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TOOLS FOR SEMIGLOBAL STABILIZATION BY PARTIAL STATE AND OUTPUT FEEDBACK*

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Abstract. We develop tools for uniform semiglobal stabilization by partial state and output feedback. We show, by means of examples, that these tools are useful for solving a variety of problems. One application is a general result on semiglobal output feedback stabilizability when global state feedback stabilizability is achievable by a control function that is uniformly completely observable. We provide more general results on semiglobal output feedback stabilization as well. Globally minimum phase input–output linearizable systems are considered as a special case. Throughout our discussion we demonstrate the usefulness of considering local convergence separate from boundedness of solutions. For the former we employ a sufficient small gain condition guaranteeing convergence. For the latter we rely on Lyapunov techniques.

Key words. semiglobal (practical) stabilizability, uniform complete observability, dynamic output feedback, high gain control, nonlinear small gain

AMS subject classifications. 93D15, 93D09

Notation.

- A function is said to be smooth if it is in C^r, i.e., r times continuously differentiable, for some integer $r \geq 1$.
- $d(t)$ is a time-varying signal contained in a compact set $D \subset \mathbb{R}^d$. It will be appropriate to denote $d(t)$ and its time derivatives $\dot{d}(t), \ddot{d}(t), \ldots$ by the same symbol d, i.e., $d = (d, \dot{d}, \ddot{d}, \ldots)$. Since this aggregated d is still assumed to lie in a compact set, in some cases we shall implicitly introduce the strong requirement that the external disturbance is smooth.
- $\dot{V}_{(0)}$ denotes the function $\frac{\partial V}{\partial x}(x) f(x,d) : \mathbb{R}^l \times D \to \mathbb{R}$ and the subscript (0) refers to equation number (0) of the differential equation

$$\dot{x} = f(x, d(t)). \tag{0}$$

- $|\cdot|$ denotes the Euclidean norm.
- $\|\cdot\|_{t_\circ}$ denotes $\text{ess–sup}_{t_\circ \leq t \leq \infty} |\cdot|$.
- A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be *of class-K* if it is continuous, strictly increasing, and satisfies $\gamma(0) = 0$. It is *of class-K_∞* if in addition $\gamma(s) \to \infty$ as $s \to \infty$.
- A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of *class-KL* if, for each fixed $t \in \mathbb{R}_{\geq 0}$, the function $\beta(\cdot, t)$ is of class-K and for each fixed $s \in \mathbb{R}_{\geq 0}$ the function $\beta(s, \cdot)$ is decreasing and

$$\lim_{t \to \infty} \beta(s,t) = 0. \tag{1}$$

- A function $f : \mho \to \mathbb{R}_{\geq 0}$, where $\mho$ is an open set of $\mathbb{R}^p$, is said to be *proper on* $\mho$ if the preimage of a compact subset of $\mathbb{R}_{\geq 0}$ is a compact subset of $\mho$.
- A function $f : \mho \to \mathbb{R}_{\geq 0}$ is said to be *positive* (*negative*) *definite on* $\mho'$, a subset of $\mho$, if $f(x)$ is strictly positive (negative) for all x in $\mho'$.

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- A solution $x(t)$ of an ordinary differential equation is said to be *captured by a set* Γ if $x(t)$ is defined on $[0,+\infty)$ and there exists $t_\circ$ such that $x(t) \in \Gamma$ for all t in $[t_\circ,+\infty)$.

1. **A motivating problem and some results.** We are interested in the semiglobal stabilization[1] problem, as it is stated in [3], for example. In subsequent sections, the following four tools for solving semiglobal stabilization problems will be presented: two "backstepping" tools, a robust observer, and a local nonlinear small gain theorem. The usefulness of these tools will be illustrated by examples throughout the paper. Initially, to give the reader a sense for what can be proved with these tools, we will state some general nonlinear output feedback stabilization results which will be proved in a stronger form and with full details in later sections.

We start by considering the output feedback stabilization problem for nonlinear systems in the general form

$$\left\{ \begin{array}{rcl} \dot{z} & = & A(z,u), \\ y & = & C(z) \ . \end{array} \right. \tag{2}$$

We will make use of the following properties.

DEFINITION 1 (stabilizability). *An equilibrium point $z = 0$ of a dynamical system*

$$\dot{z} = A(z,u) \tag{3}$$

with A a smooth function, z in $\mathbb{R}^n$, and u in $\mathbb{R}$ is said to be globally (respectively, locally exponentially and globally) stabilizable if there exists a smooth function $\bar{u}$ such that $z = 0$ is a globally asymptotically (respectively, locally exponentially and globally asymptotically) stable equilibrium of

$$\dot{z} = A(z,\bar{u}(z)) \ . \tag{4}$$

DEFINITION 2 (uniform complete observability). *A function $\bar{u}(z)$ is said to be uniformly completely observable (UCO) with respect to the dynamical system (2) if there exist two integers n_y and n_u and a C^1 function Ψ such that, for each solution of*

$$\left\{ \begin{array}{rcl} \dot{z} & = & A(z,u_0), \\ \dot{u}_0 & = & u_1, \\ & \vdots & \\ \dot{u}_{n_u} & = & v, \end{array} \right. \tag{5}$$

we have, for all t where the solution makes sense,

$$\bar{u}(z(t)) = \Psi(y(t),\ldots,y^{(n_y)}(t),u_0(t),\ldots,u_{n_u}(t)), \tag{6}$$

where $y^{(i)}(t)$ denotes the ith time derivative of y at time t.[2]

Achieving global stabilization by output feedback can be impossible for very simple systems that are globally stabilizable by state feedback even when each component

[1] See Definition 3. Depending on the authors, this type of stabilization is also called "potentially global", "on compacta," or "widely local."

[2] If u is not present in (6) we let $n_u = -1$.

of the state is uniformly completely observable. For example, it was shown in [28] that there is no continuous, finite-dimensional dynamic output feedback to globally stabilize the equilibrium point $z = 0$ of the system

$$\left\{ \begin{array}{rcl} \dot{z}_1 & = & z_2, \\ \dot{z}_2 & = & z_2^n + u, \\ y & = & z_1, \end{array} \right. \tag{7}$$

with $n \geq 3$. This is true even though the system is globally feedback linearizable and the state is related to the output by $z_1 = y$, $z_2 = \dot{y}$. For this reason we restrict our attention to the semiglobal stabilization problem.

DEFINITION 3 (semiglobal stabilizability). [3] *The equilibrium $z = 0$ of the system (2) is said to be semiglobally stabilizable by dynamic state (respectively, output) feedback if, for each compact set $\mathcal{K}_l$, a neighborhood of 0, there exists a locally Lipschitz dynamic state (respectively, output) feedback $u = \bar{u}(z, \zeta)$, $\dot{\zeta} = \theta(z, \zeta)$ (respectively, $u = \bar{u}(y, \zeta)$, $\dot{\zeta} = \theta(y, \zeta)$) and a compact set $\mathcal{K}_{\zeta l}$ such that the equilibrium $(z, \zeta) = (0, 0)$ is asymptotically stable, with basin of attraction containing $\mathcal{K}_l \times \mathcal{K}_{\zeta l}$.*

It was shown in [40] that, when each component of the state vector z is UCO, global stabilizability by state feedback implies semiglobal stabilizability by output feedback. An implication of the state being UCO is that any globally stabilizing function $\bar{u}(z)$ is UCO. One might hope that this weaker assumption, existence of a UCO globally stabilizing state feedback, would yield semiglobal stabilizability by output feedback as well. Unfortunately, some difficulties appear in this case when attempting to establish local asymptotic stability. To guarantee this local property, we will impose extra local requirements on the system (4). A sufficient condition, generalizations of which are discussed in §5, is local exponential stability.

THEOREM 1.1. *If the equilibrium point $z = 0$ of the system (2) is locally exponentially and globally stabilizable by a UCO and C^2 state feedback, then it is semiglobally stabilizable by dynamic output feedback.*

Otherwise, since the only obstruction is local, we can still achieve semiglobal practical stabilization as summarized in the next definition and theorem.

DEFINITION 4 (semiglobal practical stabilizability). *A point $z = 0$ (not necessarily an equilibrium) is said to be semiglobally practically stabilizable by dynamic state (respectively, output) feedback if, for each pair of compact sets $(\mathcal{K}_s, \mathcal{K}_l)$, neighborhoods of $(0, 0)$ with $\mathcal{K}_s \subset \mathcal{K}_l$, there exists a locally Lipschitz dynamic state (respectively, output) feedback $u = \bar{u}(z, \zeta)$, $\dot{\zeta} = \theta(z, \zeta)$ (respectively, $u = \bar{u}(y, \zeta)$, $\dot{\zeta} = \theta(y, \zeta)$) and a pair of compact sets $(\mathcal{K}_{\zeta s}, \mathcal{K}_{\zeta l})$ such that all the solutions of the closed-loop system, with initial condition in $\mathcal{K}_l \times \mathcal{K}_{\zeta l}$, are captured by the set $\mathcal{K}_s \times \mathcal{K}_{\zeta s}$.*

THEOREM 1.2. *If the equilibrium point $z = 0$ of the system (2) is globally stabilizable by a UCO and C^2 state feedback, then it is semiglobally practically stabilizable by dynamic output feedback.*

The technique for proving these theorems is to exhibit a feedback controller based on the given state feedback controller $\bar{u}$, implemented dynamically using estimates of a sufficient number of derivatives of y provided by an observer and a sufficient number of derivatives of u provided by a suitable dynamic extension. The idea of implementing $\bar{u}$ through dynamic extension comes from the work of Tornambè [43]. That such a dynamically extended state feedback controller can be constructed while retaining semiglobal (practical) stabilizability will be shown using the iterating tool of Lemma

[3] It follows from this definition that a family of feedback laws is involved. This family is indexed by $\mathcal{K}_l$.

2.3. Further, we will show that combining the dynamically extended controller and an appropriate observer still yields semiglobal (practical) stability. Our robust observer tool, Lemma 2.4, will provide the technical result for showing this can be done.

Although different from the technicalities of the proof, the intuition behind our analysis follows from considering the closed-loop behavior as having two phases. During the first phase while we are trying to find, in finite time, an exact estimate of the derivatives of y, we acknowledge that this dynamically extended, estimated state feedback makes no sense. Still we must make sure that there is no finite escape time. To solve this problem, we use the a priori information that the actual control $\bar{u}(z)$ is in a known compact set to disregard any estimation $\widehat{\bar{u}(z)}$ which would lie outside this set. Mathematically there are many ways to reject these bad estimates but a very simple and efficient way is to saturate the estimated control as proposed in [11]. Then, using the worst saturated control, we can estimate the smallest time period T which will be needed by the system to go from its initial compact set to some larger compact set on which our estimated state semiglobally stabilizing feedback is valid. This time period is the period within which we should get our exact estimate of the derivatives of y. (See [18, Rem. 5].)

In phase two, if the estimates of the derivatives of y were correct, we could apply our dynamically extended state feedback. Unfortunately we are not always able to get an exact estimate of the derivatives of y. This is due to the possible presence of unobserved states and possible uncertainty in A. However, we can obtain an arbitrarily good approximate estimate. If we have designed our (dynamic) state feedback control using Lyapunov methods we have a measure of the stability robustness achieved by our feedback controller via the derivative of the Lyapunov function. We consequently build our approximate observer to account for this robustness margin. This strategy in fact has been exactly applied in [13] but in discrete time. There, no finite escape time is possible and exact estimation in finite time is assumed.

Finally note that Theorems 1.1 and 1.2 as well as the other results on semiglobal stabilization to come are presented here only as existence results. Nevertheless, the practical significance of the dynamic output feedback we shall exhibit has been investigated in the context of robotics applications in [1] and [2].

The remainder of the paper is organized as follows. In §2 we present tools for semiglobal practical stabilization by state and output feedback. Several applications of these tools are presented including, in §3, the proof of Theorem 1.2 and generalizations. In §4 we present a small gain theorem for local asymptotic stability analysis. This tool along with the tools of §2 are used, in §5, to prove Theorem 1.1 and generalizations. In §6 several corollaries for minimum phase input–output linearizable nonlinear systems are presented. A nonminimum phase example is also discussed.

2. Tools for semiglobal practical stabilization. As exhibited by the statement of our two theorems, we have found it is very useful to decouple the local convergence analysis in robust semiglobal stabilization problems from the analysis regarding the boundedness of solutions. In this section we are concerned only with the problem of uniform semiglobal practical stabilization by partial state and output feedback. We defer study of local convergence until §4. We will present tools that will be used to construct an output feedback for proving Theorem 1.2. However, these tools have their own interest. As illustrated by examples throughout this section, they can be used to address a wide variety of control problems.

These tools provide conditions under which solutions starting in some compact set are captured by a "smaller" one. They consider systems with an interconnection structure and a state decomposed, accordingly, into two parts, say (z, x), where the $\dot{x}$ equation contains a large gain, say K. The effect of this large gain K is to introduce an exponential dichotomy between the x component and the z component. This implies the existence of a center-stable manifold which can be described by $x = H(z, K)$. It follows that the motion of the solutions can be decomposed into two stages: *convergence* to this manifold and *sliding* along this manifold. This decomposition has been a standard tool used to prove early semiglobal stabilization results (see [36] and [7] for example). Here instead, like in [4], we completely ignore this decomposition and use a Lyapunov argument showing the decrease of an energy function outside a neighborhood of the origin. More precisely what is implicitly used here is the fact that as $K \to \infty$, the manifold tends to the set $\{(z, x) : x = 0\}$. So a Lyapunov function, which is simply the sum of the energy functions of x and z separately, should be sufficient and indeed it is. The availability of a Lyapunov function is extremely useful. It makes explicit the ultimate bound on trajectories as well as the domain of attraction without the formalism of invariant manifolds. It will also allow us to use our tools consecutively.

We will present two closely related "backstepping" tools, to borrow the terminology of [17]. This will be followed by an observer tool useful for analysis when the parameter K comes from a high gain observer. These tools are based on the following technical lemma, inspired by a similar result in [4].

LEMMA 2.1. *Let S be a compact set in a product space $\mathbb{R}^m \times \mathbb{R}^n$, and denote by S_z and S_x its respective projections (i.e., $S \subset S_z \times S_x$). Let $\chi(z)$ be a continuous real function on S_z which is positive definite on the projection of the set $\{(z, x) : x = 0\} \cap S$. Let $\psi(x)$ be a continuous real function on S_x which is positive definite on $S_x \backslash \{0\}$. Let $\varphi(z, x, d)$ be a continuous real function on $S \times D$ which satisfies*

$$\varphi(z, x, d) = 0 \qquad \forall (z, x, d) \in (\{(z, x) : x = 0\} \cap S) \times D. \tag{8}$$

Let κ be a function of class-K_∞. Under these conditions, there exists a positive real number K_* such that, for all $K \geq K_*$,

$$-\chi(z) - \kappa(K)\psi(x) + \varphi(z, x, d) < 0 \qquad \forall (z, x, d) \in S \times D. \tag{9}$$

Proof. For purposes of contradiction, assume the result is false. This implies that, for each n, there exists a point (z_n, x_n, d_n) in $S \times D$ such that

$$-\chi(z_n) - \kappa(n)\psi(x_n) + \varphi(z_n, x_n, d_n) \geq 0. \tag{10}$$

Consequently, since κ is class-K_∞ and $\psi \geq 0$ we have, for each $m \geq 1$ and for all $n \geq m$,

$$-\chi(z_n) - \kappa(m)\psi(x_n) + \varphi(z_n, x_n, d_n) \geq 0. \tag{11}$$

Now, since $S \times D$ is compact, the (sub)sequence (z_n, x_n, d_n) converges to a point (z_*, x_*, d_*) in $S \times D$. By continuity, this point satisfies

$$-\chi(z_*) - \kappa(m)\psi(x_*) + \varphi(z_*, x_*, d_*) \geq 0 \tag{12}$$

for all $m \geq 1$. Then, if $x_* = 0$, (8), (12), and the properties of ψ imply $-\chi(z_*) \geq 0$ which is not possible since χ is strictly positive on the projection of the set $\{(z, x) :$

$x = 0\} \cap S$. On the other hand, if $|x_*| \neq 0$ then $\psi(x_*) > 0$ and there exists an $m_* \geq 1$ such that, for all $m \geq m_*$,

$$-\chi(z_*) - \kappa(m)\psi(x_*) + \varphi(z_*, x_*, d_*) < 0 \tag{13}$$

since $\kappa(\cdot)$ is of class-K_∞. This contradicts (12), however, and completes the proof. □

Throughout the remainder of this section we use the following assumption.

Assumption ULP (uniform Lyapunov property)[4]. For the system

$$\dot{z} = h(z, 0, d(t)), \tag{14}$$

there exists an open set $\mho_1$ in $\mathbb{R}^m$, a nonnegative real number $\vartheta < 1$, a real number $c \geq 1$, and a C^1 function $V : \mho_1 \to \mathbb{R}_{\geq 0}$ such that the set $\{z : V(z) \leq c + 1\}$ is a compact subset of $\mho_1$, and we have

$$\dot{V}_{(14)} \leq -\Phi_1(z), \tag{15}$$

where $\Phi_1(z)$ is continuous on $\mho_1$ and positive definite on the set $\{z : \vartheta < V(z) \leq c+1\}$.

Remark 2.1. In the absence of $d(t)$, if the equilibrium $z = 0$ of the system

$$\dot{z} = h(z, 0) \tag{16}$$

is locally asymptotically stable with domain of attraction $\mho_1$, the converse Lyapunov theorem [22, Thm. 7] provides a smooth Lyapunov function satisfying Assumption ULP. Further, ϑ can be chosen to be equal to zero and c can be chosen to be arbitrarily large.

We now present our backstepping tools. The first lemma shows how one can semiglobally practically stabilize from a disturbed first derivative of the control instead of the control itself. The second lemma allows, in one step, the designer to semiglobally practically stabilize from a jth disturbed derivative of the control when the perturbations have a special form.

LEMMA 2.2 (semiglobal backstepping I). *Consider the C^1 nonlinear control system*

$$\begin{cases} \dot{z} &= h(z, x, d(t)), \\ \dot{x} &= f(z, x, d(t)) + g(z, x, d(t))u, \end{cases} \tag{17}$$

where $x \in \mathbb{R}$, $z \in \mathbb{R}^m$, the sign of $g(z, x, d)$ is constant, and the magnitude of g is bounded away from zero by a strictly positive real number b

$$|g(z, x, d)| \geq b \qquad \forall (z, x, d) \in \mathbb{R}^m \times \mathbb{R} \times D. \tag{18}$$

Suppose Assumption ULP is satisfied. Given $\mu \geq 1$, we define the function

$$W(z, x) = c\frac{V(z)}{c + 1 - V(z)} + \mu\frac{x^2}{\mu + 1 - x^2} \tag{19}$$

and the set

$$\mho_2 = \{z : V(z) < c + 1\} \times \{x : x^2 < \mu + 1\}. \tag{20}$$

[4] The number "1," here and in the following, is arbitrary and could be replaced by any strictly positive real number.

