ is smooth everywhere, and, for all $x \in \mathbf{R}^n$,

$$\tilde{\omega}_i(x)L_{g_i} V(x) \leq 0, \qquad |\tilde{\omega}_i(x) - \omega_i(x)| < \varepsilon_0. \tag{47}$$

Consequently, when the controls $\tilde{\omega}_i$'s are used instead of the ω_i's, (34) becomes

$$\dot{V}_{(21)}(x, t) \leq -\frac{W(x)}{2} - \left(\sum_{i=1}^{p} |L_{g_i} V(x)|\right)(\mathscr{S}(V(x)) - |c(t)| - \varepsilon_0). \tag{48}$$

It follows that (22) and (23) are replaced by

$$|\tilde{\omega}(x(t))| \leq \beta_u(|x(s)|, t-s) + (\mathrm{Id} + \kappa_u)\left(\sup_{\tau\in[s,t)} \{c(\tau)\} + \varepsilon_0\right) + p\varepsilon_0$$
$$\leq \beta_u(|x(s)|, t-s) + (\mathrm{Id} + \kappa_u)\left(\sup_{\tau\in[s,t)} \{\tilde{c}(\tau)\}\right) \tag{49}$$
$$|x(t)| \leq \beta_x(|x(s)|, t-s) + \kappa_x\left(\sup_{\tau\in[s,t)} \{\tilde{c}(\tau)\}\right), \tag{50}$$

where $\tilde{c} = |c| + (p+1)\varepsilon_0$.

Proof of Proposition 1. By applying Lemma 3 we get a continuous feedback law $\omega(x)$ which, when applied to (1), gives a closed-loop system which can be seen as the interconnection

$$\dot{x} = f(x) + G(x)[\omega(x) + y_1], \qquad y_1 = c(x, z, \omega(x)), \tag{51}$$
$$\dot{z} = a(y_{21}, z, y_{22}), \qquad y_{21} = x,\ y_{22} = \omega(x), \tag{52}$$

where, from (7), (22), and (23),

$$|y_1(t)| \leq c_0 + \beta_c(|z(0)|, t) + \gamma_u\left(\sup_{\tau\in[0,t)} \{|y_{22}(\tau)|\}\right) + \gamma_x\left(\sup_{\tau\in[0,t)} \{|y_{21}(\tau)|\}\right), \tag{53}$$
$$|y_{22}(t)| \leq \beta_u(|x(0)|, t) + (\mathrm{Id} + \kappa_u)\left(\sup_{\tau\in[0,t)} \{|y_1(\tau)|\}\right), \tag{54}$$
$$|y_{21}(t)| \leq \beta_x(|x(0)|, t) + \kappa_x\left(\sup_{\tau\in[0,t)} \{|y_1(\tau)|\}\right). \tag{55}$$

To conclude we could apply Theorem 2.1 of [JTP] if:

—the function $\omega(x)$ were locally Lipschitz,
—we would have a one channel interconnection instead of the two channels given by y_{21} and y_{22}.

Nevertheless, if the statement of this theorem is not exactly appropriate, we can follow its proof line by line. First we can show with (16) that the outputs corresponding to any solutions are bounded on their maximal interval of definition. In particular, we have (see (80) of [JTP])

$$|y_1(t)| \leq c_0 + \beta_c(|z(0)|, t) + \gamma_u\left(\beta_u(|x(0)|, 0) + (\mathrm{Id} + \kappa_u)\left(\sup_{\tau\in[0,t)} \{|y_1(\tau)|\}\right)\right)$$
$$+ \gamma_x\left(\beta_x(|x(0)|, 0) + \kappa_x\left(\sup_{\tau\in[0,t)} \{|c(\tau)|\}\right)\right). \tag{56}$$

With (16), this yields (see (83) of [JTP])

$$\sup_{\tau\in[0,t)} \{|y_1(\tau)|\} \leq (\mathrm{Id} + \rho_2^{-1})(\beta_c(|z(0)|, 0) + \gamma_u \circ (\mathrm{Id} + \rho_1^{-1})(\beta_u(|x(0)|, 0))$$
$$+ \gamma_x \circ (\mathrm{Id} + \rho_1^{-1})(\beta_x(|x(0)|, 0)) + c_0). \tag{57}$$

With the BIBS property of both subsystems, this implies that all the solutions are defined and bounded on $\mathbf{R}_{\geq 0}$. This means that, for each $(x(t), z(t))$, there exists a positive real number s_∞ so that, for all t in $\mathbf{R}_{\geq 0}$, we have

$$|(x(t), z(t))| \leq s_\infty. \tag{58}$$

Second, we obtain, for all t in $\mathbf{R}_{\geq 0}$ (see (93) of [JTP]),

$$\begin{aligned} |y_1(t)| \leq & \left[\beta_c\left(s_\infty, \frac{t}{2}\right) + \gamma_u \circ (\mathrm{Id} + \rho_1^{-1}) \circ \beta_u\left(s_\infty, \frac{t}{4}\right) + \gamma_x \circ (\mathrm{Id} + \rho_1^{-1}) \circ \beta_x\left(s_\infty, \frac{t}{4}\right)\right] \\ & + (\mathrm{Id} + \rho_2)^{-1}\left(\sup_{\tau \in [t/4, \infty)} \{|y_1(\tau)|\}\right) + c_0. \end{aligned} \tag{59}$$

So, with Lemma A.1 of [JTP], for any function ρ_3 of class $\mathscr{K}_\infty$, we know the existence of a function $\hat{\beta}$ of class $\mathscr{KL}$ such that we have, for all t in $\mathbf{R}_{\geq 0}$,

$$|y_1(t)| \leq \hat{\beta}(s_\infty, t) + (\mathrm{Id} + \rho_2^{-1}) \circ (\mathrm{Id} + \rho_3)(c_0). \tag{60}$$

Since, with (58) and (52), (23) gives

$$|x(t)| \leq \beta_x\left(s_\infty, \frac{t}{2}\right) + \kappa_x\left(\sup_{\tau \in [t/2, \infty)} \{|y_1(\tau)|\}\right), \tag{61}$$

it follows readily that

$$\limsup_{t \to \infty} |x(t)| \leq \kappa_x \circ (\mathrm{Id} + \rho_2^{-1}) \circ (\mathrm{Id} + \rho_3)(c_0), \tag{62}$$

for *any* function ρ_3 of class $\mathscr{K}_\infty$. However, the solution $(x(t), z(t))$ is independent of ρ_3, this implies (17). ∎

4. Second Solution with a Dynamic Feedback

The solution we have proposed in the previous section relies on the use of high gain. This fact is hidden in the choice of the function $\mathscr{S}$ which has to be sufficiently large and not only for small values. This may lead to problems if other robustness problems are considered. What leads to high gain in the previous approach is the use of the matching assumption and a worst-case design. By using more *a priori* knowledge on the unmodeled dynamics it may be hoped that high gain be involved in a different way. To this purpose, we incorporate Assumption A2′ in the following result.

Proposition 6. *Assume Assumption* A1 *holds with* W *a proper function,*[5] *i.e., precisely*

$$\alpha_3(V(x)) \leq \tfrac{1}{2} W(x), \tag{63}$$

[5] With Assumption A1, we can always modify the function V to meet this requirement (see Proposition 13, for instance).

where α_3 is some function of class $\mathcal{K}_\infty$. We choose a real number μ, functions γ_{vu} and γ_c of class $\mathcal{K}$, and a function κ_1 of class $\mathcal{K}_\infty$ so that

$$\gamma_{vu} \circ (\mathrm{Id} + \rho_4) \circ \kappa_1^{-1} \circ (\mathrm{Id} + \rho_4) \circ \gamma_c \leq \mu\, \mathrm{Id} - \rho_5 \tag{64}$$

for some functions ρ_4 and ρ_5 of class $\mathcal{K}_\infty$. We assume that, with such a choice, Assumption A2′ holds with a function γ_{cu} satisfying

$$\gamma_{cu} \leq \mathrm{Id} - \kappa_1. \tag{65}$$

Under these conditions, for any functions κ_2, κ_3, and κ_4 of class $\mathcal{K}_\infty$, there exists a continuous dynamic feedback law $\omega(x, r)$ with r given by (11) such that all the solutions of the closed-loop system are bounded and their x-components satisfy

$$\limsup_{t \to +\infty} |x(t)| \leq \alpha_1^{-1} \circ (\mathrm{Id} + \kappa_3^{-1}) \circ \alpha_3^{-1} \circ (\mathrm{Id} + \kappa_4^{-1})(c_0 \kappa_2(c_0)). \tag{66}$$

Remark 7. When $c_0 = 0$, we get convergence of the x-component:

$$\lim_{t \to +\infty} |x(t)| = 0. \tag{67}$$

When $c_0 \neq 0$, since κ_2, κ_3^{-1}, and κ_4^{-1} can be chosen as arbitrarily small functions of class $\mathcal{K}_\infty$, (66) gives a practical convergence result.

