Adding Integrations, Saturated Controls, and Stabilization for Feedforward Systems

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***Abstract*—Our study relates to systems whose dynamics generalize $\dot{x} = h(y,u)$, $\dot{y} = f(y,u)$, where the state components x integrate functions of the other components y and the inputs u. We give sufficient conditions under which global asymptotic stabilizability of the y subsystem (respectively, by saturated control) implies global asymptotic stabilizability of the overall system (respectively, by saturated control). It is obtained by constructing explicitly a control Lyapunov function and provides feedback laws with several degrees of freedom which can be exploited to tackle design constraints. Also, we study how appropriate changes of coordinates allow us to extend its domain of application.**

Finally we show how the proposed approach serves as a basic tool to be used, in a recursive design, to deal with more complex systems. In particular the stabilization problem of the so-called feedforward systems is solved this way.

I. Introduction

A. Problem Statement

THE idea of backstepping, also called adding one integrator (see [28] for instance), has led to one of the basic tools proposed nowadays for designing stabilizing controllers. In [13], Krstić *et al.* give a repertory of the many procedures which can be obtained to deal with various classes of systems by combining, maybe recursively, this particular Lypunov-based design with other ones (see [12]).

This idea applies to the problem of knowing when asymptotic stabilizability for the system $\dot{y} = f(y,u)$ implies asymptotic stabilizability for the system

$$\dot{y} = f(y,x), \qquad \dot{x} = h(x,y,u). \tag{1}$$

In this paper, systems of a different class are considered. To simplify, in this introduction let us just mention that we propose a solution to the problem of knowing when global asymptotic stabilizability (respectively, by saturated control) of the system $\dot{y} = f(y,u)$ implies global asymptotic stabilizability (respectively, by saturated control) for the system

$$\dot{x} = h(y,u), \qquad \dot{y} = f(y,u). \tag{2}$$

That is, instead of making the control a state component, i.e., controlling through a differentiator, as in (1), we add state components which integrate functions of the other components. Such components are called "integrating coordinates." The knowledge of a solution for this latter problem, called here "adding one integration," allows us to deal with systems whose dynamics can be written, by using appropriate coordinates, in a specific recurrent structure called feedforward form[1]

$$\begin{cases} \dot{x}_n = f_n(x_1,\dots,x_{n-1},u), \\ \quad\vdots \\ \dot{x}_2 = f_2(x_1,u), \\ \dot{x}_1 = f_1(x_1,u). \end{cases} \tag{3}$$

In particular we shall prove that for stabilizability of the system linearized at the origin being assumed, global asymptotic stabilizability holds if $\dot{x} = f_1(x,u)$ is globally asymptotically stabilizable with local exponential stability. Systems which can be written in this feedforward form are not singularities in practice. For instance, consider the celebrated cart-pendulum system. Let:

- (M,x) be mass and position of the cart which is moving horizontally;
- (m,l,θ) be mass, length, and angular deviation from the upward position for the pendulum which is pivoting around a point fixed on the cart;
- finally, F be a horizontal force acting on the cart.

The dynamics can be written as

$$\begin{cases} (M+m)\ddot{x} + ml\cos(\theta)\ddot{\theta} = ml\dot{\theta}^2\sin(\theta) + F, \\ \ddot{x}\cos(\theta) + l\ddot{\theta} = g\sin(\theta). \end{cases} \tag{4}$$

By means of the following change of control, coordinates, and time:

$$\begin{cases} u_0 = \frac{l}{g}\frac{F + ml\dot{\theta}^2\sin(\theta) - mg\sin(\theta)\cos(\theta)}{M + m\sin(\theta)^2}, \ x_0 = \frac{x}{l}, \\ s_0 = \frac{\dot{x}}{\sqrt{gl}}, \ \theta_0 = \theta, \ \omega_0 = \dot{\theta}\sqrt{\frac{l}{g}}, \ \tau = t\sqrt{\frac{g}{l}} \end{cases} \tag{5}$$

and by denoting by "$\circ$" the derivation with respect to the new time τ, we get the equivalent but normalized dynamics

$$\begin{cases} \overset{\circ}{x}_0 = s_0\ , \quad \overset{\circ}{s}_0 = u_0\ , \quad \overset{\circ}{\theta}_0 = \omega_0 \\ \overset{\circ}{\omega}_0 = \sin(\theta_0) - u_0\cos(\theta_0) \end{cases} \tag{6}$$

which are exactly in the feedforward form (3) with

$$x_1 = (\theta_0, \omega_0), \quad x_2 = s_0, \quad x_3 = x_0. \tag{7}$$

This system will be used in Section V-C to illustrate our feedback design.

[1] Systems in the form (3) are generically not feedback linearizable. In particular, this is the case when, controllability of the system linearized at the origin being assumed, $\frac{\partial^2 f_2}{\partial u^2}\frac{\partial f_1}{\partial u} - \frac{\partial f_2}{\partial u}\frac{\partial^2 f_1}{\partial u^2}$ is not identically equal to zero on a neighborhood of the origin.

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B. The Main Sources of Our Work

The first significant results about feedforward systems have been presented by Teel in [25] (see also [24]). The main point discovered in this work is that the knowledge of the system linearized at the origin is already sufficient to propose a family of feedback laws in which we are guaranteed to find one element appropriate for the particular system under consideration. This result follows from these two facts.

1) Higher order terms (see our basic definitions) play a role only in the choice of an element in this family of feedback, not on the definition of the family itself.
2) The "integrating" coordinates—x_2 to x_n in (3)—must be selected in such a way that only higher-order terms appear in their time derivative. To meet such a constraint, a linear change of coordinates is appropriate.

Such a result can be proved by using the new concepts in interconnected systems theory that Teel has formalized in [27] (see also [26]).

Our main objective here is to propose a Lyapunov analysis counterpart to the interconnected systems point of view. This is made possible from the following remark: assume that the functions in (2) are C^2 and that we have

$$h(y, 0) \equiv 0, \ \forall y. \tag{8}$$

In this case, there exist functions h_2, f_0, and f_2 such that (2) can be rewritten as

$$\dot{x} = h_2(y, u)\, u, \quad \dot{y} = f_0(y) + f_2(y, u)\, u. \tag{9}$$

If $y = 0$ is a globally asymptotically stable solution of $\dot{y} = f_0(y)$, then the converse Lyapunov theorem [30, Th. V.19.8] guarantees the existence of a Lyapunov function $V(y)$ such that

$$u = 0 \Longrightarrow \dot{\overbrace{x^\top x + V(y)}}_{(9)} \leq -c\, V(y). \tag{10}$$

This proves that we are exactly in the context of the theory usually referred to as the Jurdjevic and Quinn approach to which many authors have contributed (see [1], [4], [9], [11], [14], and [16] and the references therein). Our goal in the following is mainly to relax (8). This is done by applying a change of coordinates which generalizes the one proposed by Teel and by translating in terms of stability margin the fact, exhibited by Teel, that higher-order terms play a minimal role.

During the preparation of the final version of this paper, we received from M. Jankovic *et al.* a preprint of their paper [10]. They propose also a Lyapunov design for feedforward systems but, instead of a change of coordinates which implicitly introduces a cross term in the Lyapunov function, they address directly the construction of such a term. Due to space limitation, we cannot go further into comparing the two methods, but the interested reader may refer to [17]. However, some of the ideas presented in that paper helped us.

- We realized that the arguments, used for the proof of our previous result [18, Proposition 2.1], were in fact powerful enough to establish Theorem III.1.
- The set of assumptions introduced in [10] will be used to illustrate our own assumptions.
- We are borrowing from [10] the dynamic solution for (30).

C. Organization of the Paper

In Section II, we first revisit the Jurdjevic and Quinn approach in a specific case. This allows us to present some technical results and make some discussions which are useful for the remainder of the paper. In Section III, we state our main result with relaxing (8). In fact, we allow h in (2) to depend on y, u, and x, but we impose a restriction on the behavior of this function for y near the origin and x going to infinity. This constraint on the dependence in y generalizes the notion of "higher order" used by Teel. In Section IV, we show how it may be possible to enforce the satisfaction of such a constraint by a change of coordinates.

In Sections V-A and V-B, by combining the Lyapunov design of Section III and the change of coordinates of Section IV, we are able to answer the question about global asymptotic stabilizability of forms generalizing (2) and (3). To help the reader in getting a better grasp on the design we propose, we apply it to the cart-pendulum system in Section V-C.

Finally Section VI contains some concluding remarks.

D. Notations and Basic Definitions

- Throughout the paper, the symbol c may be used to denote generically a strictly positive real number (i.e., $c + c * c = c$!).
- For an element X in $R^m \otimes R^n \otimes R^p$ and a vector x in R^p, we denote their contraction by $\langle X, x\rangle$. It is a matrix in $R^m \otimes R^n$ whose (i, j) entry $\langle X, x\rangle_{ij}$ is

$$\langle X, x\rangle_{ij} = \sum_{k=1}^{p} X_{(i,j,k)}\, x_k. \tag{11}$$

- Let p be a nonnegative real number. A continuous function $\mathcal{F}(x, y)$ on R^l is said to have a zero of order p at $y = 0$ if there exists a nonnegative continuous function $\tilde{\mathcal{F}}$ such that for all $(x, y) \in R^l$

$$|\mathcal{F}(x, y)| \leq \tilde{\mathcal{F}}(x, y)\, |y|^p. \tag{12}$$

- With Q a positive definite symmetric matrix, we denote

$$|x| = \sqrt{x^\top x}, \qquad |x|_Q = \sqrt{x^\top Q x}. \tag{13}$$

- For any matrix M we denote by λ_{Mi} one of its eigenvalues.
- By $\dot{V}_{(15)}$ we denote the function defined as follows when this makes sense:

$$\dot{V}_{(15)}(\mathcal{X}) = \lim_{t \to 0_+} \frac{V(\mathcal{X} + t\varphi(\mathcal{X})) - V(\mathcal{X})}{t}. \tag{14}$$

The subscript (15) refers to the equation number of the differential equation

$$\dot{\mathcal{X}} = \varphi(\mathcal{X}). \tag{15}$$

If φ is continuous and V is Lipschitz continuous, then we have (see [30, p. 3])

$$\lim_{t\to 0_+} \frac{V(\Phi(t,\mathcal{X})) - V(\mathcal{X})}{t} = \dot{V}_{(15)}(\mathcal{X}) \tag{16}$$

where $\Phi(t,\mathcal{X})$ is any of the solutions of

$$\frac{\partial \Phi}{\partial t}(t,\mathcal{X}) = \varphi(\Phi(t,\mathcal{X})), \quad \Phi(0,\mathcal{X}) = \mathcal{X}. \tag{17}$$

This property as well as [30, Th. II.8.1] will be used throughout the paper.

- For a real valued C^1 function k, we denote by k' its first derivative.

II. The Jurdjevic and Quinn Approach

A. Result

We start our analysis by restating, in a slightly more general form (see also [4, Corollary 1.6]), a result by Bacciotti in [1, Remark 10.9] which is based on the Jurdjevic and Quinn approach [11]. For this, we consider the controlled system

$$\begin{cases} \dot{x}_1 = h_0(x_1) + h_2(x_1,x_2,y,u)\,u \\ \dot{x}_2 = e_0(x_2) + e_2(x_1,x_2,y,u)\,u \\ \dot{y} = f_0(y) + f_2(x_1,x_2,y,u)\,u \end{cases} \tag{18}$$

where y is in R^n, x_1 in R^{m_1}, x_2 in R^{m_2}, u in R^q. We introduce the following assumptions.

Assumption A0: The functions h_0, h_2, e_0, e_2, f_0, and f_2 are C^0 and h_0, e_0, and f_0 are zero at the origin.

Assumption A1: There exist three positive definite and proper C^1 functions Q, S, and V so that

$$\frac{\partial Q}{\partial x_1}(x_1)h_0(x_1) = -R(x_1) \le 0 \qquad \forall x_1 \tag{19}$$

$$\frac{\partial S}{\partial x_2}(x_2)e_0(x_2) = -T(x_2) < 0 \qquad \forall x_2 \neq 0 \tag{20}$$

$$\frac{\partial V}{\partial y}(y)f_0(y) = -W(y) < 0 \qquad \forall y \neq 0. \tag{21}$$

Assumption A2: $x_1 = 0$ is the only solution of

$$\begin{cases} \dot{x}_1 = h_0(x_1), \\ \frac{\partial Q}{\partial x_1}(x_1)\,h_2(x_1,0,0,0) = 0, \frac{\partial Q}{\partial x_1}(x_1)\,h_0(x_1) = 0. \end{cases} \tag{22}$$

Theorem II.1: Under Assumptions A0, A1, and A2, for any $\bar{u}$ in $(0,+\infty]$, the origin can be made a globally asymptotically stable solution of (18) by a state feedback bounded by $\bar{u}$ and zero at the origin.

B. Discussion of the Assumptions A1 and A2

1) The peculiarity of (18) is that it is made of three decoupled subsystems when u is set to zero. Assumption A1 expresses the fact that for these three subsystems, the origin is respectively globally stable, globally asymptotically stable, and globally asymptotically stable. We shall study in Section III a case where coupling terms are present. Using the terminology of the introduction, x_1 and x_2 represent the "integrating" coordinates. With (19), the x_1 subsystem is not strictly dissipative so x_1 is a true "integrating" coordinate. With (20), this is not the case of x_2. In fact, at this stage, y and x_2 play the same role. We have distinguished them for the sake of coherence with Section III.

2) It is not easy to check when Assumption A2 holds. However, sufficient conditions implying it are known. For instance, the reader will find in [14] or [16] checkable sufficient geometric conditions. Note that without A2, asymptotic stabilization may be impossible. Indeed, consider the system

$$\begin{cases} \dot{\mathcal{X}}_1 = -\mathcal{X}_2 + \left[(\mathcal{X}_1^2 + \mathcal{X}_2^2) - 1\right]u, \\ \dot{\mathcal{X}}_2 = \mathcal{X}_1 + \left[(\mathcal{X}_1^2 + \mathcal{X}_2^2) - 1\right]u. \end{cases} \tag{23}$$

Assumption A1 holds. But with $Q(\mathcal{X}_1,\mathcal{X}_2) = \mathcal{X}_1^2 + \mathcal{X}_2^2$, Assumption A2 does not hold. In fact, asymptotic stabilization is not possible since

$$\mathcal{X}_1(t) = \cos(t), \ \mathcal{X}_2(t) = \sin(t) \tag{24}$$

is a solution whatever u may be.

This example shows that Assumption A2 is related to the property that the control is able to force any solution to leave any level set $Q(x_1) = q > 0$. From this remark we get readily the following lemma.

