Systems & Control Letters 23 (1994) 67–78
North-Holland

Technical results for the study of robustness of Lagrange stability

Z.-P. Jiang and L. Praly

Centre Automatique et Systèmes, École des Mines de Paris, 35 rue St Honoré, 77305 Fontainebleau, France

Received 11 November 1992
Revised 1 September 1993

Abstract: We study the robustness of boundedness of solutions of nonlinear dynamical systems. A sufficient coordinate-free technical condition on the characterization of unmodelled effects is given. This characterization with some analogy with the input to state stability of [16] has the potential to encompass most classical uncertainties. In this context, we establish Lagrange stability results which look very much like a small gain theorem. We illustrate the use of these technical results in the robust stabilization problem by a class of interconnected systems.

Keywords: Nonlinear dynamical systems; boundedness; unmodelled effects; robustness; input to state stability.

Definitions.
- $|\cdot|$ denotes the Euclidean norm.
- A function $V:\mathbb{R}^n \to \mathbb{R}$ is said to be positive-definite if $V(x)$ is strictly positive for all nonzero x and is zero at zero.
- A function $V:\mathbb{R}^n \to \mathbb{R}_+$ is said to be proper if $V(x) \to \infty$ as $|x| \to \infty$.
- A function $\gamma:\mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class K if it is continuous, increasing and is zero at zero.
- A function $\gamma:\mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class K_∞ if it is of class K and proper.
- A function $\beta:\mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class KL if, for each t in $\mathbb{R}_+$, $\beta(\cdot, t)$ is a function of class K and, for each s in $\mathbb{R}_+$, the function $\beta(s,\cdot)$ is decreasing and, furthermore, $\beta(s,t)$ tends to zero as t goes to $+\infty$.

1. Introduction

While the theory of nonlinear systems has greatly advanced, the corresponding robustness problem has also received the attention of many researchers. The word *robustness* generally means that some property of a system stands in the face of some perturbations (see [8, 18, 13, 15, 12]).

The purpose of this paper is to present a new way of characterizing unmodelled effects which are not well structured. For this, we seek to generalize standard total stability results [8] and singular perturbation results [13]. The price paid for this generalization is in our characterization which is at first glance solution-dependent. Nevertheless, with the help of an example, we show that, in some cases, it may be checked from the system equation.

With respect to these unstructured uncertainties, we consider the robustness of Lagrange stability. It is worth noting that the results obtained in this paper have some analogy with a small nonlinear gain theorem [14, Theorem 2] (see [11] for more details). In contrast to usual approaches such as total stability theory [8] and singular perturbation theory [13], our approach does not take full account of the structure of the uncertainties. Consequently, we can only observe that the system's solutions do not satisfy the model dynamical equations. In this circumstance, the problem is to propose bounds on the corresponding equation errors which are both tractable and meaningful. This will be done with the help of a Lyapunov function and a comparison signal, but then only a *functional* bound for the uncertainties is available.

Correspondence to: Dr. Z.-P. Jiang, Centre Automatique et Systèmes, École des Mines de Paris, 35 rue St Honoré, 77305 Fontainebleau, France. E-mail: praly@cas.ensmp.fr.

0167-6911/94/$07.00
SSDI 0167-6911(93)E0120-6

In Section 2, we present the problem formulation and state our coordinate-free characterization of unmodelled effects. Our main results are contained in Section 3. An academic example will be given in Section 4 to illustrate our assumptions. In Section 5, we prove the main results. Our conclusion is in Section 6.

2. Problem formulation and assumptions

Consider the following nonautonomous ordinary differential equation to describe what we call a system:

$$\begin{aligned} \dot{X} &= F(X, t), \quad X \in \Omega, \\ x &= H(X, t), \quad x \in \mathbb{R}^n, \end{aligned} \tag{1}$$

where Ω is an open neighborhood of the origin in $\mathbb{R}^N$ and H is an observation map. We assume that F and H are C^1 from $\Omega \times \mathbb{R}_+$ to $\mathbb{R}^N$ and to $\mathbb{R}^n$, respectively.

Moreover, a system described by the differential equation

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n \tag{2}$$

is intended as an approximate model for the observed variable x of (1).

Our problem is to seek sufficient conditions which ensure (globally and/or locally) that the Lagrange stability of (2) implies the boundedness of solutions of (1).

In the following, we first state all the assumptions needed for the global case. Then we treat the local case. What these assumptions mean will be explained in Section 4 via an illustrative example.

2.1. Global case

The following two assumptions specify the class of systems (1) we are considering.

Assumption G (*Globality*). H and Ω are such that[1]

$$H(\Omega, 0) = \mathbb{R}^n. \tag{3}$$

Equation (3) means that we are seeking results which are global in the output space of the dynamical system (1).

Assumption GEO (*Global escape observability*). For all initial conditions $X(0)$ in Ω, for the corresponding solution of (1) $X : [0, T) \to \Omega$, right maximally defined on $[0, T)$, we have

$$(X(t) \to \partial\Omega \text{ as } t \to T) \ \Rightarrow\ \limsup_{t \to T} |H(X(t), t)| = +\infty,$$

where $\partial\Omega$ stands for the boundary of Ω.