Proof. First, we remark, with (11), that $r(t)$ is nonnegative for any t in $\mathbf{R}_{\geq 0}$. Then we follow the same lines as for the proof of Lemma 3. Let $\dot{V}_{(1)}$ denote the function

$$\dot{V}_{(1)}(x, u, t) = \frac{\partial V}{\partial x}(x)(f(x) + G(x)(u + c(t))), \tag{68}$$

and let ω be chosen of the form

$$\omega_i = -\operatorname{sign}(L_{g_i} V(x))\hat{\omega}_i, \qquad \hat{\omega}_i \geq 0, \quad i = 1, 2, \ldots, p, \tag{69}$$

with functions $\hat{\omega}_i$ defined below. With (10), we obtain

$$\begin{aligned} \dot{V}_{(1)}(x, \omega, t) \leq {} & -W(x) - \sum_{i=1}^{p} |L_{g_i} V|(\hat{\omega}_i - |u_{n_i}(x)|) \\ & + \sum_{i=1}^{p} |L_{g_i} V|(c_0 + \beta(|z(0)|, t) + \gamma_{cu}(|\omega|) + \gamma_{cx}(|x|) + \gamma_c(r)). \end{aligned} \tag{70}$$

To define the functions $\hat{\omega}_i$, let κ_2 be the function of class $\mathcal{K}_\infty$ chosen in Proposition 6. We also choose a function l of class $\mathcal{K}$ and bounded by l_∞. Since, for any positive real numbers a, b, we have

$$ab \leq \frac{\kappa_2(b)b}{p} + \kappa_2^{-1}(pa)a \tag{71}$$

(consider two cases: $a \leq \kappa_2(b)/p$ and $a \geq \kappa_2(b)/p$), we obtain

$$\dot{V}_{(1)}(x, \omega, t) \leq -W(x) - \sum_{i=1}^{p} |L_{g_i} V|(\hat{\omega}_i - \gamma_{cu}(|\omega|) - \bar{b}_i(x) - \gamma_c(r)) + v_0(t)\kappa_2(v_0(t)), \tag{72}$$

where $v_0(t)$ is defined as

$$v_0(t) = \beta(|z(0)|, t) + c_0, \tag{73}$$

and, for each i,

$$\bar{b}_i(x) = |u_{n_i}(x)| + \gamma_{cx}(|x|) + \kappa_2^{-1}(p|L_{g_i}V(x)|). \tag{74}$$

Let

$$\bar{b}(x) = \max_{j \in \{1,\ldots,p\}} \{\bar{b}_j(x)\}. \tag{75}$$

We define $\hat{\omega}_i$ as

$$\hat{\omega}_i(x, r) = \theta_i(x, r)\kappa_1^{-1}(\bar{b}(x) + \gamma_c(r)), \tag{76}$$

where κ_1 is chosen in Proposition 6. As in (29), the function θ_i: $\mathbf{R}^n \times \mathbf{R} \to [0, 1]$ is introduced to enforce the continuity. It is defined as follows: For each i, let

$$\mathscr{B}_{0i} = \{(x, r)\text{: } L_{g_i}V(x) = 0, (x, r) \neq (0, 0)\} \tag{77}$$

and

$$\mathscr{B}_{1i} = \left\{(x, r)\text{: } |L_{g_i}V(x)|\kappa_1^{-1}(\bar{b}(x) + \gamma_c(r)) \geq \frac{W(x)}{2p} + \frac{l(r)}{p}, (x, r) \neq (0, 0)\right\}. \tag{78}$$

Since both W and l are positive definite, it follows that $\mathscr{B}_{0i}$ and $\mathscr{B}_{1i}$ are disjoint closed subsets of $\mathbf{R}^n \times \mathbf{R}\backslash\{(0, 0)\}$. Then we can define a continuous function θ_i: $\mathbf{R}^n \times \mathbf{R}\backslash\{(0, 0)\} \to [0, 1]$ such that

$$\theta_i(x, r) = \begin{cases} 1, & \text{if } x \in \mathscr{B}_{1i}, \\ 0, & \text{if } x \in \mathscr{B}_{0i}. \end{cases} \tag{79}$$

To obtain a definition of θ_i on $\mathbf{R}^n \times \mathbf{R}$ we simply let $\theta_i(0, 0) = 0$. Although θ_i may fail to be continuous at $(0, 0)$, the functions $\hat{\omega}_i$ and ω_i are continuous on $\mathbf{R}^n \times \mathbf{R}$ since $\bar{b}(0) = \gamma_c(0) = 0$.

Now, since condition (65) implies

$$\begin{aligned} \hat{\omega}_i(x, r) - \gamma_{cu}(|\omega(x, r)|) &\geq (-(1 - \theta_i(x, r))\mathrm{Id} + \mathrm{Id} - \gamma_{cu}) \circ \kappa_1^{-1}(\bar{b}(x) + \gamma_c(r)) \\ &\geq -(1 - \theta_i(x, r))\kappa_1^{-1}(\bar{b}(x) + \gamma_c(r)) + \bar{b}(x) + \gamma_c(r), \end{aligned} \tag{80}$$

inequality (72) becomes, with (63), (76), (75), and the definition of θ_i,

$$\begin{aligned} \dot{V}_{(1)}(x, \omega, t) &\leq -W(x) + \sum_{i=1}^{p} |L_{g_i}V|(1 - \theta_i(x, r))\kappa_1^{-1}(\bar{b}(x) + \gamma_c(r)) + v_0(t)\kappa_2(v_0(t)) \\ &\leq -\tfrac{1}{2}W(x) + v_0(t)\kappa_2(v_0(t)) + l(r) \\ &\leq -\alpha_3(V(x)) + v_0(t)\kappa_2(v_0(t)) + l(r). \end{aligned} \tag{81}$$

On the other hand, with the control law given by (76), equations (11) imply

$$\dot{r} \leq -\mu r + \gamma_{vu} \circ \kappa_1^{-1}(\bar{b}(x) + \gamma_c(r)) + \gamma_{vx}(|x|). \tag{82}$$

With condition (64), it follows that

$$\begin{aligned}\mu r - \gamma_{vu} \circ \kappa_1^{-1}(\bar{b}(x) + \gamma_c(r)) &\geq \mu r - \gamma_{vu}(\kappa_1^{-1} \circ (\mathrm{Id} + \rho_4) \circ \gamma_c(r) + \kappa_1^{-1} \circ (\mathrm{Id} + \rho_4^{-1})(\bar{b}(x))) \\ &\geq \mu r - \gamma_{vu} \circ (\mathrm{Id} + \rho_4) \circ \kappa_1^{-1} \circ (\mathrm{Id} + \rho_4) \circ \gamma_c(r) \\ &\quad - \gamma_{vu} \circ (\mathrm{Id} + \rho_4^{-1}) \circ \kappa_1^{-1} \circ (\mathrm{Id} + \rho_4^{-1})(\bar{b}(x)) \\ &\geq \rho_5(r) - \gamma_{vu} \circ (\mathrm{Id} + \rho_4^{-1}) \circ \kappa_1^{-1} \circ (\mathrm{Id} + \rho_4^{-1})(\bar{b}(x)). \end{aligned} \tag{83}$$

Let ρ_6 be a function of class $\mathscr{K}_\infty$ satisfying

$$\gamma_{vx}(|x|) + \gamma_{vu} \circ (\mathrm{Id} + \rho_4^{-1}) \circ \kappa_1^{-1} \circ (\mathrm{Id} + \rho_4^{-1})(\bar{b}(x)) \leq \rho_6(V(x)). \tag{84}$$

Such a function exists since (2) holds for all x. With (81), we have finally obtained the following system of differential inequalities:

$$\begin{cases} \dot{V}_{(1)}(x, \omega, t) \leq -\alpha_3(V(x)) + v_0(t)\kappa_2(v_0(t)) + l(r) \\ \qquad\qquad\qquad \leq -\alpha_3(V(x)) + v_0(t)\kappa_2(v_0(t)) + l_\infty, \\ \qquad\qquad \dot{r} \leq -\rho_5(r) + \rho_6(V(x)). \end{cases} \tag{85}$$