Under these conditions, $W(z,x) : \mho_2 \to \mathbb{R}_{\geq 0}$ *is proper on* $\mho_2$*. Further, if*

$$u = -K\mathrm{sgn}(g)x \tag{21}$$

then, for each strictly positive real number ρ*, there exists a positive real number* K_**, such that, for each* $K \geq K_*$*,* W *satisfies*

$$\dot{W}_{(17)} \leq -\Phi_2(z,x), \tag{22}$$

where $\Phi_2(z,x)$ *is continuous on* $\mho_2$ *and positive definite on the set* $\{(z,x) : \vartheta + \rho \leq W(z,x) \leq c^2 + \mu^2 + 1\}$.

Proof of Lemma 2.2. With $u = -K\mathrm{sgn}(g)x$ the closed loop system is

$$\left\{ \begin{array}{rcl} \dot{z} & = & h(z,x,d(t)), \\ \dot{x} & = & f(z,x,d(t)) - K\mathrm{sgn}(g)g(z,x,d(t))x. \end{array} \right. \tag{23}$$

For sake of generality we replace x^2 in (19) by $U(x)$. Now assume $W(z,x) \leq c^2+\mu^2+1$. This implies

$$V(z) \leq (c+1)\frac{c^2+\mu^2+1}{c^2+\mu^2+1+c}, \qquad U(x) \leq (\mu+1)\frac{c^2+\mu^2+1}{c^2+\mu^2+1+\mu}\ . \tag{24}$$

Now, we have

$$\dot{W}_{(23)} = \frac{c(c+1)}{(c+1-V)^2}\dot{V}_{(23)} + \frac{\mu(\mu+1)}{(\mu+1-U)^2}\dot{U}_{(23)}. \tag{25}$$

From (24), we get, when $W(z,x) \leq c^2 + \mu^2 + 1$,

$$\begin{array}{c} \dfrac{c}{c+1} \leq \dfrac{c(c+1)}{(c+1-V)^2} \leq \dfrac{(c^2+\mu^2+1+c)^2}{c(c+1)}, \\ \dfrac{\mu}{\mu+1} \leq \dfrac{\mu(\mu+1)}{(\mu+1-U)^2} \leq \dfrac{(c^2+\mu^2+1+\mu)^2}{\mu(\mu+1)}. \end{array} \tag{26}$$

Then, let us define

$$\left\{ \begin{array}{rcl} \chi(z) & = & \frac{c(c+1)}{2(c+1-V(z))^2}\Phi_1(z), \\ \psi(x) & = & \frac{\mu}{\mu+1}bx^2, \\ \varphi(z,x,d) & = & \frac{(c^2+\mu^2+1+c)^2}{c(c+1)}\left|\frac{\partial V}{\partial z}(z)[h(z,x,d)-h(z,0,d)]\right| \\ & & +\ \frac{(c^2+\mu^2+1+\mu)^2}{\mu(\mu+1)}2\left|xf(z,x,d)\right|, \\ \kappa(K) & = & K, \end{array} \right. \tag{27}$$

and consider the left-hand side of (9) in Lemma 2.1. We pick an arbitrarily small but strictly positive real number ρ and define a set S by

$$S = \{(z,x) : \vartheta + \rho \leq W(z,x) \leq c^2 + \mu^2 + 1\}. \tag{28}$$

The set S is compact from (24) and Assumption ULP. Also, from (24) the projections of S satisfy

$$S_z \subset \{z : V(z) < c+1\}, \qquad S_x \subset \{x : x^2 < \mu + 1\}. \tag{29}$$

Consequently $\chi(z)$ is continuous on S_z and $\psi(x)$ is continuous on S_x and positive definite on $S_x\backslash\{0\}$. Further, (24) also implies that $\varphi(z,x,d)$ is continuous on $S \times D$. From (27), it follows that

$$\varphi(z,x,d) = 0 \qquad \forall (z,x,d) \in (\{(z,x) : x = 0\} \cap S) \times D. \tag{30}$$

Finally, to see that $\chi(z)$ is positive definite on the projection of the set $\{(z,x) : x = 0\} \cap S$, we note

$$\{x = 0 \ , \ \vartheta + \rho \leq W(z,x)\} \quad \Longrightarrow \quad \vartheta + \rho \leq c\frac{V(z)}{c+1-V(z)}. \tag{31}$$

Further, for $0 \leq \vartheta \leq 1$,

$$\vartheta + \rho \leq c\frac{V(z)}{c+1-V(z)} \quad \Longrightarrow \quad \vartheta < V(z) \qquad \forall \rho > 0. \tag{32}$$

Then, from Assumption ULP, $\chi(z)$ is positive definite on the projection of the set $\{(z,x) : x = 0\} \cap S$. This demonstrates that the conditions of Lemma 2.1 are satisfied. It follows that there exists a positive real number K_* such that, for all $K \geq K_*$, (22) is satisfied with

$$\Phi_2(z,x) = \frac{c(c+1)}{2(c+1-V)^2}\Phi_1(z) + \frac{\mu}{\mu+1}Kbx^2. \tag{33}$$

Also, since φ is positive, it follows, from (27) and (9), that this function Φ_2 is positive definite on $\{(z,x) : \vartheta + \rho \leq W(z,x) \leq c^2 + \mu^2 + 1\}$ for all $K \geq K_*$. □

Example 2.1. The first application of Lemma 2.2 is a result for the C^1 control system:

$$\left\{ \begin{array}{rcl} \dot{z} & = & A(z,\zeta), \\ \dot{\zeta} & = & F(z,\zeta,d(t)) + G(z,\zeta,d(t))u, \end{array} \right. \tag{34}$$

$z \in \mathbb{R}^m$, $\zeta \in \mathbb{R}$, where the sign of $G(z,\zeta,d(t))$ is constant and the magnitude of G is bounded away from zero. Specifically, *if the equilibrium point $z = 0$ is semiglobally stabilizable by C^ℓ ($\ell \geq 2$) state feedback, with ζ as control, then $(z,\zeta) = (0,0)$ is semiglobally practically stabilizable by C^ℓ state feedback.*

This statement is to be added to the many results known on the stabilization via a disturbed derivative of the input ([6], [9], [12], [44]). Its proof follows.

Let $\bar{u}(z)$ represent the control law we get once the compact set $\mathcal{K}_{zl}$ of the semiglobal stabilizability property for the z subsystem is chosen. Define $x = \zeta - \bar{u}(z)$. Then we have

$$\left\{ \begin{array}{rcl} \dot{z} & = & A(z,\bar{u}(z)+x) \\ & \doteq & h(z,x) \\ \dot{x} & = & F(z,x+\bar{u}(z),d(t)) + G(z,x+\bar{u}(z),d(t))u - \frac{\partial\bar{u}}{\partial z}(z)A(z,\bar{u}(z)+x) \\ & \doteq & f(z,x,d(t)) + g(z,x,d(t))u. \end{array} \right. \tag{35}$$

From [22, Thm. 7], Assumption ULP is satisfied with $\vartheta = 0$ and a positive definite function V such that $\mathcal{K}_{zl}$ is contained in the set $\{z : V(z) \leq c\}$ for some real number $c \geq 1$.

With $\mathcal{K}_{\zeta l} \subset \mathbb{R}$, a chosen compact set, we choose μ to satisfy

$$\mu \geq \max\left\{1, \max_{\{z\in\mathcal{K}_{zl},\zeta\in\mathcal{K}_{\zeta l}\}}\{\zeta - \bar{u}(z)\}\right\}. \tag{36}$$

Similarly, let $\mathcal{K}_s$, a neighborhood of $(0,0)$, be the compact set we want the solutions of the closed-loop system to be captured by. We choose ρ to satisfy

$$0 < \rho \le \min\left\{1\,,\ \frac{1}{2}\inf_{(z,\zeta)\notin\mathcal{K}_s}\left\{\max\{V(z),(\zeta-\bar{u}(z))^2\}\right\}\right\}\ . \tag{37}$$

With these choices, the function $W(z,x)$ defined in (19) satisfies

$$W(z,\zeta-\bar{u}(z)) \le \rho \implies V(z) < 2\rho\ ,\ (\zeta-\bar{u}(z))^2 < 2\rho\ , \tag{38}$$
$$\implies (z,\zeta)\in\mathcal{K}_s\ . \tag{39}$$

Now, from Lemma 2.2, if u is chosen to be of the form

$$u = -K\text{sgn}(g)x = -K\text{sgn}(G)[\zeta-\bar{u}(z)], \tag{40}$$

then there exists a positive real number K_* such that, for each $K \ge K_*$, (22) holds with $\Phi_2(z,x)$ positive definite on the set $\{(z,x) : \rho \le W(z,x) \le c^2+\mu^2+1\}$. We conclude that, for each initial condition $(z(0),\zeta(0))$ in $\mathcal{K}_{zl}\times\mathcal{K}_{\zeta l}$, the corresponding solution of (34), (40) is captured by the set $\{(z,\zeta) : W(z,\zeta-\bar{u}(z)) \le \rho\}$ and therefore by $\mathcal{K}_s$. Since this holds for any compact sets $\mathcal{K}_{zl}$, $\mathcal{K}_{\zeta l}$, and $\mathcal{K}_s$, the semiglobal practical stabilizability result follows.

Now, if $y = C(z)$ is an output function, the discussion above and the very special structure of (40) yields the following result.

If the equilibrium point $z = 0$ is semiglobally stabilizable by C^ℓ ($\ell \ge 2$) and UCO state feedback, with ζ as control, and ζ is UCO, then the point $(z,\zeta) = (0,0)$ is semiglobally practically stabilizable by C^ℓ and UCO state feedback.

Example 2.2 (almost disturbance decoupling). A solution to the almost disturbance decoupling problem as described in [25] can be obtained by repeated application of Lemma 2.2 for systems that can be put in the following form:

$$\left\{\begin{array}{rcl} \dot{z} &=& h(z,x_1),\\ \dot{x}_1 &=& x_2+f_1(z,x_1,d(t)),\\ \dot{x}_2 &=& x_3+f_2(z,x_1,x_2,d(t)),\\ &\vdots& \\ \dot{x}_{r-1} &=& x_r+f_{r-1}(z,x_1,\ldots,x_{r-1},d(t)),\\ \dot{x}_r &=& f_r(z,x_1,\ldots,x_r,d(t))+g(z,x_1,\ldots,x_r,d(t))u, \end{array}\right. \tag{41}$$

where the equilibrium point $z = 0$ of $\dot{z} = h(z,0)$ is globally asymptotically stable and where the sign of g is constant and the magnitude of g is bounded away from zero. This is illustrated by the following example (compare with (7)):

$$\left\{\begin{array}{rcl} \dot{x}_1 &=& x_2+d_1(t),\\ \dot{x}_2 &=& x_2^3 d_2(t)+d_3(t)+u,\\ y &=& x_1, \end{array}\right. \tag{42}$$

where $d_1(t), d_2(t), d_3(t)$ are unknown bounded disturbances. The problem is to achieve $|x_1(t)| \le \wp \le 1$ asymptotically from arbitrarily large domains of attraction. Without loss of generality we assume $|d_i(t)| \le 1$.

Assume the initial conditions satisfy $|x_i(0)|^2 \le c$. We first consider the intermediate subsystem

$$\dot{x}_1 = u_1 + d_1(t). \tag{43}$$

If we choose the control $u_1 = -K_1 x_1$, and the Lyapunov function candidate $W_1(x_1) = x_1^2$, then for the intermediate closed-loop system

$$\dot{x}_1 = -K_1 x_1 + d_1(t) \tag{44}$$

we have

$$\dot{W}_{1_{(44)}} \leq -2x_1[K_1 x_1 - d], \tag{45}$$

which is negative definite on the set $\{x : \frac{1}{(K_1-1)^2} \leq W_1(x_1)\} \times \{|d| \leq 1\}$. We then choose

$$K_1 = 1 + \frac{2}{\wp} \tag{46}$$

so that $\dot{W}_{1_{(44)}}$ is negative definite on the set $\{x : \frac{\wp^2}{4} \leq W_1(x_1)\} \times \{|d| \leq 1\}$. We now make the coordinate change $\zeta = x_2 + K_1 x_1$ to get the system

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & -K_1 x_1 + \zeta + d_1(t), \\ \dot{\zeta} & = & u + (\zeta - K_1 x_1)^3 d_2(t) + K_1(\zeta - K_1 x_1 + d_1(t)) + d_3(t). \end{array} \right. \tag{47}$$

By applying Lemma 2.2 with $\vartheta = \rho = \frac{\wp^2}{4}$, we get the final control

$$u = -K_2 \zeta = -K_2 x_2 - K_2 K_1 x_1 \tag{48}$$

and a Lyapunov function candidate

$$W_2(x_1, \zeta) = \frac{\mu_1 W_1(x_1)}{\mu_1 + 1 - W_1(x_1)} + \frac{\mu_2 \zeta^2}{\mu_2 + 1 - \zeta^2}, \tag{49}$$

where $\mu_1 = c$ and μ_2 is so that the initial value of ζ satisfies $\zeta^2 \leq \mu_2$, i.e., $\mu_2 = (1 + K_1)^2 c$. We then have that the initial condition satisfies $W_2(x_1, \zeta) \leq \mu_1^2 + \mu_2^2$. Also, we know that, for K_2 large enough (see [41] for an explicit expression), the time derivative of W_2 is negative definite on the compact set

$$\Gamma_2 = \left\{ (x_1, \zeta) : \frac{\wp^2}{2} \leq W_2(x_1, \zeta) \leq \mu_1^2 + \mu_2^2 + 1 \right\}. \tag{50}$$

Therefore, the solutions, with $|x_i(0)|^2 \leq c$, are captured by the set $\{(x_1, \zeta) : W_2(x_1, \zeta) \leq \frac{\wp^2}{2}\}$, contained in the set $\{(x_1, \zeta) : |x_1| \leq \wp\}$.

It is important to note that, with our controller (48), we do not have the vanishing regions of attraction phenomenon as described in [21] and [25]. Indeed, in these papers, the same type of high gain controller is proposed but with the implicit constraint that $K_2 = K_1$. Here, instead, our iterative design leads to gains such that the ratio K_2/K_1 tends to infinity as K_1 tends to $+\infty$. However, although x_1 and ζ can be made ultimately arbitrarily small, x_2, called the peaking component, remains of unity magnitude as long as d_1 is present. For a discussion of the peaking phenomenon in feedback systems, see [36] and the references therein.

Finally, we remark that, if $\dot{d}_1(t)$ has a known bound (see our notation section), by applying our forthcoming robust observer tool, Lemma 2.4, the almost disturbance decoupling problem for the system (42) can be solved semiglobally by output feedback. (See [41].)

It is possible to handle a block of integrators in one step, instead of iterating the application of Lemma 2.2, when the system has the structure described in the following lemma.

LEMMA 2.3 (semiglobal backstepping II). *Consider the C^1 nonlinear control system*

$$\left\{\begin{array}{rcl} \dot{z} &=& h(z, x_1, d(t)), \\ \dot{x}_1 &=& x_2 + f_1(z, x_1, d(t)), \\ \dot{x}_2 &=& x_3 + f_2(z, x_1, d(t)), \\ &\vdots& \\ \dot{x}_{j-1} &=& x_j + f_{j-1}(z, x_1, d(t)), \\ \dot{x}_j &=& u + f_j(z, x_1, d(t)), \end{array}\right. \tag{51}$$

where $x = (x_1, \ldots, x_j)^T \in \mathbb{R}^j$, $z \in \mathbb{R}^m$. Suppose Assumption ULP is satisfied. Let the polynomial

$$p(s) = s^j + a_j s^{j-1} + \cdots + a_1 \tag{52}$$

be Hurwitz and let A_c be the companion form matrix corresponding to $p(s)$. Also let P_c solve the matrix equation $A_c^T P_c + P_c A_c = -I$. For $K \geq 1$ to be specified, define the variables

$$\xi_i = \frac{x_i}{K^{i-1}}, \qquad i = 1, \ldots, j. \tag{53}$$

Then given $\mu \geq 1$, define the function

$$W(z, \xi) = c\frac{V}{c + 1 - V} + \mu\frac{\xi^T P_c \xi}{\mu + 1 - \xi^T P_c \xi} \tag{54}$$

and the set

$$\mho_2 = \{z : V(z) < c + 1\} \times \{\xi : \xi^T P_c \xi < \mu + 1\}. \tag{55}$$

Under these conditions, $W(z, \xi) : \mho_2 \to \mathbb{R}_{\geq 0}$ is proper on $\mho_2$. Also, if

$$u = -K^j(a_1\xi_1 + \cdots + a_j\xi_j)\ , \tag{56}$$

then, for each strictly positive real number ρ, there exists a positive real number $K_* \geq 1$ such that, for all $K \geq K_*$, W satisfies

$$\dot{W}_{(17)} \leq -\Phi_2(z, \xi), \tag{57}$$

where $\Phi_2(z, \xi)$ is continuous on $\mho_2$ and positive definite on the set $\{(z, \xi) : \vartheta + \rho \leq W(z, \xi) \leq c^2 + \mu^2 + 1\}$.

Proof of Lemma 2.3. With the control (56) and the coordinates ξ defined in (53), the closed-loop system becomes

$$\left\{\begin{array}{rcl} \dot{z} &=& h(z, \xi_1, d(t)), \\ \dot{\xi} &=& KA_c\xi + F_K(z, \xi_1, d(t)), \end{array}\right. \tag{58}$$

where

$$F_K(z, \xi_1, d) = \begin{pmatrix} f_1(z, \xi_1, d) \\ \frac{1}{K} f_2(z, \xi_1, d) \\ \vdots \\ \frac{1}{K^{j-1}} f_j(z, \xi_1, d) \end{pmatrix}. \tag{59}$$

From here, if we replace $\xi^T P_c \xi$ in (54) by $U(\xi)$, then we can follow the proof of Lemma 2.2 with the only modifications being that, in (27) and (33), x^2 is replaced by $\frac{1}{2}\xi^T\xi$, xf by $\xi^T P_c F_K$, and $b = 1$. The fact that $F_K(z, \xi_1, d)$ depends on K is immaterial because, for $K \geq 1$, $\xi^T P_c F_K$ can be bounded by a continuous function which is independent of K. □

Remark 2.2. Lemma 2.3 is used in the same manner as Lemma 2.2 in Example 2.1. One difference is that the free parameter μ is chosen so that the initial conditions of x satisfy $\xi^T P_c \xi \leq \mu$ with ξ defined as in (53). The parameter μ thus appears to depend on K. However, for $K \geq 1$, we have

$$x^T P_c x \leq \mu \qquad \Longrightarrow \qquad \xi^T P_c \xi \leq \mu \frac{\lambda_{\max}\{P_c\}}{\lambda_{\min}\{P_c\}}, \tag{60}$$

where the left-hand side can be achieved independent of K. Nevertheless, the inequality (57) will not guarantee that x ultimately becomes small but only that (z, ξ) ultimately becomes small. As mentioned in Example 2.2, the coordinates x are called peaking coordinates.