Lemma II.2: Let Q be a C^1 function and h_0 and H be C^0 functions. If there exists a C^0 function Λ satisfying $\Lambda(x_1,0) = 0$, for all x_1, and such that the system $\dot{x}_1 = h_0(x_1) + \Lambda(x_1,H(x_1))$ has no solution remaining for ever in a fixed level set of Q except $x_1 = 0$, then $x_1 = 0$ is the only solution of

$$\dot{x}_1 = h_0(x_1), \ H(x_1) = 0, \ \frac{\partial Q}{\partial x_1}(x_1)\,h_0(x_1) = 0. \tag{25}$$

So to check if Assumption A2 holds, it is sufficient to find such a function Λ with

$$H(x_1) = \frac{\partial Q}{\partial x_1}(x_1)\,h_2(x_1,0,0,0). \tag{26}$$

As a direct consequence we have the following lemma.

Lemma II.3: Let M_1 be a matrix such that there exists a positive definite matrix Q_1 for which $Q_1 M_1 + M_1^\top Q_1$ is negative semidefinite. If (M_1, D_1) is a stabilizable pair, then $\mathcal{X}_1 = 0$ is the only solution of

$$\dot{\mathcal{X}}_1 = M_1\mathcal{X}_1, \ \mathcal{X}_1^\top Q_1 D_1 = 0, \ \mathcal{X}_1^\top Q_1 M_1 \mathcal{X}_1 = 0. \tag{27}$$

C. Proof of Theorem II.1

We have

$$\begin{aligned} &\overbrace{V(y) + Q(x_1) + S(x_2)}^{\cdot}{}_{(18)} \\ &\quad = -W(y) - R(x_1) - T(x_2) + \mathcal{G}(x_1,x_2,y,u)\,u \end{aligned} \tag{28}$$

with the notation

$$\begin{aligned} \mathcal{G}(x_1,x_2,y,u) = &\frac{\partial V}{\partial y}(y)\,f_2(x_1,x_2,y,u) \\ &+ \frac{\partial S}{\partial x_2}(x_2)\,e_2(x_1,x_2,y,u) \\ &+ \frac{\partial Q}{\partial x_1}(x_1)\,h_2(x_1,x_2,y,u). \end{aligned} \tag{29}$$

Since V, Q, and S are C^1 and f_2, e_2, and h_2 are continuous, the function $\mathcal{G}$ is continuous.

With (28), we see that global stability holds if the control u is chosen such that

$$\mathcal{G}(x_1, x_2, y, u)\, u \leq 0. \tag{30}$$

We propose two solutions for this inequality.

1) *Static Solution:*

Lemma II.4: Let $G(\xi, u)$ be a continuous function. For any strictly positive real number $\bar{u}$, there exists a function $\lambda(\xi)$, as smooth as $G(\xi, u)$ is, such that if

$$u(\xi) = -\lambda(\xi)\, G(\xi, 0)^\top \tag{31}$$

then, for all ξ, we have

$$|u(\xi)| \leq \bar{u}, \;\; G(\xi, u(\xi))\, u(\xi) \leq -\frac{1}{2}\lambda(\xi)\, |G(\xi, 0)|^2. \tag{32}$$

Moreover, if G is C^1, then λ is strictly positive on any compact set.

Proof: See Appendix A.

2) *Dynamic Solution*[2]: From (28), the system with input u, state (x_1, x_2, y), and output $\mathcal{G}(x_1, x_2, y, u)$ is passive. In this context, instead of satisfying (30) for each time, it is sufficient to meet it dynamically, i.e., it is sufficient for u to be the output of a strictly passive system with $-\mathcal{G}(x_1, x_2, y, u)$ as input. However, we cannot forget the constraint on u. By drawing our inspiration from barrier methods as they are used in optimization theory (see [7, ch. 3] for instance), we propose the following dynamic feedback:

$$u = z, \;\; \dot{z} = -z - \left(\bar{u}^2 - |z|^2\right)\mathcal{G}(x_1, x_2, y, u), \qquad |z(0)| < \bar{u}. \tag{33}$$

This system with input $-\mathcal{G}(x_1, x_2, y, u)$, state z, and output u is strictly passive and the set $\{z : |z| < \bar{u}\}$ is positively invariant. Indeed, we have

$$\overbrace{V(y) + Q(x_1) + S(x_2) - \frac{1}{2}\ln\left(\bar{u}^2 - |z|^2\right)}^{\cdot}{}_{(18),(33)} = -W(y) - R(x_1) - T(x_2) - \frac{|z|^2}{\bar{u}^2 - |z|^2}. \tag{34}$$

This equality proves the global stability of the origin of the extended system as well as the positive invariance of the set above. Moreover, we have

$$u \equiv 0 \Longrightarrow \mathcal{G}(x_1, x_2, y, 0) \equiv 0. \tag{35}$$

These two solutions give a feedback law upperbounded in norm by $\bar{u}$, as smooth as the function $\mathcal{G}$ is such that (35) holds. Also the derivatives (28) or (34) are zero if and only if

$$y = 0, \;\; x_2 = 0, \;\; R(x_1) = 0, \;\; u = 0. \tag{36}$$

It remains to prove global attractiveness. With Assumption A0, the right-hand side of the closed-loop system is only continuous. So we do not have necessarily uniqueness of solutions. To prove asymptotic stability, we use the following generalized invariance principle.

Lemma II.5 [20, Th. 2]: Consider the system

$$\dot{\mathcal{X}} = \varphi(\mathcal{X}) \tag{37}$$

with $\mathcal{X} \in R^n$ and φ a continuous function. Let $\mathcal{V}$: $R^n \to R$ be a Lipschitz continuous nonnegative function and $\mathcal{W}$: $R^n \to R$ be a nonnegative continuous function such that for all $\mathcal{X}$

$$\dot{\mathcal{V}}_{(37)}(\mathcal{X}) = -\mathcal{W}(\mathcal{X}). \tag{38}$$

Then, all the bounded maximal solutions of (37) exist on $[0, +\infty)$ and converge to the largest quasi-invariant[3] set contained in $\{\mathcal{X} \in R^n \colon \mathcal{W}(\mathcal{X}) = 0\}$.

To apply this Lemma to the closed-loop system we have obtained, we evaluate what is the largest quasi-invariant set contained in

$$\{(x_1, x_2, y)\colon y = 0, \; x_2 = 0, \; R(x_1) = 0, \; u = 0\}.$$

From (35), the definition (29) of $\mathcal{G}$, and Assumption A2, we see that this quasi-invariant set is reduced to the origin. Therefore, our feedback provides global asymptotic stability.

+Note that the derivatives $\overbrace{V(y) + Q(x_1) + S(x_2)}^{\cdot}{}_{(18)}$ or

$$\overbrace{V(y) + Q(x_1) + S(x_2) - \ln\left(\bar{u}^2 - |z|^2\right)}^{\cdot}{}_{(18),(33)}$$

are made negative definite if, for all $x_1 \neq 0$

$$\left|\frac{\partial Q}{\partial x_1}(x_1) h_0(x_1)\right| + \left|\frac{\partial Q}{\partial x_1}(x_1) h_2(x_1, 0, 0, 0)\right| \neq 0. \tag{39}$$

III. Design Tool 1: Lyapunov Design

A. Result

Let us now extend the Jurdjevic and Quinn approach to a broader class of systems. Precisely, we modify (18) by introducing coupling terms which are identically zero when y is at the origin

$$\begin{cases} \dot{x}_1 = h_0(x_1) + h_1(x_1, x_2, y)\, y + h_2(x_1, x_2, y, u)\, u \\ \dot{x}_2 = e_0(x_2) + e_1(x_1, x_2, y)\, y + e_2(x_1, x_2, y, u)\, u \\ \dot{y} = f_0(y) + f_1(x_1, x_2, y)\, y + f_2(x_1, x_2, y, u)\, u \end{cases} \tag{40}$$

where y is in R^n, x_1 in R^{m_1}, x_2 in R^{m_2}, u in R^q. For this new system, we modify Assumption A0 as follows.

[2]This idea is borrowed from [10] and is one possible interpretation of [21, Th. 1].

[3]A set $\mathcal{E}$ is said to be quasi-invariant with respect to (37) if, for each $\mathcal{X} \in \mathcal{E}$, there exists at least one maximal solution of (37), defined on $[0, +\infty)$ and remaining in $\mathcal{E}$.

Assumption A0: The functions h_0, $h_1 y$, h_2, e_0, $e_1 y$, e_2, f_0, $h_1 y$, and f_2 are C^0 and h_0, e_0, and f_0 are zero at the origin.

Within the context of Theorem II.1, the coupling terms h_1, e_1, and f_1 are inconvenient. However, global asymptotic stabilizability may hold only because of the presence of these terms. For instance, the linear system

$$\dot{X}_1 = aY, \qquad \dot{Y} = -Y + u \tag{41}$$

is stabilizable if and only if $a \neq 0$, i.e., there is a coupling term. On the other hand, we may hope that by choosing appropriate coordinates, this system can be written with no coupling terms as in the form (18). Indeed, by letting

$$x_1 = X_1 + aY, \qquad y = Y \tag{42}$$

we get the system

$$\dot{x}_1 = au, \qquad \dot{y} = -y + u \tag{43}$$

which satisfies Assumptions A1 and A2 if $a \neq 0$. Unfortunately for the general case, it may be hopeless to find a "computable" change of coordinates such that in these new coordinates, there is no coupling terms. This leads us to the question: are we allowed to replace an exact but "incomputable" change of coordinates by a "computable" but approximated one? In the following we answer positively to this question. For this, we need to introduce the following new assumption.

Assumption A3:

A3.1) There exist a function ρ which is defined, nonnegative, and continuous on $[0, +\infty)$ and a function κ which is defined, strictly positive, and continuous on $(0, +\infty)$ such that

$$\begin{aligned} &\left|\frac{\partial Q}{\partial x_1}(x_1)\, h_1(x_1, x_2, y)\, y\right| + \left|\frac{\partial S}{\partial x_2}(x_2)\, e_1(x_1, x_2, y)\, y\right| \\ &\quad \leq \sqrt{\kappa(V(y))W(y)}(1 + \rho(Q(x_1) + S(x_2))) \\ &\quad \times \left[\sqrt{\kappa(V(y))W(y)}(1 + \rho(Q(x_1) + S(x_2))) \right. \\ &\quad \left. + \sqrt{T(x_2)}\right] \end{aligned} \tag{44}$$

$$\frac{1}{1+\rho} \notin L^2([0, +\infty)) \tag{45}$$

$$\begin{aligned} &\forall c_1 > 0,\ \exists c_2\colon \{y \in R^n \backslash \{0\},\ |y| \leq c_1\} \\ &\qquad \Longrightarrow \kappa(V(y)) \left|\frac{\partial V}{\partial y}(y)\right| \leq c_2. \end{aligned} \tag{46}$$

Moreover, V, W, and f_1 satisfy[4]

$$\frac{\partial V}{\partial y}(y)\, f_1(x_1, x_2, y)\, y \leq \frac{1}{4}\, W(y). \tag{47}$$

A3.2) $\kappa(V(y)) \frac{\partial V}{\partial y}(y) f_2(x_1, x_2, y, u)$ can be extended as a continuous function on $R^n \times R^{m_1} \times R^{m_2} \times R^q$.

[4] The particular value $1/4$ in (47) is written to simplify the further notations. Any real number in $[0, 1)$ would be right.

Theorem III.1: If Assumptions A0, A1, A2, and A3 hold, then for any $\bar{u}$ in $(0, +\infty]$, the origin of (40) is globally asymptotically stabilizable by a state feedback bounded by $\bar{u}$ and zero at the origin.

B. Discussion of Assumption A3

Assumption A3.1): Assumption A3.1) introduces restrictions on the coupling terms h_1, e_1, and f_1 without which asymptotic stabilizability may be impossible (see Sections III-B-4 and III-B-5).

Inequality (47) implies that the term f_1 cannot change the asymptotic stability of $y = 0$ whatever the function $(x_1(t), x_2(t))$ is, as long as it is measurable and locally essentially bounded.

The other conditions in A3.1) limit the behavior of the functions h_1 and e_1. The starting point is (44). Being able to write such an inequality is by itself not a restriction. More precisely, from Lemma B.1 in Appendix B, and since Q and S are proper functions, there always exist nonnegative and continuous functions γ_y and γ_x satisfying

$$\begin{aligned} &\left|\frac{\partial Q}{\partial x_1}(x_1)\, h_1(x_1, x_2, y)\, y\right| + \left|\frac{\partial S}{\partial x_2}(x_2)\, e_1(x_1, x_2, y)\, y\right| \\ &\quad \leq |y|\, \gamma_y(|y|)\, (1 + \gamma_x(Q(x_1) + S(x_2))). \end{aligned} \tag{48}$$

The restriction arises with the fact that we need to find:

1) a function ρ satisfying the nonintegrability condition (45) and, for instance

$$\rho(s) \geq \sqrt{1 + \gamma_x(s)} - 1. \tag{49}$$

This is a constraint concerning the behavior of the functions h_1 and e_1 for (x_1, x_2) going to infinity;

2) a function κ satisfying the regularity condition (46) and, for instance

$$\kappa(V(y))W(y) \geq |y|\, \gamma_y(|y|). \tag{50}$$

In view of Lemma B.2, this is a constraint concerning the order of the zeros of the functions h_1 and e_1 at $y = 0$.

We shall see, in the proof of Theorem III.1, how this special kind of restriction in terms of bounding functions allows a Lyapunov analysis.

A case where our assumptions are satisfied is given by A0, A1, A2, and the set of the following three assumptions, related to the one considered in [10], assuming:

$$f_1(x_1, x_2, y) \equiv 0. \tag{51}$$

H1) There exists a nonnegative continuous function γ such that for all (x_1, x_2, y), we have[5]

$$\begin{cases} |h_1(x_1, x_2, y)| \leq (1 + |x_1|)\, \gamma(|y|), \\ |e_1(x_1, x_2, y)| \leq (1 + |x_2|)\, \gamma(|y|). \end{cases} \tag{52}$$

[5] Note that we can extract from γ a lower bound for the order of the zeros of h_1 and e_1 at $y = 0$. This function is not necessarily zero at zero.

H2) We have

$$\begin{cases} \limsup_{|x_1|\to+\infty} \frac{\left|\frac{\partial Q}{\partial x_1}(x_1)\right||x_1|}{Q(x_1)} < +\infty, \\ \limsup_{|x_2|\to+\infty} \frac{\left|\frac{\partial S}{\partial x_2}(x_2)\right||x_2|}{S(x_2)} < +\infty. \end{cases} \tag{53}$$

H3) The solution $y = 0$ of $\dot{y} = f_0(y)$ is locally exponentially stable.

