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ON THE GLOBAL DYNAMICS OF ADAPTIVE SYSTEMS: A STUDY OF AN ELEMENTARY EXAMPLE*

MARTÍN D. ESPAÑA† AND LAURENT PRALY‡

Abstract. The inherent nonlinear character of adaptive systems poses serious theoretical problems for the analysis of their dynamics. On the other hand, the importance of their dynamic behavior is directly related to the practical interest in predicting such undesirable phenomena as nonlinear oscillations, abrupt transients, intermittence or a high sensitivity with respect to initial conditions. A geometrical/qualitative description of the phase portrait of a discrete-time adaptive system with unmodeled disturbances is given. For this, the motions in the phase space are referred to normally hyperbolic (structurally stable) locally invariant sets. The study is complemented with a local stability analysis of the equilibrium point and periodic solutions. The critical character of adaptive systems under rather usual working conditions is discussed. Special emphasis is put on the causes leading to intermittence. A geometric interpretation of the effects of some commonly used palliatives to this problem is given. The "dead-zone" approach is studied in more detail. The predicted dynamics are compared with simulation results.

Key words. adaptive systems, intermittency, dynamic systems, discrete time systems, periodic solutions, invariant sets

AMS subject classifications. 93B27, 93C10, 93C40

1. Introduction. It is an already well-known fact that adaptive systems may exhibit very complicated dynamics. For instance, Anderson (1985), first showed that, although bounded, abrupt and explosive transients may occur in the presence of disturbances. Such undesirable behavior is not exclusive to adaptive control schemes inasmuch as it can occur in any system with parametric feedback (i.e., whose parameters are functions of the signals generated by the system itself). Other examples are output error and serial-parallel identification schemes. The nonlinear character of these systems poses serious theoretical problems for its dynamic analysis. However, from a practical point of view, a successful implementation is based on a thorough knowledge of the circumstances under which nonlinear oscillations, abrupt transients, or even intermittency may occur. The study of the sensitivity of the solutions with respect to the initial conditions is also of obvious importance given that, as pointed out by Bergé, Pomeau, and Vidal (1984), this circumstance happens to be intimately related with the existence of strange attractors (and chaos). The influence of external inputs on the overall behavior also needs careful attention. The above considerations have been at the origin of increasing interest during the past years on the dynamic description of adaptive systems and, more specifically, in the explanation of the occurrence of intermittent bursts. In the presence of stochastic disturbances, Anderson (1985), suggests that "bursting" is a consequence of nonpersistently exciting reference signals. However, as shown in this paper, intermittency may subsist and (as already shown by Narendra and Anaswamy (1986)) even solutions with unbounded parametric components may take place if the reference, although persistently exciting, has not enough energy. Jaïdane-Saïdane and Macchi (1988) have proposed a heuristic explanation to the intermittent phenomenon and attributed to it a "self-stabilizing" property implying bounded signals of the closed-loop adaptive linear system. The general validity of the last conclusion has, however, been already criticized by Egardt

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† NASA-Ames/Dryden Flight Research Facility, P.O. Box 273, M/S D2002, Edwards, California 93523 (mespana@mcimail.com).
‡ Centre d'Automatique et Informatique, Ecole de Mines de Paris, 35 Rue Saint-Honoré, 77305 Fontainebleau Cedex, France (praly@caia.ensmp.fr).

(1979), who has established that bounded perturbations and reference signals may produce unbounded outputs unless the parameters remain bounded. More recently, intermittent bursts have been studied extrapolating a local analysis (around critical points such as equilibria and 2-periodic solutions) performed using bifurcation techniques (Golden and Ydstie (1988); Rey, Bitmead, and Johnson (1991)). Analysis based on averaging approximations are performed in Sethares and Mareels (1991) and España (1991). The latter shows that intermittency with either a continuous change (in average) or almost fixed parameter values may take place, the latter case being associated with self-oscillating modes. The use of averaging can rigorously be justified by the existence of the attractive locally invariant set. A concept is developed in detail in this paper for a particular example (see also Praly (1990)). We think, however, that the key issue to understand, and thus to prevent, undesired dynamics, is to address the problem of the global description of the trajectories in the phase space. To our knowledge, this is the first time that a global geometrical description of the phase portrait has been given for an adaptive system. Only local results have been obtained before. They are almost all contained in Ljung and Soderstrom (1983), Anderson et al. (1986), Riedle and Kokotovic (1986), and Benveniste, Metvier, and Priouret (1987). Although this analysis is particular to our example, more general conclusions can be obtained, since we use mathematical tools as integral sets (Praly (1985), (1990); Riedle and Kokotovic (1986)) or the existence of periodic solutions (Ljung (1977); Bodson et al. (1986); Pomet, Coron, and Praly (1990)), which have been shown to be applicable in more general situations. The analysis is made in a deterministic context and special emphasis is put on the causes leading to intermittent bursting.

2. Problem formulation. A sufficiently general formulation of a linear discrete-time system in closed loop with an adaptive controller is given by the following equations (see, for instance, Pomet, Coron, and Praly (1990)):

$$y(t+1) = A(\theta)y(t) + B(\theta)w(t), \qquad \theta(t+1) = \theta(t) + \mu C(y(t), \theta(t), w(t)),$$

where μ is usually used to control the adaptation speed; $w(t)$ represents all the external inputs, including the output reference signal $r(t)$ and any unmodeled disturbances. The first equation is the regressor model and the second is the parameters updating algorithm.

In what follows, a discrete-time first-order plant with an adaptive proportional output-feedback controller is considered. The objective is to regulate the plant's output to a constant value r. Any possible mismatch between the model, used for the control purposes, and the plant is represented by the unknown and unmeasurable equation error $d(t)$:

$$d(t) = y(t) - ay(t-1) - u(t-1). \tag{1}$$

We refer to "the ideal case" when $d(t) \equiv 0$. Normally, $d(t)$ is the result of unmeasured disturbances or unmodeled dynamics. In our analysis, $d(t)$ is supposed constant. This particular situation arises in practice when, due to a (possible temporary) misalignment in the actuator, a bias exists in the effective control action applied.

A proportional controller has as an effect to shift the pole of this plant, and if the parameter a is known, any possible stable value can be assigned for it. When a is unknown, the following adaptive controller with a normalized gradient type updating parameter equation (Goodwin and Sin (1984), also called stochastic approximation by Egardt (1979)),

$$u(t) = -\theta(t)y(t) + r \tag{2}$$

$$\theta(t) = \theta(t-1) + \mu \frac{y(t-1)(y(t)-r)}{(1+y(t-1)^2)} \tag{3}$$

guarantees, for the ideal case and for any $a \in \mathbb{R}$, that $y(t) \to r$ when $t \to \infty$. Moreover, if $r \neq 0$, $\theta(t) \to a$ as $t \to \infty$. The choice of a normalized-type algorithm (plus, perhaps, a uniform bound for the parameters not considered here) is crucial in practice to assure bounded signals, particularly when (bounded) disturbances are present (Egardt (1979)).

The system (1) in closed loop with the controller (2), (3), results in

$$(\Sigma_1) \qquad \begin{aligned} y(t+1) &= -\psi(t)y(t) + d + r \\ \psi(t+1) &= \psi(t) + \mu \frac{y(t)(d - \psi(t)y(t))}{l + y(t)^2}, \end{aligned}$$

where $\psi = \theta - a$. When d is nonzero, the change of variables $x = y/d$; $\psi = \psi$; $\alpha = r/d$, transforms (Σ_1) into

$$(\Sigma) \qquad \begin{aligned} x(t+1) &= -\psi(t)x(t) + 1 + \alpha \\ \psi(t+1) &= \psi(t) + d^2\mu \frac{x(t)(1 - \psi(t)x(t))}{1 + d^2x^2(t)}. \end{aligned}$$

This variable rescaling puts in relief the role played by the reference-to-disturbance relationship α, called by Narendra and Annaswamy (1986) "persistent excitation of the reference relative to the disturbance." Moreover, it allows us to better describe the system's behavior for r and d close to zero. As discussed later, this (slightly disturbed regulation regime) is a very critical working condition. In the rescaled system, d^2 controls the adaptation speed of the algorithm. For our purposes we can thus assume that $\mu = 1$. The developments that follow can be done for the original system (Σ_1) replacing, when appropriate, the statement "d^2 sufficiently small" for "μ sufficiently small." In the first case, the slow adaptation condition is a consequence of the low level of the signals involved.

We can easily verify that (Σ) has a fixed point in $(\psi, x) = (1/\alpha, \alpha)$ if and only if α is nonzero and that it is unique if and only if $\alpha \neq -1$. In terms of the original system, this equilibrium corresponds to the output equal to the reference signal. The control objective is thus perfectly achieved at the fixed point.

The simulations show that for small values of d^2, the behavior of the solutions of (Σ) is characterized by the following stages.

(a) Explosive stage: growth of the modulus of the x-component in the "instability" set $\{|\psi| > 1\}$.

(b) Reinjection stage: decrease of the modulus of the ψ-component in the set $\{|x|$ "large", $|\psi| > 1\}$ until $|\psi| < 1$.

(c) Implosive stage: decrease of the modulus of the x-component in the "stability" set $\{|\psi| < 1\}$.

(d) Drift-ejection stage: slow growth of $|\psi|$ leading a solution from the "stability" set to the "instability" set.

(e) When (d) does not occur, the desired working condition is globally attractive.

Stages (a)–(c) are very short in time and, under certain conditions, stage (d) may be performed very slowly. In such a case, two successive occurrences of stages (a)–(c) are separated by a very long period of time. The result is an intermittent phenomenon, as studied by Pomeau and Manville (1980), characterized by a succession of "bursts" on the x-component separated by long quiescent periods. In practice, some palliatives (such as dead zone or leakage (Egardt (1979)), normalization (Praly (1983)), internal model principle (Elliott and Goodwin (1984)), filtering (Anderson et al. (1986)), etc.) are used to avoid intermittency and other nondesirable behaviors. However, if these remedies are not appropriately chosen, a qualitatively similar behavior may be observed for these more intricate cases (see Praly (1988)). The effect of some of these modifications in our example is discussed in §§ 6 and 7. To give a geometrical explanation of stages (a)–(e), the existence of two locally invariant (under the action of (Σ)) sets is demonstrated using the graph transform technique (Shub (1987)). The first one is repellent, explaining stage (a). The second one is attractive and allows us to explain stages (c), (d), and (e). Finally, stage (b) results from (a), when, during bursts, the disturbance d becomes negligible with respect to the x-component of the solutions. These locally invariant graphs are easily computed when the ψ-component remains constant. For this, we consider the set of all bounded solutions of (Σ) when $d = 0$ given by

$$S_0 \equiv \{(\psi, h(\psi)) \in \mathbb{R}^2 / h(\psi) = (1+\alpha)/(1+\psi), \psi \neq -1\}. \tag{4}$$

S_0, called the "frozen parameters invariant set," is invariant under the map $\Sigma_{d=0}$ (i.e., $\Sigma_{d=0}(S_0) \subseteq S_0$) and has exactly the properties associated with (a) and (c). It seems reasonable to expect that, when $|d|$ is not zero but still small, locally invariant graphs, approximated by S_0, still exist. The idea of using locally invariant sets or, more generally, locally integral sets, has been introduced by Riedle and Kokotovic (1986) and Praly (1985), (1990). However, their existence was only established for the "stability" set $\{|\psi| < 1\}$ and locally with respect to the x-components.