Assumption GEO means that any trajectory $X(t)$ of (1) starting from Ω cannot go to the frontier of Ω without its observation variable $x(t)$ escaping to infinity at the same time. Of course, this property implies that the trajectory $\{X(t)\}_{t \in [0, T)}$ is bounded when the trajectory $\{H(X(t), t)\}_{t \in [0, T)}$ is bounded. Note that the observation of the escape is not implied by the usual observability concept. This is in the following simple example.

Example 1. Consider the two-dimensional system

$$\dot{x} = x^3, \qquad \dot{y} = x. \tag{4}$$

[1] Note that our problem is meaningful only when $n \le N$.

For this system, we let $X = (x, y)^{\mathrm{T}}$ and the output map $H = (0, 1)$. Assumption G is satisfied with $\Omega = \mathbb{R}^2$. For each nonzero x_0 and each y_0 in $\mathbb{R}$, the solution of (4) with the initial condition (x_0, y_0) is defined on $[0, 1/2x_0^2)$ and has the following explicit expression:

$$x(t) = x_0 \sqrt{\frac{1}{1 - 2tx_0^2}}, \quad x(0) = x_0, \tag{5}$$

$$y(t) = y_0 + \frac{1}{x_0} - \frac{\sqrt{1 - 2tx_0^2}}{x_0}, \quad y(0) = y_0. \tag{6}$$

Thus, for every t in $[0, 1/2x_0^2)$, x_0 may be obtained from the output measurements $y(t)$ of (4). This shows that system (4) is observable. Indeed, $y(t) - y_0$ is a continuous and increasing function of x_0. However $y(t)$ is bounded on $[0, 1/2x_0^2)$, while $x(t)$ did not.

The next assumption restricts the model (2) to be Lagrange-stable but with the existence of an appropriate function. Precisely, we have the following assumption.

Assumption GB (*Global boundedness*). There exists a positive proper C^1 function V such that for all x in $\mathbb{R}^n$, we have

$$\frac{\partial V}{\partial x}(x) f(x) \overset{\text{def}}{=} - W(x) \leq 0. \tag{7}$$

In the case where (7) is satisfied with a positive-definite function V, V is called a *weak* Lyapunov function (see [2]).

As mentioned above, (2) is considered as a model for the observation x of system (1). Namely, with x given as the measurement from (1), i.e.

$$x(t) = H(X(t), t), \tag{8}$$

the function f is supposed to satisfy

$$\frac{\partial H}{\partial X}(X, t) F(X, t) + \frac{\partial H}{\partial t}(X, t) = f(H(X, t)), \quad \forall (X, t) \in \Omega \times \mathbb{R}_+. \tag{9}$$

In practice, for the system output x to be considered as a state (of the model), it may be useful to augment the actual system measurement with dynamics.

Anyway, because of approximation, uncertainties, etc. (9) is typically not satisfied. Quantifying the departure from the equality (9) or, in other words, characterizing the distance between the model (2) and the actual system (1) is our next objective. For this, we introduce a function and two sequences as follows:

(H) There exists a positive C^0 real function Υ defined on $\mathbb{R}^n$, with $\Upsilon(0) = 0$, which satisfies the following property:
there exist two increasing sequences of positive real numbers $\{u_i\}_{i=0}^{+\infty}$, $\{v_i\}_{i=0}^{+\infty}$ such that

$$0 \leq u_0 < v_0 < \cdots < u_i < v_i < u_{i+1} < \cdots, \quad u_i \to +\infty \tag{10}$$

and

$$W(x) \geq \sup_{y \in E_x} \Upsilon(y), \quad \forall x, \ V(x) \in [u_i, v_i], \tag{11}$$

with E_x denoting the set $\{y \in \mathbb{R}^n \colon V(y) \leq V(x)\}$.

This technical assumption is related to the fact that W dominates Υ on some level sets of V. It is satisfied, for instance, in the following two cases:

(1) there exists a positive C^0 real function $\bar{\Upsilon}$, with $\bar{\Upsilon}(0) = 0$, which is increasing on $[v_0, +\infty) \subseteq \mathbb{R}_+$ and satisfies, for all $x \in \mathbb{R}^n$,

$$\bar{\Upsilon}(V(x)) \leq W(x). \tag{12}$$

Then $\Upsilon(x) = \bar{\Upsilon}(V(x))$ satisfies (11).

(2) for all $(x, y) \in \mathbb{R}^{2n}$, $\Upsilon(y) \leq W(x)$ if $V(y) \leq V(x)$.

Now with V, W defined in (7) and Υ in (11), our assumption characterizing this distance between model and system can be stated as follows.