Now let $(x(t), r(t), z(t))$ be a solution of the closed-loop system (1), (11), (69), (76). Such a solution exists for any initial condition $(x(0), z(0))$ and has a right maximal interval of definition $[0, T)$. However, since α_3 is of class $\mathscr{K}_\infty$, V is proper, v_0 and l are bounded, and (85) implies that $x(t)$ is bounded on $[0, T)$. This, with the fact that ρ_5 is of class $\mathscr{K}_\infty$, implies that $r(t)$ is also bounded on $[0, T)$. It follows that the control

$$u(t) = \omega(x(t), r(t)) \tag{86}$$

is bounded on $[0, T)$. So with the BIBS property of the z subsystem, $z(t)$ is bounded on $[0, T)$. Hence, by contradiction, it is shown that the solution is defined and bounded on $\mathbf{R}_{\geq 0}$, i.e., for all t in $\mathbf{R}_{\geq 0}$,

$$\|(V(x(t)), r(t), z(t))\| \leq s_\infty < +\infty. \tag{87}$$

Also, as in the proof of Proposition 1, by following the same lines as in the Claims on p. 441 in [S2], for any functions ρ_v and ρ_r of class $\mathscr{K}_\infty$, with

$$\rho_v \leq \mathrm{Id}, \tag{88}$$

we can show the existence of class $\mathscr{KL}$ functions β_v and β_r such that, for all $0 \leq s \leq t$, we have

$$V(x(t)) \leq \beta_v(V(x(s)), t - s) + \alpha_3^{-1} \circ (\mathrm{Id} + \rho_v)\left(\sup_{\tau \in [s, t)} \{y_r(\tau)\} \right), \tag{89}$$

$$r(t) \leq \beta_r(r(s), t - s) + \rho_5^{-1} \circ 2\rho_6\left(\sup_{\tau \in [s, t)} \{V(x(\tau))\} \right), \tag{90}$$

where we have introduced the function

$$y_r(t) = v_0(t)\kappa_2(v_0(t)) + l(r(t)). \tag{91}$$

However, since

$$\limsup_{t\to+\infty} y_r(t) \le c_0\kappa_2(c_0) + l_\infty, \tag{92}$$

by taking $s = t/2$ in (89) and using (87), we conclude readily that

$$\limsup_{t\to\infty} V(x(t)) \le \alpha_3^{-1} \circ (\mathrm{Id} + \rho_v)(c_0\kappa_2(c_0) + l_\infty). \tag{93}$$

The facts that function ρ_v is arbitrary and the solution $(x(t), r(t), z(t))$ is independent of ρ_v imply that

$$\limsup_{t\to\infty} V(x(t)) \le \alpha_3^{-1}(c_0\kappa_2(c_0) + l_\infty), \tag{94}$$

from which it follows that

$$\limsup_{t\to+\infty} |x(t)| \le \alpha_1^{-1} \circ \alpha_3^{-1}(c_0\kappa_2(c_0) + l_\infty). \tag{95}$$

To show that this bound can be improved, we proceed as follows. We choose the function l not only of class $\mathscr{K}$ and bounded by l_∞ but also satisfying

$$l \le \rho_7, \tag{96}$$

where ρ_7 is of class $\mathscr{K}_\infty$ and is defined by

$$\rho_7 = (2(\mathrm{Id} + \kappa_4))^{-1} \circ \alpha_3 \circ (\mathrm{Id} + \kappa_3)^{-1} \circ (2\rho_6)^{-1} \circ (2\rho_5^{-1})^{-1}. \tag{97}$$

The constraint (96) can always be satisfied and the function ρ_7 depends only on known data. Indeed,

- —the functions κ_3 and κ_4 are chosen in Proposition 6,
- —the function α_3 is obtained, in order to meet (63), from the known function V and W,
- —the functions ρ_4 and ρ_5 are obtained, in order to meet (64), from the chosen quantities μ, γ_{vu}, γ_c, and κ_1,
- —the function ρ_6 is obtained, in order to meet (84), from the known or chosen functions γ_{vu}, ρ_4, κ_1, u_n, γ_{cx}, κ_2, g, γ_{vx}, and V.

Then, from (89), (91), and (90), we can consider the interconnection of a fictitious system with state x, input y_r, and output $V(x)$ with a fictitious system with state r, input $V(x)$, and output $l(r)$. Although the systems are fictitious, the proof of the Small-Gain Theorem [JTP, Theorem 2.1] still applies. This can be seen by writing, in a way similar to (59),

$$\begin{aligned} V(x(t)) \le{}& \beta_v\left(s_\infty, \frac{t}{2}\right) + \alpha_3^{-1} \circ (\mathrm{Id} + \rho_v) \circ (\mathrm{Id} + \kappa_4^{-1})\left(v_0\left(\frac{t}{2}\right)\kappa_2\left(v_0\left(\frac{t}{2}\right)\right)\right) \\ &+ \alpha_3^{-1} \circ (\mathrm{Id} + \rho_v) \circ (\mathrm{Id} + \kappa_4) \circ l\left(2\beta_r\left(s_\infty, \frac{t}{4}\right)\right) \\ &+ \alpha_3^{-1} \circ (\mathrm{Id} + \rho_v) \circ (\mathrm{Id} + \kappa_4) \circ l \circ 2\rho_5^{-1} \circ 2\rho_6\left(\sup_{\tau\in[t/4,+\infty)} \{V(x(\tau))\}\right). \end{aligned} \tag{98}$$

So, with (88) and (96), we can again apply Lemma A.1 of [JTP] to conclude that

$$\limsup_{t\to\infty} V(x(t)) \le (\mathrm{Id} + \kappa_3^{-1}) \circ (\mathrm{Id} + \rho_8) \circ \alpha_3^{-1} \circ (\mathrm{Id} + \rho_v) \circ (\mathrm{Id} + \kappa_4^{-1})(c_0\kappa_2(c_0)) \tag{99}$$

holds for any functions ρ_8 and ρ_v of class $\mathscr{K}_\infty$ with (88) satisfied. From this we get (66). ■

5. Comparison Between the Static and Dynamic Feedback Designs

Two common features of the designs we have proposed are that they require a similar small-gain condition and, when $c_0 \neq 0$, they provide practical stability for the closed-loop system with a residual set which can be made smaller at the price of introducing a high-gain feature for small values:

—Indeed, condition (16) of Proposition 1 and (64), (65) of Proposition 6 are approximately equivalent. In (16), since ρ_1, ρ_2, and κ_x are arbitrary, this condition can be interpreted as mainly requiring that the function $\mathrm{Id} - \gamma_u$ be bounded below by a function of class $\mathscr{K}_\infty$. Similarly in (64), since ρ_4 and ρ_5 are arbitrary, this condition is mainly that the real number μ and the functions γ_{vu}, κ_1, and γ_c should be chosen such that the function $\kappa_1 - (\gamma_{cu} \circ (1/\mu)\gamma_{vu})$ is bounded below by a function of class $\mathscr{K}_\infty$. Then (65) mainly requires that the function $\mathrm{Id} - (\gamma_{cu} + \gamma_c \circ (1/\mu)\gamma_{vu})$ is bounded below by a function of class $\mathscr{K}_\infty$. Our remark follows with (12).

—Concerning the size of the residual set, in the static case, i.e., in the context of Proposition 1, the solutions can be made to converge to a smaller neighborhood of the origin by choosing a smaller function κ_x. In the dynamic design, i.e., in the context of Proposition 6, it is done by choosing a smaller function κ_2. In both cases this causes the "high-gain" phenomenon at least for small values: in the static design it is necessary to choose a bigger function $\mathscr{S}$ (see (28)), while in the dynamic design the same problem occurs with the term $\kappa_2^{-1}(p|L_{g_i}V(x)|)$ in (74).

Two significant differences between the designs are that the dynamic design requires more *a priori* knowledge and that, when c_0 is known to be zero and the unmodeled effect is more dynamic, and if we do not take into account the effects of the functions θ_i's presented in both designs, the dynamic design is less demanding in high gain than the static design:

—While in the static design it is only necessary to know that Assumptions A1 and A2 hold, in the dynamic design knowledge of the real number μ and the functions γ_{vx}, γ_{vu}, γ_{cx}, and γ_c are required so that Assumption A2′ holds.

—In the static design we cannot use the *a priori* knowledge $c_0 = 0$. Indeed, in any case, the function $\mathscr{S}$, involved in $\bar{b}_i$ defined in (30), needs to satisfy the "high-gain" inequality (28). However, in the dynamic design the function κ_2, involved in $\bar{b}_i$ defined in (74), is completely arbitrary. For instance, we can take

$\kappa_2 = p\,\mathrm{Id}$ in (76) and (74). This yields

$$\bar{b}_i(x) = |u_{n_i}(x)| + \gamma_{cx}(|x|) + |L_{g_i}V(x)|, \tag{100}$$

where every term is a "raw" data of the problem. When there is no static unmodeled effect, that is, when $\gamma_{cu} = 0$, the gain function κ_1^{-1} in (76) can be taken as the identity function, so the high-gain feature is only caused by the function $\theta(x, r)$ used to make the feedback smooth. This difference between the two designs follows from the fact that the static one is definitely a worst-case design using very little *a priori* knowledge.