Indeed, (51) implies that (47) is trivially satisfied. Then, with H1) and H2), we get

$$\begin{aligned} &\left|\frac{\partial Q}{\partial x_1}(x_1)h_1(x_1,x_2,y)y\right| + \left|\frac{\partial S}{\partial x_2}(x_2)e_1(x_1,x_2,y)y\right| \\ &\quad \le \left[\left|\frac{\partial Q}{\partial x_1}(x_1)\right|(1+|x_1|) + \left|\frac{\partial S}{\partial x_2}(x_2)\right|(1+|x_2|)\right]|y|\,\gamma(|y|) \\ &\quad \le c_1[1+Q(x_1)+S(x_2)]|y|\,\gamma(|y|) \end{aligned} \tag{54}$$

where c_1 is some positive real number. Hence, to meet (44), it is sufficient:

1) to choose ρ, satisfying (45), as

$$\rho(s) = \sqrt{s+1} - 1. \tag{55}$$

2) to find a function κ satisfying (see (50) above)

$$c_1\,|y|\,\gamma(|y|) \le \kappa(V(y))\,W(y). \tag{56}$$

This last inequality generalizes the notion of higher-order terms considered in [25] and [24]. It illustrates how the behavior of h_1 and e_1 for y near the origin, quantified by γ, should be related to the stability margin of the y-subsystem, quantified by W. The smallest is the order of the zeros of h_1 and e_1 at $y = 0$ and the stronger the local attractiveness of $y = 0$ for $\dot{y} = f_0(y)$ should be.

In the case where H3) holds, an appropriate function κ meeting (46) and (56) always exists with no constraint on γ and therefore no constraint on the order of the zeros of the functions h_1 and e_1 at $y = 0$. Indeed, let us first remark that with A1 and H3), an appropriate convex combination of the function V, provided by A1, and a quadratic form, provided by the local exponential stability, gives a new Lyapunov function, still denoted V, which is C^2 on a neighborhood of the origin and such that for all y with $|y| \le \alpha_5$, we have

$$\alpha_1|y|^2 < V(y) < \alpha_2|y|^2, \ \alpha_3|y|^2 < W(y) < \alpha_4|y|^2 \tag{57}$$

$$\limsup_{y\to 0} \frac{\left|\frac{\partial V}{\partial y}(y) - y^\top \frac{\partial^2 V}{\partial y^2}(0)\right|}{W(y)} < +\infty \tag{58}$$

where the α_i's are strictly positive real numbers. Hence to meet (46), it is sufficient to choose, for $s \le 1$

$$\kappa(s) = \frac{c_2}{\sqrt{s}} \tag{59}$$

where c_2 is a strictly positive real number. With Lemma B.2, the definition of κ can be completed on $(0, +\infty)$ in such a way that (56) holds.

So we have proved that Assumptions A0–A2 and H1)–H3) imply Assumptions A0–A2 and A3.1).

Assumption A3.2): Assumption A3.2) is an extra smoothness condition for y near the origin. Let us study this point within the context of Assumptions A0–A2 and H1)–H3).

Satisfaction of A3.1) does not require $\gamma(0) = 0$. But for A3.2), we observe that if there exists a point (x_1, u) satisfying

$$f_2(x_1, 0, 0, u) \ne 0 \tag{60}$$

then with A0, (58), (46), (56), and (57), A3.2) cannot hold if we do not have $\gamma(0) = 0$. Conversely, we assume

$$\gamma(|y|) = |y|\,\tilde{\gamma}(|y|) \tag{61}$$

with $\tilde{\gamma}$ some nonnegative continuous function. This means that the zeros of $h_1 y$ and $e_1 y$ at $y = 0$ are at least of order two. Then A3.1) and A3.2) are satisfied with (59) replaced by $s \le 1$, $\kappa(s) = c_2$.

So, if the zeros of $h_1 y$ and $e_1 y$ at $y = 0$ are at least of order two and Assumptions A0–A2 and H1)–H3) hold, then Assumptions A0–A3 are satisfied. However, in this discussion, we have not exploited the positive definiteness of T, or more precisely the presence of $\sqrt{T(x_2)}$ in (44). This explains why, in some cases as in Theorem V.1, no restriction on the order of the zero of $e_1 y$ at $y = 0$ is needed.

Stability for the (x_1, x_2) Subsystem When $y = 0$ and $u = 0$: Consider the system

$$\dot{x} = m\,x + u, \qquad \dot{y} = -ay - y^3 - y^3\,u \tag{62}$$

with a a strictly positive real number. Assumptions A2 and A3 are satisfied, but Assumption A1 holds only if $m \le 0$. In fact, for all real number $m > 2a$, there is no asymptotically stabilizing feedback. Indeed, in this case, the set

$$\left\{(x,y)\left|\frac{1}{2y^2} - x + \frac{1}{m} \le 0\right.\right\}$$

is positively invariant whatever u may be. Since the origin is not in this set, it follows that $m > 2a$ implies there is no asymptotically stabilizing state feedback. We conclude that in the general case, we cannot have instability for the systems $\dot{x}_1 = h_0(x_1)$ or $\dot{x}_2 = e_0(x_2)$ without an extra assumption on the system $\dot{y} = f_0(y)$.

Restriction on the Behavior of h_1 and e_1 for (x_1, x_2) Going to Infinity: We have observed that the nonintegrability condition (45) implies a restriction on the behavior of h_1 and e_1 when (x_1, x_2) goes to infinity. To motivate this restriction, we consider the system

$$\dot{x} = \frac{u}{1+u^2} + y^2 x^n, \qquad \dot{y} = -\frac{1}{2}y \tag{63}$$

with $n \ge 1$. Assumptions A1 and A2 are satisfied.

Case $n = 1$: Assumption A3 holds with

$$Q(s) = V(s) = W(s) = \frac{1}{2}s^2, \quad \rho(s) = \sqrt{2s}, \kappa(s) = 2 \tag{64}$$

so Theorem III.1 applies. In fact, Theorem II.1 applies already after the change of coordinate

$$X = \exp(y^2)\,x. \tag{65}$$

Case $n > 1$: We first observe that there is no function ρ such that $\frac{1}{(1+\rho)^2}$ is nonintegrable and

$$|xy^2x^n| \le y^2\left(1+\rho\left(\frac{1}{2}x^2\right)\right)^2. \tag{66}$$

So we do not know how to check if Assumption A3 holds. In fact, there is no globally asymptotically stabilizing feedback. Indeed, the set

$$\left\{(x,y)\colon x > 1, y^2x^{n-1} > \frac{4}{n-1}+3\right\}$$

is positively invariant whatever u may be. Since the origin is not in the closure of this set, our conclusion follows.

Restriction on the Behavior of h_1 and e_1 for y Near the Origin: Assumption A3 limits the behavior of the functions h_1 and e_1 for y near the origin. Indeed, consider the system

$$\dot{x} = |y|^n + u, \qquad \dot{y} = -y^3 + u. \tag{67}$$

Assumptions A1 and A2 are satisfied. But, we observe that the origin is a stable solution of (67) when $u = 0$ if and only if $n > 2$. According to Lemma III.2, this implies that A3.1) does not hold when $n \le 2$. On the other hand, A3.1) holds when $n \ge 3$ with

$$\begin{aligned} Q(x) &= \frac{1}{2}x^2, \quad \rho(s) = (2s)^{\frac{1}{4}} \\ V(y) &= \frac{1}{2}y^2, \quad \kappa(s) = (2s)^{\frac{n-4}{2}}. \end{aligned} \tag{68}$$

With this choice, A3.2) holds when $n > 3$.

In fact, for $n \le 3$ this system is not globally asymptotically stabilizable by continuous dynamic state feedback. Indeed, it does not satisfy Brockett's condition (see [1, Th. 7.1]).

C. Proof

Proof of Theorem III.1: We begin with some preliminary remarks.

1) By adding 1 to κ if necessary, we can assume that this function is not in $L^1([1,+\infty))$.
2) With (46) and A0, the functions $\kappa(V(y))W(y)$ and $\kappa(V(y))\frac{\partial V}{\partial y}(y)f_1(x_1,x_2,y)y$ can be extended as continuous functions on the whole space.
3) With κ being continuous on $(0,+\infty)$ and V being C^1 and positive definite, we have, with $y(s) = y_1 + s(y_2 - y_1)$

$$\int_{V(y_1)}^{V(y_2)} \kappa(s)ds = \int_0^1 \kappa(V(y(s)))\frac{\partial V}{\partial y}(y(s))(y_2-y_1)ds \tag{69}$$

for all y_1 and y_2 in R^n and such that the origin is not in the segment $[y_1, y_2]$.
4) Condition (46) implies $\int_0^1 \kappa(V(sy))\frac{\partial V}{\partial y}(sy)y\,ds$ is a well-defined Riemann integral.

These various points allow us to conclude that the function

$$k(y) = \lim_{i\to+\infty}\int_{V(\frac{1}{i}y)}^{V(y)} \kappa(s)ds \quad \forall y \ne 0, \quad k(0) = 0 \tag{70}$$

is well defined, proper, and Lipschitz continuous on R^n and satisfies

$$k(y_2) - k(y_1) = \int_0^1 \kappa(V(y(s)))\,\frac{\partial V}{\partial y}(y(s))\,(y_2 - y_1)ds \tag{71}$$

for all y_1 and y_2 in R^n. With (14), (46), this identity, and the continuity of the functions $[f_0 + f_1 y + f_2 u]$ and $\kappa(V)\frac{\partial V}{\partial y}[f_0 + f_1y + f_2u]$, we can establish the equation

$$\begin{aligned} &\dot{k}_{(40)}(x_1,x_2,y,u) \\ &\quad = \kappa(V(y))\,\frac{\partial V}{\partial y}(y) \\ &\qquad \times [f_0(y) + f_1(x_1,x_2,y)y + f_2(x_1,x_2,y,u)u]. \end{aligned} \tag{72}$$

Now, we denote by l the function which is zero at zero, C^1, positive definite, and proper on $[0,+\infty)$ whose derivative l' is [see (45)]

$$0 < l' = \frac{1}{(1+\rho)^2}. \tag{73}$$

With these notations, we introduce the following candidate Lyapunov function:

$$U(x_1,x_2,y) = l(Q(x_1)+S(x_2)) + \frac{7}{3}\,k(y). \tag{74}$$

It is positive definite, proper, and Lipschitz continuous. Also the function $\dot{U}_{(40)}(x_1,x_2,y,u)$ is well defined and continuous. Moreover, with (44), we get

$$\begin{aligned} &\dot{U}_{(40)}(x_1,x_2,y,u) \\ &\quad \le -\frac{7}{3}\,\kappa(V(y))W(y) + \mathcal{G}(x_1,x_2,y,u)u \\ &\qquad + l'\big[-R(x_1) - T(x_2) \\ &\qquad\quad + \sqrt{\kappa(V(y))W(y)}[1+\rho]\sqrt{T(x_2)}\,\big] \\ &\qquad + \frac{7}{3}\,\kappa(V(y))\frac{\partial V}{\partial y}(y)f_1(x_1,x_2,y)y \\ &\qquad + l'\,\kappa(V(y))W(y)[1+\rho]^2 \end{aligned} \tag{75}$$

where l' and ρ are evaluated at $Q(x_1)+S(x_2)$ and with $\mathcal{G}$ the continuous function (see A0, A1, and A3.2) defined now as

$$\begin{aligned} \mathcal{G}(x_1,x_2,y,u) = {}& \frac{7}{3}\kappa(V(y))\frac{\partial V}{\partial y}(y)f_2(x_1,x_2,y,u) \\ &+ l'\left[\frac{\partial Q}{\partial x_1}(x_1)h_2(x_1,x_2,y,u)\right. \\ &\quad \left. + \frac{\partial S}{\partial x_2}(x_2)e_2(x_1,x_2,y,u)\right]. \end{aligned} \tag{76}$$

By completing the squares, we get finally

$$\begin{aligned} \dot{U}(x_1,x_2,y,u)_{(40)} \le {}& \mathcal{G}(x_1,x_2,y,u)u \\ &- l'(Q(x_1)+S(x_2))\left[R(x_1) + \frac{1}{2}\,T(x_2)\right] \\ &- \frac{1}{4}\,\kappa(V(y))\,W(y). \end{aligned} \tag{77}$$

This inequality is the key point of our analysis. It has been established by using only Assumptions A1 and A3.1), with A3.2) helping only for getting continuity of $\mathcal{G}$. It allows us to conclude the proof by following exactly the same arguments as those invoked in the proof of Theorem II.1. □

Since (77) holds under Assumptions A0, A1, and A3.1), we have proved the following.

Lemma III.2: If Assumptions A0, A1, and A3.1) hold, then the origin is a globally stable solution of (40) with u set equal to zero. Moreover, all the solutions converge to the largest quasi-invariant set contained in $\{(x_1, x_2, y): R(x_1) = 0,\ x_2 = 0,\ y = 0\}$.

Let us finally remark that the feedback depends only on $\mathcal{G}(x_1, x_2, y, u)$, where the functions e_0, e_1, h_0, h_1, f_0, and f_1 are not used. It follows that Theorem III.1 also gives a kind of robust global asymptotic stabilizability result. Indeed, with the functions Q, S, V, h_2, e_2, f_2, κ, and ρ fixed, i.e., with $\mathcal{G}(x_1, x_2, y, u)$ fixed, we have a globally asymptotically stabilizing feedback for any system of the form (40) whose functions e_0, e_1, h_0, h_1, f_0, and f_1 are such that A0 to A3 hold.

A Larger Class of Stabilizing Feedback: For proving Theorem III.1, we have used a feedback law which makes the product $\mathcal{G}(x_1, x_2, y, u)u$ nonpositive for each time, in the case of the static feedback, or in an integral sense, in the case of the dynamic feedback. This constraint of nonpositiveness follows from not taking advantage of the negativeness already provided by the term $-\frac{1}{4}\kappa(V(y))W(y)$. By using this property,[6] we shall be able to propose a broader family of feedback laws. The interesting fact about this new family is that it contains elements which can be written without the explicit knowledge of the function V. This may be helpful when Theorem III.1 is used repeatedly in a recursive design.

To show how this new family can be obtained, we work within a smoother context than for Theorem III.1. Namely, we modify Assumptions A0, A1, and A3.2) into the following.