Assumption GUEC (*Global unmodelled effects characterization*). There exist two positive real numbers μ_1, μ_2 such that, for any initial condition $X(0) \in \Omega$ with corresponding solution of (1) $X : [0, T) \to \Omega$ right maximally defined on $[0, T)$, there exists a positive real number D satisfying for all $t \in [0, T)$,

$$\left| \frac{\partial V}{\partial x}(x(t)) [\dot{x}(t) - f(x(t))] \right| \leq \mu_1 W(x(t)) + \mu_2 \sup_{0 \leq s \leq t} \Upsilon(x(s)) + D, \tag{13}$$

where $x(t) = H(X(t), t)$.

We refer to $r(t) = \sup_{0 \leq s \leq t} \Upsilon(x(s))$ as the *comparison signal*.

First of all, we note that two important features of this characterization:

(1) *Invariance by diffeomorphism*: equation (13) is model-coordinate-independent. Precisely, for any global diffeomorphism φ, (13) is satisfied with the new model state variable $\chi = \varphi(x)$.

(2) *Invariance under convex transformation*: in the case when $\Upsilon(x) \leq \alpha(V(x)) \leq W(x)$ for a function α of class K_∞, for any C^1 convex positive-definite function ψ, assumption GUEC is satisfied with the new Lyapunov function $\psi(V)$.

Remark 1. Equation (13) may be checked on-line if $\dot{x}$ is measurable. If $\dot{x}$ is not measurable, in place of (13), we need the following integral version, $\forall 0 \leq s \leq t < T$:

$$\left| V(x(t)) - V(x(s)) - \int_s^t \frac{\partial V}{\partial x}(x(\tau)) f(x(\tau)) \, d\tau \right|$$
$$\leq \mu_1 \int_s^t W(x(\tau)) \, d\tau + \mu_2 \int_s^t \sup_{0 \leq \tau' \leq \tau} \Upsilon(x(\tau')) \, d\tau + D(t - s). \tag{14}$$

This allows us to overleap the measure of X.

On the left-hand side of (13), $\dot{x}(t)$ can be replaced by $\partial H(F)/\partial x + \partial H/\partial t$ and $x(t)$ by H. So indeed (13) quantifies the error in (9). But, willing to bound this error only in terms of the model state $x(t)$ ($= H(X(t), t) =$ system measurement) and nevertheless willing to allow unmodelled dynamics, we are led to introduce the comparison signal r. The reason is that this signal memorizes in some sense the past values of the model state and, by the way, allows us to capture some dynamics. Introduction of r is the key difference from previous work (see, for example, [3, 4, 7]). It can be seen as the L^∞ norm of an 'operator' but with a very specific input Υ related to the 'stability margin' of the model and an even more specific output (multiplication by $\partial V/\partial x$). If r were not present, it would be sufficient as usual to check (4) only point, by point in the extended state space (X, t). However, due to the presence of r, dynamically defined in (7), (4) becomes an assumption to be satisfied solution by solution. However, by using the properties of the systems under consideration, it may be possible to check this assumption without the explicit knowledge of the solutions.

2.2 Local case

We have considered the global (in the model space) aspect. To look at the local case, we make the following assumptions in place of assumptions G, GEO, GB and GUEC.

First assumption G is replaced by

$$H(0,0)=0 \tag{15}$$

and there exist two open sets $\Omega_0 \subset \Omega_1 \subset \Omega$, two compact sets $\mathscr{K} \subset \mathbb{R}^n$, with a nonempty interior, and $\Gamma \subset \Omega_1$, all containing their respective origin, and three positive real numbers μ_1, μ_2, D such that the following assumptions hold.

Assumption LEO (*Local escape observability*). For any initial condition $X(0)$ in Ω_0, the corresponding right maximal solution of (1) $X:[0,T) \to \Omega_1$ satisfies

$$\exists t_0 \in [0,T)\colon X(t_0) \notin \Gamma \;\Rightarrow\; \exists t_1 \in [0,t_0)\colon H(X(t_1),t_1) \notin \mathscr{K}.$$

Assumption LAS (*Local asymptotic stability*). There exists a positive-definite, proper, C^1 function V such that

$$\frac{\partial V}{\partial x}(x) f(x) \overset{\text{def}}{=} -W(x) \le -\alpha(V(x)), \quad \forall x \in \upsilon, \tag{16}$$

where α is a function of class K and υ an open neighborhood of $0 \in \mathbb{R}^n$.

In contrast with the global case, we assume here that V and W are positive-definite. This allows us to define $\Delta_{\mathscr{K}}$, a strictly positive real number such that

$$V(x) \le \Delta_{\mathscr{K}} \;\Rightarrow\; x \in \mathscr{K}. \tag{17}$$

Assumption LUEC (*Local unmodelled effects characterization*). Any right maximal solution of (1) $X:[0,T) \to \Omega_1$ satisfies, for all $t \in [0,T)$,

$$\left|\frac{\partial V}{\partial x}(x(t))[\dot{x}(t) - f(x(t))]\right| \le \mu_1 W(x(t)) + \mu_2 \sup_{0 \le s \le t} \Upsilon(x(s)) + D, \tag{18}$$

where $x(t) = H(X(t),t)$ and $\Upsilon(x) = \alpha(V(x))$.