The paper is organized as follows. In § 3, the existence and properties of locally invariant sets are established. Critical elements and locally invariant sets are combined in § 4 to obtain theoretical results on the system global dynamics. These results are interpreted and compared with simulations in § 5. In § 6, the effects of introducing a "dead-zone" in the algorithm (3) is discussed. Finally, § 7 is dedicated to our concluding remarks. The critical elements—fixed points and periodic solutions—of (Σ) and the corresponding nearby local behavior needed to complement the global analysis are considered in the Appendix.

3. Locally invariant sets. In § 2, we observed that for $d = 0$, the set S_0 is invariant under $\Sigma_{d=0}$. Now from the definition of (Σ), when $\psi(t)$ and $x(t)$ are such that $1 + \psi(t)$ and $1 + \psi(t) + d^2 x(t)(1 + x(t))$ are nonzero we have

$$\left(x(t+1) - \frac{1+\alpha}{1+\psi(t+1)}\right) = -\psi(t)\left(x(t) - \frac{1+\alpha}{1+\psi(t)}\right) + \frac{d^2(1+\alpha)x(t)(x(t)\psi(t)-1)}{(1+\psi(t))(1+\psi(t)+d^2x(t)(1+x(t)))}.$$

The presence of d^2 in the second term on the right-hand side shows that S_0 is close to being a locally invariant set of (Σ) with d nonzero. Finally, for $d = 0$, this expression proves that (i) $S_0 \cap |(\psi, x)\,|\,|\psi| > 1\}$ is exponentially repellent and (ii) $S_0 \cap \{(\psi, x)\,|\,|\psi| < 1\}$ is exponentially attractive. These remarks lead us to look for locally invariant sets close to S_0, which are repellent in the set $\{|\psi| > 1\}$ and attractive in the set $\{|\psi| < 1\}$. These sets will be used as references for the global description of the solutions of (Σ).

3.1. The repellent locally invariant set (RLIS). Given any nonzero d, let ε be the smallest positive root of

$$\Delta(\varepsilon) = \left(\varepsilon - \frac{|d|}{1+\varepsilon}\right) - 2\sqrt{\frac{|1+\alpha|\,|d|(1+\varepsilon+|d|)}{\varepsilon}}. \tag{5}$$

For any function $M:\{|\psi|\geqq 1+\varepsilon\}\to\mathbb{R}$, we define its image by the operator T as

$$TM(\psi)=\frac{1+\alpha-M(\phi_M(\psi))}{\psi},\tag{6}$$

where ϕ_M is defined in terms of the function

$$\hat\phi_M(\psi)=\psi+d^2M(\psi)\frac{(1-\psi M(\psi))}{1+d^2M^2(\psi)},\tag{7}$$

as follows:

$$\phi_M(\psi)=\begin{cases}\sup(1+\varepsilon,\hat\phi_M(\psi)) & \text{if } \psi\geqq 1+\varepsilon,\\ \inf(-1-\varepsilon,\hat\phi_M(\psi)) & \text{if } \psi<-1-\varepsilon.\end{cases}\tag{8}$$

By definition, $\phi_M:\{|\psi|\geqq 1+\varepsilon\}\to\{|\psi|\geqq 1+\varepsilon\}$ is a continuous function and $\psi\phi_M(\psi)$ is positive. We are interested in the operator T because, if it has a fixed point H, then H satisfies the local invariance property:

$$H(\hat\phi_H(\psi))=1+\alpha-\psi H(\psi)\quad\text{if}\quad \psi\hat\phi_H(\psi)>0,\quad |\psi|>1+\varepsilon,\quad |\hat\phi_H(\psi)|>1+\varepsilon.\tag{9}$$

The graph $\{(\psi,H(\psi))/|\psi|\geqq 1+\varepsilon\}$ has two connected components in the plane (ψ,x). They are such that, with its initial condition in one of these sets, any solution of (Σ) will stay in it unless its ψ-component leaves its corresponding definition interval $\{\psi>1+\varepsilon\}$ or $\{\psi<-(1+\varepsilon)\}$. To exhibit the fixed point H, we consider the subset of $C^0(\{|\psi|\geqq 1+\varepsilon\},\mathbb{R})$:

$$\mathscr{B}=\{M|\partial(M,0)\leqq m_0 \text{ and } \operatorname{sgn}(\psi_1)=\operatorname{sgn}(\psi_2)\Rightarrow |M(\psi_1)-M(\psi_2)|\leqq m_1|\psi_1-\psi_2|\}.$$

It is a complete metric space with the distance ∂ defined as

$$\partial(M_1,M_2)=\sup_{\{|\psi|\geqq 1+\varepsilon\}}|M_1(\psi)-M_2(\psi)|.$$

The constants m_0, m_1 are

$$m_0=\frac{|1+\alpha|}{\varepsilon},\qquad m_1=\frac{2|1+\alpha|}{\varepsilon(\varepsilon-|d|/1+\varepsilon)}.\tag{10}$$

The next lemma and the uniform contraction theorem (see Hale (1980)) allow us to prove that T has a fixed point in $\mathscr{B}$.

LEMMA 1. *For any nonzero d, let ε be given by* (5), *then* (i) *T maps $\mathscr{B}$ into $\mathscr{B}$*; (ii) *For M_i, $i=1,2$, in $\mathscr{B}$, we have $\partial(TM_1,TM_2)\leqq\tau\partial(M_1,M_2)$, with*

$$\tau=\frac{1}{1+\varepsilon}+|d|m_1\left(1+\frac{|d|}{1+\varepsilon}\right)<1.$$

Proof. From (6) and definition (10), it can be easily shown that for all $M\in\mathscr{B}$, $\partial(TM,0)\leqq m_0$. The rest of the proof is obtained by showing that for all $M_1,M_2\in\mathscr{B}$, if $\operatorname{sgn}(\psi_1)=\operatorname{sgn}(\psi_2)$ then $|TM_1(\psi_1)-TM_2(\psi_2)|\leqq m_1|\psi_1-\psi_2|+\tau\partial(M_1,M_2)$. For this, use is made of (8) to write $|\hat\phi_M(\psi_1)-\hat\phi_M(\psi_2)|\geqq|\phi_M(\psi_1)-\phi_M(\psi_2)|$. The details can be consulted in España and Praly (1988). $\square$

Another important property of H is the following. Let $(\psi_0,x_0)\in\{|\psi_0|>1+\varepsilon\}\times\mathbb{R}$. We denote by $(\psi_1,x_1)=\Sigma(\psi_0,x_0)$ its image by (Σ). Suppose that $\operatorname{sgn}(\psi_1)=$

sgn (ψ_0) and $|\psi| > 1 + \varepsilon$. With Lemma 1, the definition (7), (8), and the property (9) we have

$$
\begin{aligned}
|x_0 - H(\psi_0)| &\leqq \left|\frac{H(\psi_1) - x_1}{\psi_0}\right| + \left|\frac{H(\phi_H(\psi_0)) - H(\psi_1)}{\psi_0}\right| \\
&\leqq \left|\frac{H(\psi_1) - x_1}{\psi_0}\right| + m_1\left|\frac{\hat{\phi}_H(\psi_0) - \psi_1}{\psi_0}\right| \\
&\leqq \left|\frac{H(\psi_1) - x_1}{\psi_0}\right| + \frac{m_1 d^2}{|\psi_0|}\sup_x\left\{\left|\frac{\partial}{\partial x}\left(\frac{x(1-\psi_0 x)}{1+d^2x^2}\right)\right|\right\}|H(\psi_0) - x_0| \qquad (11)\\
&\leqq \left|\frac{H(\psi_1) - x_1}{\psi_0}\right| + m_1|d|\left(1 + \frac{|d|}{1+\varepsilon}\right)|H(\psi_0) - x_0| \\
&\leqq \frac{1}{1 + (1+\varepsilon)(1-\tau)}\,|H(\psi_1) - x_1|.
\end{aligned}
$$

Hence, since τ is strictly smaller than 1, the distance from a solution to its projection, parallel to the x-axis, on the graph of H, must increase as long as its ψ-component stays in the same interval $\{\psi > 1 + \varepsilon\}$ or $\{\psi < -(1 + \varepsilon)\}$. Using the above derivations and the definitions (5), (10) it can be shown that

$$(12) \qquad |H(\phi_H(\psi_0)) - H(\psi_1)| \leqq \frac{m_1|d|(1 + |d|/(1+\varepsilon))}{1 - m_1|d|(1 + |d|/(1+\varepsilon))}\,|x_1 - H(\psi_1)|.$$

If we replace in (12) the value of m_1 given by (10) and use (5) again, we obtain

$$0 < m_1|d|(1 + |d|/(1+\varepsilon)) < \tfrac{1}{2} \Rightarrow |H(\phi_H(\psi_0)) - H(\psi_1)| < |x_1 - H(\psi_1)|.$$

Then from the following product, computed using the definition of (Σ) and the invariance property of H,

$$(x_0 - H(\psi_0))(x_1 - H(\psi_1)) = -\frac{(H(\psi_1) - x_1)^2}{\psi_0} + \frac{(H(\phi_H(\psi_0)) - H(\psi_1))(x_1 - H(\psi_1))}{\psi_0}$$

we have sgn $[(x_0 - H(\psi_0))(x_1 - H(\psi_1))] \neq$ sgn (ψ_0). To summarize, we have established the following theorem.

THEOREM 1 (the RLIS). *For any nonzero d, let ε be given by* (5). *There exist a bounded Lipschitz continuous function H, defined on $\{|\psi| \geqq 1 + \varepsilon\}$, with bound m_0 and a Lipschitz constant m_1 given by* (10), *such that*

(i) *If* $|\psi| \geqq 1 + \varepsilon$, $|\hat{\phi}_H(\psi)| \geqq 1 + \varepsilon$, sgn $(\hat{\phi}_H(\psi))$ = sgn (ψ), *then*

$$(13) \qquad H(\hat{\phi}_H(\psi)) = 1 + \alpha - \psi H(\psi).$$

(ii) *There exists ρ positive such that* $(\psi, x) \in \{|\psi| \geqq 1 + \varepsilon\} \times \mathbb{R}$, $(\phi, y) := \Sigma(\psi, x) \in \{|\psi| \geqq 1 + \varepsilon\} \times \mathbb{R}$, *and* $\psi\phi > 0$ *imply*

$$(14) \qquad \operatorname{sgn}((x - H(\psi))(y - H(\phi))) \neq \operatorname{sgn}(\psi),$$

$$(15) \qquad |y - H(\phi)| \geqq (1 + \rho)|x - H(\psi)|.$$

(iii) *Approximation of H* : $\sup_{\{|\psi| \geqq 1+\varepsilon\}} |H(\psi) - h(\psi)|/d^2$ *is bounded when* $d^2 \to 0$.