To understand the difference between our two designs better, we consider the following system:

$$\dot{x} = x^2 + u + c(z, u), \tag{101}$$

where the function c is assumed to satisfy

$$|c(z, u)| \leq c_u|u| + c_z|z| \tag{102}$$

and the unmodeled dynamics are given by

$$\dot{z} = -\delta z + u, \tag{103}$$

for some $\delta > 0$.

Assumption A1 is satisfied with $V(x) = x^2$ by taking

$$u_n(x) = -x|1 + x|. \tag{104}$$

Assumption A2 is satisfied with

$$c_0 = 0, \qquad \gamma_u(s) = \left(c_u + \frac{c_z}{\delta}\right)s, \qquad \gamma_x(s) = 0. \tag{105}$$

Finally, by choosing

$$\gamma_{vx}(s) = \gamma_{cx}(s) = 0, \qquad \gamma_{vu}(s) = s, \qquad \gamma_c(s) = hs, \tag{106}$$

Assumption A2′ is satisfied with

$$c_0 = 0, \qquad \gamma_u(s) = c_u s, \tag{107}$$

if we have

$$\mu \leq \delta, \qquad h \geq c_z. \tag{108}$$

Our static feedback is

$$u(x) = -x\left(|1 + x| + \frac{1}{k}(1 + |x|)\right), \tag{109}$$

with the parameter k to be chosen. It is obtained by taking

$$\mathcal{S}(s) = \frac{1}{k}(\sqrt{s} + s). \tag{110}$$

To obtain boundedness of the solutions and global attractivity of the origin, it is sufficient for the system to meet

$$c_u + \frac{c_z}{\delta} < 1 \tag{111}$$

and for the controller parameter k to meet

$$k < \frac{1 - (c_u + c/\delta)}{(c_u + c/\delta)}. \tag{112}$$

This shows that an upper bound for $c_u + c_z/\delta$ is needed for the design.

Our dynamic feedback is, if continuity is not enforced,

$$\dot{r} = -\mu r + |u(x, r)|, \qquad u(x, r) = -\frac{1}{k_1}(x|1 + x| + x + hr\,\mathrm{sign}(x)), \tag{113}$$

with the parameters μ, k_1, and h to be chosen. It is obtained by taking

$$\kappa_1(s) = k_1 s, \qquad \kappa_2(s) = s, \tag{114}$$

where, according to (64) and (65), k_1 should satisfy

$$\frac{h}{k_1} < \mu, \qquad c_u \leq 1 - k_1. \tag{115}$$

To obtain boundedness of the solutions and global attractivity of the origin, it is sufficient for the controller parameters (μ, h, k_1) to meet

$$\mu \leq \delta, \qquad h \geq c_z, \qquad \frac{h}{\mu} < k_1 \leq 1 - c_u. \tag{116}$$

This shows that a lower bound for δ and upper bounds for c_u and c_z are needed. Also the system must satisfy

$$c_u + \frac{c_z}{\delta} < 1. \tag{117}$$

We conclude that the restrictions on the system are the same with both controllers. However, implementation of the dynamic controller requires more *a priori* knowledge.

Concerning the gains $1/k_1$, used in the dynamic feedback, and $1/k$, used in the static feedback, we see, with (116) and (112), that their need to be high depends on c_u and c/δ. Notice, however, that the high gain occurs in different ways in the two methods: when the unmodeled effect is more static, the gain $1/k$ in the static feedback is lower than the gain $1/k_1$ used in the dynamic feedback; when the unmodeled effect is more dynamic, the gain $1/k_1$ is lower than the gain $1/k$. This can be observed in two extreme cases when $c_z = 0$ and when $c_u = 0$. When $c_z = 0$, that is, when the unmodeled effect is purely static, the static gain $1/k = c_u/(1 - c_u)$, and the dynamic gain is $1/k_1 = 1/(1 - c_u)$. If c_u gets close to 1, both $1/k$ and $1/k_1$ become high gain, but, clearly, $1/k$ is lower than $1/k_1$. This suggests that when the unmodeled effect is more static, the static feedback is more suitable than the dynamic one. When $c_u = 0$, that is, when the unmodeled effect is purely dynamic, the static gain is $1/k = c_1/(1 - c_1)$, where $c_1 = c_z/\delta$, and the dynamic gain $1/k_1$ can be taken as any number between 1 and μ/h. When c_1 gets close to one, the static gain k becomes a high gain, while the dynamic gain $1/k_1$ remains bounded. With more detailed analysis, it can be shown that if c_u is bounded away from one, then the dynamic unmodeled effect will cause the high gain the static design, while the gain used in the dynamic design remains bounded as long as c_u remains bounded. This is what should be expected, because the dynamic feedback was introduced mainly to deal with the dynamic unmodeled effect. However, to be able to carry out the dynamic design, more data on how the unmodeled dynamics affects the system is necessary.

Finally, since in this example x is in $\mathbf{R}$ and the functions a and c are linear, we can compare our second design method with the one proposed in [KSK]. This method leads to the dynamic state feedback

$$\dot{r} = -\mu r + |u(x, r)|, \qquad u(x, r) = -x|1 + x| - kx(1 + hr + |x||1 + x|), \tag{118}$$

with the parameters μ, h, and k to be chosen. It guarantees boundedness of the solutions and convergence of x to the set

$$\left\{x \colon |x| \leq \frac{\max\{c_u, c_z/h\}}{k(1 - c_u)}\right\}$$

provided that the controller parameters h and μ satisfy

$$\frac{c_z}{1 - c_u} < \mu \leq \delta, \qquad \frac{h}{\mu} < \frac{1 - c_u}{c_u}. \tag{119}$$

Hence the system should be such that

$$c_u + \frac{c_z}{\delta} < 1. \tag{120}$$

This shows that a lower bound for δ and an upper bound for c_u are needed for the design. So, in this case, as opposed to our second design method, c_z plays no explicit role in the control design. However, the convergence of the solutions to the origin is not guaranteed without further restriction.

An interesting topic for future research would be to find a way to combine the two designs leading to a dynamic controller retaining the advantages of both.

6. Extension to *One-Sided* IOpS and *One-Sided* exp-IOpS

In the case of linear systems, designs of static state feedback providing infinite gain margin are known. From Propositions 1 or 6, we get that u can be changed into ku with k in $[\varepsilon, 2 - \varepsilon]$, with a chosen $\varepsilon > 0$. Nevertheless, the property that k can be in $[\varepsilon, +\infty[$ is recovered by noting that, in fact, our results still hold if, in Assumption A2, IOpS is replaced by *one-sided* IOpS, and, in Assumption A2′, exp-IOpS is replaced by *one-sided* exp-IOpS where:

***One-sided* IOpS** is the same as IOpS except that (7) is replaced by

$$\max_{i\in\{1,\dots,p\}} \{c_{\mathrm{sided}}(t)\} \le c_0 + \beta_c(|z(0)|, t) + \gamma_u(U(t)) + \gamma_x\left(\sup_{\tau\in[0,t)} \{|x(\tau)|\}\right), \tag{121}$$

where

$$c_{\mathrm{sided}}(t) = \max\{-\mathrm{sign}(u_i(t))c_i(x(t), z(t), u(t)), 0\}. \tag{122}$$

***One-sided* exp-IOpS** is the same as exp-IOpS except that (10) is replaced by

$$\max_{i\in\{1,\dots,p\}} \{c_{\mathrm{sided}}(t)\} \le c_0 + \beta(|z(0)|, t) + \gamma_{cu}(|u(t)|) + \gamma_{cx}(|x(t)|) + \gamma_c(r(t)). \tag{123}$$

Such a fact can be proved by following exactly the same lines as for the *two-sided* case with, in particular, the fact that Lemma 3 still holds if, in (22) and (23), we replace $c(\tau)$ by $c_+(\tau)$ defined as

$$c_+(\tau) = \max_{i\in\{1,\dots,p\}} \{c_{+i}(t)\}, \tag{124}$$

where

$$c_{+i}(t) = \max\{c_i(t)\,\mathrm{sign}(L_{g_i}V(x(t))), 0\}. \tag{125}$$

With such *one-sided* properties, we see that if u is changed into ku, then c is given by

$$c(x, z, u) = (k-1)u. \tag{126}$$

It follows that we have, for all i,

$$c_{\mathrm{sided}} = \max\{1-k, 0\}|u_i|. \tag{127}$$

In this case, we obtain

$$\gamma_x = \gamma_{cx} = \gamma_c = 0, \qquad \gamma_u(s) = \gamma_{cu}(s) = \max\{1-k, 0\}s. \tag{128}$$

So, given $\varepsilon > 0$, we can design our controller so that we can allow $k \in [\varepsilon, +\infty)$.