3. Main results

We are now in a position to state two main results.

Proposition 1 (Global boundedness). *Let assumptions G, GEO and GB hold. If, for some function Υ satisfying assumption* H, *assumption GUEC holds with μ_1, μ_2 satisfying*

$$1 - \mu_1 - \mu_2 > 0, \tag{19}$$

then any solution $X(t)$ of (1) *with initial condition $X(0) \in \Omega$, whose corresponding real number D in assumption GUEC satisfies*

$$D < (1 - \mu_1 - \mu_2) \liminf_{|x| \to +\infty} W(x), \tag{20}$$

is well defined on $[0, +\infty)$, unique and bounded. Moreover, if there exists a function α of class K such that

$$\Upsilon(x) \le \alpha(V(x)) \le W(x), \quad \forall x \in \mathbb{R}^n, \tag{21}$$

then for any solution, satisfying for all $t_0 \ge 0$,

$$\max\left\{0, \left|\frac{\partial V}{\partial x}(x(t))\,[\dot{x}(t) - f(x(t))]\right| - \mu_1 W(x(t)) - \mu_2 \sup_{t_0 \le s \le t} \{\Upsilon(x(s))\}\right\} \to 0 \tag{22}$$

as $t - t_0$ tends to infinity, we have $V(x(t))$ tends to 0 as t tends to $+\infty$.

We remark that the dependence of D on the solution and therefore on the initial conditions causes a difficulty for global analysis. However, when W is proper, there is no constraint on the initial condition $X(0)$ or more precisely on $H(X(0), 0)$. The technical assumption (22) invoking uniformity with respect to t_0 means the existence of a positive time function $D(t, t_0)$, with $D(t, t_0) \to 0$ as $t - t_0 \to +\infty$, such that [13] is verified with the zero initial instant replaced by *any* initial instant t_0 and D by $D(t, t_0)$. Note also that, from assumption GUEC, $\mu_2/(1-\mu_1)$ can be seen as the L^∞ gain of the 'operator' 'unmodelled effects' $\mapsto \Upsilon$ since as discussed above these unmodelled effects are captured by r. In this case, (19) rewritten as $(1/(1-\mu_1))\mu_2 < 1$ is nothing but the small gain condition [6].

Proposition 2 (Local boundedness). *Let assumptions* (15), *LEO and LAS hold with*

$$\upsilon \supseteq H(\Omega_1, \mathbb{R}_+) \cup \mathscr{K}. \tag{23}$$

If, for some positive real numbers μ_1, μ_2, D *satisfying*

$$\frac{1}{1-\mu_1-\mu_2} D \le \alpha(\Delta_{\mathscr{K}}), \tag{24}$$

assumption LUEC holds, then there exists an open neighborhood $\Omega_2 \subseteq \Omega_0$ *of the origin such that any solution of* (1), *with* $X(0) \in \Omega_2$ *is well defined on* $[0, +\infty)$, *unique and in* Γ. *Moreover, for any such solution, satisfying for all* $t_0 \ge 0$,

$$\max\left\{0, \left|\frac{\partial V}{\partial x}(x(t))[\dot{x}(t) - f(x(t))]\right| - \mu_1 W(x(t)) - \mu_2 \sup_{t_0 \le s \le t} \{\Upsilon(x(s))\}\right\} \to 0 \tag{25}$$

as $t - t_0$ *tends to infinity, we have* $x(t)$ *tends to* 0 *as* t *tends to* $+\infty$.

4. An illustrative example

In this section, we illustrate assumptions GEO, GB and GUEC as well as their interest in the global stabilization problem via a class of interconnected systems.

Consider the nonlinear systems

$$\begin{aligned} \dot{x} &= f(x) + \omega(x, y), \\ \dot{y} &= g(y, x), \end{aligned} \tag{26}$$

where $x \in \mathbb{R}^{n_1}$, $y \in \mathbb{R}^{n_2}$, f, g and ω are smooth mappings. We assume that:

(A1) The subsystem $\dot{y} = g(y, x)$ is input-to-state-stable (ISS) with x as input in the sense that there exist a function β of class KL and a function γ of class K such that, for each initial condition $y(0)$, and for each continuous input $x: [0, T) \to \mathbb{R}$, the solution $y(t)$ exists for all $t \in [0, T)$ and satisfies

$$|y(t)| \le \beta(|y(0)|, t) + \gamma\left(\sup_{0 \le s \le t} |x(s)|\right), \quad \forall t \in [0, T). \tag{27}$$