Proof. Statements (i) and (ii) are already established. To prove (iii), we first note that h, defined in (4), belongs to $\mathscr{B}$. Then, using Lemma 1, we have $\partial(h, H) \leqq \partial(h, Th) + \partial(Th, TH) \leqq \partial(h, Th)/(1 - \tau)$. The result is finally obtained using the definition of h and T to show that $|h(\psi) - Th(\psi)| = O(d^2)$. (The details can be consulted in España and Praly (1988).) $\square$

Remarks. 1. With (i) and (ii), this theorem establishes the existence of a globally, exponentially RLIS, given by the graph of a bounded continuous function $H : \{|\psi| > 1 + \varepsilon\} \to \mathbb{R}$ which, following (iii), can be approximated by the "frozen parameter invariant set," for $|d|$ sufficiently small. It is important to emphasize at this point that the repulsiveness of this graph, expressed by (15), has a global character in the sense that it is valid for any starting point $x \in R$. This is a direct consequence of the use of a normalized updating parameter equation (see (3) and (11)).

2. According to the sign of its ψ-component, the x-component of any solution changes side or not with respect to the graph of H (see (14)).

3. Although T has a unique fixed point H in $\mathscr{B}$, H need not be the unique function in $\mathscr{B}$ satisfying (13). This nonuniqueness comes from the arbitrariness of the function ϕ_M, which is not determined by (Σ) (see the discussion about the stopping function in Praly (1990)). In § 4 (see Theorem 3(b) and related remarks) the conditions under which the whole RLIS or a portion of it is unique are established.

3.2. The attractive locally invariant set (ALIS). Let

$$d^{*2} = \frac{1}{2|1+\alpha|^2}\left(\sqrt{\frac{1+3|1+\alpha|}{1+2|1+\alpha|}} - 1\right). \tag{16}$$

Taking $|d|$ in $(0, d^*)$, let η be the smallest positive root of

$$\Delta(\eta) = \left(\eta - \frac{d^2 n_0^2}{1+d^2 n_0^2}\right) - 2\sqrt{\frac{(1+2n_0)n_0 d^2}{1+d^2 n_0^2}}, \tag{17}$$

where n_0 is defined by

$$n_0 = \frac{|1+\alpha|}{\eta}. \tag{18}$$

The constraint introduced on d assures that $\Delta(0)\Delta(1)$ is strictly negative. This implies that η is strictly smaller than 1. Now for any d, $0 < |d| < d^*$, we define an operator P acting on functions $N : \mathbb{R} \to \mathbb{R}$ by

$$PN(\phi) = \begin{cases} 1 + \alpha - \psi_N(\phi)N(\psi_N(\phi)) & \text{if } |\phi| \leqq 1 - \eta, \\ PN(1-\eta)\ (\text{resp., } PN(\eta - 1)) & \text{if } \phi \geqq 1 - \eta\ (\text{resp., } \leqq -(1-\eta)), \end{cases} \tag{19}$$

where $\psi_N(\phi)$, mapping $\{|\psi| \leqq 1 - \eta\}$ into $\{|\psi| \leqq 1 - \eta\}$, is a function implicitly defined by the next two relations:

$$\phi = \frac{\hat{\psi}_N(\phi) + d^2 N(\hat{\psi}_N(\phi))}{1 + d^2 N^2(\hat{\psi}_N(\phi))}, \tag{20}$$

$$\psi_N(\phi) = \begin{cases} \hat{\psi}_N(\phi) & \text{if } |\hat{\psi}_N(\phi)| \leqq 1 - \eta, \\ (1-\eta)\ (\text{resp., } -(1-\eta)) & \text{if } \hat{\psi}_N(\phi) > 1 - \eta\ (\text{resp., } < -(1-\eta)). \end{cases} \tag{21}$$

We are interested in the operator P because, if it has a fixed point G, then G satisfies the local invariance property:

$$G(\phi) = -\hat{\psi}_G(\phi)G(\hat{\psi}_G(\phi)) + 1 + \alpha, \quad \text{if } |\phi| \leqq 1 - \eta \quad \text{and} \quad |\hat{\psi}_G(\psi)| \leqq 1 - \eta. \tag{22}$$

As for H in the "instability" set, the graph $\{(\psi, G(\psi))/|\psi| \leqq 1-\eta\}$ defines a set in the plane (ψ, x). It is such that, with its initial condition in this set, any solution of (Σ) will stay in it unless its ψ-component leaves the set of "strict stability" $\{|\psi| \leqq 1-\eta\}$.

To exhibit the fixed point G, we consider the set of "saturated" functions:

$$(23)\qquad \mathscr{C} = \left\{ N \in C^0(\mathbb{R}, \mathbb{R}) \;\middle|\; \begin{array}{l} (1)\ \partial(N, 0) \leqq n_0 \\ (2)\ \forall \psi_1, \psi_2 \in \mathbb{R},\ |N(\psi_1) - N(\psi_2)| \leqq n_1 |\psi_1 - \psi_2| \\ (3)\ N(\psi) = \begin{cases} N(1-\eta) & \text{if } \psi \geqq 1-\eta \\ N(-(1-\eta)) & \text{if } \psi \leqq -(1-\eta) \end{cases} \end{array} \right\},$$

$$(24)\qquad n_1 := \sqrt{(1+d^2 n_0^2) n_0 / (1+2n_0) d^2},$$

which is a complete metric space with the distance

$$\partial(N_1, N_2) := \sup_{\mathbb{R}} \{|N_1(\psi) - N_2(\psi)|\}.$$

Before studying the operator P acting on $\mathscr{C}$, we must be sure that $\hat{\psi}_N(\phi)$, implicitly defined by (20), makes sense. For this we have the following lemma.

LEMMA 2. *For any d, $0 < |d| < d^*$ and η given by* (17), *there exists a function $D: \mathscr{C} \times \{|\phi| \leqq 1-\eta\} \to \mathbb{R}$ and positive numbers $b_\phi(n_0, n_1, d^2)$, $b_n(n_0, n_1, d^2)$ satisfying for (N_i, ϕ_i) in $\mathscr{C} \times \{|\phi| \leqq 1-\eta\}$ and $i = 1, 2$*

$$(25)\qquad |D(N_1, \phi_1) - D(N_2, \phi_2)| \leqq b_\phi |\phi_1 - \phi_2| + b_n \partial(N_1, N_2),$$

$$(26)\qquad |D(N_i, \phi_i)| \leqq n_0(1+n_0),$$

$$(27)\qquad D(N, \phi) = N(\phi - d^2 D)(1 - \phi N(\phi - d^2 D)).$$

Proof. The proof takes advantage of the "almost identity" character of the function defined by (20) and (21) when d^2 is sufficiently small. Moreover, it gives conditions on the sizes of d^2 and η (see (16), (17)). The details may be consulted in España and Praly (1988); a more general result is also established in Praly (1990). □

With this function D, we can rewrite (20) as follows:

$$(28)\qquad \hat{\psi}_N(\phi) = \phi - d^2 D(N, \phi).$$

The next lemma and the uniform contraction theorem (see Hale (1980)) allow us to prove that P has a fixed point in $\mathscr{C}$.

LEMMA 3. *For any d, $0 < |d| < d^*$ and η given by* (17), *we have* (i) *P maps $\mathscr{C}$ into $\mathscr{C}$.* (ii) *For all N_i, $i = 1, 2$, in $\mathscr{C}$, we have $\partial(PN_1, PN_2) \leqq \lambda \partial(N_1, N_2)$ with $\lambda < 1$.*

Proof. It can be easily checked using (18)–(21) that PN satisfies (1) and (3) of (23). By using the properties of N, D, and the fact that $|\phi| \leqq 1-\eta$, we obtain

$$(29)\qquad \begin{array}{l} |PN_1(\phi_1) - PN_2(\phi_2)| \leqq \lambda(\eta, d^2)\partial(N_1, N_2) + n_1^* |\phi_1 - \phi_2|, \\ n_1^* := ((1-\eta)n_1 + n_0)(1 + d^2 b_\phi) \leqq n_1. \end{array}$$

Now, thanks to the choice of η in (17) we can show that

$$(30)\qquad \lambda(\eta, d^2) := (1-\eta)(1 + d^2 b_n n_1) + n_0 b_n d^2 = \frac{1 - (1+2n_0) d^2 n_1}{1 + d^2 n_0^2} < 1.$$

The details can be found in España and Praly (1988). See also Lemma 1 of Praly (1990). □

We next establish an important feature of the graph of G with respect to the solutions of (Σ). Let (ψ_0, x_0) be an element of $\{|\psi| \leqq (1-\eta)\} \times \{|x| \leqq \xi\}$ and (ψ_1, x_1) its image by (Σ). Whenever $|\psi_1|$ is smaller than $1-\eta$, from (28), (22), (19) we have

$$\begin{aligned}|x_1 - G(\psi_1)| &\leqq |\psi_0|\,|G(\psi_0) - x_0| + |\psi_0|\,|G(\psi_G(\psi_1)) - G(\psi_0)| \\ &\quad + |G(\psi_G(\psi_1)|\,|\psi_G(\psi_1) - \psi_0| \\ &\leqq |\psi_0|\,|G(\psi_0) - x_0| + (n_0 + |\psi_0| n_1)|\hat{\psi}_G(\psi_1) - \psi_0|.\end{aligned}$$

On the other hand, adding and subtracting $G(\psi_G)(1-\psi_1 G(\psi_0))$ we obtain

$$\begin{aligned}|\hat{\psi}_G(\psi_1) - \psi_0| &= d^2|x_0(1 - x_0\psi_1) - G(\hat{\psi}_G(\psi_1))(1 - \psi_1 G(\hat{\psi}_G(\psi_1)))| \\ &\leqq d^2(1 + \xi + n_0)|x_0 - G(\psi_0)| + d^2(1 + 2n_0)n_1|\psi_0 - \hat{\psi}_G(\psi_1)| \\ &\leqq \frac{d^2(1 + \xi + n_0)}{1 - d^2(1 + 2n_0)n_1}|x_0 - G(\psi_0)|.\end{aligned}$$

Hence, using (17) and (24), we have established

$$|x_1 - G(\psi_1)| \leqq \sigma(\xi)|x_0 - G(\psi_0)|, \tag{31}$$

$$\sigma(\xi) = \frac{1 - (1 + 2n_0)d^2 n_1}{1 + d^2 n_0^2} + \frac{d^2 n_1}{1 + d^2 n_0^2}(\xi - n_0). \tag{32}$$

With (30), $\sigma(\xi) > 0$ is strictly smaller than 1 if

$$n_0 < \xi < n_0 + \frac{n_0^2 + n_1(1 + 2n_0)}{n_1}. \tag{33}$$

Thus, any solution staying in the set $\{|\psi| \leqq (1-\eta)\} \times \{|x| \leqq \xi\}$, with ξ satisfying (33), exponentially approaches the graph of G. Moreover, with the above derivations, we have

$$|\psi_G(\psi_1)G(\psi_G(\psi_1)) - \psi_0 G(\psi_0)| \leqq \frac{d^2(1 + \xi + n_0)n_1}{1 + d^2 n_0^2}|x_0 - G(\psi_0)|.$$

However, since the invariance property of G implies

$$\begin{aligned}\psi_0(x_1 - G(\psi_1))(x_0 - G(\psi_0)) &= -\psi_0^2(x_0 - G(\psi_0))^2 + \psi_0(x_0 - G(\psi_0)) \\ &\quad \cdot(\psi_G(\psi_1)G(\psi_G(\psi_1)) - \psi_0 G(\psi_0)),\end{aligned}$$

it follows that from $|x_0| \leqq \xi$ and $d^2(1 + \xi + n_0)n_1/(1 + d^2 n_0^2) \leqq |\psi_0| \leqq 1 - \eta$ we obtain $\operatorname{sgn}[(x_1 - G(\psi_1))(x_0 - G(\psi_0))] \neq \operatorname{sgn}(\psi_0)$. To summarize, we have established the following theorem.