7. On the Equivalence of the IOS and exp-IOS Properties

We now study the relation between IOS and exp-IOS properties. We have already mentioned that exp-IOS implies IOS. For finite-dimensional observable linear systems, the converse is true. Indeed, in this case IOS implies that the eigenvalues of any appropriate realization have strictly negative real part. For nonlinear systems, we replace observability by the strong unboundedness observability (SUO) property introduced in [JTP], i.e.:

SUO. There exist a function β^0 of class $\mathcal{KL}$, a function γ^0 of class $\mathcal{K}$, and a nonnegative real number d^0 such that, for each initial condition $z(0)$ and each measurable function (x, u): $[0, T) \to \mathbf{R}^n \times \mathbf{R}^p$, with $0 < T \le \infty$, the corresponding solution $z(t)$, right maximally defined on $[0, T')$, with T' in $(0, T]$, satisfies, for all t

in $[0, T')$,

$$|z(t)| \leq \beta^0(|z(0)|, t) + \gamma^0\left(\sup_{\tau\in[0,t)} \{|(x(\tau), u(\tau), c(x(\tau), z(\tau), u(\tau))|\}\right) + d^0. \qquad (129)$$

Indeed, by following exactly the same arguments as in the proof of Proposition 3.1 of [JTP], we see that IOS and SUO, with $d^0 = 0$, imply ISS, and if in addition $c(0, 0, 0) = 0$, then exp-ISS implies exp-IOS. Therefore if ISS and exp-ISS are equivalent properties, IOS and exp-IOS are also equivalent properties under the extra assumptions SUO, with $d^0 = 0$, and $c(0, 0, 0) = 0$.

To study this equivalence of ISS and exp-ISS, consider the following system:

$$\dot{x} = f(x, u), \qquad (130)$$

with x in $\mathbf{R}^n$ and $f\colon \mathbf{R}^n \times \mathbf{R}^p \to \mathbf{R}^n$ a locally Lipschitz function satisfying $f(0, 0) = 0$. We assume this system is ISS, i.e.:

ISS. There exist a function β of class $\mathcal{KL}$ and a function γ of class $\mathcal{K}$ such that, for each initial condition $x(0)$ and each measurable essentially bounded function $u\colon \mathbf{R}_{\geq 0} \to \mathbf{R}^p$, the corresponding solution $x(t)$ satisfies, for all t in $\mathbf{R}_{\geq 0}$,

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma(U(t)), \qquad (131)$$

where $U(t)$ is defined in (8).

In this context, the exp-ISS property is:

exp-ISS. Given $\mu > 0$, there exist a function β of class $\mathcal{KL}$, a function γ_c of class $\mathcal{K}$ which is C^1 on $\mathbf{R}_{>0}$, and a function γ_v of class $\mathcal{K}$ which is C^1 on $\mathbf{R}_{\geq 0}$, such that, for each initial condition $z(0)$ and each measurable essentially bounded function $u\colon \mathbf{R}_{\geq 0} \to \mathbf{R}^p$, the corresponding solution $x(t)$ satisfies, for all t in $\mathbf{R}_{\geq 0}$,

$$|x(t)| \leq \beta(|x(0), t) + \gamma_c(r(t)), \qquad (132)$$

where $r(t)$ is the solution of the following initial value problem:

$$\dot{r} = -\mu r + \gamma_v(|u|), \qquad r(0) = 0. \qquad (133)$$

Clearly, exp-ISS implies ISS with γ in (131) given by $\gamma_c \circ (1/\mu)\gamma_v$. The converse is also true. We have, precisely:

Proposition 8. *System* (130) *is ISS if and only if it is* exp-*ISS. This result still holds if we impose that γ_c be concave and γ_v be convex.*

To establish this statement, we need to recall the definition of the ISS-Lyapunov functions introduced in [SW].

Definition 9. A smooth function $V\colon \mathbf{R}^n \to \mathbf{R}_{\geq 0}$ is called an *ISS-Lyapunov function* for system (130) if there exist functions α_1 and α_2 of class $\mathcal{K}_\infty$ and α_3 and χ of class

$\mathcal{K}$ such that, for all x,

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \tag{134}$$

and

$$|x| \geq \chi(|u|) \quad \Rightarrow \quad \frac{\partial V}{\partial x}(x) f(x, u) \leq -\alpha_3(|x|). \tag{135}$$

One of the main results in [SW] provides the following Lyapunov characterization of ISS:

Lemma 10. *System* (130) *is ISS if and only if it admits an ISS-Lyapunov function.*

In fact, the property for a system to have an ISS-Lyapunov function can be strengthened as follows:

Lemma 11. *If a system admits an ISS-Lyapunov function V satisfying* (134) *and* (135), *then, for any $\mu > 0$, there exists a C^1 function $\mathscr{V}$ and functions $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ of class $\mathcal{K}_\infty$ such that, for all x,*

$$\tilde{\alpha}_1(|x|) \leq \mathscr{V}(x) \leq \tilde{\alpha}_2(|x|) \tag{136}$$

and

$$|x| \geq \chi(|u|) \quad \Rightarrow \quad \frac{\partial \mathscr{V}}{\partial x}(x) f(x, u) \leq -\mu \mathscr{V}(x). \tag{137}$$

Note that if $\mathscr{V}$ is smooth, then $\mathscr{V}$ is again an ISS-Lyapunov function with an associated function α_3 equal to $-\mu\mathscr{V}$.

Proof. First observe that by renaming by α_3 the function $(2/\mu)\alpha_3 \circ \alpha_1^{-1}$ which is still a function of class $\mathcal{K}$, (135) becomes

$$|x| \geq \chi(|u|) \quad \Rightarrow \quad \frac{2}{\mu}\frac{\partial V}{\partial x}(x) f(x, u) \leq -\alpha_3(V(x)). \tag{138}$$

Now consider a C^1 function a of class $\mathcal{K}$ with the property[6]

$$a(\tau) \leq \min\{\tau, \alpha_3(\tau)\}, \qquad a'(0) = 0. \tag{139}$$

For instance, we can take

$$a(\tau) = \frac{2}{\pi} \int_0^\tau \frac{\min\{s, \alpha_3(s)\}}{1 + s^2}\, ds. \tag{140}$$

Then let ρ be the function defined as

$$\begin{cases} \rho(\tau) = \exp\left(\displaystyle\int_1^\tau \frac{2\, ds}{a(s)} \right), & \forall \tau \in \mathbf{R}_{>0}, \\ \rho(0) = 0. \end{cases} \tag{141}$$

[6] a' denotes the first derivative of the real function a.

This function is continuous on $\mathbf{R}_{>0}$. Since the integral inside the exponential function diverges to $-\infty$ as τ tends to zero, and diverges to $+\infty$ as τ tends to $+\infty$, it is seen that ρ is of class $\mathscr{K}_\infty$. Furthermore, we have the following:

Lemma 12. *The function ρ can be extended as a C^1 function on $\mathbf{R}_{\geq 0}$.*

Before proving this lemma, we remark that the function $\mathscr{V}$, defined as

$$\mathscr{V} = \rho \circ V, \tag{142}$$

allows us to prove Lemma 11. Indeed, in this case, (136) holds with

$$\tilde{\alpha}_1 = \rho \circ \alpha_1, \qquad \tilde{\alpha}_2 = \rho \circ \alpha_2. \tag{143}$$

We get

$$|x| \geq \chi(|u|) \quad \Rightarrow \quad \frac{\partial \mathscr{V}}{\partial x}(x)f(x, u) = \frac{2}{a(V(x))}\mathscr{V}(x)\frac{\partial V}{\partial x}(x)f(x, u) \leq -\mu\mathscr{V}(x). \quad \blacksquare \tag{144}$$

Proof of Lemma 12. Clearly, ρ is a C^2 function on $\mathbf{R}_{>0}$. So it is enough to show

$$\rho'(0) = 0, \qquad \lim_{\tau\to 0^+} \rho'(\tau) = 0. \tag{145}$$

First note that, for τ small enough, we have the estimation

$$\rho(\tau) = \exp\left(-\int_\tau^1 \frac{2\,ds}{a(s)}\right) \leq \exp\left(-\int_\tau^1 \frac{2\,ds}{s}\right) = \exp(\ln \tau^2) = \tau^2. \tag{146}$$

It follows that $\rho'(0)$ exists and

$$\rho'(0) = 0. \tag{147}$$

To show the second point of (145), we proceed as follows: For $\tau \neq 0$, we readily obtain

$$\rho'(\tau) = \frac{2}{a(\tau)}\rho(\tau), \qquad \rho''(\tau) = \left(\frac{4}{a^2(\tau)} - \frac{2a'(\tau)}{a^2(\tau)}\right)\rho(\tau). \tag{148}$$

Since $a'(0)$ is zero, it follows that there exists some strictly positive real number δ such that

$$0 < \tau < \delta \quad \Rightarrow \quad 0 < a'(\tau) < 1. \tag{149}$$

We conclude:

- The function ρ' is positive and strictly increasing on $(0, \delta)$. This implies that $\lim_{\tau\to 0^+} \rho'(\tau)$ exists and is nonnegative.
- The function ρ'' is bounded below by $\rho'(\tau)/a(\tau)$ on $(0, \delta)$.