(A2) There exist a smooth function $V: \mathbb{R}^{n_1} \to \mathbb{R}_+$, and three functions $\alpha_1, \alpha_2, \alpha_3$ of class K_∞ such that, for each $x \in \mathbb{R}^{n_1}$,

$$\alpha_1(|x|) \le V(x) \le \alpha_2(|x|) \tag{28}$$

and

$$\frac{\partial V}{\partial x}(x) f(x) = -W(x) \le -\alpha_3(V(x)). \tag{29}$$

(A3) The coupling term $\omega(x, y)$ is restricted as follows: there exist three positive real functions $\gamma_1, \gamma_2, \gamma_3$ and three positive real numbers $\varepsilon_1, \varepsilon_2, \varepsilon_3$ such that γ_3 is nondecreasing and

$$\gamma_3 \circ 2\gamma(s) \leq \alpha_3 \circ \alpha_1(s), \quad \forall s \geq 0, \tag{30}$$

$$|\omega(x, y)| \leq \gamma_1(|x|) + \gamma_2(|y|), \quad \forall (x, y) \in \mathbb{R}^{n_1 + n_2}, \tag{31}$$

$$\left|\frac{\partial V}{\partial x}(x)\right| \gamma_1(|x|) \leq \varepsilon_1 W(x), \quad \forall x \in \mathbb{R}^{n_1}, \tag{32}$$

$$\left|\frac{\partial V}{\partial x}(x)\right| \gamma_2(|y|) \leq \varepsilon_2 W(x) + \varepsilon_3 \gamma_3(|y|), \quad \forall (x, y) \in \mathbb{R}^{n_1 + n_2}, \tag{33}$$

where γ in (30) is given by (A1).

Note that the ISS definition in (A1) is equivalent to that introduced by Sontag [16, 17], but restricted to continuous controls. Also, (30) is nothing but a small nonlinear gain condition (see [11]).

In (26), we identify $N = n_1 + n_2$, $n = n_1$ and

$$X = (x, y)^{\mathrm{T}}, \qquad H = (I_{n_1 \times n_1}, 0_{n_1 \times n_2}), \qquad \Omega = \mathbb{R}^{n_1 + n_2}.$$

Assumption G is clearly satisfied. By considering the system

$$\dot{x} = f(x) \tag{34}$$

as design model for (26), we have the following lemma.

Lemma 1. *If assumptions* (A1), (A2) *and* (A3) *hold, then assumptions GEO, GB, and GUEC are satisfied.*

Proof. First remark that assumptions GEO and GB follow directly from assumptions (A1) and (A2), respectively. Let $(x(t), y(t))$ be a solution of (26) right maximally defined on the time interval $[0, T)$. In view of assumptions (A1)–(A3), we obtain, for all $t \in [0, T)$ (using [16, eq. (12)]):

$$\begin{aligned} \left|\frac{\partial V}{\partial x}(x(t))\,\omega(x(t), y(t))\right| &\leq (\varepsilon_1 + \varepsilon_2) W(x(t)) + \varepsilon_3 \gamma_3(|y(t)|) \\ &\leq (\varepsilon_1 + \varepsilon_2) W(x(t)) + \varepsilon_3 \alpha_3(\sup\nolimits_{0 \leq s \leq t} V(x(s))) + \varepsilon_3 \gamma_3(2\beta(|y(0)|, t)). \end{aligned} \tag{35}$$

This implies assumption GUEC for system (26) with $\mu_1 = \varepsilon_1 + \varepsilon_2$ and $\mu_2 = \varepsilon_3$. $\square$

The following result is a direct application of Lemma 1 and Proposition 1.

Proposition 3. *Let assumptions* (A1), (A2) *and* (A3) *hold with*

$$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 < 1.$$

Then all the solutions $(x(t), y(t))$ *of* (26) *are well defined on* $[0, +\infty)$, *unique and bounded. Moreover, we have*

$$\lim_{t \to +\infty} (|x(t)| + |y(t)|) = 0.$$

Note that by (35) and time invariance, (22) is satisfied. In particular, we deduce a positive time function $D(t, t_0)$ (see remarks after Proposition 1) as

$$D(t, t_0) = \varepsilon_3 \gamma_3(2\beta(|y(t_0)|, t - t_0)).$$

Remark 2. Proposition 3 may be seen as a slight generalization of the well-known result on the global asymptotic stability of cascaded systems, i.e. (26) with $\omega = 0$ (see, for instance, Sontag [17]). Indeed, assumption (A3) is automatically satisfied with

$$\gamma_1 = \gamma_2 = \gamma_3 \equiv 0, \qquad \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 0.$$

Remark 3. The interest of studying homogeneous systems has been pointed out in, for example, [1, 2, 5]. We observe that assumptions (A1), (A2) and (A3) hold for a class of perturbed homogeneous systems provided that

(1) f and g are two homogeneous vector fields of degree $\lambda \geq 1$,
(2) The zero solutions of $\dot{x} = f(x)$ and $\dot{y} = g(y, 0)$ are globally asymptotically stable,
(3) $|\omega(x, y)| \leq \varepsilon_1 |x|^\lambda + \varepsilon_2 |y|^\lambda$, for two real numbers $\varepsilon_1 \geq 0$, $\varepsilon_2 \geq 0$. Note that ω may include non-homogeneous nonlinearities such as $\omega(x, y) = xy^{\lambda-1}\sin(x^2)$.