Example 2.3 (observer canonical form). We have used Lemma 2.3 as a tool in [42] to design a semiglobally stabilizing output feedback for the following class of systems:

$$\left\{ \begin{array}{rcl} \dot{z} & = & h(z, x_1), \\ \dot{x}_1 & = & x_2 + f_1(z, x_1), \\ & \vdots & \\ \dot{x}_r & = & u + f_r(z, x_1), \\ y & = & x_1 \end{array} \right. \tag{61}$$

under a global minimum phase assumption (the point $z = 0$ of the system $\dot{z} = h(z, 0)$ is globally asymptotically stable) and a small gain-based assumption which guarantees local convergence. Here, h and f_i are C^1 and u in $\mathbb{R}$. The special form of (61) permits a technique for output feedback stabilization different from the one mentioned at the end of §1 and used in the proof of Theorem 1.2. Here, on the contrary, we design the observer first, then we define the controller in such a way that the stability it provides is robust to the estimation errors. Our algorithm is inspired by the global results in [17], [26], [27], and [29]. We begin by building the dynamic compensator

$$\left\{ \begin{array}{rcl} \dot{\hat{x}}_1 & = & \hat{x}_2 + \ell_1(x_1 - \hat{x}_1), \\ & \vdots & \\ \dot{\hat{x}}_r & = & u + \ell_r(x_1 - \hat{x}_1), \end{array} \right. \tag{62}$$

where the coefficients ℓ_i are the coefficients of a Hurwitz polynomial. If we define $e_i = x_i - \hat{x}_i$ we get the error dynamics

$$\dot{e} = A_o e + F(z, x_1). \tag{63}$$

We choose to consider the dynamics

$$\dot{e} = A_o e + F(z, 0) \tag{64}$$

as an augmentation of the zero dynamics $\dot{z} = h(z, 0)$ so that the equilibrium point $(z, e) = (0, 0)$ of the augmented system

$$\left\{ \begin{array}{rcl} \dot{z} & = & h(z, 0), \\ \dot{e} & = & A_o e + F(z, 0) \end{array} \right. \tag{65}$$

is globally asymptotically stable. This follows from the cascade structure and that the state e is input-to-state stable with respect to the input z. (See [35] or Lemma 4.1.) Now we consider the complete system

$$(66)\qquad \left\{\begin{array}{rcl} \dot{z} &=& h(z,x_1),\\ \dot{e} &=& A_o e + F(z,x_1),\\ \hline \dot{x}_1 &=& \hat{x}_2 + e_2 + f_1(z,x_1),\\ \dot{\hat{x}}_2 &=& \hat{x}_3 + \ell_2 e_1,\\ &\vdots& \\ \dot{\hat{x}}_r &=& u + \ell_r e_1. \end{array}\right.$$

It is in the form (51), and we can apply Lemma 2.3 to construct a controller depending only on $x_1, \hat{x}_2, \ldots, \hat{x}_r$ and achieving bounded trajectories from a given compact set of initial conditions. Local exponential stability of the point $z = 0$ of the system $\dot{z} = h(z,0)$ is a sufficient condition to guarantee convergence to the equilibrium $(z, e, x_1, \hat{x}_2, \ldots, \hat{x}_r) = (0, \ldots, 0)$. This condition can be relaxed by using the tools of §4.

When the output feedback stabilization problem is approached from the point of view discussed in §1, a linear high gain observer is introduced to get approximations of the derivatives of the output. The high gain parameter is tuned according to size of the compact set of initial conditions and the stability robustness that would be achieved by the state feedback controller. However, the linear high gain observer introduces possibly very large values of the state estimate over a short period of time. As already noted, this means that during this short period of time, the state estimate makes no sense and should be disregarded. This was achieved in [11] by saturating the control when the estimates had a value which was known a priori to be unreachable within this period of time by the actual state. The success of this modification was demonstrated by using a singular perturbation approach. However, the result seemed to require a form of nonlocal exponential stability [11, Assump. 2]. Even the more general interconnection conditions of [30] on which this assumption is based are too restrictive for the problem of boundedness (only) of solutions from compact sets. These assumptions mix the local and nonlocal analysis while weaker assumptions can be imposed if these aspects are handled separately. The next lemma demonstrates this.

LEMMA 2.4 (robust observer [11]). *Consider the C^1 nonlinear system*

$$(67)\qquad \left\{\begin{array}{rcl} \dot{z} &=& h(z,e,d(t)),\\ \dot{e} &=& LA_o e + p(z,e,d(t)), \end{array}\right.$$

where $z \in \mathbb{R}^m$, $e \in \mathbb{R}^n$, and L is a strictly positive real number. Suppose Assumption ULP is satisfied and let

$$(68)\qquad \Gamma = \{z : V(z) \leq c+1\}\ .$$

Also assume the matrix A_o is Hurwitz and there exist positive real numbers ν_1 and ν_2 and a bounded continuous function γ with $\gamma(0) = 0$ satisfying

$$(69)\qquad \left.\begin{array}{rcl} |h(z,e,d) - h(z,0,d)| &\leq& \gamma(|e|)\\ |p(z,e,d)| &\leq& \nu_1 + \nu_2|e| \end{array}\right\}\quad \forall (z,e,d) \in \Gamma \times \mathbb{R}^n \times D.$$

Let $\mu(L)$ be a class-K_∞ function satisfying

$$\liminf_{L\to\infty} \frac{L}{\mu^4(L)} \to \infty \ . \tag{70}$$

Let P_o solve the matrix equation $A_o^T P_o + P_o A_o = -I$ and, finally, define the function

$$W(z,e) = c\frac{V(z)}{c+1-V(z)} + \mu(L)\frac{\ln(1+e^T P_o e)}{\mu(L)+1-\ln(1+e^T P_o e)} \tag{71}$$

and the set

$$\mho_2 = \{z : V(z) < c+1\} \times \{e : \ln(1+e^T P_o e) < \mu(L)+1\} \ . \tag{72}$$

Under these conditions, for each strictly positive real number L, the function $W(z,e)$: $\mho_2 \to \mathbb{R}_{\geq 0}$ is proper on $\mho_2$. Also, for each strictly positive real number ρ, there exists a positive real number L_* such that, for all $L \geq L_*$, W satisfies

$$\dot{W}_{(67)} \leq -\Phi_2(z,e), \tag{73}$$

where $\Phi_2(z,e)$ is continuous on $\mho_2$ and positive definite on the set $\{(z,e) : \vartheta + \rho \leq W(z,e) \leq c^2 + \mu^2(L) + 1\}$.

Remark 2.3. The motivation for allowing μ to depend on L, in contrast to the previous two lemmas, is to allow the initial conditions of e to possibly depend on L. If the initial conditions of e can be bounded independent of L, then

1. the bounds in (69) are not needed,
2. μ can be chosen independent of L and the function $\ln(1+e^T P_o e)$ in (71) can be replaced by $e^T P_o e$.

Examples 2.4 and 2.5 demonstrate situations where the initial condition of the observation error can be bounded independent of L.

The motivation for the choice of the function ln is that for our problem, as will be seen later in the proof of Theorem 1.2, we wish to allow initial conditions of e to be of order L^{n_y}. This requires that we choose a Lyapunov function $U(e)$ and a function $\mu(L)$ satisfying the limit (70) and such that, given a strictly positive real number λ_1, we have

$$|e| \leq \lambda_1 L^{n_y} \quad \Longrightarrow \quad U(e) \leq \mu(L) \ . \tag{74}$$

For instance, if we choose $\mu(L) = \ln(1+\lambda_2 L^{2(n_y)})$, with λ_2 any strictly positive real number, then the limit (70) is satisfied since we have

$$\lim_{s\to\infty} \frac{s}{\ln(1+\lambda_2 s^{\alpha_1})^{\alpha_2}} = \infty \qquad \forall \lambda_2, \alpha_1, \alpha_2 > 0 \ . \tag{75}$$

Then, with the appropriate choice of λ_2, (74) is satisfied if we choose $U(e) = \ln(1+e^T P_o e)$. The choice of ln in turn requires the special form of the bounds imposed in (69).

With this remark, we see that if we disregard the issue of ultimate convergence, we recover the result of [11, Thm. 2].

Proof of Lemma 2.4. We follow the lines of the proof of Lemma 2.2. We begin by replacing $\ln(1+e^T P_o e)$ in (71) by $U(e)$. Assume that $W(z,e) \leq c^2 + \mu^2(L) + 1$. From

(24) and the definition of Γ in (68), this implies, for any L, that z is in Γ. Hence from Assumption ULP and the bounds in (69) we can write

(76)
$$\left.\begin{array}{lll}\dot{V}_{(67)} & \leq & -\Phi_1(z) + \nu_3\gamma(|e|) \\ \dot{U}_{(67)} & \leq & \frac{1}{1+e^TP_oe}\left[-L|e|^2 + 2\lambda_{\max}\{P_o\}|e|\,(\nu_2|e| + \nu_1)\right]\end{array}\right\} \forall (z,e,d) \in \Gamma \times \mathbb{R}^n \times D,$$

where ν_3 is a positive real number which bounds $\frac{\partial V}{\partial z}$ on the set Γ. Such a bound exists because V is C^1 and Γ is compact. Then, from (71), (25), and (76) we can write

$$\begin{aligned}\dot{W}_{(67)} \leq {} & \tfrac{c(c+1)}{(c+1-V)^2}\left\{-\Phi_1(z) + \nu_3\gamma(|e|)\right\} \\ & + \tfrac{\mu(\mu+1)}{(\mu+1-U)^2}\,\tfrac{1}{1+e^TP_oe}\left[-L|e|^2 + 2\lambda_{\max}\{P_o\}|e|\,(\nu_2|e| + \nu_1)\right].\end{aligned} \tag{77}$$

Now fix L_{*_1} so that $\mu^2(L_{*_1}) = c^2 + c + 1$. Such an L_{*_1} exists because $\mu(L)$ is of class-K_∞. Then, using the bounds from (24) and (26), using $c \geq 1$ from Assumption ULP, and choosing $L \geq L_{*_1}$ we have

$$\frac{1}{2} \leq \frac{c(c+1)}{(c+1-V)^2} \leq 2\mu^4. \tag{78}$$

Thus we can rewrite (77) as

$$\begin{aligned}\dot{W}_{(67)} \leq {} & \tfrac{c(c+1)}{(c+1-V)^2}\Big\{-\Phi_1(z) + \nu_3\gamma(|e|) \\ & + \frac{\mu(\mu+1)}{(\mu+1-U)^2}\,\frac{1}{1+e^TP_oe}\left[-\frac{L}{2\mu^4}|e|^2 + 4\lambda_{\max}\{P_o\}|e|(\nu_2|e| + \nu_1)\right]\Big\}\end{aligned} \tag{79}$$

Since $(c(c+1))/((c+1-V)^2)$ is positive and bounded away from zero on Γ, it suffices to consider the expression

(80)
$$-\Phi_1(z) + \nu_3\gamma(|e|) + \tfrac{\mu(L)(\mu(L)+1)}{(\mu(L)+1-U(e))^2}\,\tfrac{1}{1+e^TP_oe}\left[-\tfrac{L}{2\mu(L)^4}|e|^2 + 4\lambda_{\max}\{P_o\}|e|\,(\nu_2|e| + \nu_1)\right].$$

We are interested in evaluating this expression on the set

$$\Lambda_L \doteq \{(z,e) : \vartheta + \rho \leq W(z,e) \leq c^2 + \mu^2(L) + 1\}. \tag{81}$$

We do so by considering the two sets

$$\Lambda_1 \doteq \{(z,e) : V(z) \leq c+1\,,\ 1 < U(e) < \mu(L) + 1\}, \tag{82}$$

$$\begin{aligned}\Lambda_0 \doteq {} & \{(z,e) : V(z) \leq c+1\,,\ U(e) \leq 1\} \\ & \cap \left\{(z,e) : \vartheta + \rho \leq \frac{cV(z)}{c+1-V(z)} + U(e)\right\}\end{aligned} \tag{83}$$

and by observing that Λ_L is contained in $\Lambda_1 \cup \Lambda_0$, since we have

$$\{U(e) \leq 1\,,\ \vartheta + \rho \leq W(z,e)\} \quad \Longrightarrow \quad \vartheta + \rho \leq \frac{cV(z)}{c+1-V(z)} + U(e)\ . \tag{84}$$

In the set Λ_1, observe that the limit (70) holds, z is contained in a compact set independent of L, the function $\gamma(|e|)$ is bounded, and $\frac{\mu(\mu+1)}{(\mu+1-U(e))^2}$ is bounded away from zero from (26). We do not use the upper bound on $\frac{\mu(\mu+1)}{(\mu+1-U(e))^2}$ from (26) which depends on L. Finally note that the function $\frac{|e|^2}{1+e^TP_oe}$ is positive and bounded away from 0 on Λ_1. Thus, by examination of expression (80), it follows that there exists a positive real number L_{*_2} such that, for each $L \geq L_{*_2}$, the function $\dot{W}_{(67)}$ can be upper bounded by a function of (z, e) which is negative definite on Λ_1.

In the set Λ_0, to check that $\dot{W}_{(67)}$ is negative for all $(z, e, d) \in \Lambda_0 \times D$, we apply Lemma 2.1 to the expression (80). We remark that, for $L \geq L_{*_1}$, we have

$$\min_{0 \leq U \leq 1} \left\{ \frac{\mu(\mu+1)}{(\mu+1-U)^2} \right\} = \frac{\mu}{\mu+1}, \qquad \max_{0 \leq U \leq 1} \left\{ \frac{\mu(\mu+1)}{(\mu+1-U)^2} \right\} \leq 2. \tag{85}$$

It follows that to know the sign of the expression (80), we can look at (9) by taking

$$\left\{ \begin{array}{rcl} x & = & e \,, \\ K & = & L \,, \\ \chi(z) & = & \frac{1}{2}\Phi_1(z) \,, \\ \psi(e) & = & \frac{1}{2}\,\frac{e^Te}{1+e^TP_oe} \,, \\ \varphi(z,e) & = & \nu_3\gamma(|e|) + 8\lambda_{\max}\{P_o\}|e|\,(\nu_2|e| + \nu_1) \end{array} \right. \tag{86}$$

and $\kappa(\cdot)$ any class-K_∞ function satisfying

$$\kappa(L) \leq \frac{L}{2\mu^3(L)(\mu(L)+1)} \,. \tag{87}$$

Such a function exists because $L/(2\mu^3(L)(\mu(L)+1)) > 0$ for $L > 0$ and (70) holds. The set S in Lemma 2.1 is given by Λ_0. It is independent of L and compact. The respective projections satisfy

$$S_z \subset \{z : V(z) \leq c+1\} = \Gamma, \qquad S_e \subset \{e : U(e) \leq 1\} \,. \tag{88}$$

Then, from (86) and the properties of Φ_1, $\chi(z)$ is continuous on S_z and $\psi(e)$ is continuous on S_e. Clearly, $\psi(e)$ is positive definite on $S_e \backslash \{0\}$. Also, from the continuity of γ and the fact that $\gamma(0) = 0$, $\varphi(z, e)$ is continuous on S and

$$\varphi(z,e) = 0 \qquad \forall (z,e) \in \{(z,e) : e = 0\} \cap S. \tag{89}$$

To see that $\chi(z)$ is positive definite on the projection of the set $\{(z,e) : e = 0\} \cap S$, we have, with $0 \leq \vartheta \leq 1$ and $\rho > 0$,

$$\left\{ e = 0 \,,\; \vartheta + \rho \leq \frac{cV}{c+1-V} + U(e) \right\} \quad \Longrightarrow \quad \vartheta < V(z). \tag{90}$$

So from Assumption ULP, $\chi(z)$ is positive definite on the projection of the set $\{(z, e) : e = 0\} \cap S$. It follows that there exists a positive real number L_{*_3} such that, for each $L \geq L_{*_3}$, the expression (80) can be upper bounded on S by the function

$$-\tfrac{1}{2}\Phi_1(z) - \tfrac{1}{2}\kappa(L)\frac{e^Te}{1+e^TP_oe},$$

which is negative definite on Λ_0 since φ is positive. We then take $L_* = \max\{L_{*_1}, L_{*_2}, L_{*_3}\}$. ◻

Example 2.4 (mechanical systems). We consider the multi-input nonlinear system

$$\left\{ \begin{array}{rcl} \dot{q} & = & r \\ \dot{r} & = & f(q,r) + g(q,r)u, \end{array} \right. \tag{91}$$

where $q \in \mathbb{R}^n$, $r \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ is the input and f and g are C^1. This system could represent a robot model, for example. We assume the existence of a (dynamic) compensator

$$\left\{ \begin{array}{rcl} \dot{v} & = & C(q,r,u), \\ u & = & \bar{u}(q,r,v), \end{array} \right. \tag{92}$$

with $v \in \mathbb{R}^l$ such that the closed-loop system

$$\left\{ \begin{array}{rcl} \dot{q} & = & r, \\ \dot{r} & = & f(q,r) + g(q,r)\bar{u}(q,r,v), \\ \dot{v} & = & C(q,r,\bar{u}(q,r,v)), \end{array} \right. \tag{93}$$

which we rewrite, with $z = (q^T, r^T, v^T)^T$, as

$$\dot{z} = h(z,0) \ , \tag{94}$$

satisfies Assumption ULP for some neighborhood $\mho_1$ and some function V, proper on $\mho_1$, with $\vartheta = 0$ and c arbitrarily large. Assumption ULP is satisfied if, for example, the equilibrium $(q,r,v) = (0,0,0)$ is made (locally) asymptotically stable by the compensator (92). To implement the compensator (92) without measurement of r we build the observer

$$\left\{ \begin{array}{rcl} \dot{\hat{q}} & = & \hat{r} + L\ell_1(q - \hat{q}), \\ \dot{\hat{r}} & = & L^2\ell_2(q - \hat{q}), \end{array} \right. \tag{95}$$

where L is an adjustable parameter and ℓ_1, ℓ_2 are coefficients of a Hurwitz polynomial. We implement the compensator

$$\left\{ \begin{array}{rcl} \dot{v} & = & C(q,\Delta(\hat{r}),v,u), \\ u & = & \bar{u}(q,\Delta(\hat{r}),v), \end{array} \right. \tag{96}$$

where

$$\Delta(\hat{r}) = \hat{r} \min\left\{1, \frac{r_{\max}}{|\hat{r}|}\right\} \tag{97}$$

and $r_{\max}$ is the maximum value of $|r|$ on the set $\Gamma = \{(q,r,v) = z : V(z) \leq c+1\}$, where $V(z)$ and c come from Assumption ULP. This idea for the modification of the compensator is based on the idea in [11]. We choose to saturate the state $\hat{r}$ rather than the entire control u and compensator C because the state r has physical significance and thus determining $r_{\max}$ in the region of interest should be quite natural. Compare with equations (116) and (139) in the proof of Theorem 1.2. If we define the error state

$$e_q \doteq L(q - \hat{q}) \ , \quad e_r \doteq r - \hat{r}, \tag{98}$$

we have the error dynamics

$$\left\{ \begin{array}{rcl} \dot{e}_q & = & Le_r - L\ell_1 e_q, \\ \dot{e}_r & = & -L\ell_2 e_q + f(q,r) + g(q,r)\bar{u}(q,\Delta(r - e_r),v) \end{array} \right. \tag{99}$$

and we can apply Lemma 2.4. The bounds in (69) can be readily checked and follow from the introduction of Δ in the compensator (96). Consequently, by choosing c large enough, the modified compensator (96) together with the observer (95) can be used to yield bounded trajectories from the compact set of initial conditions $\mathcal{K}_l \times \mathcal{K}_{(\hat{q},\hat{r})\,l} \subset \mathbb{R}^{2n+l} \times \mathbb{R}^{2n}$, where $\mathcal{K}_l$ is any compact subset of $\mho_1$.