Assumption A0′: The functions f_0, f_1, e_1, and h_1 are C^0, the functions f_2, e_0, e_2, h_0, and h_2 are C^1, and h_0, e_0, and f_0 are zero at the origin.

Assumption A1′: There exist functions Q, S, and V satisfying A1 and of class C^2.

Assumption A3.2′: $\kappa(V(y))\frac{\partial V}{\partial y}(y)f_2(x_1, x_2, y, u)$ is a C^1 function on $R^n \times R^{m_1} \times R^{m_2} \times R^q$.

We introduce the following compact notations:

$$\begin{cases} \Gamma_v(x_1, x_2, y) = \frac{7}{3}\frac{\partial V}{\partial y}(y) f_2(x_1, x_2, y, 0), \\ \Gamma(x_1, x_2, y) = l'(Q(x_1) + S(x_2)) \\ \qquad \times \left[\frac{\partial Q}{\partial x_1}(x_1)h_2(x_1, x_2, y, 0) + \frac{\partial S}{\partial x_2}(x_2)e_2(x_1, x_2, y, 0)\right]. \end{cases} \tag{78}$$

Note that Γ_v depends on V and

$$\Gamma_v(x_1, x_2, 0) = 0. \tag{79}$$

The function Γ, on the other hand, does not depend on V but depends on l'. However, l' can be determined, via (73), (48), and Lemma B.1 from the data of the (x_1, x_2)-subsystem only.

[6] For the sake of simplicity, we do not take advantage of the nonpositiveness of $-l'(Q(x_1) + S(x_2))T(x_2)$.

The new Assumptions A0′, A1′, and A3.2′ imply that the function $\mathcal{G}$, defined in (76), is C^1. It follows that there exists a continuous function $\tilde{\mathcal{G}}$ satisfying

$$\langle \tilde{\mathcal{G}}(x_1, x_2, y, u), u\rangle = \mathcal{G}(x_1, x_2, y, u) - \mathcal{G}(x_1, x_2, y, 0). \tag{80}$$

With these notations, (76) and (77) become simply

$$\mathcal{G}(x_1, x_2, y, u) = \kappa(V)\Gamma_v + \Gamma + \langle\tilde{\mathcal{G}}, u\rangle \tag{81}$$

$$\begin{aligned} \dot{U}(x_1, x_2, y, u)_{(40)} \le -l'\left[R + \frac{1}{2}T\right] - \frac{1}{4}\kappa(V)W \\ + \left[\kappa(V)\Gamma_v + \Gamma + \langle\tilde{\mathcal{G}}, u\rangle\right]u. \end{aligned} \tag{82}$$

So, global asymptotic stability can be concluded if u satisfies the constraints

$$\begin{aligned} |y| \neq 0 \Longrightarrow -l'\left[R + \frac{1}{2}T\right] - \frac{1}{4}\kappa(V)W \\ + \left[\kappa(V)\Gamma_v + \Gamma + \langle\tilde{\mathcal{G}}, u\rangle\right]u < 0 \end{aligned} \tag{83}$$

$$\{y = 0,\ \Gamma \neq 0\} \Longrightarrow \left[\Gamma + \langle\tilde{\mathcal{G}}, u\rangle\right]u < 0 \tag{84}$$

$$\{y = 0,\ \Gamma = 0\} \Longrightarrow u = 0. \tag{85}$$

Proposition III.3: Assume (40) satisfies Assumptions A0′, A1′, A2, A3.1), and A3.2′. Under this condition, for any $\bar{u}$ in $(0, +\infty]$, the origin can be made a globally asymptotically stable solution by a static state feedback bounded by $\bar{u}$ and of the form

$$\begin{aligned} u(x_1, x_2, y) = -\beta(x_1, x_2, y) \\ \times [\alpha(x_1, x_2, y)\kappa(V(y))\Gamma_v(x_1, x_2, y) \\ + \Gamma(x_1, x_2, y)]^{\top} \end{aligned} \tag{86}$$

where α and β are any continuous functions satisfying

$$\beta(x_1, x_2, y) \ge 0 \tag{87}$$

$$\Gamma(x_1, x_2, 0) \neq 0 \Longrightarrow \beta(x_1, x_2, 0) > 0 \tag{88}$$

$$\begin{aligned} W(y) \ge 3\beta(x_1, x_2, y)[\alpha(x_1, x_2, y) - 1]^2 \\ \times \kappa(V(y))|\Gamma_v(x_1, x_2, y)|^2 \end{aligned} \tag{89}$$

$$|\tilde{\mathcal{G}}(x_1, x_2, y, u(x_1, x_2, y))|\beta(x_1, x_2, y) \le \frac{1}{3} \tag{90}$$

and β is such that $|u(x_1, x_2, y)|$ is upperbounded by $\bar{u}$.

Proof: We first remark that (79), (88), and (90) imply (84) and (85). So it remains only to establish (83). Since, for any real numbers α, β, b, and c, we have

$$-(b + c)(\alpha b + c) = \frac{(\alpha - 1)^2}{4}b^2 - \left(\frac{\alpha + 1}{2}b + c\right)^2 \tag{91}$$

with (86), (83) is implied by

$$\begin{aligned} |y| \neq 0 \Longrightarrow -\frac{1}{4}\kappa(V)W + \beta^2|\tilde{\mathcal{G}}||\alpha\kappa(V)\Gamma_v + \Gamma|^2 \\ + \beta\left[\frac{(\alpha - 1)^2}{4}|\kappa(V)\Gamma_v|^2 \right. \\ \left. - \left|\frac{\alpha + 1}{2}\kappa(V)\Gamma_v + \Gamma\right|^2\right] < 0. \end{aligned} \tag{92}$$

The function $\kappa(V)W$ being positive definite in y, with (87), (89), and (92) holds if

$$|y| \neq 0 \Longrightarrow \beta|\tilde{\mathcal{G}}||\alpha\kappa(V)\Gamma_v + \Gamma|^2 - \frac{1}{8}[\alpha-1]^2\kappa(V)^2|\Gamma_v|^2 - \left|\frac{\alpha+1}{2}\kappa(V)\Gamma_v + \Gamma\right|^2 \leq 0. \quad (93)$$

Since (90) implies we have here a negative semidefinite quadratic form in $(\kappa(V)\Gamma_v \quad \alpha\kappa(V)\Gamma_v + \Gamma)$, our conclusion follows. □

An important aspect of the expression (86) is that if we can set $\alpha \equiv 0$, the feedback law becomes simply [see (78)]

$$u(x_1, x_2, y) = -\beta(x_1, x_2, y)l'(Q(x_1) + S(x_2)) \times \left[\frac{\partial Q}{\partial x_1}(x_1)h_2(x_1, x_2, y, 0) + \frac{\partial S}{\partial x_2}(x_2)e_2(x_1, x_2, y, 0)\right]. \quad (94)$$

Therefore if an expression, not depending on V, can be found for β, we have reached our objective of finding a feedback law not requiring the explicit knowledge of V. So our new task is to find such a function β.

From Proposition III.3, this function β must satisfy (87)–(90). With (88) and (89) and $\alpha \equiv 0$, a necessary condition for the existence of such a function β is

$$\Gamma(x_1, x_2, 0) \neq 0 \Rightarrow \liminf_{y \to 0} \frac{W(y)}{\kappa(V(y))|\Gamma_v(x_1, x_2, y)|^2} > 0 \quad (95)$$

which, with the definition (78), is guaranteed if

$$\liminf_{y \to 0} \frac{W(y)}{\kappa(V(y))\left|\frac{\partial V}{\partial y}(y)\right|^2} > 0. \quad (96)$$

In fact this latter condition is also sufficient. To see this, let R and $\bar{u}$ be two strictly positive real numbers, and let us introduce two functions independent of V.

1) Let φ_R be a smooth nonnegative function onto $[0, 1]$ such that

$$\varphi_R(0) = 1, \qquad \varphi_R(|y|^2) = 0 \quad \forall y\colon |y| \geq R. \quad (97)$$

2) Let $\psi_{R,\bar{u}}$ be a smooth function satisfying

$$\psi_{R,\bar{u}}(x_1, x_2) \geq \max\left\{1, \sup_{\substack{|u| \leq \bar{u} \\ |y| \leq R}} \{\hat{\psi}(x_1, x_2, y, u)\}\right\} \quad (98)$$

with the function $\hat{\psi}$ defined as

$$\hat{\psi}(x_1, x_2, y, u) = \frac{7}{3}\left|\frac{f_2(x_1, x_2, y, u) - f_2(x_1, x_2, y, 0)}{u}\right| + \left|\frac{\frac{\partial Q}{\partial x_1}(x_1)[h_2(x_1, x_2, y, u) - h_2(x_1, x_2, y, 0)]}{u}\right| + \left|\frac{\frac{\partial S}{\partial x_2}(x_2)[e_2(x_1, x_2, y, u) - e_2(x_1, x_2, y, 0)]}{u}\right| \quad (99)$$

which makes sense since the functions f_2, h_2, and e_2 are C^1 (see A0′).

Proposition III.4: Assume (40) satisfies Assumptions A0′, A1′, A2, A3.1), and A3.2′. Under this condition, if

$$\liminf_{y \to 0} \frac{W(y)}{\kappa(V(y))\left|\frac{\partial V}{\partial y}(y)\right|^2} > 0 \quad (100)$$

then, for any $\bar{u}$ in $(0, +\infty)$, there exists a positive real number $\mu^\star$ in $(0, \bar{u}]$ so that the origin of (40) can be made a globally asymptotically stable solution by a static state feedback bounded by $\bar{u}$ and of the form

$$u(x_1, x_2, y) = -\frac{\mu\varphi_R(|y|^2)\Gamma(x_1, x_2, y)}{\psi_{R,\bar{u}}(x_1, x_2)(1 + |\Gamma(x_1, x_2, y)|)(1 + |f_2(x_1, x_2, y, 0)|^2)} \quad (101)$$

where μ is any real number in $(0, \mu^\star]$ and Γ, φ_R, and $\psi_{R,\bar{u}}$ are defined in (78), (97), and (98), respectively.

Proof: Since (101) is obtained from (86) by letting

$$\begin{cases} \alpha(x_1, x_2, y) = 0, \\ \beta(x_1, x_2, y) = \frac{\mu\varphi_R(|y|^2)}{\psi_{R,\bar{u}}(x_1,x_2)(1+|\Gamma(x_1,x_2,y)|)(1+|f_2(x_1,x_2,y,0)|^2)} \end{cases} \quad (102)$$

it is sufficient to check that (87)–(90) hold. Clearly (87) and (88) are satisfied and, with $\mu \leq \bar{u}$, we have

$$|u(x_1, x_2, y)| \leq \bar{u}. \quad (103)$$

To check that (89) holds, we observe that since W is positive definite, (100) implies the following real number ξ_W is strictly positive:

$$\min_{|y| \leq R} \frac{W(y)}{\kappa(V(y))\left|\frac{\partial V}{\partial y}(y)\right|^2} = \xi_W > 0. \quad (104)$$

Then, since $\alpha \equiv 0$ and (102) implies that $\beta(x_1, x_2, y)$ is zero when $|y|$ is larger than R, the Schwartz inequality and (78) yield

$$3\beta(x_1, x_2, y)\kappa(V(y))|\Gamma_v(x_1, x_2, y)|^2 \leq \frac{49\mu}{3\xi_W}W(y). \quad (105)$$

Hence (89) holds if $\mu < \frac{3}{49}\xi_W$.

To check that (90) holds, we observe that the definition of $\tilde{\mathcal{G}}$, (80), (76), (73), (98), and (103) imply

$$|y| \leq R \Longrightarrow |\tilde{\mathcal{G}}(x_1, x_2, y, u(x_1, x_2, y))| \leq \left(\kappa(V(y))\left|\frac{\partial V}{\partial y}(y)\right| + 2\right)\psi_{R,\bar{u}}(x_1, x_2). \quad (106)$$

This yields, for all (x_1, x_2, y)

$$\psi_{R,\bar{u}}(x_1, x_2)\beta(x_1, x_2, y) \geq \xi_{\tilde{\mathcal{G}}}|\tilde{\mathcal{G}}(x_1, x_2, y, u((x_1, x_2, y)))|\beta(x_1, x_2, y) \quad (107)$$

with the notation

$$\xi_{\tilde{\mathcal{G}}} = \min_{|y| \leq R}\left\{\frac{1}{\kappa(V(y))\left|\frac{\partial V}{\partial y}(y)\right| + 2}\right\}. \quad (108)$$

But (102) gives

$$\psi_{R,\bar{u}}(x_1,x_2)\beta(x_1,x_2,y) = \frac{\mu\varphi_R(|y|^2)}{(1+|\Gamma(x_1,x_2,y)|)(1+|f_2(x_1,x_2,y,0)|^2)} \le \mu. \tag{109}$$

So μ satisfying

$$\mu \le \frac{1}{3}\xi_{\bar{\mathcal{G}}} \tag{110}$$

is sufficient for (90) to hold.

Finally, by taking

$$0 < \mu \le \mu^\star = \min\left\{\frac{3}{49}\xi_W, \frac{1}{3}\xi_{\bar{\mathcal{G}}}, \bar{u}\right\} \tag{111}$$

the assumptions of Proposition III.3 are met which implies the conclusion of Proposition III.4 holds. □

Remark III.5:

1) As already mentioned, the interest of the feedback law (101) is that it does not depend on the functions V and k. But we have the parameter μ to be tuned.
2) Requirement (100) is satisfied in the context of the discussion in Section III-B, i.e., if Assumptions A0′, A1′, A2, and H1) to H3) hold with γ satisfying (61).
3) If f_2, e_2, and h_2 do not depend on u, then $\bar{\mathcal{G}}$ is zero and (101) can be simplified to

$$u(x_1,x_2,y) = -\frac{\mu\varphi_R(|y|^2)\Gamma(x_1,x_2,y)}{(1+|\Gamma(x_1,x_2,y)|)(1+|f_2(x_1,x_2,y,0)|^2)}. \tag{112}$$

IV. Design Tool 2: Change of Coordinates

A. The Context

With Assumption A3, we have defined a context within which the Jurdjevic and Quinn approach can be applied to (40). Our task now is to investigate if there exists an appropriate change of coordinates so that the modified coupling terms f_1, h_1, and e_1 satisfy A3. Stated this way, the problem is difficult. Today, we have no general answer. To solve it here, we limit the field of investigation to a particular subclass of systems in the form (40). In [17], another subclass is considered.