Indeed, according to [8, Theorem 57.4], there exists a homogeneous Lyapunov function $V_1(y)$ of degree v_1 for the homogeneous system $\dot{y} = g(y, 0)$ such that for $c_1 > 0$, $c_2 > 0$ and $c_3 > 0$,

$$c_1 |y|^{v_1} \leq V_1(y) \leq c_2 |y|^{v_1} \tag{36}$$

and

$$\frac{\partial V_1}{\partial y}(y) g(y, 0) \leq -c_3 |y|^{v_1+\lambda-1}, \quad \forall y. \tag{37}$$

Observe that
(P1) $\partial V_1/\partial y(y)$ is homogeneous of degree $v_1 - 1$.
In fact, since $V_1(y)$ is homogeneous of degree v_1, we have

$$V_1(cy) = c^{v_1} V_1(y), \quad \forall c \in \mathbb{R}.$$

Then, by differentiating with respect to y, we get

$$\frac{\partial V_1}{\partial y}(cy) = c^{v_1-1} \frac{\partial V_1}{\partial y}(y).$$

Also, since g is homogeneous of degree λ, we observe that
(P2) $(g(y, x) - g(y, 0))/|x|$ is homogeneous of degree $\lambda - 1$.
With these two properties at hand, since $\partial V_1(y)/\partial y$ is bounded on the unit sphere $\{y \mid |y| = 1\}$, there exists a strictly positive real number d_1 such that

$$\frac{\partial V_1}{\partial y}(y) = |y|^{v_1-1} \frac{\partial V_1}{\partial y}(y/|y|) \leq d_1 |y|^{v_1-1}, \quad \forall y. \tag{38}$$

On the other hand, since $(g(y, x) - g(y, 0))/|x|$ is bounded on the sphere $\{(x, y) \mid |x|^{\lambda-1} + |y|^{\lambda-1} = 1\}$, there exists a strictly positive real number d_2 such that

$$\frac{g(y/\rho, x/\rho) - g(y/\rho, 0)}{|x/\rho|} \leq d_2, \quad \forall x, y, \tag{39}$$

where $\rho := (|x|^{\lambda-1} + |y|^{\lambda-1})^{1/(\lambda-1)}$ if $\lambda > 1$ and $\rho := 1$ if $\lambda = 1$.
Thus, (37)–(39) give

$$\begin{aligned} \frac{\partial V_1}{\partial y}(y) g(y, x) &= \frac{\partial V_1}{\partial y}(y) g(y, 0) + \frac{\partial V_1}{\partial y}(y) [g(y, x) - g(y, 0)] \\ &\leq -c_3 |y|^{v_1+\lambda-1} + d_1 d_2 |x| (|x|^{\lambda-1} + |y|^{\lambda-1}) |y|^{v_1-1}. \end{aligned} \tag{40}$$

By Young's inequality [10], (40) implies the existence of $c_4 > 0$ and $c_5 > 0$ such that

$$\frac{\partial V_1}{\partial y}(y) g(y, x) \leq -|y|^{v_1-1}(c_4 |y|^\lambda - c_5 |x|^\lambda). \tag{41}$$

From the algorithm to check if a system is ISS, proposed in [16, Proof of Theorem 1], with (36) and (41), we can find a function β of class KL and a strictly positive real number k such that

$$|y(t)| \leq \beta(|y(0)|, t) + k \sup_{[0,t]}\{|x(s)|\}, \tag{42}$$

i.e. assumption (A1) holds with $\gamma(s) = ks$.

To check assumption (A2), we remark that there exists a homogeneous Lyapunov function $V(x)$ of degree v for the homogeneous system $\dot{x} = f(x)$ such that for a positive real number c.

$$\frac{\partial V}{\partial x}(x) f(x) \leq -c|x|^{v+\lambda-1}. \tag{43}$$

Thus, there exist three positive real numbers l_i ($i = 1, 2, 3$) such that assumption (A2) holds with

$$\alpha_1(|x|) = l_1|x|^v, \qquad \alpha_2(|x|) = l_2|x|^v, \qquad \alpha_3(v) = l_3 v^{(v+\lambda-1)/v}. \tag{44}$$

Finally, assumption (A3) is verified with

$$\varepsilon_1 = k_1\varepsilon_1, \qquad \varepsilon_2 = \frac{k_1(v-1)}{c(v+\lambda-1)}\varepsilon_2, \qquad \varepsilon_3 = \frac{k_1\lambda(2k)^{v+\lambda-1}}{(v+\lambda-1)l_3 l_1^{(v+\lambda-1)/v}}\varepsilon_2,$$

$$\gamma_1(s) := \varepsilon_1 s^\lambda, \qquad \gamma_2(s) := \varepsilon_2 s^\lambda, \qquad \gamma_3(s) := l_3 l_1^{(v+\lambda-1)/v}(2k)^{1-v-\lambda} s^{v+\lambda-1},$$

where $k_1 > 0$ depends only on V.