Now to obtain a contradiction we assume that $\lim_{\tau\to 0^+} \rho'(\tau)$ is strictly positive. In this case there exists some strictly positive real number c such that

$$\tau \in (0, \delta) \quad \Rightarrow \quad \rho'(\tau) \geq c, \tag{150}$$

$$\Rightarrow \quad \rho''(\tau) \geq \frac{c}{a(\tau)} \geq \frac{c}{\tau}. \tag{151}$$

However, with

$$\rho'(\tau) = \rho'(\delta) - \int_\tau^\delta \rho''(s)\, ds, \tag{152}$$

this implies

$$\lim_{\tau \to 0^+} \rho'(\tau) = -\infty. \tag{153}$$

This contradicts the fact that ρ' is positive on $\mathbf{R}_{>0}$. So ρ' must be continuous on $\mathbf{R}_{\geq 0}$. ■

In proving Lemma 11, we have also reestablished the following statement which can be found, for example, in Theorem 3.6.10 of [LL] but is rarely used:

Proposition 13. *If a system $\dot{x} = f(x)$ admits a C^1 Lyapunov function V, that is, there exist functions α_1 and α_2 of class $\mathscr{K}_\infty$ and α_3 of class $\mathscr{K}$, such that we have, for all x,*

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|), \qquad \frac{\partial V}{\partial x}(x) f(x) \leq -\alpha_3(|x|), \tag{154}$$

then, for each $\mu > 0$, the system also admits a C^1 Lyapunov function $\mathscr{V}$ satisfying, for all x,

$$\tilde{\alpha}_1(|x|) \leq \mathscr{V}(x) \leq \tilde{\alpha}_2(|x|), \qquad \frac{\partial \mathscr{V}}{\partial x}(x) f(x) \leq -\mu \mathscr{V}(x), \tag{155}$$

for some functions $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ of class $\mathscr{K}_\infty$.

We are now ready to prove Proposition 8.

Proof of Proposition 8. We know already that exp-ISS implies ISS. We now show that ISS implies exp-ISS. Assume that system (130) is ISS. Then, by Lemma 11, there exists some C^1 function $\mathscr{V}$ satisfying (136) and (137). We define on $\mathbf{R}_{\geq 0}$ the function γ_v as follows:

$$\gamma_v(s) = s + \max_{|x| \leq \chi(|u|),\, |u| \leq s} \left\{ \frac{\partial \mathscr{V}}{\partial x}(x) f(x, u) + \mu \mathscr{V}(x) \right\}. \tag{156}$$

It is of class $\mathscr{K}_\infty$ and, from (137), we readily get (see also [SW] for more detailed reasoning), for all (x, u),

$$\frac{\partial \mathscr{V}}{\partial x}(x) f(x, u) \leq -\mu \mathscr{V}(x) + \gamma_v(|u|). \tag{157}$$

Now pick any measurable essentially bounded function $u: \mathbf{R}_{\geq 0} \to \mathbf{R}^p$ and any initial condition $x(0)$ in $\mathbf{R}^n$. By (157), the corresponding solution $x(t)$ satisfies, for all t in $\mathbf{R}_{\geq 0}$,

$$\overbrace{\mathscr{V}(x(t))}^{\bullet} \leq -\mu\mathscr{V}(x(t)) + \gamma_v(|u(t)|). \tag{158}$$

It follows that

$$\mathscr{V}(x(t)) \leq \exp(-\mu t)\mathscr{V}(x(0)) + r(t), \tag{159}$$

where $r(t)$, defined here as

$$r(t) = \int_0^t \exp(-\mu[t-s])\gamma_v(|u(s)|)\, ds, \tag{160}$$

is the unique solution of the initial value problem (133). With (136), we have obtained

$$|x(t)| \leq \tilde{\alpha}_1^{-1}(\exp(-\mu t)\mathscr{V}(x(0)) + r(t)) \leq \beta(s, t) + \gamma_c(r(t)), \tag{161}$$

where

$$\beta(s, t) = \tilde{\alpha}_1^{-1}(2\exp(-\mu t)\tilde{\alpha}_2(s)), \qquad \gamma_c(s) = \tilde{\alpha}_1^{-1}(2s). \tag{162}$$

To complete the proof of the proposition, we need to show that γ_c and γ_v can be restricted to being concave and convex, respectively, with the desired continuous differentiability. To this purpose, we need the following:

Lemma 14. *For any function γ of class $\mathscr{K}$, there exist a convex function γ_v, of class $\mathscr{K}$ and C^1 on $\mathbf{R}_{\geq 0}$, and a concave function γ_c, of class $\mathscr{K}$ and C^1 on $\mathbf{R}_{>0}$, such that*

$$\gamma_c \circ \gamma_v \geq \gamma. \tag{163}$$

Proof. Let $[0, S]$ (where $S \leq +\infty$) be the image by γ of $\mathbf{R}_{\geq 0}$ and let

$$s_0 = \min\left\{1, \frac{S}{2}\right\}. \tag{164}$$

We define

$$\gamma_c^{-1}(s) = \begin{cases} \displaystyle\int_0^s \gamma^{-1}(\tau)\, d\tau, & \forall s \leq s_0, \\ \gamma_c^{-1}(s_0) + (s - s_0)\gamma^{-1}(s_0), & \forall s_0 < s. \end{cases} \tag{165}$$

Since γ^{-1} is increasing and continuous on $[0, s_0]$, the function γ_c^{-1} is convex, of class $\mathscr{K}$ and C^1 on $\mathbf{R}_{\geq 0}$. So the function γ_c is concave, of class $\mathscr{K}$ and C^1 on $\mathbf{R}_{>0}$. We also have

$$\gamma_c^{-1}(s) \leq s\gamma^{-1}(s) \leq \gamma^{-1}(s), \qquad \forall s \leq s_0. \tag{166}$$

This implies

$$\gamma_c(s) \geq \gamma(s), \qquad \forall s \leq \gamma^{-1}(s_0). \tag{167}$$

Now we define a function γ_v as

$$\gamma_v(s) = \frac{\gamma^{-1}(s_0)}{s_0} \int_0^{2s} \gamma(\tau)\, d\tau + s. \tag{168}$$

This function γ_v is convex, of class $\mathcal{K}$ and C^1 on $\mathbf{R}_{\geq 0}$ and we have

$$\gamma_v(s) \geq \frac{\gamma^{-1}(s_0)}{s_0} s\gamma(s) + s. \tag{169}$$

Then, for $s \leq \gamma^{-1}(s_0)$, we have, with (167) and (169),

$$\gamma_c(\gamma_v(s)) \geq \gamma_c\left(\frac{\gamma^{-1}(s_0)}{s_0} s\gamma(s) + s\right) \geq \gamma(s). \tag{170}$$

For $s \geq \gamma^{-1}(s_0)$, we have, with (166) and (169),

$$\gamma_v(s) \geq \gamma^{-1}(s_0) \geq \gamma_c^{-1}(s_0). \tag{171}$$

So, in this case, we can use the second definition in (165) to evaluate $\gamma_c(\gamma_v(s))$. With (169), this yields

$$\begin{aligned} \gamma_c(\gamma_v(s)) &\geq \frac{((\gamma^{-1}(s_0)/s_0)s\gamma(s) + s) + \gamma^{-1}(s_0)s_0 - \gamma_c^{-1}(s_0)}{\gamma^{-1}(s_0)} \\ &\geq \frac{(\gamma^{-1}(s_0)/s_0)s\gamma(s)}{\gamma^{-1}(s_0)} \geq \gamma(s). \quad \blacksquare \end{aligned} \tag{172}$$