So, now we restrict our attention to the set of assumptions considered in Section III-B.1 and for a system where the undriven x_1 and x_2 subsystems are linear and there is no coupling term f_1

$$\begin{cases} \dot{X}_1 = M_1X_1 + H_1(X_1,X_2,Y)Y + H_2(X_1,X_2,Y,u)u \\ \dot{X}_2 = M_2X_2 + E_1(X_1,X_2,Y)Y + E_2(X_1,X_2,Y,u)u \\ \dot{Y} = F_0(Y) + F_2(X_1,X_2,Y,u)u \end{cases} \tag{113}$$

where Y is in R^n, X_1 in R^{m_1}, X_2 in R^{m_2}, u in R^q. We use capital letters to distinguish the initial coordinates (X_1,X_2,Y) from the transformed ones (x_1,x_2,y) in which Theorem III.1 is applied. We assume the following.

Assumption B0: The functions H_1, H_2, E_1, E_2, F_0, and F_2 are C^3, and F_0 is zero at the origin.

Following the discussion in Section III-B, we know that H_1Y, at least, should have a zero at $Y=0$ of order strictly larger than one and possibly of order two. This is generically not satisfied. So the problem we are addressing now is to find a global diffeomorphism

$$(X_1,X_2,Y) \mapsto (x_1,x_2,y)$$

such that in the new coordinates, the coupling term h_1 is of largest possible order or even that it is absent as it was the case for (41).

B. Change of the X_1-Coordinate

For (113), since with Assumption B0 the function H_1Y is C^3, it can be decomposed as

$$H_1(X_1,X_2,Y)Y = H_{10}(Y) + H_{11}(Y)X_1 + H_{12}(X_1,X_2,Y) \tag{114}$$

where H_{12} is a C^2 function and

$$\begin{aligned} H_{10}(Y) &= H_1(0,0,Y)Y, \\ H_{11}(Y) &= \left\langle \frac{\partial H_1}{\partial X_1}(0,0,Y), Y \right\rangle. \end{aligned} \tag{115}$$

To simplify our task, we look for an appropriate change of coordinates for an auxiliary system

$$\dot{X}_1 = M_1X_1 + H_{10}(Y) + H_{11}(Y)X_1, \quad \dot{Y} = F_0(Y). \tag{116}$$

To preserve linearity in X_1, we restrict ourselves with the following class of transformation:

$$\begin{pmatrix} x_1 \\ y \end{pmatrix} = \begin{pmatrix} \exp(-P_2(Y))[X_1 + P_1(Y)] \\ Y \end{pmatrix} \tag{117}$$

where the matrix function P_2 and the vector P_1 are to be chosen. With these new coordinates (116) is rewritten

$$\dot{x}_1 = M_1x_1 + h_{10}(y) + h_{11}(y)x_1, \quad \dot{y} = f_0(y) \tag{118}$$

where, by using the identity[7]

$$\overbrace{\exp(-P_2(y))}^{\cdot} = -\exp(-P_2(y)) \times \int_0^1 \exp(P_2(y)s)\overbrace{P_2(y)}^{\cdot}\exp(-P_2(y)s)\,ds \tag{119}$$

we have

$$\begin{aligned} h_{11}(y) = &-M_1 + \exp(-P_2(y)) \\ &\times \left[M_1 + H_{11}(y) - \int_0^1 \exp(P_2(y)s) \right. \\ &\left. \times \left\langle \frac{\partial P_2}{\partial y}(y), F_0(y) \right\rangle \exp(-P_2(y)s)\,ds \right] \\ &\times \exp(P_2(y)) \end{aligned} \tag{120}$$

[7] Which is obtained from the identity

$$\frac{d}{d\tau}\exp(-P_2(y)\tau) = -P_2(y)\exp(-P_2(y)\tau).$$

$$h_{10}(y) = \exp(-P_2(y)) \times \left[H_{10}(y) + \frac{\partial P_1}{\partial y}(y)F_0(y) - (M_1 + H_{11}(y))P_1(y)\right] \quad (121)$$

$$f_0(y) = F_0(y). \quad (122)$$

This leads us to the following questions.

1) Given two functions H_{11} and F_0, what can be done on h_{11} defined in (120) with a function P_2?
2) Given four functions H_{11}, F_0, H_{10}, and P_2, what can be done on h_{10} defined in (121) with a function P_1?

Lemma IV.1: Assume B0 holds and let A denote

$$A = \frac{\partial F_0}{\partial Y}(0). \quad (123)$$

If the spectra of A and M_1 are such that

$$\lambda_{A_i} + \lambda_{M_{1j}} \neq \lambda_{M_{1k}} \quad (124)$$

$$\lambda_{A_i} \neq \lambda_{M_{1k}} \quad (125)$$

for any (i, j, k), then there exist smooth functions P_1 and P_2 which give h_{11}, in (120), and h_{10}, in (121), having zeros of order two[8] at $y = 0$, i.e., there exists a C^1 function $\bar{h}_{1y}$ satisfying

$$h_{10}(y) + h_{11}(y)x_1 = \langle \bar{h}_{1y}(x_1, y), y\rangle y. \quad (126)$$

Proof[9]: We remark that by denoting

$$\nabla H_{11(k,i,j)} = \frac{\partial H_{11(i,j)}}{\partial y_k}(0) \quad (127)$$

by letting $(\mathcal{P}_{2(k,i,j)})$ be the solution of the linear system

$$0 = \nabla H_{11(k,i,j)} + \sum_l \left[M_{1(i,l)}\mathcal{P}_{2(k,l,j)} - \mathcal{P}_{2(k,i,l)}M_{1(l,j)} - \mathcal{P}_{2(l,i,j)}A_{(l,k)}\right] \quad (128)$$

and by defining the matrix P_2 as (see [15, I (10.10)])

$$(\exp(P_2(y)) - I)_{(i,j)} = \frac{\sum_k \mathcal{P}_{2(k,i,j)}y_k}{1 + \sum_{i,j}\left|\sum_k \mathcal{P}_{2(k,i,j)}y_k\right|} \quad (129)$$

we obtain a function h_{11} with a zero of order two at $y = 0$. Arguments similar to those used in [2, Proof of Lemma 1.1] show that (128) can be solved in general if and only if (124) holds.

More simply, the function h_{10} in (121) has a zero of order two at $y = 0$ if we choose[10]

$$P_1(y) = Py \quad (130)$$

where P is the solution of the linear system

$$PA - M_1P + \frac{\partial H_{10}}{\partial y}(0) = 0 \quad (131)$$

which can be solved in general if and only if (125) holds (see [8, Sec. 8.1]). □

When $H_{11}(y) = 0$, in (121), we get $h_{10}(y) = 0$ if we can find a function P_1 solving the following partial differential equation:

$$-M_1P_1(y) + H_{10}(y) + \frac{\partial P_1}{\partial y}(y)F_0(y) = 0. \quad (132)$$

This case is interesting since the Jurdjevic and Quinn approach, i.e., Theorem II.1, applies in the new coordinates. We remark that if (132) holds, then the graph $\{(X_1, Y): X_1 + P_1(Y) = 0\}$ is an invariant set of

$$\dot{X}_1 = M_1X_1 + H_{10}(Y), \qquad \dot{Y} = F_0(Y). \quad (133)$$

If the matrix A is asymptotically stable, this graph is a subset of the stable manifold of the origin and even the stable manifold itself if all the eigenvalues of M_1 have zero real part. So, in this latter case, the partial differential equation (132) has a solution at least on a neighborhood of the origin. In fact, here we can exploit the triangular structure of (133) to prove that when the following integral makes sense:

$$P_1(Y) = \int_0^\infty \exp(-sM_1)H_{10}(\Phi(s, Y))ds \quad (134)$$

and is C^1, with $\Phi(t, Y)$ the solution of

$$\frac{\partial \Phi}{\partial t}(t, Y) = F_0(\Phi(t, Y)), \quad \Phi(0, Y) = Y \quad (135)$$

then $P_1(Y)$ is a solution of (132) [see (249)]. This has been remarked by Yang in [29]. Precisely, given (134) and out of the context of this paper, we have the following.

Lemma IV.2: If the matrix $\frac{\partial F_0}{\partial Y}(0)$ is asymptotically stable, the matrix $-M_1$ is stable and the function H_{10} is C^1, then P_1, given by (134), makes sense and is C^1 and a solution of (132).

Proof: See Appendix C.

Remark IV.3:

1) When M_1 is equal to zero, (134) gives

$$\overbrace{P_1(Y)}^{\cdot}{}_{(133)} = -H_{10}(Y) \quad (136)$$

meaning that $H_{10}(Y)$ is a total derivative. This property is of prime importance in the applications. It will be extensively used for the cart-pendulum system in Section V-C.

2) The existence result, given by Lemma IV.2, has been exploited by Yang in [29], and Sontag and Sussmann in [23], to prove global asymptotic stabilizability by saturated feedback of globally null-controllable linear systems via a Lyapunov technique similar to the one used in Theorem II.1 (with x replaced by $X + P_1(Y)$).

3) When X is of dimension 1 with $M_1 = 0$, the matrix A is asymptotically stable and the function $H_{11}(Y)$ is C^1, the (scalar) function P_2, given as

$$P_2(Y) = -\int_0^\infty H_{11}(\Phi(s, Y))ds \quad (137)$$

[8] By exploiting Poincaré normal form theory one can make these functions of higher orders by introducing other nonresonance conditions (see the proof of [2, Th. 3.1]).

[9] The formulas (129) and (130) may not be the most appropriate for practice. The only important point to retain about the functions P_1 and P_2 is that $\frac{\partial P_1}{\partial y}(0)$ and $\frac{\partial P_2}{\partial y}(0)$ are imposed.

[10] (130) is the change of coordinates proposed by Teel in [26, (21)].

with $\Phi(s, Y)$, the solution of (135), well defined (see Appendix C). In this case, we get [see (120)]

$$h_{11}(y) = 0. \tag{138}$$

We remark also that (137) gives

$$\overbrace{P_2(Y)}^{\cdot}_{(133)} = H_{11}(Y) \tag{139}$$

meaning that $H_{11}(Y)$ is a total derivative.

V. Applications

With the tools we have proposed in the previous sections, we are now equipped to grapple with various global asymptotic stabilization problems as those stated in the first section. We begin with a generalization of the question of adding one integration [see (2)]. This will be followed by a solution for feedforward systems [see (3)]. Finally, we shall illustrate the various aspects of the proposed design by studying the cart-pendulum system.

A. Adding Integration

Result: We consider (113) again under Assumption B0, with the notation (123) and the decomposition (114). We introduce the following assumptions.

Assumption B1:

B1.1) The point $Y = 0$ is a globally asymptotically stable equilibrium point of the Y-subsystem when u is set to zero.

B1.2) The matrices A and M_2 are asymptotically stable, the matrix M_1 is stable, and the spectra of these matrices are such that for any (i, j, k)

$$\lambda_{A_i} + \lambda_{M_{1j}} \neq \lambda_{M_{1k}}, \quad \lambda_{A_i} \neq \lambda_{M_{1k}}. \tag{140}$$

Assumption B1.2) implies the existence of positive definite symmetric matrices Q_1 and Q_2 satisfying

$$Q_1 M_1 + M_1^\top Q_1 \leq 0, \quad Q_2 M_2 + M_2^\top Q_2 < 0 \tag{141}$$

and of P and $\mathcal{P}_2$, solutions of, respectively, (131) and (128). (See Lemma IV.1). Then let $\mathcal{H}_2$ be

$$\begin{aligned}\mathcal{H}_2(X_1)u = &-\langle \mathcal{P}_2, F_2(X_1, 0, 0, 0)u\rangle X_1 \\ &+ [H_2(X_1, 0, 0, 0) + PF_2(X_1, 0, 0, 0)]u. \end{aligned}\tag{142}$$

Assumption B2: $X_1 = 0$ is the only solution of

$$\dot{X}_1 = M_1 X_1, \; X_1^\top Q_1 M_1 X_1 = 0, \; X_1^\top Q_1 \mathcal{H}_2(X_1) = 0. \tag{143}$$

Assumption B3: The functions H_{12} and E_1 are such that there exists a nonnegative continuous function γ satisfying

$$\begin{aligned}H_{12}(X_1, X_2, Y) &\leq |Y|[|X_2| + |Y|(1 + |X_1| + |X_2|)]\gamma(Y) \\ E_1(X_1, X_2, Y) &\leq (1 + |X_1| + |X_2|)\gamma(Y). \end{aligned}\tag{144}$$

Theorem V.1: If[11] assumptions B0 to B3 hold, then for any $\bar{u}$ in $(0, +\infty]$, the origin of (113) can be made a globally asymptotically stable solution by a C^3 state feedback bounded by $\bar{u}$ and zero at the origin. Moreover, if the linearization of (113) is stabilizable, the linearized closed-loop system is asymptotically stable. Finally, in the case where the X_1 component is not present, the origin of (113) with $u = 0$ is globally asymptotically stable.

Discussion:

1) Assumptions B1 and B3 give guidelines on how to decompose the "integrating" coordinates X into X_1 and X_2. First, the coupling terms H_1 and E_1 can grow at most linearly in X at infinity. Second, the decomposition must be done so that matrix M_2 is asymptotically stable and matrix M_1 is only stable but satisfying the spectral separation (140). Finally, the remainder H_{12} in (114), when divided by $|Y|$, should vanish with X_2 and Y.
2) We observe that Assumption B2 involves the function $\mathcal{H}_2(X_1)$ and not the function $H_2(X_1, 0, 0, 0)$ as would be the case with A2. This is a consequence of the change of coordinates. Unfortunately, there is no guarantee that B2 holds if A2 holds and vice versa. For instance, consider the following system proposed in [10]:

$$\dot{X}_1 = Y^3 - Y + u, \quad \dot{Y} = -Y + u. \tag{145}$$

Assumption A2 is satisfied but not A3. The change of coordinates given by Lemma IV.1 is

$$x_1 = X_1 - y, \quad y = Y. \tag{146}$$

Then the system is rewritten as

$$\dot{x} = y^3, \quad \dot{y} = -y + u. \tag{147}$$

This time Assumption A3 holds but not A2.[12]

3) Another important constraint imposed by Assumption B1.1) is the asymptotic stability of A. It is known to be superfluous in some cases. In our general context, the properties of A are used:
   a) to make H_{10} and H_{11} have a zero of order two at $y = 0$, as discussed in Lemma IV.1. But in this case, we need only the nonresonance condition (140);
   b) to guarantee that (57) holds to make sure that we can find a function κ satisfying the requirements in Assumption A3. But, if we can make the change of coordinates (117) so that H_{10} and H_{11} have a zero of high order at $y = 0$, then Assumption A3 may hold without the need of asymptotic stability of A. However, the existence of this particular change of coordinates will involve more nonresonance conditions than simply (140).