For want of space, no example has been given to illustrate the assumptions LEO, LAS and LUEC in the local case. The reader could find examples in [11].

5. Proof of the main results

Since the proofs of Propositions 1 and 2 are similar, we will only give a sketch of the proof of Proposition 2 while the proof of Proposition 1 is complete.

Proof of Proposition 1. Since the function F is of class C^1, for each initial condition $X(0) \in \Omega$, there exists a unique solution of (1) $X : [0, T) \to \Omega$ right maximally defined on $[0, T)$ and C^1. Then let $x(t) = H(X(t), t)$ be its output and consider the time function $V(x(t))$.

By assumptions GB, GUEC and (H), the time derivative of V along this solution satisfies

$$\dot{\overline{V(x(t))}} \leq -(1-\mu_1)W(x(t)) + \mu_2 \sup_{0 \leq s \leq t} \Upsilon(x(s)) + D, \quad \forall t \in [0, T). \tag{45}$$

Let the positive real number V^* be defined by

$$V^* = \max\left\{V(x(0)), \sup\left\{V(x) \mid x \in \mathbb{R}^n \text{ and } W(x) \leq \frac{D}{1-\mu_1-\mu_2}\right\}\right\}. \tag{46}$$

From (20), V^* is well defined. With $\{v_i\}_{i=0}^{+\infty}$ the sequence given in (H), let i be the smallest integer such that $V^* < v_i$ and let us define a set S by

$$S = \{x \in \mathbb{R}^n \mid V(x) \leq \max\{V^*, u_i\}\}. \tag{47}$$

Note that u_i depends on $x(0)$ and cannot be infinite. Since V is proper and $x(0) \in S$, S is compact and nonempty. We claim that $x(t) \in S$ for all $t \in [0, T)$.

If it is false, there exists an $\varepsilon \in (0, v_i - \max\{V^*, u_i\})$ such that the set

$$\{t \in [0, T) \mid V(x(t)) \geq \max\{V^*, u_i\} + \varepsilon\}$$

is nonempty. Then, let

$$t_3 = \inf\{t \in [0, T) \mid V(x(t)) \geq \max\{V^*, u_i\} + \varepsilon\}. \tag{48}$$

By continuity, we have

$$V(x(t_3)) = \sup_{0 \leq s \leq t_3} V(x(s)) \in [u_i, v_i]. \tag{49}$$

Note that by (11) and (49) we have

$$\sup_{0 \leq s \leq t_3} \Upsilon(x(s)) - W(x(t_3)) \leq 0. \tag{50}$$

Let δ be defined by

$$2\delta = W(x(t_3)) - \frac{D}{1 - \mu_1 - \mu_2}. \tag{51}$$

Since $x(t_3)$ is not in S, δ is strictly positive. Then if we choose $\eta \in (0, \delta(1 - \mu_1 - \mu_2)/\mu_2)$, continuity of $\Upsilon(x(t))$ and $W(x(t))$ implies the existence of two time instants $t_1 < t_3 < t_2$ in $[0, T)$ such that for all $t \in [t_1, t_2]$,

$$\sup_{0 \leq s \leq t} \Upsilon(x(s)) - W(x(t)) \leq \eta, \quad W(x(t)) \geq \frac{D}{1 - \mu_1 - \mu_2} + \delta. \tag{52}$$

Thus, for all $t \in [t_1, t_2]$, we have

$$\overline{\dot{V(x(t))}} \leq -(1 - \mu_1) W(x(t)) + \mu_2 W(x(t)) + \mu_2 \eta + D < 0. \tag{53}$$

This implies $V(x(t_1)) > V(x(t_3))$ which contradicts (49). Therefore, $x(t) \in S$ for all $t \in [0, T)$.

Assumption GEO implies that $X(t)$ is bounded on $[0, T)$ and does not go to $\partial\Omega$ as $t \to T$. From the theorem on continuation of solutions of differential equations (see [9]), this implies that $T = +\infty$.

Now, we consider the case where (21) and (22) hold. Then, instead of (45), we have

$$\overline{\dot{V(x(t))}} \leq -(1 - \mu_1) W(x(t)) + \mu_2 \sup_{t_0 \leq s \leq t} \alpha(V(x(s))) + D(t, t_0), \tag{54}$$

with $0 \leq D(t, t_0) \leq D$ and $D(t, t_0) \to 0$ as $t - t_0 \to +\infty$. We wish to show that $V(x(t)) \to 0$ as $t \to +\infty$. Let $T_0 = 0$. Since from above $x(t) \in S$ and V is continuous, there exists a $V_0 \in \mathbb{R}_+$ such that

$$\alpha(V(x(t))) \leq \alpha(V_0), \quad \forall t \in [T_0, +\infty). \tag{55}$$