THEOREM 2 (the ALIS). *For any d, $0 < |d| < d^*$, and η given by* (17), *there exists a bounded Lipschitz continuous function G with bound n_0 and Lipschitz constant n_1 given, respectively, by* (18) *and* (24), *such that*

(i) *If $|\phi| \leqq 1 - \eta$ and $|\hat{\psi}_G(\phi)| \leqq 1 - \eta$, then*

$$G(\phi) = 1 + \alpha - \psi G(\psi) \quad \textit{and} \quad \phi = \psi + d^2\frac{G(\psi)(1 - \psi G(\psi))}{1 + d^2 G(\psi)^2}. \tag{34}$$

(ii) *Let ξ satisfy*

$$n_0 < \xi < n_0 + \frac{n_0^2 + n_1(1 + 2n_0)}{n_1},$$

then there exists $\sigma(\xi) < 1$ *such that* $(\psi, x) \in \{|\psi| < 1 - \eta\} \times \{|x| \leqq \xi\}$ *and* $(\phi, y) := \Sigma(\psi, x) \in \{|\phi| \leqq 1 - \eta\} \times \mathbb{R}$ imply

$$|y - G(\phi)| \leqq \sigma(\xi)|x - G(\psi)|. \tag{35}$$

Moreover, if

$$\frac{d^2(1 + \xi + n_0)n_1}{1 + d^2 n_0^2} \leqq |\psi|,$$

then $\operatorname{sgn}((y - G(\phi))(x - G(x))) \neq \operatorname{sgn}(\psi)$.

(iii) *Approximation of* G: $\sup_{\{|\psi|<(1-\eta)\}} |G(\psi) - h(\psi)|/d^2$ *is bounded for* $d^2 \to 0$.

Proof. Statement (i) is a direct consequence of Lemma 3. Statement (ii) follows from (31). To prove (iii), we first note that h, defined in (4), belongs to $\mathscr{C}$. Now, since G is the fixed point of P, Lemma 3(ii) gives $\partial(h, G) \leqq \partial(h, Ph)/(1 - \lambda(\eta, d^2))$. The result is finally obtained using the definition of h and P to show that (see details in España and Praly (1988))

$$\partial(h, G) \leqq d^2 n_0(n_0 + n_1)(n_0 + 1)/(1 - \lambda(\eta, d^2)). \qquad \square \tag{36}$$

Remarks. 1. This theorem establishes the existence of an (exponentially) ALIS given by the graph of a bounded continuous function $G : \{|\psi| < 1 - \eta\} \to \mathbb{R}$ which, following (iii), can be approximated by the "frozen-parameter invariant set," when $|d|$ is sufficiently small.

2. If its ψ-component is larger than

$$\frac{d^2(1 + \xi + n_0)n_1}{1 + d^2 n_0^2},$$

respectively, smaller than

$$-\frac{d^2(1 + \xi + n_0)n_1}{1 + d^2 n_0^2},$$

the x-component of any solution changes side with respect to (respectively, remains on the same side of) the graph of G.

3. Even though G is the unique fixed point of P in $\mathscr{C}$, its graph need not be the only one satisfying (22). The nonuniqueness of the ALIS comes from the arbitrariness of the definition (21), which is not determined by (Σ).

3.3. Additional characterization of the locally invariant sets. The existence Theorems 1 and 2 do not give enough information about the location of the locally invariant sets in the phase space. For this we have the following useful property.

Property 1. In their respective domains of definition, the functions H and G, determined by Theorems 1 and 2, belong to the family of functions F_α whose elements satisfy

(i) If $1/\alpha < -1$, then

$$\psi < 1/\alpha \Rightarrow M(\psi)(1 - \psi M(\psi)) \leqq 0, \quad \text{and} \quad \psi > 1/\alpha \Rightarrow M(\psi)(1 - \psi M(\psi)) \geqq 0.$$

(ii) If $1/\alpha > -1$, then

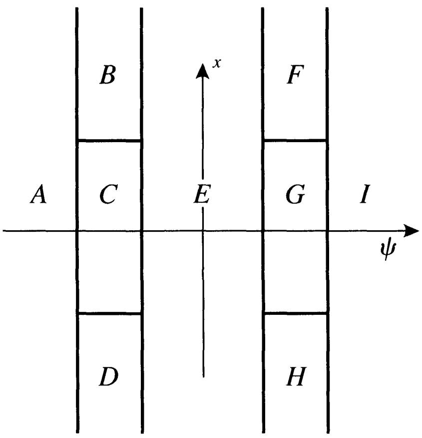
$$\psi < 1/\alpha \Rightarrow M(\psi)(1 - \psi M(\psi)) \geqq 0, \quad \text{and} \quad \psi > 1/\alpha \Rightarrow M(\psi)(1 - \psi M(\psi)) \leqq 0.$$

(iii) If $\alpha = -1 \Rightarrow M(\psi) \equiv 0$.

Proof. Statements (i) and (ii) are demonstrated by showing that, in the corresponding domain of definition, F_α is a closed subset of $\mathscr{B}$ (respectively, $\mathscr{C}$) such that $TF_\alpha \subseteq$

F_α (respectively, $PF_\alpha \subseteq F_\alpha$) (see España and Praly (1988)). Note that F_α does not depend on d^2. Statement (iii) results from a continuity argument.

4. Global behavior of the solutions: Technical results. Knowing the existence of critical elements and locally invariant sets, we are now in position for studying the behavior of the solutions. We decompose the plane (ψ, x) into nine subsets:



$$A = \{(\psi, x)/\psi \leqq -(1+\varepsilon)\},$$
$$B = \{(\psi, x)/-(1+\varepsilon) < \psi < -(1-\eta) \text{ and } \chi \leqq x\},$$
$$C = \{(\psi, x)/-(1+\varepsilon) < \psi < -(1-\eta) \text{ and } |x| < \chi\},$$
$$D = \{(\psi, x)/-(1+\varepsilon) < \psi < -(1-\eta) \text{ and } x \leqq -\chi\},$$
$$E = \{(\psi, x)/-(1-\eta) \leqq \psi \leqq (1-\eta)\},$$
$$F = \{(\psi, x)/1-\eta < \psi < 1+\varepsilon \text{ and } \chi \leqq x\},$$
$$G = \{(\psi, x)/1-\eta < \psi < 1+\varepsilon \text{ and } |x| < \chi\},$$
$$H = \{(\psi, x)/1-\eta < \psi < 1+\varepsilon \text{ and } x \leqq -\chi\},$$
$$I = \{(\psi, x)/1+\varepsilon \leqq \psi\},$$

with ε given by (5), η by (17), and $\chi > 1/(1-\eta)$. The global behavior of the solutions can be understood by looking at their evolution in each of these sets on the locally invariant graphs and outside them. We call $A \cup I$ the "strict instability" set, E the "strict stability" set, and $B \cup C \cup D \cup F \cup G \cup H$ the "critical stability" set.

4.1. Solutions in the locally invariant sets (RLIS and ALIS).

THEOREM 3(a) (the stationary solution). (i) *For any nonzero d and ε given by* (5), *the* (*unique*) *equilibrium point of* (Σ) *belongs to the* RLIS *if and only if* $|1/\alpha| \geqq 1+\varepsilon$.

(ii) *For any d, $0 < |d| < d^*$, and with d^*, η given by* (16), (17), *the* (*unique*) *equilibrium point of* (Σ) *is in the* ALIS *if and only if* $|1/\alpha| \leqq 1-\eta$.

(iii) *When* $\alpha = -1$, $H(\psi) \equiv 0$, $G(\psi) \equiv 0$ *and any point in the* RLIS *or* ALIS *is an equilibrium point.*

Proof. Given the global repulsiveness (respectively, attractiveness) of the RLIS (respectively, ALIS), the fixed point must be in the RLIS (respectively, ALIS) if it is in $\{|\psi| > 1+\varepsilon\}$ (respectively, $\{|\psi| < 1-\eta\}$). The rest follows from Theorem A in the Appendix and Property 1. □

THEOREM 3(b) (the nonstationary solutions). *If $\psi(t)$ is the ψ-component of any solution of* (Σ) *with initial condition in a locally invariant set, then*

(i) *if* $1/\alpha > -1 \Rightarrow (\psi(t) - 1/\alpha)/(\psi(t+1) - 1/\alpha) > 1$.

(ii) *if* $1/\alpha < -1 \Rightarrow (\psi(t+1) - 1/\alpha)/(\psi(t) - 1/\alpha) > 1$.

Proof. Theorem 3(b) is a consequence of Property 1 and the definition of (Σ). □

Remarks. From Theorems 3(a) and 3(b), if $1/\alpha < 109{-}1$ (respectively, $1/\alpha > -1$), the ψ-component of the solutions on any locally invariant set moves monotonically away from (respectively, toward) the value $1/\alpha$.

1. If $|1/\alpha| > 1$ and $(\psi, x) \in$ RLIS with $\psi \in (-\infty, 1/\alpha) \cup (1+\varepsilon, \infty)$, then it can be seen, with (8), that $\phi_H(\psi) = \hat{\phi}_H(\psi)$. Hence, the portion of the RLIS defined in $(-\infty, 1/\alpha) \cup (1+\varepsilon, \infty)$ is unique since the "stopping mechanism" is not active here. Note that the whole RLIS is unique if $1/\alpha > 1+\varepsilon$; moreover, in this case, the RLIS and the stable manifold of the fixed point coincide over $\{|\psi| > 1+\varepsilon\}$ (see Iooss (1979)).

