Systems & Control Letters 21 (1993) 19–33
North-Holland

Stabilization by output feedback for systems with ISS inverse dynamics

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Received 15 August 1992
Revised 8 January 1993

Abstract: We consider the subclass of the set of systems which admit a global normal form where only the output and not its time derivatives appear in the nonlinearities. We prove that, when the inverse dynamics are "input-to-state stable" (ISS) and a finite gain condition is satisfied, global asymptotic stability can be achieved by dynamic output feedback.

Keywords: Stabilization; dynamic feedback; output feedback; minimum phase systems; "input-to-state stability"

1. Definitions and notations

- $|\cdot|$ denotes the Euclidian norm and, for a vector $\omega = (\omega_i)_{i\in\{1,\dots,r\}}$, we denote $\|\omega\| = \sup_{i\in\{1,\dots,r\}}\{|\omega_i|\}$.
- For a real function $f(x)$, we denote by f' its first derivative $\mathrm{d}f/\mathrm{d}x$.
- A function $\gamma:\mathbb{R}_+\to\mathbb{R}_+$ is said to be of class $\mathscr{K}$ if it is continuous, nondecreasing[1] and satisfies $\gamma(0)=0$.
- A function $\beta:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ is said to be of class $\mathscr{KL}$ if, for each t in $\mathbb{R}_+$, $\beta(\cdot,t)$ is a function of class $\mathscr{K}$ and, for each s in $\mathbb{R}_+$, the function $\beta(s,\cdot)$ is nonincreasing and we have

$$\lim_{t\to+\infty}\beta(s,t)=0. \tag{1}$$

- Given a strictly convex C^1 function φ of class $\mathscr{K}$, we denote by $\ell\varphi$ its Legendre–Fenchel transform, i.e.

$$\ell\varphi(x)=\int_0^x(\varphi')^{-1}(s)\,\mathrm{d}s, \tag{2}$$

  where the function $(\varphi')^{-1}$ is the inverse function of the first derivative of φ. The interest of this definition is in Young's inequality [5, Theorem 156], i.e.

$$xy\le\varphi(|x|)+\ell\varphi(|y|)\quad\forall(x,y)\in\mathbb{R}^2. \tag{3}$$

- A function $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ is said to be of class $\mathscr{LF}^m$ if it is C^1, strictly convex, of class $\mathscr{K}$, has a Legendre–Fenchel transform $\ell\varphi$ as defined above and the function $\ell\varphi(|x|)/x$ is of class C^m and zero at zero.

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[1] Usually, one requires strictly increasing function.

2. Problem statement and main result

Consider a single input u, single output y dynamical system which can be described globally in the following normal form:

$$\begin{aligned} \dot{z} &= h(z, x_1), \\ \dot{x}_i &= x_{i+1} + f_i(z, x_1) + g_i(x_1), \quad i \in \{1, \ldots, r-1\}, \\ \dot{x}_r &= u + f_r(z, x_1) + g_r(x_1), \\ y &= x_1, \end{aligned} \tag{4}$$

where z is in $\mathbb{R}^n$ and f_i's, g_i's and h are C^∞ functions, with

$$f_i(0, 0) = 0, \qquad g_i(0) = 0, \qquad h(0, 0) = 0. \tag{5}$$

In the following, it will be useful to denote by $\Phi(z, y)$ the following vector:

$$\Phi(z, y) = (f_1(z, y), \ldots, f_r(z, y))^{\mathrm{T}}. \tag{6}$$

Remarks.
(1) Note the restriction that the f_j's and h are not allowed to depend on x_i, $i > 1$.
(2) The system

$$\dot{z} = h(z, y) \tag{7}$$

corresponds to the minimal dynamics of an inverse of (4) (see [6]).

For such a system, our problem is to design an output feedback making the origin a globally asymptotically stable equilibrium point.

Conditions under which a nonlinear system can be written in a form close to the one in (4) are discussed in [2, 11]. In the case where the system is linear, if it is also minimum phase, then a $(r-1)$-dimensional output feedback with high gain is sufficient to guarantee global asymptotic stabilization. This result has been extended to the nonlinear case in [9] when the functions f_i's and h are globally Lipschitz-continuous and the zero solution of

$$\dot{z} = h(z, 0) \tag{8}$$

is globally exponentially stable. When this Lipschitz condition is not satisfied, one obstruction to global stabilization by output feedback is due to the fact that even a vanishing time function $y(t)$ may be destabilizing for the system (see [19, 15, 16]):

$$\dot{z} = h(z, y(t)). \tag{9}$$

To overcome this difficulty, we strengthen the global asymptotic stability of (8) into the property of being input to state, as defined in [17]. Namely, we assume[2] the following.