As pointed out in Remark 2.3, the bounds in (69) are required because the initial conditions of e grow with L. Specifically, $e_q = L(q-\hat{q})$. However, observe that it may be reasonable to initialize the value of $\hat{q}$ such that $\hat{q}(0) = q(0)$ since q is measured. In this case, the initial condition of e is $(e_q(0) = 0, e_r(0) = r(0) - \hat{r}(0))$ which is independent of L. As mentioned in Remark 2.3, in this case the bounds in (69), and hence the function Δ in (96), are not needed. Nevertheless, if this initialization cannot be done exactly, then the function Δ should be retained.

It would also be possible to build a reduced-order observer for this system. Consider the state $s = r - Lq$. We have

$$\dot{s} = f(q,r) + g(q,r)u - Lr \tag{100}$$

$$= f(q,r) + g(q,r)u - Ls - L^2 q. \tag{101}$$

If we build the observer

$$\left\{ \begin{array}{rcl} \dot{\hat{s}} &=& -L\hat{s} - L^2 q, \\ \hat{r} &=& \hat{s} + Lq, \end{array} \right. \tag{102}$$

then for the error $e_r = r - \hat{r}$, we have

$$\dot{e}_r = f(q,r) + g(q,r)u - Le_r. \tag{103}$$

If we don't specify the initial value of $\hat{s}$, then we choose the modified compensator in (96). If $\hat{s}(0)$ is chosen so that $\hat{s}(0) = -Lq(0)$ then $e_r(0) = r(0)$ and the function Δ is not needed. Let us also remark that the linear operator $q \mapsto \hat{r}$ defined by (102) is output strictly passive. This important property has been exploited in [5].

In all cases, if the original compensator (92) is locally exponentially stabilizing then the conditions of Lemma 4.1 will be satisfied and asymptotic stability is also achieved.

As mentioned earlier, the ideas presented here have been investigated further in [1] and [2].

Example 2.5 (the ball and beam). This example summarizes the result of [37]. Consider the ball-and-beam system

$$\left\{ \begin{array}{rcl} \dot{x}_1 &=& x_2, \\ \dot{x}_2 &=& -G\sin(x_3) + x_1 x_4^2, \\ \dot{x}_3 &=& x_4, \\ \dot{x}_4 &=& \frac{1}{Mx_1^2+J}[\tau - 2Mx_1x_2x_4 - MGx_1\cos(x_3)], \end{array} \right. \tag{104}$$

with three strictly positive real numbers G, M, and J, four state variables x_1 to x_4, and one control τ. See [14] for an interpretation of the state variables and a derivation of the dynamics. We wish to stabilize the system using measurement of x_1 and x_3 only. It can be shown that there exists a semiglobally stabilizing, and locally exponentially stabilizing, control $\bar{u}(x_1,x_2,x_3,x_4)$. See [39] for the case when M, J are known. For the case where M, J are unknown but have known bounds, the procedure is to use the results of [39] to get a result for the (x_1,x_2,x_3) subsystem and then apply Lemma 2.2

and Lemma 4.1 to get a result for the full system. See [37] for a complete discussion. From the results of [22, Thm. 7], Assumption ULP is satisfied with $\vartheta = 0$ for the closed-loop system with $\bar{u}(x_1, x_2, x_3, x_4)$ as the control. To implement this control, we build the observer

$$\left\{ \begin{array}{rcl} \dot{\hat{x}}_1 & = & \hat{x}_2 + L\ell_1(x_1 - \hat{x}_1), \\ \dot{\hat{x}}_2 & = & -G\sin(x_3) + L^2\ell_2(x_1 - \hat{x}_1), \\ \dot{\hat{x}}_3 & = & \hat{x}_4 + L\ell_1(x_3 - \hat{x}_3), \\ \dot{\hat{x}}_4 & = & L^2\ell_2(x_3 - \hat{x}_3), \end{array} \right. \tag{105}$$

and we let

$$\tau = \bar{u}(x_1, \hat{x}_2, x_3, \Delta(\hat{x}_4)) \tag{106}$$

where Δ is defined as in Example 2.4. Note that Δ does not need to act on $\hat{x}_2$ because, coincidentally, $\bar{u}$ can be chosen so that the x_2 dependence is already bounded. Again we choose to saturate the state $\hat{x}_4$ instead of the entire control τ because the state x_4 has physical significance. If we define the observer error

$$e_1 = L(x_1 - \hat{x}_1)\ , \quad e_2 = x_2 - \hat{x}_2\ , \quad e_3 = L(x_3 - \hat{x}_3)\ , \quad e_4 = x_4 - \hat{x}_4\ , \tag{107}$$

we have the error dynamics

$$\left\{ \begin{array}{rcl} \dot{e}_1 & = & Le_2 - L\ell_1 e_1, \\ \dot{e}_2 & = & -L\ell_2 e_1 + x_1 x_4^2, \\ \dot{e}_3 & = & Le_4 - L\ell_1 e_3, \\ \dot{e}_4 & = & -L\ell_2 e_3 + \frac{1}{Mx_1^2 + J}[\tau - 2Mx_1x_2x_4 - MGx_1\cos(x_3)]. \end{array} \right. \tag{108}$$

The bounds in (69) are satisfied and we can achieve bounded solutions from any compact set of initial conditions $(x, \hat{x})$. Furthermore, since $\bar{u}$ is locally exponentially stabilizing, asymptotic stability is also achieved.

Note that, as for the system in Example 2.4, we could choose the initial conditions of $\hat{x}_1$ and $\hat{x}_3$ so that $e_1(0) = 0$ and $e_3(0) = 0$. This is possible because x_1 and x_3 are measured. This, in turn, would remove the need for introducing the function Δ in the control $\bar{u}$. Building a reduced-order observer is also possible.

3. A generalized version of Theorem 1.2.

3.1. Assumptions and results. The proof of Theorem 1.2 follows from an appropriate application of Lemmas 2.3 and 2.4. An even more general case can be considered. Indeed, let the control system be[5]

$$\left\{ \begin{array}{rcl} \dot{z} & = & A(z, u, d(t)), \\ y & = & C(z, d(t)). \end{array} \right. \tag{109}$$

We assume only that the point $z = 0$ is semiglobally practically stabilizable by UCO static state feedback, as in the following assumption.

Assumption S-GPS. There exist two integers N_y and N_u so that, for each pair of compact sets $(\mathcal{K}_{zs}, \mathcal{K}_{zl})$, neighborhoods of 0 and with $\mathcal{K}_{zs} \subset \mathcal{K}_{zl}$, we can find

[5] May be augmented with the dynamics of a controller in the case of a dynamically stabilizable system.

1. a positive C^1 function V, zero at 0, which is defined on $\mho$, an open set containing $\mathcal{K}_{zl}$, and such that there exist three positive real numbers ϑ_l, c_s, and c_l satisfying

$$c_s < c_l, \qquad \{z : V(z) \le \vartheta_l\} \subset \mathcal{K}_{zs}, \qquad \mathcal{K}_{zl} \subset \{z : V(z) \le c_s\} \tag{110}$$

and so that the set $\{z : V(z) \le c_l\}$ is compact and contained in $\mho$.

2. a C^2 function $\bar{u}(z)$ which is zero at 0, is defined on $\mho$, and is UCO (i.e. (115) holds) with $n_y \le N_y$, $n_u \le N_u$, such that, for the system

$$\dot{z} = A(z, \bar{u}(z), d(t)), \tag{111}$$

we have

$$\dot{V}_{(111)} \le -\Phi(z), \tag{112}$$

where $\Phi(z)$ is continuous on $\mho$ and positive definite on $\{z : \vartheta_s \le V(z) \le c_l\}$ for some real number ϑ_s satisfying

$$0 < \vartheta_s < \vartheta_l. \tag{113}$$

The meaning of this assumption, as we shall make precise later, is that, once a pair of compact sets $(\mathcal{K}_{zs}, \mathcal{K}_{zl})$ is chosen, we know the existence of a UCO control law $\bar{u}$ so that Assumption ULP holds for the system (14). We shall prove the following proposition.

PROPOSITION 3.1. *If Assumption S-GPS holds then the point $z = 0$ of the system (109) is semiglobally practically stabilizable by dynamic output feedback.*

Proof of Theorem 1.2. If the equilibrium $z = 0$ of the system (2) is globally stabilizable by a C^2 state feedback $\bar{u}(z)$, i.e., $z = 0$ is a globally asymptotically stable equilibrium of

$$\dot{z} = A(z, \bar{u}(z)), \tag{114}$$

then, according to the converse Lyapunov theorem [22, Thm. 7], there exists a C^1 function V defined on R^n which is positive definite on $\mathbb{R}^n \backslash \{0\}$ and proper on $\mathbb{R}^n$ so that $\dot{V}_{(114)}$ is negative definite on $\mathbb{R}^n \backslash \{0\}$. It follows that point 1, (112), and (113) in Assumption S-GPS hold for any pair of compact sets $(\mathcal{K}_{zs}, \mathcal{K}_{zl})$. Therefore, if $\bar{u}(z)$ is also UCO, Assumption S-GPS holds. Thus Theorem 1.2 follows from Proposition 3.1. □

3.2. Proof of Proposition 3.1. Our idea for proving Proposition 3.1 is, instead of using $\bar{u}(z)$ which cannot be "measured", to use an approximation $\widehat{\bar{u}}$. To get this approximation, we use the fact that $\bar{u}$ is UCO, i.e.,[6]

$$\bar{u}(z) = \Psi\left(y, y^{(1)}, \ldots, y^{(n_y)}, u, u^{(1)}, \ldots, u^{(n_u)}\right) . \tag{115}$$

Following [43], the control u and its n_u derivatives can be obtained if we augment the dynamics of the controller. But for y and its n_y derivatives, we shall need an observer. Our proof is made in three steps. The first two steps—dirty derivatives of y and dynamic extension—concern the dynamic output feedback design. In the third step, we shall establish practical stability.

[6] Note the strong requirement that Φ is independent of d.

For the first two steps of this proof, the compact sets $\mathcal{K}_{zs}$ and $\mathcal{K}_{zl}$ are arbitrary but fixed. So from Assumption S-GPS, $\bar{u}$, V, and $\mho$ are given. Then, the following real number is well defined:

$$\bar{u}_{\max} = \max_{\{z: V(z) \leq c_l\}} \{|\bar{u}(z)|\} \ . \tag{116}$$

And, by picking ϑ_1 as an arbitrary real number in $(0, 1/8)$, let κ be a C^1 class-K_∞ function satisfying

$$\kappa(\vartheta_s) = \vartheta_1 \ , \quad \kappa(\vartheta_l) \geq 8\vartheta_1 \ , \quad \kappa(c_s) \geq 1 \ , \quad \kappa(c_l) > 1 + \kappa(c_s). \tag{117}$$

This function exists since our assumption gives

$$0 < \vartheta_s < \vartheta_l \leq c_s < c_l. \tag{118}$$

Then we let

$$V_1(z) = \kappa(V(z)), \qquad c_1 = \kappa(c_s). \tag{119}$$

So Assumption ULP is satisfied and we have

$$\{z : V_1(z) \leq 8\vartheta_1\} \subset \mathcal{K}_{zs}, \qquad \mathcal{K}_{zl} \subset \{z : V_1(z) \leq c_1\} \, . \tag{120}$$

Let us also pick ρ as

$$\rho = \frac{\vartheta_1}{2}. \tag{121}$$

3.2.1. Dirty derivatives of the measurement. With n_y the number of derivatives of y needed in (115) to reconstruct $\bar{u}$, we see, by induction on the order of derivation (see also the notation section), that there exist $n_y + 1$ smooth functions C_i and an integer $m_u \leq n_y$ such that, for each solution of

$$\left\{ \begin{array}{lcl} \dot{z} & = & A(z, u_0, d(t)), \\ \dot{u}_0 & = & u_1, \\ & \vdots & \\ \dot{u}_{m_u - 1} & = & u_{m_u}, \end{array} \right. \tag{122}$$

we have, for all t where the solution makes sense,

$$y^{(i)} = C_i\left(z(t), u_0(t), \ldots, u_{m_u}(t), d(t)\right), \qquad i = 1, \ldots, n_y + 1. \tag{123}$$

Then, for the system

$$\left\{ \begin{array}{lcl} \dot{y} & = & y^{(1)}, \\ & \vdots & \\ \dot{\overline{y^{(n_y)}}} & = & C_{n_y+1}\left(z(t), u_0(t), \ldots, u_{m_u}(t), d(t)\right), \end{array} \right. \tag{124}$$

we can propose the following approximate observer:

$$\left\{ \begin{array}{lclcl} \dot{\widehat{y}}_0 & = & \widehat{y}_1 & + & L\ell_0\,(y - \widehat{y}_0)\,, \\ & \vdots & & & \\ \dot{\widehat{y}}_{n_y - 1} & = & \widehat{y}_{n_y} & + & L^{n_y}\ell_{n_y - 1}\,(y - \widehat{y}_0)\,, \\ \dot{\widehat{y}}_{n_y} & = & & & L^{n_y+1}\ell_{n_y}\,(y - \widehat{y}_0) \ + \ C_{n_y+1}\left(0, u_0, \ldots, u_{m_u}, 0\right), \end{array} \right. \tag{125}$$

with the ℓ_i's chosen as the coefficients of a Hurwitz polynomial associated with a matrix A_o and the real number L to be specified later. It is important to note that (125) is not a true observer since $(y, y^{(1)}, \ldots, y^{(n_y)})$ is not a solution of (125).

3.2.2. Dynamic extension. To reconstruct $\bar{u}$, we need to know n_u derivatives of u. Similarly, to implement the above observer, we need m_u derivatives of u. So, by letting[7]

$$l_u = \max\{n_u, m_u\} + 1 \ , \tag{126}$$

we see that, by adding l_u integrators to the system (109) to be controlled, we shall have u and its required derivatives as measured state components of the system

$$\left\{ \begin{array}{lcl} \dot{z} & = & A(z, u_0, d(t)), \\ \dot{u}_0 & = & u_1, \\ & \vdots & \\ \dot{u}_{l_v - 1} & = & v. \end{array} \right. \tag{127}$$

To design the control v for this augmented system we can use Lemma 2.3. By letting

$$\xi_1 = u_0 - \bar{u}(z) \ , \quad \xi_i = \frac{u_{i-1}}{K^{i-1}} \qquad i = 1, \ldots, l_u \ , \tag{128}$$

with K a positive real number to be specified later, we get the system

$$\left\{ \begin{array}{lcl} \dot{z} & = & A(z, \bar{u}(z) + \xi_1, d(t)) \ \doteq \ h(z, \xi_1, d(t)), \\ \dot{\xi}_1 & = & K\,\xi_2 - \frac{\partial \bar{u}}{\partial z}(z) A(z, \bar{u}(z) + \xi_1, d(t)), \\ \dot{\xi}_2 & = & K\,\xi_3, \\ & \vdots & \\ \dot{\xi}_{l_u} & = & K^{1-l_u}\, v, \end{array} \right. \tag{129}$$

which is in the form of the system (51) written in the ξ_i's coordinates. As we mentioned earlier, Assumption ULP is satisfied by the system

$$\dot{z} = h(z, 0, d(t)) \ . \tag{130}$$

Then we choose coefficients, a_i's, of a Hurwitz polynomial associated with a matrix A_c. Let P_c be the solution of

$$A_c^T P_c + P_c\, A_c = -I \ . \tag{131}$$

Let $\mathcal{K}_{\xi l}$ be an arbitrary compact set where we choose to initialize ξ. We let

$$\mu_1 = \max\left\{1, \max_{\xi \in \mathcal{K}_{\xi l}} \{\xi^T P_c\, \xi\}\right\} \ . \tag{132}$$

Then Lemma 2.3 gives the bound K_*, the intermediate control

$$\begin{aligned} v &= -K^{l_u}(a_1\xi_1 + \ldots + a_{l_u}\xi_{l_u}) \\ &= -K^{l_u}\left(a_1\left[u_0 - \bar{u}(z)\right] + a_2\xi_2 + \ldots + a_{l_u}\xi_{l_u}\right) \end{aligned} \tag{133}$$

and the intermediate Lyapunov function

$$W_1(z, \xi) = \frac{c_1 V_1(z)}{c_1 + 1 - V_1(z)} + \frac{\mu_1 \xi^T P_c \xi}{\mu_1 + 1 - \xi^T P_c \xi} \ . \tag{134}$$

[7] In fact the result holds with $l_u = \max\{n_u, m_u\}$ but such a choice leads to more complicated notation.

We have

$$W_1(z,\xi) \le c_1^2 + \mu_1^2 \qquad \forall (z,\xi) \in \mathcal{K}_{zl} \times \mathcal{K}_{\xi l} \tag{135}$$

and, for $K \ge K_*$,

$$\dot{W}_{1\,(127,133)} \le -\Phi_1(z,\xi), \tag{136}$$

where $\Phi_1(z,\xi)$ is positive definite on $\{(z,\xi) : \vartheta_1 + \rho \le W_1(z,\xi) \le c_1^2 + \mu_1^2 + 1\}$.

For future use we define the set

$$\Gamma = \{(z,\xi) : W_1(z,\xi) \le c_1^2 + \mu_1^2 + 1\}\ . \tag{137}$$

This set is compact. (See (24).) We also define the real number $c_2 = c_1^2 + \mu_1^2$. To summarize, by denoting by Z the state vector $(z^T, \xi^T)^T$, we can write the system (127), (133) as

$$\dot{Z} = H_0(Z, d(t)) \tag{138}$$

and we have that Assumption ULP is satisfied for this system with $V = W_1$, $\vartheta = \vartheta_1 + \rho$, and $c = c_2$.