2. If $1/\alpha < -1$ the ψ-component of the trajectories in the RLIS with $\psi(0) \in (-\infty, 1/\alpha) \cup (1/\alpha, \infty)$ are asymptotically unbounded.

3. If the fixed point lies in the ALIS (i.e., $|1/\alpha| < 1 - \eta$)), it is a global attractor inside the ALIS. If it lies in the RLIS, the solutions in the ALIS leave the set $\{|\psi| < 1 | \eta\}$ in a finite time through the boundary $\psi = 1 - \eta$. An estimation of the "traveling" speed of the ψ-component in the ALIS for this case will be of interest in our analysis and is given by the next theorem.

THEOREM 3(c). *If the fixed point is not in the* ALIS, *the solutions in it leave the "strict stability" set through the boundary $\psi = 1 - \eta$ in a finite time.*

Proof. From Property 1, $\psi G(\psi) < 1$ and $G(\psi) > 0$. Now, G being continuous on the compact set $\{|\psi| \leqq 1 - \eta\}$, there exists η_1, strictly positive, such that: $G(\psi) > \eta_1$ and $1 - \psi G(\psi) > \eta_1$. With the definition of (Σ) this implies that for all ψ; $|\psi| < 1 - \eta$,

$$n_0(1 + n_0) > \frac{\phi - \psi}{d^2} = \frac{G(\psi)(1 - \psi G(\psi))}{1 + d^2 G(\psi)^2} > \frac{\eta_1^2}{1 + d^2 n_0^2}. \tag{37}$$

Therefore, in the ALIS, $\psi(t)$ moves with positive speed of the order of d^2, thus leaving the set $\{|\psi| < 1 - \eta\}$ in a finite time through the boundary $\psi = 1 - \eta$. □

4.2. Solutions in the "strict instability" set outside the RLIS.

THEOREM 4. *For any $d \neq 0$ and with ε given by* (5), *we have*

(i) *Global repulsiveness. While the solution remains in $\{\psi \geqq 1 + \varepsilon\}$ (respectively, $\{\psi \leqq -(1 + \varepsilon)\}$), it exponentially diverges from the* RLIS *crossing it at each time t (respectively, remaining on the same side of the* RLIS).

(ii) *Injection. If for some time t_0, a solution satisfies $|\psi(t_0)| \geqq 1 + \varepsilon$, $x(t_0) \neq H(\psi(t_0))$, then there exists a finite time $t_1 > t_0$, such that $|\psi(t_1)| < 1 + \varepsilon$. Hence, there is no solution satisfying every $x \neq H(\psi)$ and $|\psi| \geqq 1 + \varepsilon$.*

Proof. Statement (i) is a direct consequence of (14), (15). To prove (ii), we first note that if $|x| > 1$ and $|\psi| > 1 + \varepsilon$ then

$$0 < \frac{d^2x^2 + |\psi x|}{(1 + d^2x^2)|\psi x|} < 1 - \frac{\varepsilon d^2}{(1 + \varepsilon)(1 + d^2)} < 1. \tag{38}$$

Hence, by Theorem 1, if for all s in $[t_0, t]$, $\psi(s) \geqq 1 + \varepsilon$ (respectively, $\leqq -(1 + \varepsilon)$) then

$$|x(t) - H(\psi(t))| \geqq (1 + \rho)^{t - t_0} |x(t_0) - H(\psi(t_0))|. \tag{39}$$

Now since $H(\psi)$ is bounded, there exists a first time t_1 (depending on $x(t_0)$, $\psi(t_0)$) such that either $|\psi(t_1)| \leqq 1 + \varepsilon$ or $|x(t_1)| \geqq 1$. In the latter case, from (38) and the definition of (Σ), we have

$$|\psi(t)| \leqq \left[1 - \frac{\varepsilon d^2}{(1 + \varepsilon)(1 + d^2)}\right] |\psi(t - 1)| \qquad \forall t > t_1, \tag{40}$$

which means that there exists $t_2 > t_1$ such that $|\psi(t_2)| < 1 + \varepsilon$. □

4.3. Solutions in the "strict stability" set outside the ALIS.

THEOREM 5. *For any d, $0 < |d| \leqq d^*$, with d^*, η given by* (16), (17), *we have*

(i) *Global attractiveness. Any solution in $\{|\psi| \leqq 1 - \eta\} \times \mathbb{R}$ exponentially approaches the* ALIS. *Moreover, a solution starting in $\{|\psi| \leqq 1 - \eta\} \times \mathbb{R}$ remains in this set as long as it remains in the set $\{|x| \geqq 1/(1 - \eta)\}$.*

(ii) *Drift/Ejection. If $|1/\alpha| > 1$ and for some time t_0, a solution satisfies $|\psi(t_0)| \leqq 1 - \eta$, then there exists a finite time t_1 such that $|\psi(t_1)| > 1 - \eta$. Hence, for $|1/\alpha| > 1$, there is no solution satisfying every $|\psi| \leqq 1 - \eta$.*

(iii) *Moreover, while a solution remains in the set*

$$\left\{\frac{d^2(1+\xi+n_0)n_1}{1+d^2n_0^2} \leqq \psi \leqq 1-\eta\right\} \times \{|x| \leqq \xi\},$$

respectively, in the set

$$\left\{-(1-\eta) \leqq \psi \leqq -\frac{d^2(1+\xi+n_0)n_1}{1+d^2n_0^2}\right\} \times \{|x| \leqq \xi\},$$

it crosses the ALIS *at each time t* (*respectively, it remains on the same side*).

Proof. (i) Since $|\psi| \leqq 1-\eta$ and $|x| \geqq 1/(1-\eta)$, then $|(\psi + d^2x)/(1+d^2x^2)| \leqq 1-\eta$, a solution starting in $\{|\psi| \leqq 1+\eta\} \times \mathbb{R}$ remains in this set at least while it remains in the set $\{|x| \geqq 1/(1-\eta)\}$. To complete the proof of (i), with property (35), we only need to show that any solution remaining in $\{|\psi| \leqq 1-\eta\} \times \mathbb{R}$ enters the set $\{|x| < \xi\}$, (with ξ satisfying (33)) in a finite number of steps. In fact, let the constant $\xi' := (n_0 + \xi)/2$ be $n_0 < \xi' < \xi$ and $|x(t)| \geqq \xi'$; then, from the equations of (Σ) and (18), we have

$$\left|\frac{x(t+1)}{x(t)}\right| \leqq \left|\frac{1+\alpha}{x(t)}\right| + (1-\eta) < 1-\eta+\frac{2n_0}{\xi+n_0}\eta < 1, \tag{41}$$

i.e., the absolute value of the x-component decreases exponentially as long as $|x| \geqq \xi'$.

(ii) We have

$$\begin{aligned}\psi(t+1) &= \psi(t) + \frac{d^2G(\psi(t))(1-\psi(t)G(\psi(t)))}{1+d^2G(\psi(t))^2} \\ &\quad + d^2[x(t)-G(\psi(t))]\frac{1+\psi(t)(x(t)+G(\psi(t)))}{(1+d^2G(\psi(t))^2)(1+d^2x(t)^2)}.\end{aligned} \tag{42}$$

From (i), either the ψ-component of the solution leaves the interval $[-(1-\eta), 1-\eta]$ or, after a finite time, $x(t) - G(\psi(t))$ will be as small as we want. The result follows from a continuity argument and Theorem 3(c).

Statement (iii) is a direct consequence of Theorem 2(ii). □

Remark. As in the discussion following Theorem 1, the global character of the attractiveness of this graph is a direct consequence of the use of a normalized algorithm. In general, nonnormalized algorithms, as treated by Praly (1990), may not lead to this kind of global result.

4.4. Solutions in the "critical stability" set.

THEOREM 6 (solutions in the sets B, D, F, H).

(i) *As long as a solution remains in the set* $\{(\psi, x)/1-\eta < |\psi| < 1+\varepsilon$ *and* $|x| > \chi\}$, *the absolute value* $|\psi|$ *decays exponentially.*

(ii) *Any solution starting in the set* $F \cup H$ (*respectively*, $B \cup D$) *either enters the set* G (*respectively*, C) *or goes into the set* E *in a finite time.*

Proof. Statement (i) follows exactly along the same lines as in (38)–(40).

(ii) From

$$\phi = \left(\frac{\psi x + d^2x^2}{1+d^2x^2}\right)\frac{1}{x},$$

we easily obtain

$$(\psi, x) \in F \Rightarrow \{\psi x > 1, x > 0\} \Rightarrow \phi > 1/x > 0,$$
$$(\psi, x) \in H \Rightarrow \{\psi x < -1, x < 0\} \Rightarrow \phi > 1/x \geqq -(1-\eta),$$
$$(\psi, x) \in D \Rightarrow \{\psi x > 1, x < 0\} \Rightarrow \phi < 1/x < 0,$$
$$(\psi, x) \in B \Rightarrow \{\psi x < -1, x > 0\} \Rightarrow \phi < 1/x < 1-\eta,$$

and the claim follows since from (i), $|\psi|$ decreases exponentially while $(\psi, x) \in B \cup D \cup F \cup H$. □

THEOREM 7 (solutions in the set G). *For $|d|$ small enough, if $|1/\alpha|$ is larger than $1+\varepsilon$, a 2-periodic orbit exists such that at least one of its points lies in the set*

$$G = \{(\psi, x)/1-\eta < \psi < 1+\varepsilon \text{ and } |x| < \chi\}.$$

Proof. According to Theorem A1 in the Appendix, for $|d|$ small enough, a 2-periodic orbit exists with its ψ component such that $1-\psi = O(d^2)$. This implies that the 2-periodic orbit is contained in $F \cup G \cup H$ for a small enough d^2. The result follows since from Theorem 6 the orbit cannot be entirely contained in $F \cup H$. □

4.5. Boundedness of solutions.

THEOREM 8. *If $1/\alpha > -1$, all the solutions of (Σ) are bounded.*

Proof. With Theorem 4.1 of Egardt (1979) it is sufficient to prove that the sequence $\{|\psi(t)|\}$ is bounded for any solution of (Σ). For this, we first show, from the second equation of (Σ) that when $|\psi| > |d|/2\sqrt{2}$, $|\psi(t+1)| > |\psi(t)|$ if and only if $\psi(t)x(t) \in (0, 1)$. The proof then follows by showing that, for $1/\alpha > -1$, there exists $\gamma > |d|/2\sqrt{2} > 0$ such that the points of the set $\Gamma \equiv \{(\psi, x)/|\psi| > \gamma; \psi x \in (0, 1)\}$ have no preimage in Γ (the trajectories starting in Γ leave it in one sampling time). This is combined with the relationship $|\psi(t+1)| < |\psi(t)| + |d|/2$ to show that if $\psi(0)$ satisfies

$$|\psi(0)| \leqq \max\left\{\frac{1+2\max\{|\alpha|, |1-\alpha|\}}{\max\{|\alpha|, |1+\alpha|\}}, \frac{|d|}{2\sqrt{2}}\right\} + |d|,$$

then $\psi(t)$ satisfies the same inequality for all $t \geqq 0$. If, on the other hand, $\psi(0)$ does not satisfy the above inequality, there exists a finite time T such that $\psi(T)$ does satisfy it (see details in España and Praly (1988)). □

5. Global behavior of the solutions: Qualitative description and simulation results. Using the technical results of the previous sections (Theorems 1–8), we can explain the five stages of the solutions' behavior observed in simulation and mentioned in § 2. For this, use is made of the phase plane decomposition introduced at the beginning of § 4. Figures 1–5 are used to illustrate the system's dynamic behavior. The function h given by (4) has been plotted in the phase portrait part of each figure. As shown by Theorems 1(iii) and 2(iii), its graph, denoted by "hg," approximates the RLIS and the ALIS, respectively, in their domain of definition.