Lemma 15. *For any functions γ_2 and γ_3 of class $\mathcal{K}$, there exist a convex function γ_v, of class $\mathcal{K}$ and C^1 on $\mathbf{R}_{\geq 0}$, and a concave function γ_c, of class $\mathcal{K}$ and C^1 on $\mathbf{R}_{>0}$, such that*

$$\gamma_2\left(\int_0^t \exp(-\mu(t-\tau))\gamma_3(|u(\tau)|)\, d\tau\right) \leq \gamma_c\left(\int_0^t \exp(-\mu(t-\tau))\gamma_v(|u(\tau)|)\, d\tau\right). \tag{173}$$

Proof. From Lemma 14, we know the existence of functions γ_{v3} and γ_{c3} with the desired properties so that

$$\gamma_3 \leq \gamma_{c3} \circ \gamma_{v3}. \tag{174}$$

Let

$$f(t) = \frac{1 - \exp(-\mu t)}{\mu} \leq \frac{1}{\mu}. \tag{175}$$

Then, with Jensen's inequality and concavity, we get

$$\begin{aligned} \int_0^t \exp(-\mu(t-\tau))\gamma_3(|u(\tau)|)\, d\tau &\leq f(t)\gamma_{c3}\left(\frac{1}{f(t)}\int_0^t \exp(-\mu(t-\tau))\gamma_{v3}(|u(\tau)|)\, d\tau\right) \\ &\leq \frac{1}{\mu}\gamma_{c3}\left(\mu \int_0^t \exp(-\mu(t-\tau))\gamma_{v3}(|u(\tau)|)\, d\tau\right). \end{aligned} \tag{176}$$

However, again there exist functions γ_{v2} and γ_{c2} with the desired properties so that

$$\gamma_2 \circ \frac{1}{\mu}\gamma_{c3} \leq \gamma_{c2} \circ \gamma_{v2}. \tag{177}$$

So we get

$$\begin{aligned}
&\gamma_2\left(\int_0^t \exp(-\mu(t-\tau))\gamma_3(|u(\tau)|)\,d\tau\right)\\
&\quad\leq \gamma_2 \circ \frac{1}{\mu}\gamma_{c3}\left(\mu\int_0^t \exp(-\mu(t-\tau))\gamma_{v3}(|u(\tau)|)\,d\tau\right)\\
&\quad\leq \gamma_{c2} \circ \gamma_{v2}\left(\mu\int_0^t \exp(-\mu(t-\tau))\gamma_{v3}(|u(\tau)|)\,d\tau\right)\\
&\quad\leq \gamma_{c2} \circ \mu f(t)\gamma_{v2}\left(\frac{1}{f(t)}\int_0^t \exp(-\mu(t-\tau))\gamma_{v3}(|u(\tau)|)\,d\tau\right)\\
&\quad\leq \gamma_{c2}\left(\mu\int_0^t \exp(-\mu(t-\tau))\gamma_{v2} \circ \gamma_{v3}(|u(\tau)|)\,d\tau\right).
\end{aligned} \tag{178}$$

Hence, we can take

$$\gamma_c(s) = \gamma_{c2}(s), \qquad \gamma_v(s) = \gamma_{v2} \circ \gamma_{v3}(s). \quad \blacksquare \tag{179}$$

Proof of Proposition 8 (continued). From (161), we get

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma_c\left(\int_0^t \exp(-\mu(t-\tau))\gamma_v(|u(\tau)|)\,d\tau\right). \tag{180}$$

By Lemma 15, there exist a concave function $\tilde{\gamma}_c$ of class $\mathcal{K}$ and a convex function $\tilde{\gamma}_v$ of class $\mathcal{K}$ with all the desired properties such that

$$\gamma_c\left(\int_0^t \exp(-\mu(t-\tau))\gamma_v(|u(\tau)|)\,d\tau\right) \leq \tilde{\gamma}_c\left(\int_0^t \exp(-\mu(t-\tau))\tilde{\gamma}_v(|u(\tau)|)\,d\tau\right). \tag{181}$$

The conclusion of Proposition 8 follows readily. $\blacksquare$

The advantage of the exp-ISS is that it allows us to replace the L_∞ norm with a memory fading L_1 norm in the ISS estimation. However, one may worry whether the exp-ISS will lead to more conservative results. Our objective of the following example is to show that this is not necessarily the case if some care is taken in choosing the real number μ and the functions γ_c and γ_v.

Consider the system

$$\dot{x} = -ax^3 + \gamma_0(|u|), \qquad a > 0, \tag{182}$$

where γ_0 is a function of class $\mathcal{K}$. This system is ISS and its gain function γ can be taken as any function of class $\mathcal{K}$ satisfying

$$\gamma > \left(\frac{\gamma_0}{a}\right)^{1/3}. \tag{183}$$

To obtain an estimation on γ_v and γ_c, we let, for each integer k strictly larger than $3a$ and $\mu/2$,

$$V_k(x) = \alpha_k(|x|), \tag{184}$$

where, for each k, α_k is a C^1 function of class $\mathscr{K}_\infty$ defined as

$$\alpha_k(s) = \begin{cases} \exp\left(\frac{k}{a}\left[1 - \frac{1}{s^2}\right]\right), & \text{if} \quad 0 \le s \le 1, \\ s^{2k/a}, & \text{if} \quad s > 1. \end{cases} \tag{185}$$

Then, for $|x|$ in $(0, 1]$, we have

$$\begin{aligned} \dot{V}_{k(182)}(x) &= -\mu V_k(x) - (2k - \mu) V_k(x) + \frac{2kV_k(x)}{a|x|^3}\gamma_0(|u|) \\ &\le -\mu V_k(x) + \max_{|x| \le \chi(|u|)} \left\{\frac{2kV_k(x)}{a|x|^3}\gamma_0(|u|)\right\} \\ &\le -\mu V_k(x) + (2k - \mu) V_k(\chi_k(|u|)), \end{aligned} \tag{186}$$

where χ_k is a function of class $\mathscr{K}$ defined as

$$\chi_k(s) = \left(\frac{2k}{2k - \mu}\right)^{1/3} \gamma(s) > \left[\frac{2k\gamma_0(s)}{(2k - \mu)a}\right]^{1/3}, \tag{187}$$

with γ given in (183). To obtain (186), we used the fact that $V_k(x)/|x|^3$ is an increasing function in $|x|$ for all k strictly larger than $3a$.

When $|x|$ is in $(1, +\infty)$, by applying the same arguments, we have

$$\begin{aligned} \dot{V}_{k,(182)} &= -\mu x^2 V_k(x) - (2k - \mu)x^2 V_k(x) + \frac{2kx^2 V_k(x)}{a|x|^3}\gamma_0(|u|) \\ &\le -\mu V_k(x) + (2k - \mu)\chi_k^2(|u|) V_k(\chi_k(|u|)). \end{aligned} \tag{188}$$

Thus, for any solution $x(t)$ of (182), we have

$$V_k(x(t)) = V_k(x(0)) \exp(-\mu t) + r(t), \tag{189}$$

with $r(t)$ the solution of

$$\dot{r} = -\mu r + \gamma_{vk}(|u|), \qquad r(0) = 0, \tag{190}$$

where, for each k,

$$\gamma_{vk}(s) = \begin{cases} (2k - \mu) V_k(\chi_k(s)), & \text{if} \quad \chi_k(s) \le 1, \\ (2k - \mu)\chi_k^2(s) V_k(\chi_k(s)), & \text{if} \quad \chi_k(s) > 1. \end{cases} \tag{191}$$

From (189), we get

$$|x(t)| \le \beta_k(|x_0|, t) + \gamma_{ck}(r(t)), \tag{192}$$

for some function β_k of class $\mathscr{KL}$ and with γ_{ck} given as

$$\gamma_{ck}(s) = \alpha_k^{-1}\left(\frac{2k}{2k - \mu}s\right). \tag{193}$$

We now prove that, as k is going to $+\infty$, the function $\gamma_{ck} \circ (1/\mu)\gamma_{vk}$ approaches γ. When $\chi_k(s)$ is in $(1, +\infty)$, $V_k(\chi_k(s))$ is strictly larger than one. Thus we have

$$\begin{aligned} \gamma_{ck} \circ \frac{1}{\mu}\gamma_{vk}(s) &\le \alpha_k^{-1}\left(\frac{2k}{\mu}\chi_k^2(s)V_k(\chi_k(s))\right) \\ &\le \left(\frac{2k}{\mu}\right)^{a/2k} \chi_k(s)^{(1+a/k)} \\ &\le \left(\frac{2k}{\mu}\right)^{a/2k}\left(\frac{2k}{2k-\mu}\right)^{(1/3)(1+a/k)} \gamma(s)^{(1+a/k)}. \end{aligned} \tag{194}$$