[11] If there is no term $H_{12}(X_1, X_2, Y)$ in the decomposition (114), we can replace C^3 by C^2 in B0. The resulting feedback is C^2 in that case.

[12] A possible solution is simply to change the control as $u = Y - Y^3 + v$ and to apply Theorem II.1.

Proof of Theorem V.1:

1) *Global asymptotic stability*: To prove the first point of Theorem V.1, we check that after a change of coordinates, Theorem III.1 applies

Using (114) and the fact that H_1 is C^3, B3 implies the existence of continuous functions H_{12Y} and H_{12X} such that

$$H_{12}(X_1, X_2, Y) = [\langle H_{12Y}(X_1, X_2, Y), Y\rangle + \langle H_{12X}(X_1, X_2, Y), X_2\rangle]Y. \quad (148)$$

Then, since the spectra of A and M_1 verify (140), Lemma IV.1 gives functions P_1 and P_2 so that by applying the change of coordinates, linear in (X_1, X_2)

$$\begin{pmatrix} x_1 \\ x_2 \\ y \end{pmatrix} = \begin{pmatrix} \exp(-P_2(Y))(X_1 + P_1(Y)) \\ X_2 \\ Y \end{pmatrix} \quad (149)$$

(113) can be rewritten, with (126) and (148), in the form

$$\begin{cases} \dot{x}_1 = M_1 x_1 + \langle h_{1y}(x_1, x_2, y), y\rangle y \\ \qquad + \langle h_{1x}(x_1, x_2, y), x_2\rangle y + h_2(x_1, x_2, y, u)u \\ \dot{x}_2 = M_2 x_2 + e_1(x_1, x_2, y)y + e_2(x_1, x_2, y, u)u \\ \dot{y} = f_0(y) + f_2(x_1, x_2, y, u)u \end{cases} \quad (150)$$

where h_2, e_2, and f_2 are C^3 and all the other functions are at least continuous so that Assumption A0 holds.

Then, we have that B1.1) and (141) imply that A1 holds with, as mentioned in Section III-B 1, functions V and W satisfying (57) and V of class possibly C^4.

Second, we remark that P_1 and P_2, given by Lemma IV.1, satisfy

$$P_1(Y) = PY, \quad \frac{\partial P_2}{\partial Y}(0) = \mathcal{P}_2. \quad (151)$$

It follows that A2 in the new coordinates is nothing but B2 with:

$$Q(x_1) = |x_1|^2_{Q_1}, \quad h_0(x_1) = M_1 x_1, \\ h_2(x_1, 0, 0, 0) = \mathcal{H}_2(x_1) \quad (152)$$

where $\mathcal{H}_2$ is defined in (142).

Third, we see, with B3 and the linearity in (X_1, X_2) of (149), that there exists a nonnegative continuous function $\bar{\gamma}$ such that

$$|h_{1y}(x_1, x_2, y)| \le (1 + |x_1| + |x_2|)\bar{\gamma}(y) \quad (153)$$
$$|h_{1x}(x_1, x_2, y)| \le \bar{\gamma}(y) \quad (154)$$
$$|e_1(x_1, x_2, y)| \le (1 + |x_1| + |x_2|)\bar{\gamma}(y). \quad (155)$$

This yields

$$\begin{aligned} &\left|x_1^\top Q_1 \langle h_{1y}(x_1, x_2, y), y\rangle y\right| \\ &\quad \le c|y|^2\bar{\gamma}(y)|x_1|(1 + |x_1| + |x_2|) && (156) \\ &\quad \le c|y|^2\bar{\gamma}(y)\left(1 + \sqrt{|x_1|^2_{Q_1} + |x_2|^2_{Q_2}}\right)^2 && (157) \\ &\left|x_1^\top Q_1 \langle h_{1x}(x_1, x_2, y), x_2\rangle y\right| \\ &\quad \le c|y|\bar{\gamma}(y)|x_1||x_2| && (158) \\ &\quad \le c|y|\bar{\gamma}(y)|x_2|\left(1 + \sqrt{|x_1|^2_{Q_1} + |x_2|^2_{Q_2}}\right) && (159) \end{aligned}$$

$$\left|x_2^\top Q_2 e_1(x_1, x_2, y)y\right| \le c|y|\bar{\gamma}(y)|x_2|\left(1 + \sqrt{|x_1|^2_{Q_1} + |x_2|^2_{Q_2}}\right). \quad (160)$$

So by taking

$$\rho(s) = \sqrt{s} \quad (161)$$
$$\kappa(s) \ge 1 + c \sup_{\{y: V(y) \le s\}} \left\{ \frac{|y|^2\left[\bar{\gamma}(y)^2 + \bar{\gamma}(y)\right]}{W(y)} \right\} \quad (162)$$

the main inequality (44) of A3.1) holds. We also have the following.

a) The function ρ is nonnegative and continuous on $[0, +\infty)$ and satisfies (45).

b) With (57) and the fact that W is positive definite, the function κ can be chosen nonnegative and C^3 on $[0, +\infty)$ and to satisfy (46) and A3.2.

We conclude that A3 holds.

Theorem III.1 applies and guarantees the existence of a C^3 globally stabilizing feedback law. Note that since B1 implies that (100) holds, a possible feedback law is (101). Let us finally recall that (74) gives an appropriate Lyapunov function with negative definite time derivative if

$$\left|x_1^\top Q_1 M_1 x_1\right| + \left|x_1^\top Q_1 \mathcal{H}_2(x_1)\right| \ne 0 \quad \forall x_1 \ne 0. \quad (163)$$

2) *Local exponential stability:* To prove the asymptotic stability of the linearized closed-loop system, we write the linearization of (150) at the origin

$$\dot{x}_1 = M_1 x_1 + \mathcal{D}_1 u, \quad \dot{x}_2 = M_2 x_2 + \mathcal{D}_2 u, \quad \dot{y} = Ay + Bu \quad (164)$$

with the notations

$$B = f_2(0, 0, 0, 0), \quad \mathcal{D}_1 = h_2(0, 0, 0, 0) \\ \mathcal{D}_2 = e_2(0, 0, 0, 0). \quad (165)$$

To prove that the linearization of the control given, for instance by (86), is stabilizing this system, we proceed in two steps.

a) We apply Theorem II.1 to obtain a linear feedback u_L for this linear system (164).

b) We check that this linear feedback u_L is nothing but the linearization at the origin of (86).

Step 1: We first remark that (164) is of the form (18). Then Assumption B1.2) implies that A1 holds. Also, the assumed stabilizability of the linearization of (113) implies the stabilizability of the pair $(M_1, \mathcal{D}_1)$. This fact with Lemma II.3 implies A2 holds. From (86) in Proposition III.3, the following linear feedback globally asymptotically stabilizes the origin of (164):

$$u_L(x, y) = -\beta_0\left(\kappa_0\alpha_0 B^\top \frac{\partial^2 V}{\partial y^2}(0)y + l_0\mathcal{D}^\top Qx\right) \quad (166)$$

with the notations

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \mathcal{D} = \begin{pmatrix} \mathcal{D}_1 \\ \mathcal{D}_2 \end{pmatrix}, \quad Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix} \quad (167)$$

and where:

a) V is the Lyapunov function satisfying (21) and (57) and therefore

$$\frac{\partial^2 V}{\partial y^2}(0)A + A^\top \frac{\partial^2 V}{\partial y^2}(0) < 0 \tag{168}$$

b) κ_0 and l_0 are any strictly positive real numbers;

c) α_0 and β_0 are any real numbers satisfying

$$\begin{cases} \beta_0 > 0, \\ -\frac{\lambda_{\min}\left\{\frac{\partial^2 V}{\partial y^2}(0)A + A^\top \frac{\partial^2 V}{\partial y^2}(0)\right\}}{\kappa_0 \lambda_{\max}\left\{\frac{\partial^2 V}{\partial y^2}(0)BB^\top \frac{\partial^2 V}{\partial y^2}(0)\right\}} \geq 3\beta_0[\alpha_0 - 1]^2. \end{cases} \tag{169}$$

Step 2: With (87)–(90), we can take

$$\begin{aligned} \kappa_0 &= \frac{7}{3}\kappa(0), & l_0 &= l'(0) \\ \alpha_0 &= \alpha(0,0,0), & \beta_0 &= \beta(0,0,0) \end{aligned} \tag{170}$$

to satisfy (169). So, the linear approximation of (86) at the origin is equal to (166).

3) *No X_1 component:* When X_1 is not present and u is set to zero, (141) and (144) imply that the X_2 subsystem with Y as input is convergent input bounded state (CIBS) as defined in [22]. It follows that the last point of Theorem V.1 is a direct consequence of [22, Th.] and that asymptotic stability of A is in fact not needed in this case. □

B. Feedforward System

Theorem V.1 can be used repeatedly to prove that the following system is globally asymptotically stabilizable:

$$\begin{cases} \dot{x}_n = h_{0n}(y_{n-1})x_n + h_{1n}(y_{n-1}) + h_{2n}(x_n, y_{n-1}, v)v \\ \vdots \\ \dot{x}_1 = h_{01}(y_0)x_1 + h_{11}(y_0) + h_{21}(x_1, y_0, v)v \\ \dot{y}_0 = \bar{f}_0(y_0) + \bar{f}_{20}(y_0, v)v \end{cases} \tag{171}$$

where y_0 is in R^n, x_i in R^{m_i}, u in R^q, and with

$$y_i = \left(x_i^\top, x_{i-1}^\top, \ldots, x_1^\top, y_0^\top\right)^\top. \tag{172}$$

We introduce the following assumption.

Assumption C0: The functions $\bar{f}_0$, h_{0i}, h_{1i} are C^2, the functions $\bar{f}_{20}$ and h_{2i} are C^3, and h_{1i} and $\bar{f}_0$ are zero at the origin.

This assumption allows us to introduce the following notations:

$$\begin{cases} M_i = h_{0i}(0), \quad C_i = \frac{\partial h_{1i}}{\partial y_{i-1}}(0), \quad D_i = h_{2i}(0,0,0), \\ A_0 = \frac{\partial \bar{f}_0}{\partial y_0}(0), \quad B_0 = \bar{f}_{20}(0,0). \end{cases} \tag{173}$$

Theorem V.2: Assume C0 holds and:

C1) There exists a C^2 feedback law $v_0(y_0)$, with $v_0(0) = 0$, which globally asymptotically stabilizes the origin of the y_0-subsystem of (171) and so that the linearized closed-loop system is asymptotically stable.

C2) For any i in $\{1, \cdots, n\}$, the matrix M_i is such that there exists a positive definite matrix Q_i satisfying

$$Q_i M_i + M_i^\top Q_i = 0. \tag{174}$$

C3) The pair

$$\left(\begin{array}{cccc} M_n & \boxed{\qquad C_n \qquad} & & \\ 0 & M_{n-1} & \boxed{\quad C_{n-1} \quad} & \\ & & \ddots & \\ & & M_1 & \boxed{C_1} \\ & & 0 & A_0 \end{array}\right), \quad \begin{pmatrix} D_n \\ D_{n-1} \\ \vdots \\ D_1 \\ B_0 \end{pmatrix}$$

is stabilizable.

C4) For any i in $\{1, \cdots, n\}$, the function h_{2i} satisfies, for all (x_i, y_{i-1}, v)

$$\frac{\partial h_{2i}}{\partial x_i}(x_i, 0, 0) = 0, \quad \frac{\partial^2 h_{2i}}{\partial x_i^2}(x_i, y_{i-1}, v) = 0. \tag{175}$$

Under these conditions, for any $\bar{u}$ in $(0, +\infty]$, the origin can be made a globally asymptotically stable solution of the system (171) by a C^2 state feedback bounded by $\bar{u} + \sup_{y_0}\{|v_0(y_0)|\}$, and the linearized closed-loop system is asymptotically stable.

Remark V.3:

1) Going back to (3), we see that the assumptions of Theorem V.2 are satisfied if $\dot{x} = f_1(x, u)$ is globally asymptotically stabilizable with local exponential stability and the linearization at the origin of (3) is stabilizable.

2) Theorem V.2 is just one of the many statements which can be obtained by repeatedly applying Theorem III.1. Note in particular that (174) is restrictive. This is made with the purpose of verifying more easily that the spectral condition B1.2) holds.

Proof of Theorem V.2: We prove this theorem by induction. We will call i-system the subsystem of (171) whose state vector is $y_i = (y_0, x_1, \ldots, x_i)$ and which we rewrite in more compact form as

$$\dot{y}_i = \bar{f}_i(y_i) + \bar{f}_{2i}(y_i, v)v. \tag{176}$$

Induction Assumption: The functions $\bar{f}_i$ and $\bar{f}_{2i}$ are C^2, and the origin can be made a globally asymptotically stable solution of the system (176) by a C^2 state feedback $v_i(y_i)$ bounded by $\frac{i}{n}\bar{u} + \sup_{y_0}\{|v_0(y_0)|\}$, with $v_i(0) = 0$, and the linearized closed-loop system is asymptotically stable.

This assumption is satisfied for $i = 0$ thanks to assumptions C0 and C1). To prove that it holds also for $i + 1$, we first remark that the functions

$$\begin{cases} \bar{f}_{i+1}(y_{i+1}) = \begin{pmatrix} h_{0i+1}(y_i)x_{i+1} + h_{1i+1}(y_i) \\ \bar{f}_i(y_i) \end{pmatrix} \\ \bar{f}_{2i+1}(y_{i+1}, v) = \begin{pmatrix} h_{2i+1}(x_{i+1}, y_i, v) \\ \bar{f}_{2i}(y_i, v) \end{pmatrix} \end{cases} \tag{177}$$

are C^2 as a direct consequence of the induction assumption and C0. Then let us show that Theorem V.1 applies.