5.1. The turbulent phase.

5.1.1. Explosive stage. According to Theorem 4(i), a solution in the sets A or I, either remains in the RLIS, which is the graph of a bounded function of ψ, or diverges exponentially from it (and, in practice, from its approximation (4)). This explains an exponential growth of the x-component, which becomes and remains large. Moreover, for a solution in the set I, at each time t, the x-component changes side with respect to

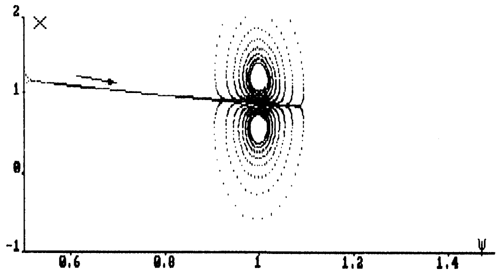


FIG. 1(a). *Phase portrait, two stable focus of* $\Sigma^2(\alpha = 0.8, d^2 = 0.005)$.

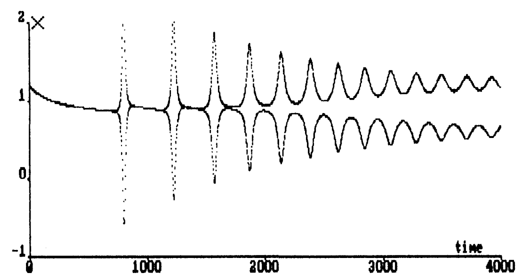


FIG. 1(b). *Time response converging to a period-2 stable orbit* $(\alpha = 0.8, d^2 = 0.005)$.

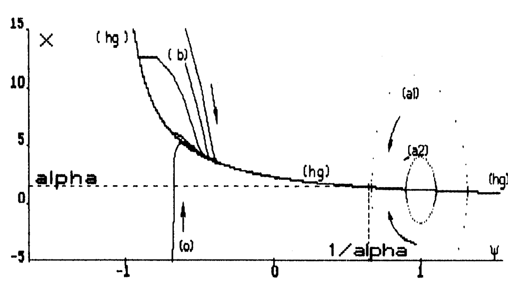


FIG. 2(a). *Phase portrait with a stable node* $(\alpha = 1.5, d^2 = 0.005)$.

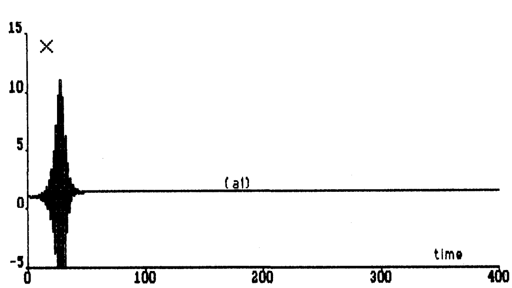


FIG. 2(b). *Time response of solution* (a1) $(\alpha = 1.5, d^2 = 0.005)$.

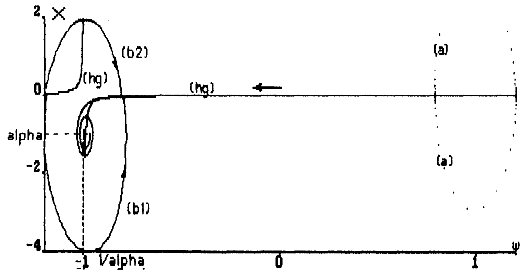


FIG. 3(a). *Phase portrait with a stable focus* ($\alpha = -1.008$, $d^2 = 0.005$).

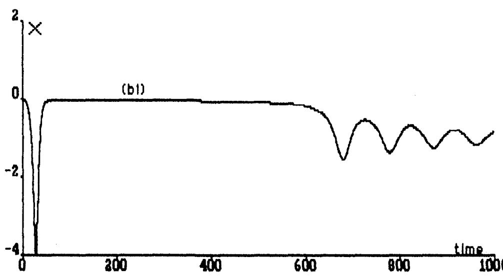


FIG. 3(b). *Time response of solution* (b1) ($\alpha = -1.008$, $d^2 = 0.005$).

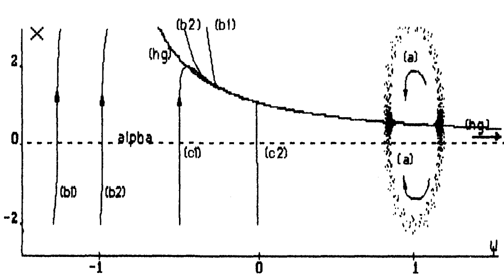


FIG. 4(a). *Phase portrait with a saddle as equilibrium point* ($\alpha = 0.1$, $d^2 = 0.005$).

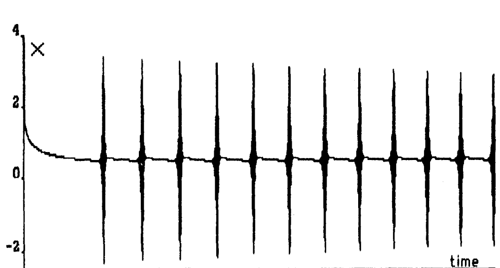


FIG. 4(b). *Time response of solution* (a) ($\alpha = 0.1$, $d^2 = 0.005$).

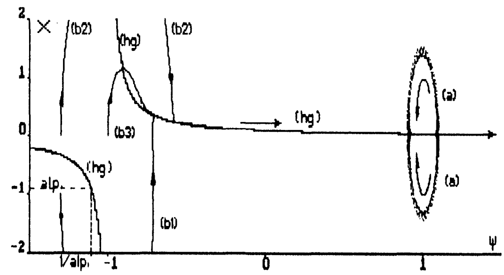


FIG. 5(a). *Phase portrait with an unstable node* ($\alpha = -0.9$, $d^2 = 0.005$).

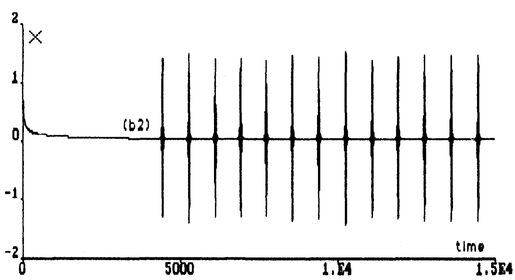


FIG. 5(b). *Time response of solution* (b2) ($\alpha = 0.9$, $d^2 = 0.005$).

this graph. This explains the "bursts" with very high frequency content on x (see solutions (a) in Figs. 2, 4, and 5). Conversely, for a solution in the set A, the x-component remains on the same side of the graph. It corresponds to a burst without oscillations (see solutions (b) in Figs. 3–5). We conclude that the explosive stage (a) takes place in the set A or I for any value of the disturbance and the reference (and therefore also in the ideal case).

5.1.2. Reinjection stage. Following Theorem 4(ii), a solution in the set A or I with a large x-component or in the set $B \cup D$ or $F \cup H$ has its ψ-component exponentially decaying. This occurs for any value of the disturbance and the reference and explains the reinjection of the solutions into the set E (see solutions (a) and (b) in Figs. 2–5).

5.1.3. Implosive stage. Theorem 5(i) states that, at least for a sufficiently small disturbance, as soon as a solution enters the set E, it is exponentially attracted toward the ALIS, which is the graph of a bounded function of ψ approximated by the set hg. This explains the exponential decrease of the x-component and, were it present, the fast decay of its high frequency content (see solutions (a) in Figs. 2, 4, and 5). Consequently, at least for small values of the disturbance, this stage occurs for any value of the reference and takes place in the set E.

5.2. The laminar phase (or the drift/ejection stage). Following Theorem 5(ii), when $|1/\alpha| > 1$, all the solutions leave the set E in a finite time. However, if before leaving the set E, they become close to the ALIS (in Figs. 1–5 one can see how the solutions practically converge to the graph (hg) approximating the ALIS), they finally leave that set "drifting" over the ALIS while its ψ-component grows with a speed of the order of d^2 (see also Theorem 3(c)). The solutions, very likely, enter the set G.

After entering the set G, a solution may either remain in it (see Fig. 1), go to the set I, thus restarting the explosive stage and possibly initiating the intermittent phenomenon, or go to the set $F \cup H$. In the latter case, Theorem 6 shows that the solution may either be reinjected into the set E, restarting the implosive stage, or returned to the set G. Intermittency may also take place in this case.

5.3. Possible 2-periodic orbits or limit cycles as ω-limits. According to Theorems A1 (in the Appendix) and 7, for a reference-to-disturbance ratio strictly smaller than 1 and for a disturbance sufficiently small, a 2-periodic orbit exists with at least one point in G and the other in $F \cup G \cup H$. Each point of this orbit is an attractive focus of Σ^2 if reference and disturbance have the same signs and a repellent focus in the case of opposite signs. In the former case, intermittency may disappear asymptotically while the solutions converge toward a 2-periodic orbit (see Fig. 1 and solution (a) in Fig. 4). When the reference and the disturbance have different signs, it is hypothesized that the solutions either exhibit a permanent intermittency (see Fig. 5) or, due to the occurrence of a supercritical Hopf bifurcation of the 2-periodic orbit, have an ω-limit comprised of two limit cycles of Σ^2 each surrounding a fixed point of Σ^2.