3.2.3. A dynamic output feedback. To summarize our design, we can propose the following dynamic output feedback for the system (109):

$$\left\{\begin{array}{ll}
\dot{\widehat{y}}_0 & = \widehat{y}_1 + L\ell_0\,(y - \widehat{y}_0)\,, \\
 & \vdots \\
\dot{\widehat{y}}_{n_y-1} & = \widehat{y}_{n_y} + L^{n_y}\ell_{n_y-1}\,(y - \widehat{y}_0)\,, \\
\dot{\widehat{y}}_{n_y} & = \quad L^{n_y+1}\ell_{n_y}\,(y - \widehat{y}_0) + C_{n_y+1}\,(0, u, K\xi_2, \ldots, K^{m_u}\xi_{m_u+1}, 0)\,, \\
\hline
\dot{u} & = K\,\xi_2, \\
 & \vdots \\
\dot{\xi}_{l_u-1} & = K\,\xi_{l_u}, \\
\dot{\xi}_{l_u} & = K^{1-l_u}\,v, \\
\hline
v & = -K^{l_u}\left(a_1[u - \Delta(\widehat{\widehat{u}})] + a_2\xi_2 + \ldots + a_{l_u}\xi_{l_u}\right),
\end{array}\right. \tag{139}$$

where

$$\widehat{\widehat{u}} = \Psi\left(y, \widehat{y}_1, \ldots, \widehat{y}_{n_y}, u, K\xi_2, \ldots, K^{n_u}\xi_{n_u+1}\right), \tag{140}$$

and

$$\Delta(s) = s\,\min\left\{1, \frac{\bar{u}_{\max}}{|s|}\right\} \tag{141}$$

with $\bar{u}_{\max}$ given in (116). This function Δ, already encountered in Examples 2.4 and 2.5, is one of the many possible ways of disregarding estimates when they are not in a given compact set and therefore make no sense. More specifically, this function guarantees that the assumption of Lemma 2.4 holds. And, in particular, we have

$$|\Delta(s_1) - \Delta(s_2)| \le \min\{|s_1 - s_2|\,, 2\bar{u}_{\max}\}\ . \tag{142}$$

It would also be possible to saturate each component $\widehat{y}_i$ independently. See Examples 2.4 and 2.5.

For the controller (139), we have chosen the $\widehat{y}_i$s and ξ_is as coordinates for its realization. Indeed, it is for this state that we shall be able to prove the practical stability.

Finally, note that we cannot implement the controller with ξ_1 as one of its state components since $\dot{\xi}_1$ involves unknown quantities.

3.2.4. Practical stability. To study the closed-loop system (109), (139), we use the coordinates $Z = (z^T, \xi^T)^T$ and e where

$$e_i = L^{n_y - i}\left(y^{(i)} - \widehat{y}_i\right). \tag{143}$$

The closed-loop dynamics can be written

$$\begin{cases} \dot{Z} &= H(Z, e, d(t)), \\ \dot{e} &= L A_o e + \Xi_e(z, \xi, d(t)), \end{cases} \tag{144}$$

where $\Xi_e(z, \xi, d)$ is a vector whose components are zero except the last one which is

$$\begin{aligned} \Xi_e(z, \xi, d)_{n_y} = {} & C_{n_y+1}\left(z, \xi_1 + \bar{u}(z), K\xi_2, \ldots, K^{m_u}\xi_{m_u+1}, d\right) \\ & - C_{n_y+1}\left(0, \xi_1 + \bar{u}(z), K\xi_2, \ldots, K^{m_u}\xi_{m_u+1}, 0\right). \end{aligned} \tag{145}$$

This system is in the form (67) considered in Lemma 2.4 with the Z dynamics playing the role of the z dynamics in that lemma.

From the conclusion of the dynamic extension stage and the facts

$$e = 0 \implies \widehat{\bar{u}} = \bar{u}(z), \qquad Z \in \Gamma \implies H(Z, 0, d(t)) = H_0(Z, d(t)), \tag{146}$$

which follow from

$$Z \in \Gamma \implies V(z) \le c_l, \tag{147}$$

$$\implies \Delta(\bar{u}(z)) = \bar{u}(z), \tag{148}$$

Assumption ULP is satisfied. We also have

$$H(Z, e, d) - H(Z, 0, d) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ K\, a_1 \left[\Delta(\widehat{\bar{u}}) - \Delta(\bar{u}(z))\right] \end{bmatrix}. \tag{149}$$

But, with (123), (128), and the compactness of Γ and D, the following function $\overline{\Psi}$ on $\mathbb{R}^2_{\ge 0}$ is well defined:

$$\begin{aligned} \overline{\Psi}(K, E) = \sup_{(z,e,d,\theta,i,L) \in \mathcal{S}} \Bigg\{ \Bigg| & \frac{\partial \Psi}{\partial y^{(i)}}\Big(y, y^{(1)} - \frac{\theta}{L^{n_y - 1}} e_1, \ldots, y^{(n_y)} \\ & - \theta e_{n_y}, \xi_1 + \bar{u}(z), K\xi_2, \ldots, K^{n_u}\xi_{n_u+1}\Big)\Bigg|\Bigg\} \end{aligned} \tag{150}$$

where

$$\mathcal{S} = \Gamma \times \{e : |e| \le E\} \times D \times [0, 1] \times \{1, \ldots, n_y\} \times [1, \infty). \tag{151}$$

Then, from (115), (140), and (142), we have, for $L \geq 1$ and all (Z, e, d) in $\Gamma \times \mathbb{R}^{(n_y+1)} \times D$,

$$|H(Z,e,d) - H(Z,0,d)| \leq K\,|a_1|\,\min\left\{|\widehat{\bar{u}} - \bar{u}(z)|\,,\,2\bar{u}_{\max}\right\}\,, \tag{152}$$

$$\leq K\,|a_1|\,\min\left\{\overline{\Psi}(K,|e|)|e|\,,\,2\bar{u}_{\max}\right\}\,, \tag{153}$$

$$\leq \gamma(|e|), \tag{154}$$

for some bounded and continuous function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfying $\gamma(0) = 0$. Note that γ depends on K which is fixed at this stage. Also $\Xi_e(z, \xi, d)$, given in (145), is bounded on the compact set $\Gamma \times D$ by a positive real number ν_1, independent of L but dependent on K. Now, let P_o satisfy the matrix equation

$$A_o^T P_o + P_o A_o = -I. \tag{155}$$

Also let $\mathcal{K}_{yl}$ be a compact set where we choose to initialize the estimated derivatives of y. Since, from (123), the $y^{(i)}$s are bounded on $\mathcal{K}_{zl} \times \mathcal{K}_{\xi l} \times D$, the following positive real number is well defined and depends on K but not on L:

$$k = \sup_{(z,\xi,(\widehat{y_i}),d) \in \mathcal{K}_{zl} \times \mathcal{K}_{\xi l} \times \mathcal{K}_{yl} \times D} \left\{|y^{(i)} - \widehat{y}_i|\right\}\,. \tag{156}$$

If we then choose

$$\mu_2(L) = \ln(1 + k\lambda_{\max}\{P_o\}L^{2n_y}) \tag{157}$$

we have, for the initial condition $e(0)$,

$$\ln(1 + e^T(0)P_o e(0)) \leq \mu_2(L) \tag{158}$$

and the limit (70) is satisfied. So Lemma 2.4 gives us a bound L_*, depending on K, and the final Lyapunov function

$$W_2(Z,e) = \frac{c_2 W_1(Z)}{c_2 + 1 - W_1(Z)} + \frac{\mu_2(L)\ln(1 + e^T P_o e)}{\mu_2(L) + 1 - \ln(1 + e^T P_o e)} \tag{159}$$

so that for $L \geq L_*$, we have

$$\mu_2(L) \geq 1 \tag{160}$$

and

$$\dot{W}_{2\,(144)} \leq -\Phi_2(Z,e), \tag{161}$$

where $\Phi_2(Z, e)$ is positive definite on $\{(Z, e) : \vartheta_1 + 2\rho \leq W_2 \leq c_2^2 + \mu_2(L)^2 + 1\}$. Since the set $\mathcal{K}_{zl} \times \mathcal{K}_{\xi l} \times \mathcal{K}_{yl}$ is contained in $\{(Z, e) : W_2 \leq c_2^2 + \mu_2(L)^2\}$ and $\rho = \frac{\vartheta_1}{2}$, we conclude that the solutions initialized in $\mathcal{K}_{zl} \times \mathcal{K}_{\xi l} \times \mathcal{K}_{yl}$ remain forever in the set $\{(Z, e) : W_2(Z, e) \leq c_2^2 + \mu_2(L)^2\}$ and are captured by the set $\{(Z, e) : W_2(Z, e) \leq 2\vartheta_1\}$. Then we remark that, c_1, μ_1, μ_2 being larger than 1, we have

$$(W_2(Z,e) \leq 2\vartheta_1) \Longrightarrow \left(e^T P_o e \leq \exp(4\vartheta_1) - 1,\ \xi^T P_c \xi \leq 8\vartheta_1,\ V_1(z) \leq 8\vartheta_1\right). \tag{162}$$

Since the real number ϑ_1 can be chosen arbitrarily small and (120) holds, we have proved the following.

For any pair of compact sets $(\mathcal{K}_{zs}, \mathcal{K}_{zl})$, neighborhoods of 0, with $\mathcal{K}_{zs} \subset \mathcal{K}_{zl}$, we can find compact sets $(\mathcal{K}_{ys}, \mathcal{K}_{\xi s})$ and $(\mathcal{K}_{yl}, \mathcal{K}_{\xi l})$; gains ℓ_is; a_is; a bound K_*; a bounding function $L_*(K)$; integers l_u, n_y; and functions $\bar{u}$, Δ, Ψ so that, for each $K \geq K_*$, $L \geq L_*(K)$, the dynamic output feedback (139) *in closed loop with the system* (109) *makes all the solutions, with initial condition in $\mathcal{K}_{zl} \times \mathcal{K}_{yl} \times \mathcal{K}_{\xi l}$, be captured by the set $\mathcal{K}_{zs} \times \mathcal{K}_{ys} \times \mathcal{K}_{\xi s}$.*

4. The small gain theorem for asymptotic stability. Up to this point we have focused on boundedness of solutions only . However, we have constructed Lyapunov functions to guarantee that, in appropriate coordinates, the states become ultimately arbitrarily small. Now, if the linear approximation in these coordinates is exponentially stable we are effectively done. If the linear approximation is not exponentially stable, then the problem reduces to studying the local stability on the center manifold. See [8]. Because the center manifold analysis can be quite involved, we choose to develop a sufficient condition, other than exponential stability, that can be checked a priori.

Our approach will be to appeal to the notion of "small gain." We will state here a version of the small nonlinear gain theorem, expressed in terms closely related to the nonlinear L_∞-gain from input to state. This is inspired by Sontag's input-to-state (ISS) stability definition [33]. We start with the following definition and give an illustrative fact.

DEFINITION 5. *The system*

$$\dot{x} = h(x, u, t) \tag{163}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, *and* $t \in \mathbb{R}_{\geq 0}$ *is said to be* uniformly (ϵ, δ) input-to-state stable (*uniformly* (ϵ, δ) *ISS*) *if there exist a class-KL function* β, *a class-K function* γ, *called the gain, and strictly positive real numbers* δ, ϵ *such that, for each* $t_\circ \geq 0$, *for each initial state* $x(t_\circ) = x_\circ$ *satisfying* $|x_\circ| \leq \delta$ *and for each measurable control* $u(\cdot)$ *satisfying* $||u||_{t_\circ} \leq \epsilon$, *the solution of* (163) *exists for each* $t \geq t_\circ$ *and satisfies*

$$|x(t)| \leq \beta(|x_\circ|, t - t_\circ) + \gamma(||u||_{t_\circ}). \tag{164}$$

FACT 4.1 ([35],[20],[45]). *For the system* (163),

1. *if* h *does not depend explicitly on time and the equilibrium point* $x = 0$ *of the system*

$$\dot{x} = h(x, 0) \tag{165}$$

is locally asymptotically stable, then the system is (*uniformly*) (ϵ, δ) *ISS.*

2. *if* $\frac{\partial h}{\partial x}(x, 0, t)$ *is bounded for sufficiently small* x *uniformly in* t, $h(x, u, t)$ *is locally Lipschitz in* (x, u) *uniformly in* t, *and the equilibrium point* $x = 0$ *of the system*

$$\dot{x} = h(x, 0, t) \tag{166}$$

is uniformly locally asymptotically stable, then the system (163) *is uniformly* (ϵ, δ) *ISS. Moreover, if* $x = 0$ *of* (166) *is locally exponentially stable then* γ *can be taken to be of the form* $\gamma(s) = ks$, *for some positive real number* k, *and* β *can be taken to be of the form* $\beta(s, t) = bse^{-at}$, *for some strictly positive real numbers* b *and* a.

Proof. See the appendix. □

Remark 4.1. For the local exponential stability case, this result was presented in [45]. For the time invariant case, the result is essentially contained in [35, Thm. 2]. For the case where h is differentiable, the proof of this fact can be constructed from theorems in [20, §4.5.2].

The local asymptotic stability of the equilibrium point of interconnected uniformly (ϵ, δ) ISS subsystems can then be analyzed using the following result.

LEMMA 4.1 (small gain). *Consider the feedback interconnection*

$$\left\{ \begin{array}{rclrcl} \dot{x}_1 & = & h_1(x_1, u_1, v, t), & u_1 & = & x_2, \\ \dot{x}_2 & = & h_2(x_2, u_2, v, t), & u_2 & = & x_1, \end{array} \right. \tag{167}$$

with $x_i \in \mathbb{R}^{n_i}$ *for* $i = 1, 2$ *and* $v \in \mathbb{R}^m$. *Define* $x = (x_1^T, x_2^T)^T$. *Assume* h_i *is locally Lipschitz in* (x_i, u_i, v) *and piecewise continuous in* t. *Assume the ith subsystem is uniformly* (ϵ_i, δ_i) *ISS with respect to both* u_i *and* v *(characterized by* δ_i, ϵ_i^u, ϵ_i^v, β_i, γ_i^u *and* γ_i^v*).*[8]

Suppose there exist strictly positive real numbers ω *and* λ *such that*[9]

$$\left. \begin{array}{l} (1+\lambda)\gamma_1^u \circ (1+\lambda)\gamma_2^u(s) \le s \\ (1+\lambda)\gamma_2^u \circ (1+\lambda)\gamma_1^u(s) \le s \end{array} \right\} \quad \forall s \in [0, \omega] \ . \tag{168}$$

Under these conditions, the feedback interconnection is uniformly (ϵ, δ) *ISS.*

More specifically, define

$$\left\{ \begin{array}{rcl} \phi_1(s) & = & (1+\lambda^{-1}) \left(\beta_1(s,0) + \gamma_1^u \left((1+\lambda^{-1})(1+\lambda^{-1})(\beta_2(s,0))\right)\right), \\ \phi_2(s) & = & (1+\lambda^{-1}) \left(\beta_2(s,0) + \gamma_2^u \left((1+\lambda^{-1})(1+\lambda^{-1})(\beta_1(s,0))\right)\right), \\ \phi(s) & = & \phi_1(s) + \phi_2(s) \end{array} \right. \tag{169}$$

and

$$\left\{ \begin{array}{rcl} r_1(s) & = & (1+\lambda^{-1})(\gamma_1^v + \gamma_1^u \circ (1+\lambda^{-1})(1+\lambda)(\gamma_2^v))(s), \\ r_2(s) & = & (1+\lambda^{-1})(\gamma_2^v + \gamma_2^u \circ (1+\lambda^{-1})(1+\lambda)(\gamma_1^v))(s), \\ r(s) & = & r_1(s) + r_2(s). \end{array} \right. \tag{170}$$

Then, for any pair (ϵ, δ) *satisfying*

$$\epsilon \le \min\{\epsilon_1^v, \epsilon_2^v\} \quad , \qquad \phi(\delta) + r(\epsilon) < \min\{\delta_1, \delta_2, \epsilon_1^u, \epsilon_2^u, \omega\} \tag{171}$$

and for each class-K_∞ *function* α *there exists a class-KL function* β *such that, for each* $t_\circ \ge 0$, *for each initial state satisfying* $|x(t_\circ)| \le \delta$, *and for each measurable input* $v(\cdot)$ *satisfying* $||v||_{t_\circ} \le \epsilon$, *the solution of* (167) *exists for each* $t \ge t_\circ$ *and satisfies*

$$|x(t)| \le \beta(|x(t_\circ)|, t - t_\circ) + (r + \alpha)(||v||_{t_\circ}). \tag{172}$$

If each subsystem is uniformly globally *ISS and inequality* (168) *holds for all* $s \in [0, \infty)$, *then inequality* (172) *holds for all initial conditions and all measurable inputs* $v(\cdot)$.

Proof. See the appendix. □

Remark 4.2. 1. Notice that when $||v||_{t_\circ} = 0$, the lemma provides an asymptotic stability result. For the local case, this lemma can be seen as a generalization of [7, Lem. 4.13] where, there, $\gamma_2 \equiv 0$. In the global case, this lemma is a generalization of the result that the cascade of an ISS system and a globally asymptotically stable (GAS) system is GAS.

2. This lemma is a form of the small nonlinear gain theorem (see [10]) which includes explicitly the effects of initial conditions. Its condition (168) was introduced in [24]. For other purely input–output results see [24] and [32] and the references

[8] For example, $|x_1(t)| \le \beta_1(|x_1(t_\circ)|, t - t_\circ) + \gamma_1^u(||u_1||_{t_\circ}) + \gamma_1^v(||v||_{t_\circ})$.

[9] See Fact A.2

therein. In [16], a generalization of this lemma is presented dealing, in particular, with practical stability and the input–output case.

To make this small gain result more efficient we remark that [20, Thm. 4.10], reproduced here, gives us a way to compute effectively the gain function γ.

LEMMA 4.2. *Let B_r^n be the set $\{x \in \mathbb{R}^n : |x| \leq r\}$, $V : \mathbb{R}_{\geq 0} \times B_r^n \to \mathbb{R}$ be a C^1 function, α_1, α_2, α_3, and α_5^{-1} be class-K functions defined on $[0,r]$, and $h : B_r^n \times B_\epsilon^m \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ be piecewise continuous in t and locally Lipschitz in (x,u). Assume ϵ satisfies*

$$\epsilon \leq \alpha_5^{-1}(\alpha_2^{-1}(\alpha_1(r))) \tag{173}$$

and, for all $t \geq 0$, for all (x,u) in $B_r^n \times B_\epsilon^m$, we have

$$\alpha_1(|x|) \leq V(t,x) \leq \alpha_2(|x|) \tag{174}$$

and

$$|x| \geq \alpha_5(|u|) \quad \Longrightarrow \quad \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} h(x,u,t) \leq -\alpha_3(|x|) \ . \tag{175}$$

Under these conditions, the system

$$\dot{x} = h(x,u,t) \tag{176}$$

is uniformly (ϵ,δ) ISS with

$$\delta = \alpha_2^{-1}(\alpha_1(r)) \ , \quad \gamma(s) = \alpha_1^{-1}(\alpha_2(\alpha_5(s))). \tag{177}$$

Furthermore, if $\alpha_i(s) = k_i s^2$, for $i = 1,\ldots,3$ and some $k_i > 0$ then

$$\beta(s,t) = \sqrt{\frac{k_2}{k_1}}\, s \, \exp\left(-\frac{k_3}{2k_2}\, t\right) . \tag{178}$$

Example 4.1. Let us consider the system

$$\left\{ \begin{array}{rcl} \dot{z} & = & -z^3 + y, \\ \dot{y} & = & u - z|z|^j, \end{array} \right. \tag{179}$$

where j is some nonnegative real number. We can apply Lemma 2.2 to deduce that the point $(0,0)$ is semiglobally practically stabilizable by the output feedback

$$u = -Ky \tag{180}$$

with K large enough. To study whether we have asymptotic stability, we check to see if Lemma 4.1 applies.