1) With v_i the feedback law given by the induction assumption, we introduce following notations:

$$\begin{cases} u = v - v_i(y_i) \\ h_0(y_i) = h_{0i+1}(y_i) + \left\langle \frac{\partial h_{2i+1}}{\partial x_{i+1}}(0, y_i, v_i(y_i)), v_i(y_i) \right\rangle \\ h_1(y_i) = h_{1i+1}(y_i) + h_{2i+1}(0, y_i, v_i(y_i))v_i(y_i) \\ h_2(x_{i+1}, y_i, u) = h_{2i+1}(x_{i+1}, y_i, v_i(y_i) + u) \\ \qquad + \langle \left(\int_0^1 \frac{\partial h_{2i+1}}{\partial u}(x_{i+1}, y_i, lu + v_i(y_i)) dl \right), v_i(y_i) \rangle \\ f_0(y_i) = \bar{f}_i(y_i) + \bar{f}_{2i}(y_i, v_i(y_i))v_i(y_i) \\ f_2(y_i, u) = \bar{f}_{2i}(y_i, u + v_i(y_i)) \\ \qquad + \langle \left(\int_0^1 \frac{\partial \bar{f}_{2i}}{\partial u}(y_i, lu + v_i(y_i)) dl \right), v_i(y_i) \rangle. \end{cases} \tag{178}$$

Also note that f_2 does not depend on x_{i+1}. All the functions appearing on the left-hand side are C^2. Since (175) implies that h_{2i+1} is linear in x_{i+1}, the $i+1$ system can be written in the form

$$\begin{cases} \dot{x}_{i+1} = h_0(y_i) + h_1(y_i)x_{i+1} + h_2(x_{i+1}, y_i, u)u \\ \dot{y}_i = f_0(y_i) + f_2(y_i, u)u \end{cases} \tag{179}$$

which is a simpler version of (113), with, in particular

$$M = M_{i+1}, \quad C = C_{i+1} + D_{i+1}K_i, \quad D = D_{i+1} \tag{180}$$

and

$$\begin{cases} A = \begin{pmatrix} M_i & & C_i \\ & \ddots & \\ & & M_1 & C_1 \\ & & 0 & A_0 \end{pmatrix} + \begin{pmatrix} D_i \\ \vdots \\ D_1 \\ B_0 \end{pmatrix} K_i \\ B = f_2(0,0) = \begin{pmatrix} D_i \\ \vdots \\ D_1 \\ B_0 \end{pmatrix} \end{cases} \tag{181}$$

where

$$K_i = \frac{\partial v_i}{\partial y_i}(0). \tag{182}$$

2) Assumption B1.1) follows from the induction assumption.
3) Since the matrix A is given by the linearization of the closed-loop system

$$\dot{y}_i = \bar{f}_i(y_i) + \bar{f}_{2i}(y_i, v_i(y_i))v_i(y_i) \tag{183}$$

its asymptotic stability is given by the induction assumption. So with (174) Assumption B1.2) holds.
4) Assumption B2 is a consequence of Assumption C3), Lemma II.2, and the fact that (175) and f_2, not depending on x_{i+1}, imply

$$\mathcal{H}_2(x_{i+1}) = h_{2i+1}(x_{i+1}, 0, 0) = D. \tag{184}$$

5) Assumption B3 is trivially satisfied.

So, with Theorem V.1, we get $u_{i+1}(y_i, x_{i+1})$, a C^2 state feedback, bounded by $\frac{1}{n}\bar{u}$, such that the induction assumption is satisfied for the $i+1$ system. This completes the proof of Theorem V.2. □

C. The Cart-Pendulum System

The cart-pendulum system, whose dynamics can be expressed as in (6), is a good example to illustrate some aspects of the designs which can be done by combining the tools described in Sections II, III, and IV.

The procedure proposed in Section V-B goes with first stabilizing the (θ_0, ω_0) subsystem. Then we add one integration for the stabilization of the $(s_0, \theta_0, \omega_0)$ subsystem. Finally, a last integration will give us the full system. However, we remark that for the first step, the (θ_0, ω_0) subsystem is living in the cylinder $S^1 \times R$. The topology of this manifold as well as the presence of $\cos(\theta_0)$ multiplying u_0 lead us to restrict our attention to $(-\frac{\pi}{2}, \frac{\pi}{2}) \times R$. Then to make our problem a global stabilization problem we let

$$t_0 = \tan(\theta_0), \qquad r_0 = (1 + t_0^2)\omega_0. \tag{185}$$

This allows us to rewrite (6) as

$$\begin{cases} \overset{\circ}{x}_0 = s_0, \quad \overset{\circ}{s}_0 = u_0, \quad \overset{\circ}{t}_0 = r_0 \\ \overset{\circ}{r}_0 = \frac{2t_0r_0^2}{1+t_0^2} + t_0\sqrt{1+t_0^2} - u_0\sqrt{1+t_0^2} \end{cases}. \tag{186}$$

Following Section IV, at each step of adding an integration, an appropriate change of coordinates will be needed. With Remark IV.3, we know that this change of coordinates is easily found when total derivatives are known. So let us start by writing a repertory of some total derivatives not depending on the control

$$\begin{cases} \overset{\circ}{x}_0 = s_0, \quad \overset{\circ}{t}_0 = r_0, \\ \overset{\circ}{\overbrace{s_0 + \frac{r_0}{\sqrt{1+t_0^2}}}} = t_0\left(1 + \frac{r_0^2}{(1+t_0^2)^{\frac{3}{2}}}\right) \\ \overset{\circ}{\overbrace{\ln\left(t_0 + \sqrt{1+t_0^2}\right)}} = \frac{r_0}{\sqrt{1+t_0^2}}. \end{cases} \tag{187}$$

The (t_0, r_0) Subsystem: Since the control u_0 is integrated in (186), we propose the following feedback which, from the list of (187), is a total derivative:

$$u_0(t_0, r_0) = 2t_0\left(1 + \frac{r_0^2}{(1+t_0^2)^{\frac{3}{2}}}\right) + r_0. \tag{188}$$

It is stabilizing as can be seen with

$$\begin{cases} V_0(t_0, r_0) = r_0^2 + \frac{1}{3}\left((1+t_0^2)^{\frac{3}{2}} - 1\right) + r_0t_0, \\ \overset{\circ}{V}_{0((186),(188))} \le -\frac{1}{2}(r_0^2 + t_0^2)\sqrt{1+t_0^2}. \end{cases} \tag{189}$$

The (s_0, t_0, r_0) Subsystem: To write this subsystem in the form (18), we let, using (187)

$$\begin{cases} x_1 = x_0 + 2\ln\left(t_0 + \sqrt{1+t_0^2}\right), \\ s_1 = s_0 + 2\frac{r_0}{\sqrt{1+t_0^2}} + t_0, \\ t_1 = t_0, \quad r_1 = r_0, \\ u_1 = u_0 - 2t_0\left(1 + \frac{r_0^2}{(1+t_0^2)^{\frac{3}{2}}}\right) - r_0. \end{cases} \tag{190}$$

This yields a (s_1, t_1, r_1) subsystem in the form (18)

$$\begin{cases} \overset{\circ}{x}_1 = s_1 - t_1, \quad \overset{\circ}{s}_1 = -u_1, \quad \overset{\circ}{t}_1 = r_1 \\ \overset{\circ}{r}_1 = -(t_1 + r_1)\sqrt{1+t_1^2} - u_1\sqrt{1+t_1^2} \end{cases}. \tag{191}$$

So the technique of Theorem II.1 applies. In particular, an appropriate Lyapunov function is

$$V_1(s_1,t_1,r_1) = V_0(t_1,r_1) + l\big(s_1^2\big). \tag{192}$$

However, once again, in designing the feedback u_1, we have to think of the next step where $s_1 - t_1$ is integrated in the system (191). Then we remark that by letting

$$u_1(s_1,t_1,r_1) = \frac{1}{10}s_1 \tag{193}$$

we get a new total derivative, depending this time on the feedback we use

$$\overset{\circ}{s}_1 = -\frac{1}{10}s_1. \tag{194}$$

This feedback is stabilizing as can be seen by taking

$$l\big(s_1^2\big) = 5s_1^2 + \frac{1}{6}|s_1|^3. \tag{195}$$

Indeed, in this case, we get

$$\begin{aligned}\overset{\circ}{V}_{1((191),(193))} \le& -\frac{1}{2}\big(r_1^2+t_1^2\big)\sqrt{1+t_1^2} - \left[s_1^2 + \frac{1}{20}|s_1|^3\right]\\ &+ \frac{1}{10}[2r_1+t_1]\sqrt{1+t_1^2}s_1. \end{aligned}\tag{196}$$

But we have, using Young's inequality

$$\begin{aligned}
&|2r_1+t_1|\sqrt{1+t_1^2}s_1 &(197)\\
&= |2r_1+t_1|\left[\sqrt{1+t_1^2}-1\right]s_1 + |2r_1+t_1|s_1 &(198)\\
&\le \left(\frac{2\sqrt{5}}{3}\right)^{\frac{3}{2}}\left[r_1^2+t_1^2\right]^{\frac{3}{4}}\left[\sqrt{1+t_1^2}-1\right]^{\frac{3}{2}}\\
&\quad + \frac{1}{2}|s_1|^3 + \frac{1}{4}\left[r_1^2+t_1^2\right] + 5s_1^2 &(199)\\
&\le \left(\frac{2\sqrt{5}}{3}\right)^{\frac{3}{2}} \frac{\left[r_1^2+t_1^2\right]\sqrt{1+t_1^2}|t_1|^3}{\sqrt{1+t_1^2}\left[\sqrt{1+t_1^2}+1\right]^{\frac{3}{2}}\left[r_1^2+t_1^2\right]^{\frac{1}{4}}}\\
&\quad + \frac{1}{2}|s_1|^3 + \frac{1}{4}\left[r_1^2+t_1^2\right] + 5s_1^2 &(200)\\
&\le \left[\left(\frac{2\sqrt{5}}{3}\right)^{\frac{3}{2}} + \frac{1}{4}\right]\\
&\quad \times \left[r_1^2+t_1^2\right]\sqrt{1+t_1^2} + 5s_1^2 + \frac{1}{2}|s_1|^3. &(201)
\end{aligned}$$

We conclude

$$\begin{aligned}-W_1(s_1,t_1,r_1) =& \overset{\circ}{V}_{1((191),(193))}\\ \le& -\frac{1}{4}\big(r_1^2+t_1^2\big)\sqrt{1+t_1^2} - \frac{1}{2}s_1^2. \end{aligned}\tag{202}$$

The (x_0,s_0,t_0,r_0) System: From Lemma IV.2, we know there exist coordinates allowing us to write this system in the form (18). But we have not found the explicit expression for the change of coordinates. Instead, we look for obtaining the form (118). For this, inspired by (116) and Lemma IV.1, and using (187) and (194), we let

$$\begin{cases} x_2 = x_1 + 10s_1 + \left(s_0 + \frac{r_0}{\sqrt{1+t_0^2}}\right),\\ s_2 = s_1, \quad t_2 = t_1, \quad r_2 = r_1,\\ u_2 = u_1 - \frac{1}{10}s_1. \end{cases}\tag{203}$$

We get

$$\begin{cases} \overset{\circ}{x}_2 = -10u_2 + \frac{t_2r_2^2}{\left(1+t_2^2\right)^{\frac{3}{2}}},\\ \overset{\circ}{s}_2 = -\frac{1}{10}s_2 - u_2,\\ \overset{\circ}{t}_2 = r_2,\\ \overset{\circ}{r}_2 = -\big(t_2+r_2+\frac{1}{10}s_2\big)\sqrt{1+t_2^2} - u_2\sqrt{1+t_2^2}. \end{cases}\tag{204}$$

This is the form (118) with a third order term $\frac{t_2r_2^2}{(1+t_2^2)^{\frac{3}{2}}}$. Using (202), we get

$$\left|\frac{t_2r_2^2}{\left(1+t_2^2\right)^{\frac{3}{2}}}\right| \le \frac{4}{3}W_1(s_2,t_2,r_2) \tag{205}$$

which follows from the implication:

$$\begin{aligned}&\left\{\frac{3}{4} \le \left(t^2-|t|+1\right), 0 \le (1-|t|)^4\right\}\\ &\quad\Longrightarrow \left\{3|t| \le 4|t|\left(t^2-|t|+1\right) \le \left(1+t^2\right)^2\right\}. \end{aligned}\tag{206}$$

So the technique of Theorem III.1 applies. In particular, a possible Lyapunov function is

$$V_2(x_2,s_2,t_2,r_2) = 2V_1(s_2,t_2,r_2) + \int_0^{|x_2|}\sigma(s)ds \tag{207}$$

where σ is any continuous odd function satisfying

$$\begin{cases}\sigma(0)=0, \quad 0 < s\sigma(s) \le |s| \ \forall s \ne 0,\\ \liminf_{s\to+\infty}\sigma(s) > 0.\end{cases}\tag{208}$$

By using (205), we get

$$\begin{aligned}\overset{\circ}{V}_2 \le& -\frac{2}{3}W_1(s_2,t_2,r_2)\\ &- \left[2\left([2r_2+t_2]\sqrt{1+t_2^2} + s_2\left[10+\frac{1}{2}|s_2|\right]\right)\right.\\ &\left.+ 10\sigma(x_2)\right]u_2. \end{aligned}\tag{209}$$

So, by choosing the function $u_2(x_2,s_2,t_2,r_2)$ with sign opposite to the sign of

$$-\left[2\left([2r_2+t_2]\sqrt{1+t_2^2} + s_2\left[10+\frac{1}{2}|s_2|\right]\right) + 10\sigma(x_2)\right]$$

when the term between brackets is not zero, we finally get a globally stabilizing feedback for (186) as

$$u = u_0(t_0,r_0) + u_1(s_1,t_1,r_1) + u_2(x_2,s_2,t_2,r_2). \tag{210}$$

For the cart-pendulum system this implies that asymptotic stability can be guaranteed provided the initial deviation from the upward position for the pendulum is strictly less than 90°.

Remark V.4:

1) This result of asymptotic stabilization on the upper half space is not new and can be found at least in [3].
2) For the sake of clarity, we have not introduced any parameters in our feedback law. By introducing them we would get degrees of freedom allowing us to modify the behavior.
3) We have tried to take full advantage of the dynamics of the system with the systematic use of total derivatives in our change of coordinates. But, for both the (s_0, t_0, r_0) and the (x_0, s_0, t_0, r_0) systems, we could also have solved the corresponding linear equation (131) and used the feedback law given by (101) which, in this particular case where h_1, h_2, and f_2 depend only on y, can be simplified in

$$u(x_1, y) = -\mu\varphi_R(|y|^2)\Gamma(x_1, y) \tag{211}$$

with Γ defined in (78) where, using (208), we can choose $l'(s) = \frac{\sigma(\sqrt{s})}{\sqrt{s}}$. Proceeding this way the following feedback law can be obtained:

$$\begin{aligned} u = {} & u_0(t_0, r_0) + \mu_s\varphi_{R_s}\left(t_0^2 + r_0^2\right) \\ & \times \sigma_s(s_0 + 2r_0 + t_0) + \mu_x\varphi_{R_x}\left(s_0^2 + t_0^2 + r_0^2\right) \\ & \times \sigma_x\left(x_0 + 2t_0 + \frac{1}{\mu_s}(s_0 + 2r_0 + t_0) + s_0 + r_0\right) \end{aligned} \tag{212}$$

where the functions φ_{R_s} and φ_{R_x} satisfy (97) with R_s and R_x any strictly positive real numbers, the functions σ_s and σ_x satisfy (208), and the real numbers μ_s and μ_x are to be chosen strictly positive and not too large.