5.4. High sensitivity with respect to the initial conditions. From simulations and the approximations given in Theorems 1(iii) and 2(iii), it seems that the ALIS and the RLIS are smoothly connected through the set G (see Figs. 4 and 5). From Remark 1 following Theorem 3(b), if the fixed point lies in the RLIS, the portion of this set defined for $\psi > 1 + \varepsilon$ is unique. Its intersection with the boundary $\psi = 1 + \varepsilon$ being transverse, we expect that it is uniquely extended by an ALIS inside the strict stability region. Using this conjecture as a working hypothesis, the more a solution approaches the ALIS while it is in E, the more its evolution will be similar to the solutions in the RLIS when entering the set I. However, according to Theorem 3(b) and the remarks that follow, for a reference-to-disturbance ratio strictly smaller than 1 in absolute value and negative, the solutions in the RLIS starting in $\psi \in (1 + \varepsilon, \infty)$ are unbounded (the same as those starting in $(-\infty, 1/\alpha)$). On the other hand, the bigger its ψ-component is, the more the x-component of a solution in the set I, but not in the RLIS, is "pushed-away" (exponentially) from this invariant set. This reasoning shows the possibility of a very high sensitivity to initial conditions of solutions starting near the ALIS or, with Theorem 2, close to the graph of the function hg given in (4).

5.5. The desired behavior. Theorem A1 (see the Appendix) shows that, for a sufficiently small disturbance and a reference-to-disturbance ratio strictly larger than 1, the fixed point is exponentially stable and there is no other periodic solution. On the other hand, and under the same conditions, with Theorem 8, each solution remains in a compact set. This suggests that the fixed point is a global attractor. In this case, intermittency should not take place and the desired working conditions should be attained (see Figs. 2 and 3). Qualitatively speaking, this case most resembles the ideal case.

Summarizing, according to the reference-to-disturbance ratio α, three qualitatively different behaviors of the solutions of (Σ) can be predicted:

1. $|\alpha| > 1$ (high level excitation): bounded solutions, no intermittency, no periodic solution, a global attractive fixed point is conjectured, behavior similar to the ideal case.
2. $0 < \alpha < 1$ (low level excitation): bounded solutions, stable periodic solutions exist and are conjectured to be global attractors, the fixed point is a saddle, intermittency may occur but is conjectured to gradually disappear while converging asymptotically to a 2-periodic solution.
3. $-1 < \alpha \leqq 0$ (low level excitation): unbounded solutions exist, unstable 2-periodic solutions exist, intermittency and/or possible nonlinear oscillations are present, the fixed point is an unstable node.

Since α is a relative quantity, drastic qualitative changes of the system's behavior may be expected when both r and d are close to zero, which is the natural working condition for an adaptive linear regulator.

6. A means to prevent intermittency: The dead zone. We study here the effects of an empirical modification to the second equation of (Σ) (and (Σ_1)). For some $\delta > 0$, we call the set $D_\delta = \{x / |x - \alpha| < \delta\}$ the δ-dead zone and substitute μ in (Σ) by:

$$\mu = \begin{cases} 0 & \text{iff } x \in D_\delta, \\ 1 & \text{iff } x \notin D_\delta. \end{cases}$$

It is expected that this modification will interrupt the drift stage of the solutions near the ALIS when $|1/\alpha| > 1$. To examine the validity of this, let us first introduce the following definition:

$$\nu(\eta, d^2) := \frac{d^2 n_0(n_0 + n_1)(n_0 + 1)}{(1 - \lambda(\eta, d^2))},$$

where η and λ are given respectively by (17) and (30). We now make the following assertion.

Assertion. (i) If $|1/\alpha| > 1$ and d and δ are such that

$$\delta < \frac{1 - \alpha + \eta\alpha}{2 - \eta} - \nu(\eta, d^2),$$

there is no solution of (Σ) with δ-dead zone satisfying for all t: $|\psi(t)| < 1 - \eta$.

(ii) There is no solution leaving the set $\{|\psi| \leqq 1\}$ if and only if $\delta \geqq 1 + |\alpha|$.

Proof. (i) From the exponential attractiveness property of the ALIS and its approximation by the graph of h given by (4), (see Theorem 3 and (36)), we see that if $\alpha + \delta < h(1 - \eta) - \nu(\eta, d^2)$, any solution remaining in $\{|\psi| < 1 - \eta\}$ enters a set (the band around h of radius $\nu(\eta, d^2)$) whose intersection with the dead zone is empty for $|\psi| < 1 - \eta$. In this set, then, $\mu = 1$, the ψ-component is strictly increasing (Theorem 4(c) and (37)), and the solutions necessarily leave the domain $\{|\psi| < 1 - \eta\}$ in a finite time.

(ii) From the definition of (Σ) when $\mu = 1$ we have the following implications:

$$\{|\psi(t)| < 1 \text{ and } \psi(t+1) > 1\} \Leftrightarrow \{1 > \psi(t) > 1 - d^2 x(1 - x)\},$$

$$\{|\psi(t)| < 1 \text{ and } \psi(t+1) < -1\} \Leftrightarrow \{-1 < \psi(t) < -1 - d^2 x(1 + x)\}.$$

Thus, the solutions of (Σ) with dead zone will not leave the domain $\{|\psi| \leqq 1\}$ if and only if the right hand side of both implications are realized inside the dead zone only. But this is clearly the case when $\delta > |\alpha| + 1$. $\square$

Remarks. The dead-zone modification may fail to work if an upper limit of the disturbance is not known. When it works, i.e., when δ is bigger than, say, some $\delta^* > (1 - \alpha + \eta\alpha)/(2 - \eta) + \nu(\eta, d^2)$, the original ALIS is transformed into a new one (possibly not given by the graph of a continuous function any more) containing a portion of the graph of h. Since h is a monotonically decreasing function in $|\psi| < 1 - \eta$, the part of the graph of h coinciding with the new ALIS is confined to the right of the interval $|\psi| < 1 - \eta$. However, the movement of the solutions over the ALIS is also in the sense of growing ψ's. Consequently, in this case the modified scheme will very likely stop the solutions' drift near the ALIS before they enter into the "instability domain."

7. Concluding remarks. The ALIS and the RLIS play a key role in the qualitative and geometrical description of the phase portrait of our example. It can be shown that these two sets exist in more general systems since their definitions rest on some general properties of the adaptive systems (see, for instance, Praly (1990)). We can thus state that the intermittent behavior is the result of the absence of a global attractor in the ALIS combined with a property of the algorithm of maintaining all the signals bounded despite a model mismatch (L^∞-robustness). In fact, since the ALIS is normally defined in a bounded open set of the parameter space (in general, the set of parameters mapped into the open unitary circle by the eigenvalues of the regressor model), the lack of a global attractor in it implies that some trajectories approaching the ALIS will eventually leave the strict stability set of parameters. This may be a very slow "quasi-stable" process. On the other hand, the L^∞-robustness is responsible for the "reinjection mechanism" into the domain of attraction of the ALIS, provoking the abandon of the turbulent phase and the restart of the cycle. Since this reinjection is guaranteed by the (desirable) robustness, we can thus say that intermittency is essentially conditioned by the dynamics in the ALIS and thus, that any palliative to this phenomenon passes by an "adequate" modification of the dynamics in this set. However, the dynamics in the ALIS (and the ALIS itself) depend on the exogenous signals. Consequently, for any "good" modification of the ALIS dynamics it may be possible to find a "counter-example" given by a particular combination of model mismatch and reference signals. Moreover, the modifications introduced (to the algorithm and to the ALIS) may even exacerbate the situation. For instance, Rey, Bitmead, and Johnson (1989), reported that a previous ("unhelped") nonintermittent system may become intermittent after the addition of leakage. This possibility seems less likely when a dead zone is used since its working principle consists in transforming part of the ALIS into the corresponding set of attractive bounded solutions of the "frozen system." When the parameters are frozen, the drift phase is necessarily eliminated at least while the dead zone is active.

Clearly, more research must be done to find algorithms assuring a robust global attractor in the ALIS for (at least) a specified family of disturbances or model mismatches with a practical meaning. A possible general approach could be to stop the adaptation when some "ad-hoc" mechanism detects the drift phase. This decision can be taken, for instance, when the calculated increment in the parameters (possibly averaged) is smaller than a prespecified threshold. Note that here is not the error that counts, as in the dead zone, but its correlation with the regressor vector. This correlation should be viewed as an approximation of the gradient of the mean value of the error with respect to the parameters. When the reference signal is sufficiently persistently exciting or, more precisely, using a concept coined by Ioannou and Kokotovik (1983), persistently dominantly exciting, the ALIS has a natural global attractor in it; it corresponds to the desired working conditions. This attractor is hyperbolic (see Anderson et al. (1986)) and thus, structurally robust. However, for large disturbances (or model-mismatch), this attractor may cease to be globally attractive, disappear, or be pushed out of the ALIS in the RLIS region; this last circumstance motivates a bifurcation analysis. Any of these conditions can be at the origin of an intermittent behavior with persistent excitation, but in particular, when the attractor leaves the ALIS, the desired working conditions can never be attained even if by some means intermittency could ever be avoided. Clearly we can "solve the problem" by adding excitation in the dominant frequency range to the reference signals, but this does not imply that the original objective will be satisfied. On the other hand, when there is no persistent excitation (or there is but with a very low energy level or a bad frequency content), the functioning regime becomes very critical since the desired working conditions may not correspond to an hyperbolically stable set. In this case, a very weak robustness of many properties is to be expected. Indeed, very different qualitative behaviors can be close to one another and the system may easily switch from the desired working condition to an intermittent behavior or a self-oscillating mode. With respect to the last mode of operation, we see, for the example considered, that a 2-periodic orbit (self-oscillating model) appears as a result of the bifurcation provoked by the expulsion of the desired working conditions out of the ALIS. This 2-periodic oscillation has also been encountered by other authors in a similar example (Sethares and Mareels (1991), Rey, Bitmead, and Johnson (1991)). Actually, this is a particularity of the example (one-dimensional parameter space) and of the excitation condition (constant reference and disturbance signals). In general, these self-oscillating modes, which determine the frequency content of the "bursts," may have any period or not exist at all (see España, (1990), (1991)). When the ALIS has no attractor in it and there are no self-oscillations or there are but they are not attractive (see $1/\alpha < -1$ in our example), a nonperiodic permanent intermittent regime is likely to take place. This is favored by the fact that, while reentering in the "stability-domain," the trajectories are attracted toward a region (near the ALIS) where the solutions are highly sensitive with respect to the initial conditions. As pointed out by Bergé, Pomeau, and Vidal (1984), this, together with the absence of a periodic or quasi periodic attractive solution, seems to be at the origin of strange attractors.

Summarizing, we may conjecture that the antidotes for intermittency, like leakage or internal model principle, whose active principle is not based in stopping the solutions in its drift phase near the ALIS (they just modify the dynamics in it), could provoke the undesired effect if combined with a particular excitation and/or model mismatch. The schemes based on freezing the parameters upon detecting the drift phase (the dead-zone approach included) may also need a priori information of the disturbances to succeed, but, even if they can possibly fail to do the job, they are less likely to provoke the undesired effect. They are based on a characteristic of the phenomenon that is independent of the excitation conditions. Nevertheless, none of them solve the problem of the attractive self-oscillations. Strongly dominantly exciting references signals produce robust desired working conditions. Otherwise, if no precautions are taken, the system may easily switch among very different qualitative behaviors.