First we consider the system

$$\dot{x}_1 = -x_1^3 + u_1. \tag{181}$$

To get an expression for γ_1, we apply Lemma 4.2. We have

$$\dot{\widehat{\tfrac{1}{2}x_1^2}}_{(181)} = -x_1^4 + x_1\, u_1, \tag{182}$$

$$\leq -\tfrac{7}{8}x_1^4 - \tfrac{1}{8}|x_1|\left(|x_1|^3 - 8|u_1|\right) . \tag{183}$$

It follows that

$$\gamma_1(s) = 2s^{\frac{1}{3}}, \qquad \epsilon_1 = \delta_1 = +\infty \ . \tag{184}$$

Similarly, for the system

$$\dot{x}_2 = -Kx_2 + u_2|u_2|^j \ , \tag{185}$$

we get

$$\gamma_2(s) = \frac{1}{K-1}|s|^{j+1}, \qquad \epsilon_2 = \delta_2 = +\infty \ . \tag{186}$$

Therefore by choosing K large enough, we can meet the constraint (168) for some λ strictly positive and $\omega = 1$ if

$$j \geq 2 \ . \tag{187}$$

In this condition, we know that the equilibrium $(0,0)$ of (179) and (180) is locally asymptotically stable.

In fact the condition (187) given by Lemma 4.1 is not necessary. Indeed, we have

$$\left.\overbrace{\frac{|z|^{j+2}}{j+2} + \frac{y^2}{2}}^{\cdot}\right|_{(179)-(180)} = -Ky^2 - |z|^{j+4} \ . \tag{188}$$

This implies global asymptotic stability for all nonnegative j.

Example 4.2 (A continuation of Example 2.1). Consider again the system (34) of Example 2.1. We have seen that the semiglobal stabilizability of the z subsystem, the definite sign of G, as well as the existence of a lower bound for G, are sufficient conditions for the existence of semiglobally practically stabilizing feedback.

We study now whether we have not only practical stability but also asymptotic stability when

$$A(0,0) = 0 \ , \quad \bar{u}(0) = 0 \ , \quad F(0,0,d) = 0 \qquad \forall d \in D \ . \tag{189}$$

For this study and with the notation of Example 2.1, consider the system

$$\dot{x} = f(z,x,d(t)) + g(z,x,d(t))(-K\text{sgn}(g)x) \tag{190}$$

with input z and a disturbance d. We consider the analysis on the set

$$B(\delta) \doteq \{(z,x) : \ \max\{|z|,|x|\} \leq \delta\} \ , \tag{191}$$

where δ is some strictly positive real number. Because of smoothness, compactness of D, and the definition of f, we can write

$$|f| \leq \gamma_f(|z|) + k_1|x| \qquad \forall((z,x),d) \in B(\delta) \times D, \tag{192}$$

where γ_f is any class-K function satisfying

$$\gamma_f(s) \geq \max_{|z|\leq s,\, d\in D} \{|f(z,0,d)|\} \tag{193}$$

and k_1 is some positive real number independent of K. Recall also that $b \leq |g| = |G|$.

We show now that the system (190) is locally asymptotically stable when $z = 0$ and that it has the uniform (ϵ, δ) ISS property with respect to z. Indeed we have, for $((z, x), d) \in B(\delta) \times D$,

$$\frac{1}{2}\dot{\widehat{x^2}}_{(190)} \leq -Kbx^2 + |x|[k_1|x| + \gamma_f(|z|)], \tag{194}$$

$$\leq -x^2 - |x|\left[(Kb - k_1 - 1)|x| - \gamma_f(|z|)\right]. \tag{195}$$

So, from Lemma 4.2, we have established that, for $K > \frac{k_1+2}{b}$, the system (190) is uniformly (ϵ_x, δ_x) with

$$\left\{\begin{array}{rcl} \delta_x & = & \delta \,, \\ \gamma_f(\epsilon_x) & < & \delta \,, \\ \gamma_x(s) & = & \frac{1}{Kb-k_1-1}\gamma_f(s) \,, \\ \beta_x(s,t) & = & s\exp(-t) \,. \end{array}\right. \tag{196}$$

On the other hand, we notice with Fact 4.1 that the asymptotic stability assumption for the z subsystem implies the existence of a class-K function γ_z and two strictly positive real numbers δ_z and ϵ_z such that the system (see (35))

$$\dot{z} = A(z, \bar{u}(z) + x) \tag{197}$$

with x as input is (ϵ_z, δ_z) ISS with gain function γ_z and class-KL function β_z.

So let us assume the existence of strictly positive real numbers λ, M, ϖ such that

$$(1 + \lambda)\gamma_z \circ \frac{1}{M}\gamma_f(s) \leq s \quad \forall s \in [0, \varpi] \,. \tag{198}$$

Then by imposing the constraint:

$$K \geq \max\left\{\frac{(1+\lambda)M + 1 + k_1}{b} \,,\, \frac{k_1 + 2}{b}\right\} \,, \tag{199}$$

the conditions of Lemma 4.1 are satisfied with

$$\omega = \min\{\varpi \,,\, (1 + \lambda)\gamma_z(\varpi)\} \,. \tag{200}$$

This result gives that the system

$$\left\{\begin{array}{rcl} \dot{z} & = & A(z, \bar{u}(z) + x), \\ \dot{x} & = & f(z, x, d(t)) + g(z, x, d(t))(-K\mathrm{sgn}(g)x) \end{array}\right. \tag{201}$$

has a basin of attraction for local asymptotic stability. Precisely, as shown with full details in §5.2.3, there exists a strictly positive number ϑ_0 independent of K, such that the basin of attraction contains the set

$$\mathcal{A} = \{(z, x) : |(z, x)| < \vartheta_0\} \,. \tag{202}$$

To complete our proof of semiglobal stabilizability under the condition in (198), it remains to establish that the solutions of the closed-loop system are captured by $\mathcal{A}$. But this follows easily by choosing, in the design, the compact set $\mathcal{K}_s$ so that $\mathcal{K}_s \subset \mathcal{A}$ and picking K large enough.

5. A generalized version of Theorem 1.1.

5.1. Assumptions and results. As was done for Theorem 1.2, we prove here a proposition from which Theorem 1.1 follows directly. We consider again the system (109) under the following assumptions (see (145)).

$$\Xi_e(0,0,d) = 0 \qquad A(0,0,d) = 0 \quad \forall d \in D \tag{203}$$

and

Assumption S. We can find

1. a strictly positive real number c_l and a positive C^1 function V which is zero at 0, defined on $\mho$, an open neighborhood of 0, so that the set $\{z : V(z) \leq c_l\}$ is a neighborhood of 0, compact and contained in $\mho$,

2. a C^2 function $\bar{u}(z)$ which is zero at 0, is defined on $\mho$, and is UCO (i.e., (6) holds), such that

$$\dot{V}_{(111)} \leq -\Phi(z) \tag{204}$$

where $\Phi(z)$ is continuous on $\mho$ and positive definite on $\{z : V(z) \leq c_l\} \setminus \{0\}$.

PROPOSITION 5.1. *Suppose the system* (109) *is so that Assumption S and* (203) *hold and there exist strictly positive real numbers* λ, M, ϖ *such that*

$$(1+\lambda)\gamma_z \circ \frac{1}{M}\gamma_{0,(\xi,e)}(s) \leq s \qquad \forall s \in [0,\varpi], \tag{205}$$

where $\gamma_{0,(\xi,e)}$ *is a class-K function satisfying* (*see* (145))

$$\gamma_{0,(\xi,e)}(s) \geq \max_{(z,d):\,|z|\leq s,\, d\in D} \left\{ \left| \frac{\partial \bar{u}}{\partial z}(z) A(z,\bar{u}(z),d) \right| \,,\, |\Xi_e(z,0,d)| \right\} \tag{206}$$

and γ_z *is the* (ϵ,δ) *gain, with respect to* u *and uniform in* d*, of the system*

$$\dot{z} = A(z, \bar{u}(z) + u, d(t)) \ . \tag{207}$$

Under these conditions, there exists a dynamic output feedback making the origin of the closed-loop system uniformly asymptotically stable with a basin of attraction such that its projection contains any strict compact subset of $\{z : V(z) \leq c_l\}$.

Proof of Theorem 1.1. As already mentioned in the proof of Theorem 1.2, there exists a C^1 function V defined and proper on $\mathbb{R}^n$ and positive definite on $\mathbb{R}^n\setminus\{0\}$ and a C^2 UCO control law so that Assumption S holds for any strictly positive real number c_l. Also, this control being locally exponentially stabilizing, it follows from Fact 4.1 that $\gamma_z(s)$ in (205) is linearly bounded on a neighborhood of 0. On the other hand, with the functions involved in (206) being at least C^1, the function $\gamma_{0,(\xi,e)}(s)$ can be chosen linear on a neighborhood of 0. Inequality (205) follows readily. The conclusion of Theorem 1.1 then follows from Proposition 5.1. □

5.2. Proof of Proposition 5.1.

5.2.1. Practical stabilization. Let us first notice that Assumption S implies the existence of a class-K function α_1 so that

$$V(z) \leq c_l \quad \Longrightarrow \quad \alpha_1(|z|) \leq V(z). \tag{208}$$

Then, let us pick three strictly positive real numbers ϑ_s, ϑ_l, and c_s so that

$$\vartheta_s < \vartheta_l \leq c_s < c_l \tag{209}$$

and define the following compact sets:

$$\mathcal{K}_{zs} = \{z : V(z) \leq \vartheta_l\}\,, \qquad \mathcal{K}_{zl} = \{z : V(z) \leq c_s\}\,. \tag{210}$$

We are in the condition where the controller design in the proof of Theorem 1.2 applies. So, for any strictly positive real number ϑ_1, and any compact sets $(\mathcal{K}_{yl}, \mathcal{K}_{\xi l})$, we can find, in particular, a real number K_{*1}, a compact set

$$\Gamma \supset \mathcal{K}_{zl} \times \left\{ \xi : \xi^T P_c \xi \leq \sup_{\xi \in \mathcal{K}_{\xi l}} \{\xi^T P_c \xi\} \right\}, \tag{211}$$

and positive functions $L_{*1}(K)$, $\mu_2(K)$, so that, for each $K \geq K_{*1}$, $L \geq L_{*1}(K)$, the dynamic output feedback (139) in closed-loop with the system (109) makes all the solutions, with initial condition in $\mathcal{K}_{zl} \times \mathcal{K}_{yl} \times \mathcal{K}_{\xi l}$, remain in the set $\{(z, \widehat{y}, \xi) : (z, \xi) \in \Gamma,\ e^T P_o e \leq \exp(\mu_2(L) + 1) - 1\}$ and be captured by

$$\mathcal{R} = \left\{(z, \widehat{y}, \xi) \;:\; |z| \leq \alpha_1^{-1}(\vartheta_l)\,,\ e^T P_o\, e \leq \exp(4\vartheta_1) - 1\,,\ \xi^T P_c \xi \leq 8\vartheta_1 \right\}, \tag{212}$$

where P_o is given by (155) and P_c is given by (131).

To study under which condition we have attractivity of a single point, we remark that the closed-loop system is made of the interconnection of

$$\dot{z} = A(z, \bar{u}(z) + \xi_1, d(t)) \tag{213}$$

with:

$$\left\{ \begin{array}{rcl} \dot{\xi} & = & K\, A_c\, \xi + \Xi_\xi(z, \xi, e, d(t)), \\ \dot{e} & = & L\, A_o\, e + \Xi_e(z, \xi, d(t)), \end{array} \right. \tag{214}$$

where Ξ_e is defined in (145) and

$$\Xi_\xi(z, \xi, e, d) = \begin{pmatrix} -\frac{\partial \bar{u}}{\partial z}(z) A(z, \bar{u}(z) + \xi_1, d) \\ 0 \\ \vdots \\ 0 \\ K a_1 \left[\Delta(\widehat{u}) - \bar{u}(z)\right] \end{pmatrix}. \tag{215}$$

From Assumption S, Fact 4.1 applies. So there exist a class-K function γ_z and two strictly positive real numbers δ_z and ϵ_z such that the system

$$\dot{z} = A(z, \bar{u}(z) + u, d(t)) \tag{216}$$

is uniformly (ϵ_z, δ_z) ISS stable with gain function γ_z. It follows then from Lemma 4.1 that local asymptotic stability can be proved if the (ξ, e)-subsystem is also uniformly $(\epsilon_{(\xi,e)}, \delta_{(\xi,e)})$ ISS for some strictly positive real numbers $\epsilon_{(\xi,e)}, \delta_{(\xi,e)}$, and gain $\gamma_{(\xi,e)}$ satisfying a small gain condition like (168).

5.2.2. Input-to-state stability of the (ξ, e)-subsystem. With (115), (140) and the function $\overline{\Psi}$ defined in (150), we have, for all e, $K \geq 0$, and $L \geq 1$,

$$|\widehat{u} - \bar{u}(z)| \leq \overline{\Psi}(K, |e|)\, |e| \qquad \forall ((z, \xi), d) \in \Gamma \times D\,. \tag{217}$$

Also there exists a positive real number ν_4 satisfying, for all $((z,\xi),d)$ in the compact set $\Gamma \times D$,

$$\left|\frac{\partial \bar{u}}{\partial z}(z)A(z,\bar{u}(z)+\xi_1,d) - \frac{\partial \bar{u}}{\partial z}(z)A(z,\bar{u}(z),d)\right| \leq \nu_4\,|\xi|. \tag{218}$$

So, with (142), (148), for all $((z,\xi),d)$ in $\Gamma \times D$ and all e, we have

$$|\Xi_\xi(z,\xi,e,d) - \Xi_\xi(z,0,0,d)| \leq \nu_4\,|\xi| + \overline{\Psi}(K,|e|)\,K\,|a_1|\,|e|\;. \tag{219}$$

Similarly, from (145), we see that there exists a positive function $\overline{C}$ satisfying, for $K \geq 1$,

$$|\Xi_e(z,\xi,d) - \Xi_e(z,0,d)| \leq \overline{C}(K)\,K^{m_u}\,|\xi| \qquad \forall((z,\xi),d) \in \Gamma \times D\;. \tag{220}$$

With (203), let $\gamma_{0,(\xi,e)}$ be any class-K function satisfying

$$\gamma_{0,(\xi,e)}(s) \geq \max_{(z,d):\,|z|\leq s,\,d\in D}\left\{\left|\frac{\partial \bar{u}}{\partial z}(z)A(z,\bar{u}(z),d)\right|\;,\;|\Xi_e(z,0,d)|\right\}. \tag{221}$$

We have, for all $((z,\xi),d)$ in $\Gamma \times D$ and all e,

(222)

$$\begin{cases} \dot{\overline{\xi^T P_c\,\xi}} & \leq \quad -K\xi^T\xi + 2\lambda_{\max}\{P_c\}|\xi|\left[\nu_4\,|\xi| + \gamma_{0,(\xi,e)}(|z|) + \overline{\Psi}(K,|e|)\,K\,|a_1|\,|e|\right], \\ \dot{\overline{e^T P_o\,e}} & \leq \quad -Le^Te + 2\lambda_{\max}\{P_o\}|e|\left[\overline{C}(K)\,K^{m_u}\,|\xi| + \gamma_{0,(\xi,e)}(|z|)\right]. \end{cases}$$

Then, from the properties of V, there exists a strictly positive real number $\epsilon_{(\xi,e)}$ satisfying

$$|z| \leq \epsilon_{(\xi,e)} \qquad \Longrightarrow \qquad z \in \mathcal{K}_{zl}. \tag{223}$$

Let also

$$r = \sqrt{\frac{\sup_{\xi\in\mathcal{K}_{\xi l}}\{\xi^T P_c \xi\}}{\lambda_{\max}\{P_c\}}}. \tag{224}$$

Then, with (211), we have

$$\left(|z| \leq \epsilon_{(\xi,e)}\;,\;\left|\begin{pmatrix}\xi\\ e\end{pmatrix}\right| \leq r\right) \qquad \Longrightarrow \qquad ((z,\xi)\in\Gamma\;,\;|e|\leq r)\,. \tag{225}$$

Finally, define[10]

$$\begin{aligned} K_{*2} &= 2\max\Big\{\lambda_{\max}\{P_c\}\,(3+4\nu_4)\,, \\ &\qquad 1 + \tfrac{\max\{\lambda_{\max}\{P_c\},\lambda_{\max}\{P_o\}\}\,(\lambda_{\max}\{P_c\}+\lambda_{\max}\{P_o\})}{r^2\min\{\lambda_{\min}\{P_c\},\lambda_{\min}\{P_o\}\}}\,\gamma_{0,(\xi,e)}\big(\epsilon_{(\xi,e)}\big)^2\Big\}, \qquad (226)\\ L_{*2}(K) &= 2\lambda_{\max}\{P_o\} + K + 2\tfrac{\left(\lambda_{\max}\{P_c\}K|a_1|\overline{\Psi}(K,r)\right)^2 + \left(\lambda_{\max}\{P_o\}K^{m_u}\,\overline{C}(K)\right)^2}{\lambda_{\max}\{P_c\}}. \end{aligned}$$

[10] The second argument in the max guarantees the condition (173) of Lemma 4.2 holds.