VI. Concluding Remarks

We have proposed a Lyapunov design for deriving a state feedback law for a class of systems in the form $\dot{x} = h(x, y, u), \dot{y} = f(y, u)$, assuming global asymptotic stabilizability for the y subsystem. We have also shown that if a saturated control is sufficient for this subsystem, the same holds for the overall. We have called our technique *adding integration*, since the required assumptions on the x subsystem are mainly implying that the x components integrate functions of y and u.

This key technical tool can be used in combination with others. In particular, the availability of a Lyapunov function makes it very well suited for association with the technique of adding one integrator or for the design of adaptive feedback (see [19]) or output feedback. For instance, in [17], the problem of stabilization of the VTOL aircraft is solved with position measurement only.

We have applied this tool repeatedly to prove global asymptotic stabilizability for systems having a special recurrent structure called feedforward form and which are generically not feedback linearizable.

Due to space limitations we have concentrated our attention mainly on presenting a new technique. But the interested reader will find in [18] an application to a problem of stabilization of a partially linear composite system and in [5] an application to the problem of orbit transfer for a satellite. More generally, the dissertation [17] contains many results, extensions, and applications about the technique proposed here.

Appendix

A. Proof of Lemma II.4

We need the following technical lemma.

Lemma A.1: Let G be a C^0 function. For each integer k, we can find two functions γ_0 and γ_1, strictly increasing, continuous, and onto $[0, +\infty)$ such that for all (ξ, u), we have

$$|G(\xi, u) - G(\xi, 0)| \leq \gamma_0(|u|)\left[1 + \gamma_1(|\xi|^2 + |u|^2)\right] \tag{213}$$

and moreover γ_1 is C^k.

Proof: Let

$$\tilde{G}(\xi, u) = |G(\xi, u) - G(\xi, 0)|. \tag{214}$$

We define two functions $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ as follows: for $s > 0$, we let

$$\begin{cases} \tilde{\gamma}_0(s) = \sup_{\{(\xi,u):|u|\leq s\}}\left\{\frac{\tilde{G}(\xi,u)}{1+|\xi|+\tilde{G}(\xi,u)^2}\right\}, \\ \tilde{\gamma}_1(s) = \sup_{\{(\xi,u):|\xi|^2+|u|^2\leq s\}}\left\{1 + |\xi| + \tilde{G}(\xi, u)^2\right\} \end{cases} \tag{215}$$

and for $s = 0$, we let

$$\tilde{\gamma}_0(0) = 0, \qquad \tilde{\gamma}_1(0) = 1. \tag{216}$$

Since $\tilde{G}$ is a continuous function and $\frac{\tilde{G}(\xi,u)}{1+|\xi|+\tilde{G}(\xi,u)^2}$ goes to zero as ξ goes to ∞, the functions $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ are well defined, nondecreasing, and continuous at $s = 0$. This allows us to define two new functions

$$\begin{cases} \bar{\gamma}_0(s) = \frac{1}{s}\int_s^{2s}\tilde{\gamma}_0(t)dt + s \geq \tilde{\gamma}_0(s) + s, \\ \bar{\gamma}_1(s) = \frac{1}{s}\int_s^{2s}(\tilde{\gamma}_1(t) - 1)dt + s \geq (\tilde{\gamma}_1(s) - 1) + s. \end{cases} \tag{217}$$

They are strictly increasing, continuous, onto $[0, +\infty)$ and satisfy, for all (ξ, u)

$$\begin{aligned} \tilde{G}(\xi, u) &\leq \frac{\tilde{G}(\xi, u)}{1 + |\xi| + \tilde{G}(\xi, u)^2}\left[1 + |\xi| + \tilde{G}(\xi, u)^2\right] \\ &\leq \bar{\gamma}_0(|u|)\left[1 + \bar{\gamma}_1(|\xi|^2 + |u|^2)\right]. \end{aligned} \tag{218}$$

Finally, we let

$$\begin{cases} \gamma_0(s) = (1 + \bar{\gamma}_1(1))\bar{\gamma}_0(s), \\ \gamma_1(s) = \int_0^{2s}\int_0^{2s_1}\cdots\int_0^{2s_{k-1}}\frac{\bar{\gamma}_1(s_k)}{1+\bar{\gamma}_1(1)}ds_k ds_{k-1}\cdots ds_1. \end{cases} \tag{219}$$

The function γ_1 is C^k, and (213) holds since we have

$$\bar{\gamma}_0(|u|)\left[1 + \bar{\gamma}_1(|\xi|^2 + |u|^2)\right] \leq \gamma_0(|u|)\left[1 + \gamma_1(|\xi|^2 + |u|^2)\right] \tag{220}$$

which follows from the fact that for all $s \geq 1$, we have

$$\gamma_1(s) \geq \frac{\bar{\gamma}_1(s)}{1 + \bar{\gamma}_1(1)}. \tag{221}$$

□

Remark A.2: If G is C^1, we have

$$|G(\xi,u)-G(\xi,0)| \leq |u|\int_0^1 \left|\frac{\partial G}{\partial u}(\xi,su)\right| ds. \tag{222}$$

This yields

$$\begin{cases}\tilde{\gamma}_0(s)=s,\\ \tilde{\gamma}_1(s)=\max_{\{(\xi,u):|\xi|^2+|u|^2\leq s\}}\left\{1+\int_0^1\left|\frac{\partial G}{\partial u}(\xi,\tau u)\right|d\tau\right\}.\end{cases} \tag{223}$$

We now prove Lemma II.4. Since G is continuous, Lemma A.1 gives two functions γ_0 and γ_1, such that with (31), we have

$$\begin{aligned}G(\xi,u(\xi))u(\xi) \leq &-\lambda(\xi)|G(\xi,0)|^2\\ &+\gamma_0(\lambda(\xi)|G(\xi,0)|)\ \lambda(\xi)|G(\xi,0)|\\ &\times\left[1+\gamma_1(|\xi|^2+\lambda(\xi)^2|G(\xi,0)|^2)\right].\end{aligned} \tag{224}$$

To get (32), it is therefore sufficient to find λ a solution of

$$\begin{aligned}&\lambda(\xi)|G(\xi,0)|\\ &\quad\leq \gamma_0^{-1}\left(\frac{1}{2}\frac{|G(\xi,0)|}{1+\gamma_1(|\xi|^2+\lambda(\xi)^2|G(\xi,0)|^2)}\right).\end{aligned} \tag{225}$$

When G is C^1, using (223), (225) gives

$$\lambda(\xi) \leq \frac{1}{2c}\frac{1}{1+\gamma_1(|\xi|^2+\lambda(\xi)^2|G(\xi,0)|^2)}. \tag{226}$$

Since γ_1 is strictly increasing, a possible solution is

$$\lambda(\xi)=\lambda_0\frac{1}{1+\gamma_1(|\xi|^2+\lambda_0^2|G(\xi,0)|^2)}\frac{1}{1+|G(\xi,0)|^2}. \tag{227}$$

with

$$\lambda_0=\min\left\{\frac{1}{c},2\bar{u}\right\}. \tag{228}$$

Since γ_1 can be obtained as smooth as we want, this function λ is as smooth as $G(\xi,0)$ is. Moreover, the strict positiveness of λ and (32) hold.

When G is C^0, a possible solution for (225) which satisfies (32) is

$$\begin{aligned}\lambda(\xi)=\min\Bigg\{&\frac{\bar{u}}{1+|G(\xi,0)|},\\ &\gamma_0^{-1}\left(\frac{1}{2}\frac{|G(\xi,0)|}{1+\gamma_1(|\xi|^2+\bar{u}^2)}\right)\frac{1}{1+|G(\xi,0)|}\Bigg\}.\end{aligned} \tag{229}$$

B. About (44)

To check if (44) holds, the following two Lemmas may be useful.

Lemma B.1: For any function $\mathcal{F}(x,y)$ with a zero of order p at $y=0$, with p possibly zero, we can find two nonnegative and continuous functions γ_y and γ_x such that for all (x,y)

$$|\mathcal{F}(x,y)| \leq |y|^p\gamma_y(|y|)[1+\gamma_x(|x|)]. \tag{230}$$

Proof: Since $\mathcal{F}$ has a zero of order p at $y=0$, there exists a positive continuous function $\tilde{\mathcal{F}}$ such that for all (x,y)

$$|\mathcal{F}(x,y)| \leq \tilde{\mathcal{F}}(x,y)|y|^p. \tag{231}$$

At this point, we could conclude the proof by using the same construction as in the proof of Lemma A.1. But this leads to a too conservative function γ_x. Another solution is to define the following nonnegative and continuous function:

$$\gamma_y(s)=1+\sup_{\{(\mathcal{X},\psi):|\mathcal{X}|\leq|\psi|\leq s\}}\{\tilde{\mathcal{F}}(\mathcal{X},\psi)\}. \tag{232}$$

Then, since we have

$$\frac{\tilde{\mathcal{F}}(x,y)}{\gamma_y(|y|)} \leq \max\left\{1,\sup_{\{(\mathcal{X},\psi):|\psi|\leq|\mathcal{X}|\leq|x|\}}\{\tilde{\mathcal{F}}(\mathcal{X},\psi)\}\right\} \tag{233}$$

the following function is well defined, nonnegative, continuous, and such that (230) holds

$$\gamma_x(s)=\sup_{\{(\mathcal{X},\psi):|\mathcal{X}|\leq s\}}\left\{\max\left\{0,\frac{\tilde{\mathcal{F}}(\mathcal{X},\psi)}{\gamma_y(|\psi|)}-1\right\}\right\}. \tag{234}$$

□

Lemma B.2: Let V and W be continuous functions such that V is positive definite and proper and W is positive definite. Let γ be a nonnegative continuous function satisfying

$$\limsup_{y\to 0}\left\{\frac{\gamma(|y|)}{W(y)}\right\} < +\infty. \tag{235}$$

Under these conditions, there exists κ, a positive definite and continuous function on $[0,+\infty)$ which satisfies

$$\kappa(V(y))W(y) \geq \gamma(|y|). \tag{236}$$

Proof: The requirement (235) imposed on γ, the continuity of this function, and the fact that V is a positive definite and proper function guarantee the existence of a strictly positive real number c such that

$$\frac{\gamma(|y|)}{W(y)} \leq c \qquad \forall y\colon V(y)\leq c. \tag{237}$$

We define on $[0,+\infty)$ a function $\bar{\kappa}$ as follows:

$$\bar{\kappa}(v)=\sup_{\{y:V(y)\leq v\}}\left\{\max\left\{c,\frac{\gamma(|y|)}{W(y)}\right\}\right\}. \tag{238}$$

It is positive, nondecreasing, and constantly equal to c, a sufficiently large positive real number on a neighborhood of the origin. So we may define another positive function on $[0,+\infty)$ by

$$\kappa(v)=\int_v^{v+1}\bar{\kappa}(s)ds. \tag{239}$$

This function is continuous, positive definite, proper, and satisfies (236). □

C. Proof of Lemma IV.2

Let Θ be an open bounded subset of R^n. We first remark that since the matrix $A = \frac{\partial F_0}{\partial Y}(0)$ is asymptotically stable, the function Ψ defined as

$$\Psi(s,Y) = \exp\left(-\frac{1}{2}As\right)\Phi(s,Y) \tag{240}$$

is bounded on $[0,+\infty) \times \Theta$ as well as is $\frac{\partial \Psi}{\partial Y}$. This implies in particular

$$\lim_{s\to+\infty} s^2|\Phi(s,Y)| < +\infty \tag{241}$$

uniformly in $Y \in \Theta$. Also, since $H_{10}(Y)$ is C^1, zero at the origin, and $|\exp(-Ms)|$ is bounded, we have

$$\lim_{s\to+\infty} s^2|\exp(-sM)H_{10}(\Phi(s,Y))| < +\infty \tag{242}$$

uniformly in $Y \in \Theta$. This yields

$$\int_0^{+\infty} |\exp(-sM)H_{10}(\Phi(s,Y))| < +\infty \tag{243}$$

which proves that P_1 given by (134) is well defined.

To prove that this function is C^1, we check that the assumptions of [6, Th. 3.150] hold.

1) For each $s \in [0,+\infty)$, $\exp(-sM)H_{10}(\Phi(s,Y))$ is a continuously differentiable function in $Y \in \Theta$.
2) We have

$$\begin{aligned}\frac{\partial}{\partial Y}&[\exp(-sM)H_{10}(\Phi(s,Y))]\\ &= \exp(-sM)\frac{\partial H_{10}}{\partial Y}(\Phi(s,Y))\frac{\partial \Phi}{\partial Y}(s,Y) \quad (244)\\ &= \exp(-sM)\frac{\partial H_{10}}{\partial Y}(\Phi(s,Y))\\ &\quad\times \exp\left(\frac{1}{2}As\right)\frac{\partial \Psi}{\partial Y}(s,Y) \quad (245)\end{aligned}$$

where $|\exp(-sM)\frac{\partial H_{10}}{\partial Y}(\Phi(s,Y))|$ and $|\frac{\partial \Psi}{\partial Y}(s,Y)|$ are functions bounded on $R \geq 0 \times \Theta$. This implies the existence of a strictly positive real number c such that

$$\left|\frac{\partial}{\partial Y}[\exp(-sM)H_{10}(\Phi(s,Y))]\right| \leq c\left|\exp\left(\frac{1}{2}As\right)\right| \tag{246}$$

for any (s,Y) in $[0,+\infty) \times \Theta$.

3) The function $|\exp(\frac{1}{2}As)|$ is integrable on $[0,+\infty)$.

These three points imply that the function P_1 is C^1 with

$$\frac{\partial P_1}{\partial Y}(Y) = -\int_0^{+\infty} \exp(-sM)\frac{\partial H_{10}}{\partial Y}(\Phi(s,Y))\frac{\partial \Phi}{\partial Y}(s,Y)ds. \tag{247}$$

We remark also that

$$\Phi(s,\Phi(t,Y)) = \Phi((s+t),Y) \tag{248}$$

for all (s,t) implies

$$\frac{\partial \Phi}{\partial Y}(s,Y)f(Y) = f(\Phi(s,Y)) = \frac{\partial \Phi}{\partial s}(s,Y). \tag{249}$$

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