Appendix. Local behavior near the equilibrium point and period-2 solutions. The analysis of the local behavior of (Σ) near its equilibrium $(\psi, x) = (1/\alpha, \alpha)$ is based in the Jacobian matrix of (Σ) at this point:

$$J = \begin{pmatrix} -1/\alpha & -\alpha \\ -d^2/(1+d^2\alpha^2) & 1/(1+d^2\alpha^2) \end{pmatrix}.$$

The product of its eigenvalues is $P = -1/\alpha$ and their sum is given by

$$\text{(A)} \qquad S = P + 1 - \frac{d^2}{P^2 + d^2}.$$

In terms of S and P, exponential stability of the equilibrium point is given by

$$\text{(B)} \qquad 1 - S + P > 0$$

$$\text{(C)} \qquad 1 + S + P > 0,$$

$$\text{(D)} \qquad 1 - P > 0,$$

and the eigenvalues are real if

$$\text{(E)} \qquad S^2 - 4P \geqq 0.$$

Thus, the equilibrium point is exponentially stable if and only if $1/\alpha \in (-1, p)$, where $p \in (\frac{1}{2}, 1)$ is the unique solution of $2(1-p)(p^2+d^2) = d^2$. Consequently, unless $1/\alpha \in (-1, p)$, the "desired working conditions" do not correspond to a stable equilibrium point. The points of intersection of curves (A) and (E) correspond to the transition from real to complex eigenvalues and vice versa. They all occur for α negative, and we may have a stable ($\alpha < -1$) or an unstable ($\alpha > -1$) equilibrium point and a node or a focus depending on d^2.

Independently of the value of d^2, for α near -1, the equilibrium point passes from being an attractive focus ($\alpha < -1$) to a repellent one ($\alpha > -1$) verifying the conditions for a postcritical Hopf bifurcation (see Iooss (1979)). For the particular value $\alpha = -1$ the whole ψ-axis is a set of fixed points implying a global bifurcation.

For α positive, we have either a stable node ($1/\alpha < p$) or a saddle ($1/\alpha > p$). When $1/\alpha$ crosses the value p, an eigenvalue passes through -1, and a stable period-2 solution bifurcates from the stable fixed point while the latter becomes unstable (see Arnold (1983) or Iooss (1979)). The 2-periodic solutions can be determined evaluating the roots of the equation $\Sigma^2(\psi, x) - (\psi, x) = 0$, or, equivalently, for d nonzero, by computing the solution of

$$\text{(A.1)} \quad \begin{aligned} F_x(x, \psi, d) &:= -(\psi + d^2\phi)(-\psi x + 1 + \alpha) + (1 + \alpha - x) = 0 \\ F_\psi(x, \psi, d) &:= \phi + \frac{(-\psi x + 1 + \alpha)[1 - (\psi + d^2\phi)(-\psi x + 1 + \alpha)]}{1 + d^2(-\psi x + 1 + \alpha)^2} = 0, \end{aligned}$$

where $\phi = x(1 - \psi x)/(1 + d^2 x^2)$. For $d = 0$, the above system has three solutions:

$$\text{(A.2)} \qquad (\psi_0 = \alpha^{-1}, x_0 = \alpha), \qquad \left(\psi_{1,2} = 1, x_{1,2} = \frac{1 + \alpha \pm \sqrt{1 - \alpha^2}}{2}\right).$$

The first one is the equilibrium point; the two others exist if and only if the disturbance-to-reference ratio $1/\alpha$ is larger (in modulus) than 1. We make the following observation.

LEMMA (Existence of periodic solutions; Pomet, Coron, and Praly (1990)). *A necessary condition for* $(\psi_{\text{per}}(t, d), x_{\text{per}}(t, d))$ *to be a period-T solution of* (Σ) *that remains bounded as d goes to* 0 *is that the accumulation point of its initial condition be one of the three points in* (A.2).

To show that the existence of zeros $(\psi_{1,2}, x_{1,2})$ of (A.1) given by (A.2) is also sufficient for having period-2 solutions, use is made of the implicit function theorem with the following expression of the Jacobian matrix of (A.1) (nonsingular for $|\alpha| < 1$):

$$\partial F(x_{1,2}; \psi_{1,2}; 0) = \begin{bmatrix} 0 & \mp\sqrt{1 - \alpha^2} \\ \mp\sqrt{1 - \alpha^2} & * \end{bmatrix},$$

where the (2, 2)-term is unimportant. Compared with our discussion on the stability of the fixed point of (Σ) we note that for $|d|$ small enough, the period-2 solutions emerge not only when an eigenvalue of J passes through $-1 (\alpha \approx +1)$, from the stability side (for period-2 bifurcation condition), but also when a pair of conjugate eigenvalues of J crosses the unit circle at $\alpha = -1$. The latter corresponds to a global bifurcation. To summarize, we have the following theorem.

THEOREM A1 (Critical elements, España and Praly (1988)). (i) *The system* (Σ) *has a unique fixed point for all* α *different from* 0 *or* -1. *It is the solution corresponding to the control objective. It is exponentially stable for* $1/\alpha \in (-1, p)$ *and exponentially unstable for* $1/\alpha \notin [-1, p]$. (ii) *For any* $|\alpha| < 1$ *we can find a strictly positive constant*

d_0 such that if $|d| \leqq d_0$, there exists two locally unique period-2 solutions that can be approximated by

$$\text{(A.3)}\quad \psi_{1,2} = 1 - \frac{\alpha d^2}{2}\frac{1+\alpha \mp \sqrt{1-\alpha^2}}{2} + O(d^4), \qquad x_{1,2} = \frac{1+\alpha \pm \sqrt{1-\alpha^2}}{2} + O(d^2).$$

These solutions are foci of Σ^2, exponentially stable for $\alpha > 0$, exponentially unstable for $\alpha < 0$, and with a pseudoperiod approximated by $T = 2\pi / d(2(1-\alpha^2))^{1/2}$.

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REFERENCES

B. D. O. ANDERSON (1985), *Adaptive systems, lack of persistence of excitation and bursting phenomena*, Automatica, 21, pp. 247–258.

B. D. O. ANDERSON, R. R. BITMEAD, C. R. JOHNSON, P. V. KOKOTOVIC, R. L. KOSUT, I. M. Y. MAREELS, L. PRALY, AND B. D. RIEDLE (1986), *Stability of Adaptive Systems: Passivity and Averaging Analysis*, MIT Press, Cambridge, MA.

V. I. ARNOLD (1983), *Geometrical Methods in the Theory of Ordinary Differential Equations*, Springer-Verlag, New York, Berlin.

A. BENVENISTE, M. METIVIER, AND P. PRIOURET (1987), *Algorithmes Adaptatifs et Approximations Stochastiques: Théorie et Applications*, Masson, Paris.

M. BODSON, S. SASTRY, B. D. O. ANDERSON, M. Y. MAREELS, AND R. R. BITMEAD (1986), *Nonlinear averaging theorems, and the determination of parameter convergence rates in adaptive control*, Systems Control Lett., 7, pp. 145–157.

P. BERGE, Y. POMEAU, AND CH. VIDAL (1984), *L'Ordre dans le Chaos*, Hermann, Paris.

B. EGARDT (1979), *Stability of Adaptive Controllers*, Springer-Verlag, New York, Berlin.

H. ELLIOT AND G. C. GOODWIN (1984), *Adaptive implementation of the internal model principle*, in Proceedings of the 23rd IEEE Conference on Decision and Control, Las Vegas, NV.

M. ESPAÑA (1991), *Intermittent phenomena in adaptive systems: A case study*, Automatica., 27, pp. 717–720.

——— (1991), *Intermittency and self-oscillations in adaptive systems*, in Proceedings, VI IFAC Symposium on Automation of Mining, Mineral, and Metal Processing, Buenos Aires, Argentina, October.

M. ESPAÑA AND L. PRALY (1988), *On the Global Dynamics of Adaptive Systems*, Internal Report CAI, Fontainebleau, France.

G. C. GOODWIN AND K. S. SIN (1984), *Adaptive Filtering, Prediction and Control*, Prentice-Hall, Englewood Cliffs, NJ.

M. P. GOLDEN AND B. E. YDSTIE (1988), *Bifurcations in model reference adaptive control systems*, Systems Control Lett., 11, pp. 413–430.

J. K. HALE (1980), *Ordinary Differential Equations*, Krieger Publishing Company, Huntington, NY.

P. A. IOANNOU AND P. V. KOKOTOVIC (1983), *Adaptive Systems with Reduced Models*, Lecture Notes in Control and Inform. Sci., Vol. 47, Springer-Verlag, New York, Berlin.

G. IOOSS (1979), *Bifurcation of Maps and Applications*, North-Holland, Amsterdam.

M. JAÏDANE-SAÏDANE AND O. MACCHI (1988), *Quasi-periodic self-stabilization of adaptive* ARMA *predictors*, Internat. J. Adaptive Control Signal Processing, March.

L. LJUNG (1977), *Analysis of recursive stochastic algorithms*, IEEE Trans. Automat. Control, August.

L. LJUNG AND T. SÖDERSTRÖM (1983), *Theory and Practice of Recursive Identification*, MIT Press, Cambridge, MA.

K. S. NARENDRA AND A. ANNASWAMY (1986), *Robust adaptive control in the presence of bounded disturbances*, IEEE Trans. Automat. Control, April, pp. 306–315.

Y. POMEAU AND P. MANVILLE (1980), *Intermittent transition to turbulence in dissipative dynamical systems*, Comm. Math. Phys., 74, pp. 189–197.

J.-B. POMET, J. M. CORON, AND L. PRALY (1990), *On the periodic solutions of adaptive systems in the presence of periodic forcing terms*, Math. Control Signals Systems, 3, pp. 373–399.

L. PRALY (1985), *A geometric approach for the local analysis of a one-step-ahead adaptive controller*, in Proc. of the 4th Yale Workshop on Applications of Adaptive Systems Theory, New Haven, CT, June.

——— (1988), *Oscillatory behavior and fixes in adaptive linear control: a worked example*, in Proc. of the IFAC Workshop on Robust Adaptive Control, Newcastle, Australia, August.

——— (1990), *Topological orbital equivalence with asymptotic phase for a two time-scales discrete-time system*, Math. Control Signals Systems, 3, pp. 225–253.

G. J. REY, R. BITMEAD, AND C. R. JOHNSON (1991), *The dynamics of bursting in simple adaptive feedback systems with leakage*, IEEE Trans. Circuits and Systems, 38, pp. 426–488.

B. D. RIEDLE AND P. V. KOKOTOVIC (1986), *Integral manifold of slow adaptation*, IEEE Trans. Automat. Control, April.

M. SHUB (1987), *Global Stability of Dynamical Systems*, Springer-Verlag, New York, Berlin.

W. A. SETHARES AND M. Y. MAREELS (1991), *Dynamics of an adaptive hybrid*, IEEE Trans. Circuits and Systems, 38, pp. 1–11.