When $\chi_k(s)$ is in $[0, 1]$ but $(2k/\mu)[V_k(\chi_k(s))]$ is still strictly larger than one, we have

$$\gamma_{ck} \circ \frac{1}{\mu}\gamma_{vk}(s) \le \alpha_k^{-1}\left(\left(\frac{2k}{\mu}\right)V_k(\chi_k(s))\right) \tag{195}$$

$$\le \left(\frac{2k}{\mu}\right)^{a/2k}\sqrt{\exp\left(1 - \frac{1}{\chi_k^2(s)}\right)}$$

$$\le \left(\frac{2k}{\mu}\right)^{a/2k} \chi_k(s) \tag{196}$$

$$\le \left(\frac{2k}{\mu}\right)^{a/2k}\left(\frac{2k}{2k-\mu}\right)^{1/3} \gamma(s). \tag{197}$$

When both $\chi_k(s)$ and $(2k/\mu)[V_k(\chi_k(s))]$ are in $[0, 1]$, we have

$$\gamma_{ck} \circ \frac{1}{\mu}\gamma_{vk}(s) \le \alpha_k^{-1}\left(\frac{2k}{\mu} V_k(\chi_k(s))\right) \tag{198}$$

$$\le \frac{1}{\sqrt{1 - (a/k)\ln((2k/\mu)V_k(\chi_k(s)))}} \tag{199}$$

$$\le \frac{1}{\sqrt{1 - (a/k)\ln(2k/\mu) - (a/k)\ln(\exp((k/a)(1 - 1/\chi^2(s))))}} \tag{200}$$

$$\le \frac{\chi_k(s)}{\sqrt{1 - (a/k)\chi_k(s)^2\ln(2k/\mu)}} \tag{201}$$

$$\le \frac{1}{\sqrt{1 - (a/k)\ln(2k/\mu)}}\left(\frac{2k}{2k-\mu}\right)^{1/3} \gamma(s). \tag{202}$$

Combining (194), (197), and (202), we see that, for any strictly positive real number ε, there exists some integer K such that, for any $k \ge K$,

$$\gamma_{ck} \circ \frac{1}{\mu}\gamma_{vk}(s) \le \begin{cases} (1+\varepsilon)\gamma(s), & \text{if } \ \gamma(s) \le 1, \\ (1+\varepsilon)(\gamma(s))^{1+\varepsilon}, & \text{if } \ \gamma(s) > 1. \end{cases} \tag{203}$$

With (203), we conclude that, for the system (182), by working with the exp-ISS gain function instead of the ISS gain function, we can obtain results which are as equally conservative as we want on any compact set.

8. Conclusion

Consider the system

$$\begin{cases} \dot{x} = f(x) + \sum_{i=1}^{p} g_i(x)[u_i + c_i(x, z, u)], \\ \dot{z} = a(x, z, u). \end{cases} \tag{204}$$

Under the following conditions

—the system

$$\dot{x} = f(x) + \sum_{i=1}^{p} g_i(x)u_i \tag{205}$$

is globally asymptotically stabilizable by a feedback law $(u_{ni}(x))$ (see Assumption A1),

—the system

$$\begin{cases} \dot{z} = a(x, z, u), \\ y_i = c_i(x, z, u) \end{cases} \tag{206}$$

has appropriate input-to-state and input-to-output properties (see Assumptions A2 or A2′),

we have shown how to modify the feedback u_n into a static or a dynamic feedback in order to guarantee that all the solutions of (204) are bounded and their x-components are captured by an arbitrarily small neighborhood of the origin. This result belongs to the broad class of results known on uncertain systems (see [C] for a survey and [Q1], [Q2], [KSK], [KK], and [JMP] for some recent developments).

The modifications we have proposed for the control law u_n are based on Lyapunov design and gain assignment techniques as introduced in [JTP]. The analysis of the properties of the closed-loop system is based on the application of the Small-Gain Theorem [JTP, Theorem 2.1]. The assumptions on the z-subsystem are written in terms of the notion of IOS introduced in [JTP] which is an extension of the notion of ISS as introduced by Sontag in [S2].

To carry out our design of a dynamic feedback, we have been led to introduce a new notion of ISS systems called exp-ISS. We have shown that for finite-dimensional systems the two notions are equivalent. For this we have used the link between the ISS property and the existence of an appropriate Lyapunov function which has been established in [LSW] and [SW].

An important feature of the system (204) is that the unmodeled effects are in the "range" of the input. This is the well-known matching assumption. By using arguments similar to those used for propagating the ISS property through integrators in Corollary 2.3 of [JTP], this matching assumption can be relaxed for systems and uncertainties having a recurrent so-called feedback structure.

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Appendix A. On the Nonexistence of a Stabilizing Feedback for (13)

We prove that there is no control law $u(t)$ that can drive to zero the x-component of any solution of (13), starting from $(x_0, 1)$ with $x_0 > M\exp(1)$. To do this, we assume that such a control exists. By the uniqueness property, the corresponding solution $(x(t), z(t))$ of (13) remains in $\mathbf{R}^2_{>0}$ for all positive time. Moreover, since

$$x(t) = \exp\left(\int_0^t (x(s) - u(s) + \gamma(z(s)))\,ds\right)x_0, \qquad \forall t \geq 0, \tag{207}$$

we have, necessarily,

$$\lim_{t\to\infty}\int_0^t (u(s) - \gamma(z(s)) - x(s))\,ds = +\infty. \tag{208}$$

On the other hand, we have

$$\frac{dz}{z^2} = (u(t) - z(t))\,dt. \tag{209}$$

With (14) and the fact that $z(t)$ is strictly positive, (209) yields

$$-\frac{1}{z(t)} + 1 = \int_0^t (u(s) - z(s))\,ds \geq \int_0^t (u(s) - M - \gamma(z(s)))\,ds, \tag{210}$$

and

$$\frac{1}{z(t)} \leq 1 - \int_0^t (u(s) - \gamma(z(s)) - x(s))\,ds - \int_0^t x(s)\,ds + Mt. \tag{211}$$

Now we define

$$t_1 = \inf\left\{t \geq 0 \colon \int_0^t (u(s) - \gamma(z(s)) - x(s))\,ds \geq 1\right\}. \tag{212}$$

This real number is well defined (see (208)) and is positive. By continuity, we get

$$x(t_1) = x_0\exp(-1), \tag{213}$$

$$\begin{aligned}\frac{1}{z(t_1)} &\leq 1 - \int_0^{t_1} (u(s) - \gamma(z(s)) - x(s))\,ds - \int_0^{t_1} x(s)\,ds + Mt_1\\ &\leq -\frac{x_0 t_1}{\exp(1)} + Mt_1.\end{aligned} \tag{214}$$

By the choice of x_0, this yields that $1/z(t_1) < 0$ which contradicts the fact that $z(t) > 0$ for all positive t.

Appendix B. An Explicit Expression for θ_i

Working within the context of the proof of Lemma 3, we propose here an explicit expression for the function θ_i.

First we define a function $\hat{\theta}_i$ as

$$\hat{\theta}_i(x) = \begin{cases} \dfrac{\sqrt{(W(x)/2p)^2 + 3(|L_{g_i}V(x)|\,\varphi_i(x))^2} - W(x)/2p}{|L_{g_i}V(x)|\,\varphi_i(x)}, & \text{if } \ |L_{g_i}V(x)| \neq 0, \\ 0, & \text{if } \ |L_{g_i}V(x)| = 0, \end{cases} \tag{215}$$

where

$$\varphi_i(x) = \mathscr{S}(V(x)) + |u_{n_i}(x)|. \tag{216}$$

According to the arguments in the proof of Theorem 1 of [S1], this function is continuous on $\mathbf{R}^n \backslash \{0\}$. Moreover, we have

$$x \in \mathscr{B}_{1i} \quad \Rightarrow \quad \left(\frac{W(x)}{2p}\right)^2 + 3(|L_{g_i}V(x)|\,\varphi_i(x))^2 \geq \left(\frac{W(x)}{2p} + |L_{g_i}V(x)|\,\varphi_i(x)\right)^2. \tag{217}$$

It follows that $\hat{\theta}_i(x)$ is larger than one on $\mathscr{B}_{1i}$. This allows us to define θ_i on $\mathbf{R}^n \backslash \{0\}$ as

$$\theta_i(x) = \operatorname{sat}(\hat{\theta}_i(x)), \tag{218}$$

where sat: $\mathbf{R}_{\geq 0} \to [0, 1]$ is the saturation function

$$\operatorname{sat}(r) = \begin{cases} r, & \text{if } \ r \in [0, 1], \\ 1, & \text{if } \ r > 1. \end{cases} \tag{219}$$

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