Since $D(t, 0)$ tends to 0, for some ρ in $(0, 1 - \mu_1 - \mu_2)$, there exists a $T_{0,1} > 0$ such that

$$D(t, 0) \leq \rho\alpha(V_0), \quad \forall t \geq T_{0,1}. \tag{56}$$

Thus, from (21), (54) and (55), we obtain

$$\overline{\dot{V(x(t))}} \leq -(1 - \mu_1)\alpha(V(x(t))) + \mu_2\alpha(V_0) + \rho\alpha(V_0), \quad \forall t \in [T_{0,1}, +\infty). \tag{57}$$

As in [16, Proof of Theorem 1], this implies the existence of $T_1 \geq T_{0,1}$ such that

$$\alpha(V(x(t))) \leq \frac{\mu_2 + \rho}{1 - \mu_1} \alpha(V_0), \quad \forall t \geq T_1. \tag{58}$$

Since α is a function of class K, by letting

$$\alpha(V_1) = \frac{\mu_2 + \rho}{1 - \mu_1} \alpha(V_0), \tag{59}$$

we have established

$$\alpha(V(x(t))) \leq \alpha(V_1), \quad \forall t \in [T_1, +\infty). \tag{60}$$

Then pursuing the same reasoning on the interval $[T_1, +\infty)$ instead of $[T_0, +\infty)$, we get that there exist two numbers $T_2 \geq T_{0,2} \geq T_1$ such that

$$D(t, T_1) \leq \rho\alpha(V_1), \quad \forall t \geq T_{0,2} \tag{61}$$

and

$$\alpha(V(x(t))) \leq \frac{\mu_2 + \rho}{1 - \mu_1}\alpha(V_1), \quad \forall t \geq T_2. \tag{62}$$

Thus, we have

$$\alpha(V(x(t))) \leq \left(\frac{\mu_2 + \rho}{1 - \mu_1}\right)^2 \alpha(V_0), \quad \forall t \geq T_2. \tag{63}$$

Continuing this procedure, we find a nondecreasing sequence of positive real numbers T_n such that

$$\alpha(V(x(t))) \leq \left(\frac{\mu_2 + \rho}{1 - \mu_1}\right)^n \alpha(V_0), \quad \forall t \geq T_n. \tag{64}$$

Since α is of class K and by (64), we conclude that $V(x(t))$ tends to zero as $t \to +\infty$. □

Proof of Proposition 2. Thanks to assumption (15), we define an open neighborhood of the origin Ω_2 by

$$\Omega_2 = \{X \in \mathbb{R}^N \mid V(H(X, 0)) < \Delta_{\mathscr{X}}\} \cap \Omega_0 \subset \Omega_1. \tag{65}$$

F being C^1, for any $X(0) \in \Omega_2$, there exists a unique solution $X: [0, T) \to \Omega_1$ right maximally defined on $[0, T)$. Let $x(t) = H(X(t), t)$ be its output. With (23), it is a C^{12} time function which takes values in υ. Then, by assumptions LAS and LUEC, along this solution, the time derivative of V satisfies

$$\overline{V(x(t))}^{\,\cdot} \leq -(1 - \mu_1) W(x(t)) + \mu_2 \sup_{0 \leq s \leq t} \Upsilon(x(s)) + D, \quad \forall t \in [0, T). \tag{66}$$

Since $\Upsilon(x) = \alpha(V(x)) \leq W(x)$ and α is increasing, it is not difficult to verify that (11) is satisfied by this Υ, with $u_0 = 0$ and $v_0 = \sup_{x \in \upsilon} V(x)$. Then consider the set S as defined in (47). Note that, since $X(0) \in \Omega_2$, $V(x(0)) < \Delta_{\mathscr{X}}$. Also clearly, (16) and (24) imply that for all x satisfying

$$W(x) \leq \frac{D}{1 - \mu_1 - \mu_2} \leq \alpha(\Delta_{\mathscr{X}}),$$

we have

$$V(x) \leq \Delta_{\mathscr{X}}.$$

It follows that S is contained in $\mathscr{X}$ which is contained in υ. Therefore, by repeating the same arguments as after equation (47), we can prove that $x(t) \in \mathscr{X}$ for all $t \in [0, T)$. Finally, assumption LEO implies that $X(t)$ belongs to the compact subset Γ of Ω_1 for all $t \in [0, T)$. The theorem on continuation of solutions of differential equations implies that $T = +\infty$. Since V is positive-definite, the last assertion of Proposition 2 follows as in the proof of Proposition 1. □

6. Conclusion

This paper is intended to give some technical results for establishing the robustness of Lagrange stability with respect to unmodelled effects which are created by perturbations with no specific structure. A sufficient condition is our so-called unmodelled effects characterization. It has some analogy with the input to state stability (see [16]). This characterization has the potential to encompass many usual but more structured uncertainties (see [11]). We showed that global asymptotic stability of homogeneous systems considered in

[1, 2, 5] holds in the face of some regular perturbations. Finally, we note that Propositions 1 and 2 have some analogy with a small gain theorem (see [14]).

Acknowledgements

We would like to thank the anonymous referees for their very detailed and constructive comments that led to a smoother presentation.

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