We have established that the condition

$$d \in D\ , \quad |z| \le \epsilon_{(\xi,e)} \text{ and } \left|\begin{pmatrix} \xi \\ e \end{pmatrix}\right| \le r$$

implies, for all $K \ge K_{*2}$ and $L \ge L_{*2}(K)$,

$$\overbrace{\xi^T P_c \xi + e^T P_o e}^{\cdot} \le -\frac{K}{2}\left(\xi^T \xi + e^T e\right) + \left(\lambda_{\max}\{P_c\} + \lambda_{\max}\{P_o\}\right) \gamma_{0,(\xi,e)}(|z|)^2. \tag{227}$$

Then by applying Lemma 4.2, we see that, for $K \ge K_{*2}$ and $L \ge L_{*2}(K)$, the (ξ, e)-subsystem is uniformly $(\epsilon_{(\xi,e)}, \delta_{(\xi,e)})$ ISS with

(228)

$$\left\{\begin{array}{rcl} \gamma_{(\xi,e)}(s) & = & \sqrt{\frac{2}{K-2}}\sqrt{\frac{\max\{\lambda_{\max}\{P_c\},\lambda_{\max}\{P_o\}\}\,(\lambda_{\max}\{P_c\}+\lambda_{\max}\{P_o\})}{\min\{\lambda_{\min}\{P_c\},\lambda_{\min}\{P_o\}\}}}\,\gamma_{0,(\xi,e)}(|z|)\ , \\ \delta_{(\xi,e)} & = & \sqrt{\frac{\min\{\lambda_{\min}\{P_o\}\,,\,\lambda_{\min}\{P_c\}\}}{\max\{\lambda_{\max}\{P_o\},\,\lambda_{\max}\{P_c\}\}}}\, r, \\ \beta_{(\xi,e)}(s,t) & = & \sqrt{\frac{\max\{\lambda_{\max}\{P_o\}\,,\,\lambda_{\max}\{P_c\}\}}{\min\{\lambda_{\min}\{P_o\}\,,\,\lambda_{\min}\{P_c\}\}}}\, s\, \exp\left(-\frac{1}{\max\{\lambda_{\max}\{P_o\}\,,\,\lambda_{\max}\{P_c\}\}}\, t\right). \end{array}\right.$$

5.2.3. Uniform asymptotic stability. With Lemma 4.1, we can now conclude that the origin is a uniformly asymptotically stable equilibrium point of the closed-loop system under consideration with domain of attraction containing

$$\mathcal{P} = \left\{(z,\xi,e) : \phi(|(z,\xi,e)|, K) < \min\{\delta_{(\xi,e)}, \delta_z, \epsilon_{(\xi,e)}, \epsilon_z, \varpi, (1+\lambda)\gamma_z(\varpi)\}\right\} \tag{229}$$

if

1. there exist strictly positive real numbers λ, M, ϖ such that

$$(1+\lambda)\gamma_z \circ \frac{1}{M}\gamma_{0,(\xi,e)}(s) \le s \qquad \forall s\ \in [0, \varpi] \tag{230}$$

2. K and L are chosen to satisfy

(231)

$$\begin{array}{l} K \ge \max\left\{K_{*1}\,,\ K_{*2}\,,\ 2 + 2\frac{\max\{\lambda_{\max}\{P_c\},\lambda_{\max}\{P_o\}\}\,(\lambda_{\max}\{P_c\}+\lambda_{\max}\{P_o\})}{\min\{\lambda_{\min}\{P_c\},\lambda_{\min}\{P_o\}\}}\, M^2\,(1+\lambda)^2\right\}, \\ L \ge \max\left\{L_{*1}(K), L_{*2}(K)\right\}. \end{array}$$

In (229), the function $\phi(s, K)$ is obtained from (169) as

$$\left\{\begin{array}{rl} \phi_1(s) & = (1+\lambda^{-1})\left(\beta_z(s,0) + \gamma_z\left((1+\lambda^{-1})(1+\lambda^{-1})(\beta_{(\xi,e)}(s,0))\right)\right), \\ \phi_2(s,K) & = (1+\lambda^{-1})\left(\beta_x(s,0) + \gamma_{(\xi,e)}\left((1+\lambda^{-1})(1+\lambda^{-1})(\beta_z(s,0))\right)\right), \\ \phi(s,K) & = \phi_1(s) + \phi_2(s,K), \end{array}\right. \tag{232}$$

where $\gamma_{(\xi,e)}$, dependent on K but not on L, and $\beta_{(\xi,e)}$ are given by (228). From (228), we see that there exists a class-K function ϱ independent of K and L satisfying

$$\phi(s,K) \le \varrho(s) \tag{233}$$

for all $s \ge 0$ and K, L satisfying (231). It follows from (229) and (233) that there exists a strictly positive number ϑ_0, independent of K and L, such that the set

$$\mathcal{A} = \left\{(z,\xi,e) : \max\left\{|z|, \xi^T P_c \xi, e^T P_o e\right\} < \vartheta_0\right\} \tag{234}$$

is contained in $\mathcal{P}$ and therefore in the domain of attraction for all K and L satisfying (231). Then since, in the controller (139), the gains ℓ_i and a_i and the bound $u_{\max}$ are chosen independent of ϑ_1 and ϑ_l, ϑ_0 does not depend on ϑ_1 and ϑ_l. Therefore, we can choose ϑ_1 and ϑ_l strictly positive and such that

$$\vartheta_1 < \min\left\{\frac{1}{4}\ln(1+\vartheta_0), \frac{\vartheta_0}{8}\right\}, \qquad \vartheta_l < \alpha_1(\vartheta_0). \tag{235}$$

With such a choice, we are guaranteed that $\mathcal{A}$ contains $\mathcal{R}$ defined in (212). This implies that the solutions are captured by the set $\mathcal{A}$.

6. Other examples.

6.1. Minimum phase i/o linearizable systems. Many results in the spirit of Theorems 1.1 and 1.2 can be formulated for minimum phase i/o linearizable systems using the tools developed in this paper. Consider the C^1 system

$$\left\{\begin{array}{rcl} \dot{z} & = & h(z,x,\zeta), \\ \dot{x}_1 & = & x_2, \\ & \vdots & \\ \dot{x}_r & = & \zeta, \\ \dot{\zeta} & = & f(z,x,\zeta,d(t)) + g(z,x,\zeta,d(t))u, \\ y & = & x_1, \end{array}\right. \tag{236}$$

$y \in \mathbb{R}$, $u \in \mathbb{R}$, $z \in \mathbb{R}^m$ and $x = (x_1,\dots,x_r)^T \in \mathbb{R}^r$ and $\zeta \in \mathbb{R}$. We assume a well-defined relative degree and the global minimum phase property as follows.

Assumption HFG. The sign of g is constant and the magnitude of g is bounded away from zero.

Assumption MP. $z = 0$ is a globally asymptotically stable equilibrium for the system

$$\dot{z} = h(z,0,0). \tag{237}$$

We will also assume semiglobal stabilizability for the origin of the (z,x) subsystem:

Assumption RSE. The equilibrium point (0,0) of the (z,x) subsystem, with ζ as input, is semiglobally stabilizable by C^ℓ ($\ell \geq 2$) partial state feedback depending only x. Furthermore, this feedback locally exponentially stabilizes the origin of the x subsystem.

Note that, with Assumption MP, Assumption RSE holds in the following cases:

1. [34], [31] the state z remains bounded for all "disturbances" x and ζ which converge to zero. A special case is when the z subsystem is globally ISS with respect to x and ζ.
2. [36, Thm. 6.2] h is globally Lipschitz.
3. [7], [36] h depends only on z and x_1.
4. [38] h depends on only one component of the vector $(x_1,\dots,x_r,\zeta)^T$.

Then, using Example 2.1 and Proposition 3.1, respectively, we have the following results.

COROLLARY 6.1. *If Assumptions HFG, MP, and RSE hold, then the origin of the system* (236) *is semiglobally practically stabilizable by C^ℓ ($\ell \geq 2$) and UCO state feedback.*

COROLLARY 6.2. *If Assumptions HFG, MP, and RSE hold, then then the origin of* (236) *is semiglobally practically stabilizable by dynamic output feedback.*

The feedback used to prove Corollary 6.1 is of the form $u = -\mathrm{sgn}(g)K(\zeta - \bar{u}(x))$, where $\bar{u}$ is the feedback given by assumption RSE. See Example 2.1. We then remark that the dynamic extension of §3.2.2 is not needed in Corollary 6.2 because the practically stabilizing feedback of Corollary 6.1 is UCO without using u or its derivatives.

Local exponential stability for the origin of the x subsystem is not used in either of these results. It will be used, together with the next assumption, to guarantee asymptotic stabilizability.

Assumption LSG. With γ_z the local gain function of the z subsystem with (x, ζ) as input, there exist a class-K function γ_f and positive real numbers λ, M, and ϖ such that

$$(1+\lambda)\gamma_z \circ \frac{1}{M}\gamma_f(s) \le s \qquad \forall s \in [0, \varpi], \tag{238}$$

$$\gamma_f(s) \ge \sup_{|z|\le s, d\in D} \{|f(z,0,0,d)|\}. \tag{239}$$

We remark that local exponential stability of the origin of the system (237) and $f(0,0,0,d) = 0$ for all $d \in D$ are sufficient to guarantee that Assumption LSG holds.

COROLLARY 6.3. *If Assumptions HFG, MP, RSE, and LSG hold, then the origin of the system* (236) *is semiglobally stabilizable by* C^ℓ $(\ell \ge 2)$ *and UCO state feedback.*

Proof. Define $\xi = \zeta - \bar{u}(x)$. With the feedback law mentioned above, the (x, ξ) subsystem has the form

(240)

$$\begin{aligned} \dot{x} &= E(x, \bar{u}(x) + \xi), \\ \dot{\xi} &= f(z, x, \bar{u}(x)+\xi, d(t)) - |g(z, x, \bar{u}(x)+\xi, d(t))|K\xi - \tfrac{\partial \bar{u}}{\partial x}(x)E(x, \bar{u}(x)+\xi), \end{aligned}$$

where

$$|E(x, \bar{u}(x)+\xi) - E(x, \bar{u}(x))| \le |\xi| \,. \tag{241}$$

We will show that, for K sufficiently large, the (x, ξ) subsystem with z as input is uniformly (ϵ, δ) ISS with

$$\beta(s,t) = k_1 s \exp(-k_2 t), \qquad \gamma(s) = \frac{k_3}{\sqrt{K}}\gamma_f(s) \tag{242}$$

for some positive real numbers ϵ, δ, k_1, k_2, k_3, which can all be taken independent of K. The equilibrium point $x = 0$ of

$$\dot{x} = E(x, \bar{u}(x)) \tag{243}$$

is locally exponentially stable. This guarantees the existence of a function $V(x)$ and strictly positive real numbers c_1, c_2, c_3, and r such that, for all $|x| \le r$,

$$\begin{aligned} c_1|x|^2 &\le V(x) \le c_2|x|^2, \\ \dot{V}_{(243)} &\le -|x|^2, \\ \left|\tfrac{\partial V}{\partial x}\right| &\le c_3|x| \,. \end{aligned} \tag{244}$$

We restrict our analysis to the set

$$B(\bar{\delta}) \doteq \left\{(z, x, \xi) : \max\{|z|, |x|, |\xi|\} \le \bar{\delta}\right\}, \tag{245}$$

where $\bar{\delta} \leq r$ is some strictly positive real number. Because of smoothness and compactness of D we can write

$$\left| f(z, x, \bar{u}(x) + \xi, d) - \frac{\partial \bar{u}}{\partial x}(x) E(x, \bar{u}(x) + \xi) \right| \leq \gamma_f(|z|) + c_4|x| + c_5|\xi| \qquad \forall ((z, x, \xi), d) \in B(\bar{\delta}) \times D, \tag{246}$$

where γ_f is defined in (239) and c_4 and c_5 are some positive real numbers independent of K. With assumption HFG, let $0 < b \leq |g|$. Then, for all $((z, x, \xi), d) \in B(\bar{\delta}) \times D$,

$$\dot{\overbrace{V(x) + \xi^2}}_{(240)} \leq -|x|^2 + c_3|x||\xi| - 2Kb|\xi|^2 + 2|\xi|[\gamma_f(|z|) + c_4|x| + c_5|\xi|], \tag{247}$$

$$\leq -\tfrac{1}{2}|x|^2 - (Kb - 4c_4^2 - c_3^2 - 2c_5)|\xi|^2 + \tfrac{1}{Kb}\gamma_f^2(|z|). \tag{248}$$

For K sufficiently large, the uniform (ϵ, δ) ISS property, with β and γ of the form (242), follows by applying Lemma 4.2. Compare with (227), (228).

From Assumption MP and Fact 4.1, the z subsystem is (ϵ_z, δ_z) ISS with respect to (x, ζ) with gain function γ_z. To identify the gain function with respect to (x, ξ) observe that, $\bar{u}$ being at least C^1, there exists a positive real number k_4 such that

$$|x| \leq \epsilon_z \quad \Longrightarrow \quad \bar{u}(x) \leq k_4|x|. \tag{249}$$

Then, we can take

$$\gamma_z^{(x,\xi)}(s) = \gamma_z((1 + k_4)s). \tag{250}$$

Finally, applying Lemma 4.1 to the interconnection of the z and (x, ξ) subsystems, one finds that condition (238) of Assumption LSG is sufficient to guarantee local asymptotic stability for K sufficiently large. Moreover, as in Example 4.2, Lemma 4.1 demonstrates that a neighborhood $\mathcal{A}$ can be described, independent of K, which is contained in the basin of attraction for all K sufficiently large. Then, semiglobal stabilizability follows from Corollary 6.1. □

COROLLARY 6.4. *If Assumptions HFG, MP, RSE, and LSG hold, then the origin of the system* (236) *is semiglobally stabilizable by dynamic output feedback.*

Sketch of proof. The proof is the same as that of the previous corollary. In this instance, the closed-loop system has the state (z, x, ξ, e) and the (x, ξ, e) subsystem is uniformly (ϵ, δ) ISS with β and γ again of the form (242). The conclusion follows from the small gain theorem and Corollary 6.2 with K chosen large enough.

Weaker versions of this last corollary have been published. In [15, §4.7] a similar local result is established for systems with locally exponentially stable zero dynamics. In [19], a similar global result is established for globally Lipschitz nonlinearities. More recently, for the case where the $\dot{z}$ equation in (236) is linear in z, it has been shown in [11] that the equilibrium point $(z, x) = (0, 0)$ is locally stabilizable by output feedback. This result provided estimates for the region of attraction but did not guarantee arbitrarily large domains of attraction. In all of these cases, Assumption LSG is automatically satisfied. When the system does not have zero dynamics it was shown in [18] that the equilibrium point $x = 0$ is semiglobally stabilizable by output feedback. For these results, high gain observers are used. In the special case where only x_1 appears in h, f is generated by differentiation of nonlinearities that depend only on x_1 and z, and $g(x_1)$ is known (see (61)), it has been shown in [42] that the system (236) is semiglobally stabilizable by output feedback, under an assumption similar to Assumption LSG, but without requiring high gain observers. See Example 2.3. If, in addition, the inverse dynamics satisfy an input-to-state stability property with respect to x_1 then, as was shown in [29], the system (236) is *globally* stabilizable by output feedback. This generalized the results of [17] and [26], [27] where it was required that the system be linear up to output injection.

6.2. A nonminimum phase i/o linearizable system. Consider the non-minimum phase system on $\mathbb{R}^3$ with y as the output

$$\begin{cases} \dot{z}_1 &= -z_1 + z_2 - z_1 y^2, \\ \dot{z}_2 &= z_2^2 + y + z_2 z_1^2, \\ \dot{y} &= u + z_2. \end{cases} \tag{251}$$

The origin of the zero dynamics

$$\begin{cases} \dot{z}_1 &= -z_1 + z_2, \\ \dot{z}_2 &= z_2^2 + z_2 z_1^2 \end{cases} \tag{252}$$

is unstable. Indeed, any solution with initial condition satisfying $z_2(0) > 0$ exhibits finite escape time. Instead of a decomposition into z and y subsystems, we view the system (251) as in Lemma 2.3 with z_1 playing the role of z and (z_2, y) as the block of integrators. Although the assumptions of Lemma 2.3 cannot be satisfied because of the presence of y in the $\dot{z}_1$ equation, the result is still valid. Namely, for K_1 large enough, the control

$$u = -K_1^2 \left(z_2 + \frac{y}{K_1} \right) \tag{253}$$

is semiglobally stabilizing. This can be checked by looking at the time derivative of

$$W = c \frac{z_1^2}{c + 1 - z_1^2} + \mu \frac{\frac{3}{2} z_2^2 + z_2 \frac{y}{K_1} + (\frac{y}{K_1})^2}{\mu + 1 - (\frac{3}{2} z_2^2 + z_2 \frac{y}{K_1} + (\frac{y}{K_1})^2)} . \tag{254}$$

Local exponential convergence follows from the exponential stability of the undriven z_1 subsystem as discussed above. To conclude semiglobal output feedback stabilizability from Propositions 3.1 and 5.1, it remains to verify that the control (253) is UCO. This property holds trivially since we have

$$z_2 = \dot{y} - u. \tag{255}$$

7. Conclusion. We have developed tools for semiglobal stabilization by partial state and output feedback with, as a main application, semiglobal output feedback stabilization for nonlinear systems that admit a uniformly completely observable stabilizing function. Our approach for this problem uses the observer idea of [11] and the dynamic extension of [43]. This result can be seen as an extension of the result given in [40].

An important feature in our approach is to consider the issue of bounded solutions separate from convergence to the equilibrium. To guarantee convergence we have imposed a sufficient, but not necessary, small gain condition which generalizes local exponential stability assumptions.

We have given several applications illustrating our tools:

- We have shown that semiglobal stabilizability by uniformly completely observable state feedback is a sufficient condition for semiglobal practical stabilization by output feedback. Stabilization itself is obtained if an extra local small gain property is satisfied. We have applied this result to input–output linearizable systems.
- We have given output feedback solutions for certain robotics problems (Example 2.4), for the ball and beam (Example 2.5), and for a nonminimum phase system (§6.2).
- We have applied our semiglobal stabilization design to the almost disturbance decoupling problem to eliminate the vanishing regions of the attraction problem discussed in [21] and [25]. See Example 2.2.

Appendix A. Appendix.

A.1. Proof of Fact 4.1. For a strictly positive real number r and a positive integer n, define the set B_r^n, a closed subset of $\mathbb{R}^n$, by

$$B_r^n = \{x \in \mathbb{R}^n : |x| \le r\} \ . \tag{256}$$

CLAIM. *Under the conditions of Fact 4.1, there exist a strictly positive real number r, a C^1 function $V : \mathbb{R}_{\geq 0} \times B_r^n \to \mathbb{R}_{\geq 0}$, class-$K$ functions α_1, α_2, α_3, and α_5^{-1} defined on $[0,r]$ such that, for all $t \geq 0$ and all (x,u) in $B_r^n \times B^m_{\alpha_5^{-1}(r)}$, we have*

$$\alpha_1(|x|) \le V(t,x) \le \alpha_2(|x|) \tag{257}$$

and

$$|x| \ge \alpha_5(|u|) \qquad \Longrightarrow \qquad \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} h(x,u,t) \le -\alpha_3(|x|) \ . \tag{258}$$

In the time-invariant case, V can be taken independent of t.

Proof. We start this proof by defining a function $d_\circ : [0,r] \to \mathbb{R}_{\geq 0}$ as follows:

1. For the time-invariant case, as in [35, Eqs. (13), (14)] but assuming only LAS of $x = 0$ for $\dot{x} = h(x,0)$, we know that there exist a strictly positive real number r, a C^1 function $V : B_r^n \to \mathbb{R}_{\geq 0}$, and functions α_1, α_2, α_3 of class-K defined on $[0,r]$ such that

$$\alpha_1(|x|) \le V(x) \le \ \alpha_2(|x|) \forall x \in B_r^n, \tag{259}$$

$$\frac{\partial V}{\partial x}(x) h(x,0) + \alpha_3(|x|) \ < \ 0 \qquad \forall x \in B_r^n \backslash \{0\}. \tag{260}$$

Then following [35, Proofs of Lem. 3.1 and 3.2], there exists a piecewise constant function $d_\circ : [0,r] \to \mathbb{R}_{\geq 0}$ such that

$$\left\{ \begin{array}{lcll} d_\circ(0) & = & 0 \ , & \\ d_\circ(s) & > & 0 & \forall s \in (0,r] \ , \\ |u| \le d_\circ(|x|) & \Rightarrow & \frac{\partial V}{\partial x}(x) h(x,0) < -\alpha_3(|x|). & \end{array} \right. \tag{261}$$

2. For the time-variant case, from the assumptions on h, there exist strictly positive real numbers r and L, a C^1 function $V : \mathbb{R}_{\geq 0} \times B_r^n \to \mathbb{R}_{\geq 0}$, and class-$K$ functions α_1, α_2, α_3, and α_4 defined on $[0,r]$ such that, for all (x,t) in $B_r^n \times \mathbb{R}_{\geq 0}$,

$$\left\{ \begin{array}{rcl} \alpha_1(|x|) & \le & V(t,x) \le \alpha_2(|x|), \\ \frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x) h(x,0,t) & \le & -2\alpha_3(|x|), \\ |\frac{\partial V}{\partial x}(t,x)| & \le & \alpha_4(|x|), \end{array} \right. \tag{262}$$

(see [20, Thm. 4.7] for example) and

$$(263)\qquad |h(x,u,t) - h(x,0,t)| \le L|u| \qquad \forall (x,u,t) \in B_r^n \times B_r^m \times \mathbb{R}_{\ge 0}\ .$$

From (262) and (263) it follows that

$$(264)\qquad \begin{array}{l} \frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x)h(x,u,t) \\ \le -2\alpha_3(|x|) + \alpha_4(|x|)L|u| \qquad \forall (x,u,t) \in B_r^n \times B_r^m \times \mathbb{R}_{\ge 0}. \end{array}$$

In this case, we let the function $d_\circ$ be

$$(265)\qquad \left\{ \begin{array}{rcl} d_\circ(0) &=& 0, \\ d_\circ(s) &=& \frac{\alpha_3(s)}{L\alpha_4(s)} \quad \forall s \in (0,r]. \end{array} \right.$$

From our definitions of $d_\circ$, we have

$$(266)\qquad \inf_{\sigma \in [\tau,r]} \{\sigma, d_\circ(\sigma)\} > 0 \qquad \forall \tau \in (0,r].$$

So let θ be the function defined on $\mathbb{R}_{\ge 0}$ by

$$(267)\qquad \left\{ \begin{array}{rcl} \theta(0) &=& 0\ , \\ \theta(s) &=& \frac{1}{s}\int_0^s \left(\frac{\tau}{1+\tau} \inf_{\sigma\in[\tau,r]} \{\sigma, d_\circ(\sigma)\} \right) d\tau \qquad \forall s \in (0,r]\ , \\ \theta(s) &=& \theta(r)\,\frac{s}{r} \qquad \forall s \in (r,+\infty)\ . \end{array} \right.$$

This function is of class-K and the definitions of $d_\circ$ imply, for all (x,u) in $B_r^n \times B_r^m$,

$$(268)\qquad |u| \le \theta(|x|) \quad \Longrightarrow \quad \frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x)h(x,u,t) \le -\alpha_3(|x|)\ .$$

So, the claim follows by defining $\alpha_5 = \theta^{-1}$.

If the equilibrium point is locally exponentially stable, then from the assumptions on h, it is well known (see [20, Thm. 4.5]) that the functions α_i, for $i = 1,2,3$, can be taken to be of the form $\alpha_i(s) = k_i r^2$ for some $k_i > 0$. Furthermore, α_4 can be taken of the form $\alpha_4(s) = k_4 s$ for some $k_4 > 0$. It follows that α_5 has the form $\alpha_5(s) = \frac{Lk_4}{k_3}s$.

Finally, Fact 4.1 is established using Lemma 4.2 and the fact that, when $\alpha_i(s) = k_i s^2$, for $i = 1,\ldots,3$, $\alpha_5 = \frac{Lk_4}{k_3}s$ implies

$$(269)\qquad \alpha_1^{-1}(\alpha_2(\alpha_5(s))) = \sqrt{\frac{k_2}{k_1}\frac{Lk_4}{k_3}}\,s. \qquad \square$$

A.2. Proof of Lemma 4.1. We will make use of the following facts (see [24]).

FACT A.1. *Let γ be a function of class-K and let g be a function of class-K_∞. We have*

$$(270)\qquad \gamma(a+b) \le \gamma \circ (Id+g)(a) + \gamma \circ (Id + g^{-1})(b)\ .$$

Proof. The proof follows from considering the two cases $b \le g(a)$ and $b \ge g(a)$. $\square$

FACT A.2. *Let γ_1 and γ_2 be functions of class-K and let λ and ω be positive real numbers. Then*

$$(271)\qquad \begin{aligned} &(1+\lambda)\gamma_1 \circ (1+\lambda)\gamma_2(s) \le s \qquad \forall s \in [0,\omega] \\ &\implies (1+\lambda)\gamma_2 \circ (1+\lambda)\gamma_1 \le s \qquad \forall s \in [0,(1+\lambda)\gamma_2(\omega)]. \end{aligned}$$

Proof. We prove Fact A.2 by contradiction. Assume there exists an $s' \in [0,(1+\lambda)\gamma_2(\omega)]$ such that

$$(272)\qquad (1+\lambda)\gamma_2 \circ (1+\lambda)\gamma_1(s') > s'.$$

Since $(1+\lambda)\gamma_2$ is of class-K, this inequality implies that $((1+\lambda)\gamma_2)^{-1}(s')$ is well defined and that

$$(273)\qquad (1+\lambda)\gamma_1(s') > ((1+\lambda)\gamma_2)^{-1}(s') \doteq t.$$

This further implies

$$(274)\qquad (1+\lambda)\gamma_1 \circ (1+\lambda)\gamma_2(t) > t,$$

which is a contradiction since $t \in [0,\omega]$. □

We now prove Lemma 4.1.

CLAIM. *For the class-K functions ϕ and r defined in* (169) *and* (170), *for positive real numbers δ and ϵ satisfying* (171), *and for each $t_\circ \ge 0$, we have*

$$(275)\qquad \{|x(t_\circ)| \le \delta\ ,\ ||v||_{t_\circ} \le \epsilon\} \quad \implies \quad \{|x(t)| \le \phi(|x(t_\circ)|) + r(||v||_{t_\circ})\ \forall t \ge t_\circ\}\ .$$

Proof. Define

$$(276)\qquad \bar\delta = \min\{\delta_1, \delta_2, \epsilon_1^u, \epsilon_2^u, \omega\}.$$

The positive real numbers on the right-hand side come from the uniform (ϵ_i, δ_i) ISS assumption on each subsystem and condition (168). Notice, from (164) and (169), that for any pair (δ, ϵ) satisfying (171) we have

$$(277)\qquad \delta < \bar\delta.$$

Then, from the assumptions on the system (167), for any initial condition $x(t_\circ)$ satisfying $|x(t_\circ)| \le \delta$ and any measurable function $v(\cdot)$ satisfying $||v||_{t_\circ} \le \min\{\epsilon_1^v, \epsilon_2^v\}$, one can find a strictly positive real number T, possibly infinite, corresponding to a maximal interval $[t_\circ, t_\circ + T)$ such that there exists a unique solution to the feedback interconnection satisfying $|x_i(t)| < \bar\delta$ for all $t \in [t_\circ, t_\circ + T)$. Define

$$(278)\qquad ||x_i||_{t_\circ}^T = \sup_{t_\circ \le \tau < t_\circ + T} |x_i(\tau)|.$$

For ease of notation, we take

$$(279)\qquad \gamma_i = \gamma_i^u, \qquad d_i = \gamma_i^v(||v||_{t_\circ}^T).$$

From the uniform (ϵ_i, δ_i) ISS assumption and causality of the feedback interconnection, for all $t \in [t_\circ, t_\circ + T)$, we have

$$\left\{ \begin{array}{lcl} |x_1(t)| & \leq & \beta_1(|x_1(t_\circ)|, t - t_\circ) + \gamma_1(||x_2||_{t_\circ}^T) + d_1, \\ |x_2(t)| & \leq & \beta_2(|x_2(t_\circ)|, t - t_\circ) + \gamma_2(||x_1||_{t_\circ}^T) + d_2. \end{array} \right. \tag{280}$$

Now, using Fact A.1 and (168), we get

$$||x_1||_{t_\circ}^T \leq \beta_1(|x(t_\circ)|, 0) + \gamma_1\left(\beta_2(|x(t_\circ)|, 0) + \gamma_2(||x_1||_{t_\circ}^T) + d_2\right) + d_1 \tag{281}$$

$$\begin{aligned} &\leq \beta_1(|x(t_\circ)|, 0) + \gamma_1\left((1+\lambda^{-1})(\beta_2(|x(t_\circ)|, 0) + d_2)\right) \\ &\quad + \gamma_1\left((1+\lambda)\gamma_2(||x_1||_{t_\circ}^T)\right) + d_1 \end{aligned} \tag{282}$$

$$\begin{aligned} &\leq \beta_1(|x(t_\circ)|, 0) + \gamma_1\left((1+\lambda^{-1})(1+\lambda^{-1})(\beta_2(|x(t_\circ)|, 0))\right) \\ &\quad + \gamma_1\left((1+\lambda^{-1})(1+\lambda)(d_2)\right) + (1+\lambda)^{-1}||x_1||_{t_\circ}^T + d_1 \; . \end{aligned} \tag{283}$$

From this we conclude:

$$\begin{aligned} ||x_1||_{t_\circ}^T \leq{} & (1+\lambda^{-1})\left[\beta_1(|x(t_\circ)|, 0) + \gamma_1((1+\lambda^{-1})(1+\lambda^{-1})(\beta_2(|x(t_\circ)|, 0))\right] \\ & + (1+\lambda^{-1})\left(d_1 + \gamma_1\left((1+\lambda^{-1})(1+\lambda)(d_2)\right)\right). \end{aligned} \tag{284}$$

Using the definition of ϕ_1 in (169), r_1 in (170), and d_i in (279), we get

$$||x_1||_{t_\circ}^T \leq \phi_1(|x(t_\circ)|) + r_1(||v||_{t_\circ}^T). \tag{285}$$

We can repeat the analysis for x_2 obtaining the class-K functions ϕ_2 and r_2 defined in (169) and (170), respectively. Then choose

$$\phi(s) = \phi_1(s) + \phi_2(s), \qquad r(s) = r_1(s) + r_2(s). \tag{286}$$

Then, since δ and ϵ are strictly positive real numbers satisfying (171), by contradiction it is easy to see that if $|x(t_\circ)| \leq \delta$ and $||v||_{t_\circ} \leq \epsilon$, T must be infinite which establishes the claim. Note for the global case, there are no restrictions on $|x(t_\circ)|$ or $||v||_{t_\circ}$.

CLAIM. *Let (ϵ, δ) be an arbitrary pair satisfying* (171). *For each strictly positive real number σ and each $(x(t_\circ), v)$ satisfying*

$$|x(t_\circ)| \leq \delta \; , \quad ||v||_{t_\circ} \leq \epsilon \; , \tag{287}$$

there exists a time T so that the corresponding solution satisfies

$$|x(t)| \leq \sigma + r(||v||_{t_\circ}) \qquad \forall t \geq t_\circ + T. \tag{288}$$

Moreover

1. *T depends only on σ and m defined as*

$$m = \phi(|x(t_\circ)|) + r(||v||_{t_\circ}). \tag{289}$$

2. *T is zero for any σ if m is zero.*
3. *For any fixed $\sigma > 0$, T increases with m.*
4. *For any fixed $m > 0$, T decreases with σ.*

Proof. Given a pair (ϵ, δ) satisfying (171), let $x(t_\circ)$ and v satisfy

$$|x(t_\circ)| \leq \delta, \qquad ||v||_{t_\circ} \leq \epsilon. \tag{290}$$

Then, given the strictly positive real number σ, we pick $t_1(\sigma, m)$ to satisfy

$$\beta_1(m, t_1) + \gamma_1\left((1+\lambda^{-1})^2 \beta_2(m, t_1)\right) \leq \frac{\frac{1}{2}\sigma}{(1+\lambda^{-1})}\ . \tag{291}$$

It is possible to pick such a t_1 because β_i is of class-KL and γ_1 is of class-K. We will show that T given as

$$T = 2t_1 \mathcal{I}\left(\frac{\ln\left(\frac{m}{\frac{1}{2}\sigma}\right)}{\ln(1+\lambda)}\right), \tag{292}$$

where $\mathcal{I}(s)$ is the smallest nonnegative integer greater than or equal to s, is sufficient to establish (288). From (291), (292) this choice satisfies points 1–4 of the claim.

From the uniform (ϵ_i, δ_i) ISS property and the previous claim, we have

$$|x_2(t)| \leq \beta_2(m, t_1) + \gamma_2(||x_1||_{t_s}) + d_2 \qquad \forall t \geq t_1 + t_s \tag{293}$$

for each $t_s \geq t_\circ$. Using this information, we can establish that

$$|x_1(t)| \leq \beta_1(m, t_1) + \gamma_1\left(\beta_2(m, t_1) + \gamma_2(||x_1||_{t_s}) + d_2\right) + d_1 \qquad \forall t \geq 2t_1 + t_s \tag{294}$$

for each $t_s \geq t_\circ$. Using the choice of t_1 in (291), the definition of d_i in (279), and Fact A.1 we get

$$|x_1(t)| \leq \frac{\frac{1}{2}\sigma}{(1+\lambda^{-1})} + (1+\lambda)^{-1}||x_1||_{t_s} + (1+\lambda^{-1})^{-1} r(||v||_{t_\circ}) \quad \forall t \geq 2t_1 + t_s \tag{295}$$

for each $t_s \geq t_\circ$. From this we get

$$||x_1||_{2t_1+t_s} \leq \frac{\frac{1}{2}\sigma}{(1+\lambda^{-1})} + (1+\lambda)^{-1}||x_1||_{t_s} + (1+\lambda^{-1})^{-1} r(||v||_{t_\circ}) \quad \forall t_s \geq t_\circ. \tag{296}$$

Since, from the previous claim, we have

$$||x_1||_{t_\circ} \leq m\ , \tag{297}$$

it follows by induction that, for any positive integer j,

$$||x_1||_t \leq (1+\lambda)^{-j} m + \frac{1}{2}\sigma + r(||v||_{t_\circ}) \qquad \forall t \geq j2t_1 + t_\circ. \tag{298}$$

Thus, with T defined in (292), we have obtained

$$||x_1||_t \leq \sigma + r(||v||_{t_\circ}) \qquad \forall t \geq T + t_\circ. \tag{299}$$

The analysis can be repeated for $x_2(t)$ to establish the claim. □

With the previous claim established, by following the same lines as in [23, Lem. 2.1.4], we can construct a family of mappings $\{T_m\}_{m>0}$ with

1. for each fixed $m > 0$, $T_m : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is continuous, strictly decreasing, and onto;

2. for each fixed $\sigma > 0$, $T_m(\sigma)$ is increasing as m increases, and $\lim_{m\to\infty} T_m(\sigma) = \infty$;

such that, for each pair (ϵ, δ) satisfying (171), we have

$$(300)\ \{|x(t_\circ)| \le \delta,\ ||v||_{t_\circ} \le \epsilon,\ m > 0\} \Longrightarrow \{|x(t)| \le \sigma + r(||v||_{t_\circ})\ \forall t \ge T_m(\sigma) + t_\circ\}\,,$$

$$(301)\qquad \{|x(t_\circ)| = ||v||_{t_\circ} = 0\} \Longrightarrow \{|x(t)| = 0 \quad \forall t \ge t_\circ\}\,,$$

where m is defined in (289).

The discussion now follows the proof of [23, Prop. 2.1.5] closely. For each $m > 0$, denote $\psi_m \doteq T_m^{-1}$. Then for each $m > 0$, $\psi_m : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is continuous, strictly decreasing, and onto. And, for each fixed $t > 0$, $\psi_m(t)$ is nondecreasing as m increases. We also write $\psi_m(0) = \infty$ which is consistent with ψ_m being strictly decreasing and onto. Finally, we extend this family with

$$(302)\qquad \psi_0(t) = 0 \quad \forall t \in \mathbb{R}_{\ge 0}.$$

To summarize the situation, we have established the following implications, for each pair (ϵ, δ) satisfying (171):

$$(303)\qquad \{|x(t_\circ)| \le \delta\,,\ ||v||_{t_\circ} \le \epsilon\} \Longrightarrow \{|x(t)| \le \psi_m(t - t_\circ) + r(||v||_{t_\circ})\ \forall t \ge t_\circ\}\,,$$

$$(304)\qquad \{|x(t_\circ)| \le \delta\,,\ ||v||_{t_\circ} \le \epsilon\} \Longrightarrow \{|x(t)| \le m\ \forall t \ge t_\circ\}\,,$$

with m given in (289).

Now, as in the proof of [23, Prop. 2.1.5], for any $s \ge 0$, $d \ge 0$, and $t \ge 0$, let

$$(305)\qquad \bar\psi(s, d, t) \doteq \min\{\psi_{\phi(s)+r(d)}(t), \phi(s)\},$$

where ϕ may need to be extended to be defined for all $s \ge 0$. Since ϕ and r are increasing, for any fixed d, t, $\bar\psi(\cdot, d, t)$ is a nondecreasing function and, for any fixed s, t, $\bar\psi(s, \cdot, t)$ is a nondecreasing function. Similarly, since, for any fixed m, $\psi_m(\cdot)$ decreases to 0 as $t \to \infty$, the same holds, for fixed s, d, for $\bar\psi(s, d, \cdot)$. Finally, if the pair (ϵ, δ) satisfies (171), we have

$$(306)\qquad \{|x(t_\circ)| \le \delta\,,\ ||v||_{t_\circ} \le \epsilon\} \\ \Longrightarrow \quad \left\{|x(t)| \le \bar\psi(|x(t_\circ)|, ||v||_{t_\circ}, t - t_\circ) + r(||v||_{t_\circ})\ \forall t \ge t_\circ\right\}.$$

Now, for any class-K_∞ function α, we can find a class-K_∞ function α_1 such that

$$(307)\qquad \phi \circ \alpha_1 \le \alpha.$$

Then, for each $t \ge 0$, we have

$$(308)\qquad \bar\psi(s, d, t) \le \bar\psi(\alpha_1(d), d, t) + \bar\psi(s, \alpha_1^{-1}(s), t).$$

This follows from considering the two cases, $s \le \alpha_1(d)$ and $d \le \alpha_1^{-1}(s)$, and by using the monotonicity properties of $\bar\psi$. But from the definition of $\bar\psi$, it is clear that

$$(309)\qquad \bar\psi(\alpha_1(d), d, t) \le \phi(\alpha_1(d)) \le \alpha(d).$$

Finally, the term $\bar\psi(s, \alpha_1^{-1}(s), t)$ can be bounded by a class-KL function $\beta(s, t)$ as in the proof of [23, Prop. 2.1.5]. Combining these manipulations, we have (172). □

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