Assumption IIS (*ISS-inverse system*). There exist two real functions β, of class $\mathcal{KL}$, and γ, of class $\mathcal{K}$, such that for any initial condition z_0 and any C^0 time function $y: [0, T) \to \mathbb{R}$, there exists a unique solution $z(t)$ of

$$\dot{z} = h(z, y(t)), \quad z(0) = z_0. \tag{10}$$

It is right maximally defined on $[0, T)$ and satisfies, for all t in $[0, T)$,

$$|z(t)| \leq \beta(|z_0|, t) + \gamma\left(\sup_{0 \leq s \leq t} \{|y(s)|\} \right). \tag{11}$$

[2] This definition, although written differently, is strictly equivalent to the one given in [17] but restricted to C^0 controls.

An algorithm to check if a system is ISS and to obtain the corresponding functions β and γ can be found in [17, Proof of Theorem 1].

With assumption IIS, the output feedback stabilization problem has been solved in [11, 8, 12] when (4) has the following special form:

$$\begin{aligned} \dot{z} &= Hz + G(x_1), \\ \dot{x}_i &= x_{i+1} + g_i(x_1), \quad i \in \{1, \ldots, r-1\}, \\ \dot{x}_r &= u + Fz + g_r(x_1), \\ y &= x_1, \end{aligned} \tag{12}$$

where F and H are matrices of appropriate dimensions, H being strictly Hurwitz, and the g_i's and G are C^∞ functions. When z does not appear linearly as is the case in (4), we have the following key intermediate result.

Lemma 2.1. *Under assumption IIS, for any strictly positive real number ε, we can find a continuous dynamic output feedback depending on ε which makes all the solutions of* (4) *bounded and converging to a compact set Γ_ε.*

This result is not completely satisfactory. Indeed, it will appear from its proof (see (91)) that, as ε goes to zero, coordinates depending on ε go to 0. But, coordinates independent of ε, like $(x_1, \ldots, x_r)$, may go to infinity, except for $x_1 = y$ which is guaranteed to go to 0. This incompleteness of the result is due to a singularity occurring at the equilibrium point. A sufficient condition for this singularity not to be an obstruction can be written in terms of a local small-gain condition. Indeed, we have the following theorem.

Theorem 2.2. *Suppose that IIS assumption is satisfied. Assume also that, for some invertible proper function θ of class $\mathcal{K}$ and some strictly positive real numbers κ, χ and C, with $\kappa \le 1$,* we have for all $x \in [0, \chi]$,

$$\gamma_y(x) + \gamma_z(2\gamma(x)) \le \begin{cases} Cx & \text{if } r > 1 \\ Cx^\kappa & \text{if } r = 1, \end{cases} \tag{13}$$

where

$$\gamma_y(x) = \sup_{\{(z,y) \mid \theta(|z|) \le |y| \le x\}} |\Phi(z, y)|, \qquad \gamma_z(x) = \sup_{\{(z,y) \mid \theta^{-1}(|y|) \le |z| \le x\}} |\Phi(z, y)|\,. \tag{14}$$

Under these conditions, the origin can be made a globally asymptotically stable equilibrium point by a continuous dynamic output feedback.

Remark. Since Φ is a C^∞ function, by taking

$$\theta(|z|) = |z|, \tag{15}$$

we can guarantee that γ_y and γ_z are at least linear locally. In this case, the first inequality in (14) can be reduced to

$$\frac{1}{C}\gamma_z(2\gamma(x)) \le x\,. \tag{16}$$

This condition is nothing but the small-gain condition of [10]. Also, it follows from [20], for example, that, if $\partial h/\partial z(0, 0)$ is a strictly Hurwitz matrix, i.e. the zero dynamics (8) are locally exponentially stable, the function γ is also linear locally. Therefore, in this case, condition (13) is satisfied.

3. Proofs

To prove Theorem 2.2 and, more precisely, to design explicitly a dynamic output feedback solving our problem, we follow the same procedure as in [8, 12]:

Step 1: we choose the following "observer":[3]

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + g_1(y) + k_1(y - \hat{x}_1), \\ \dot{\hat{x}}_2 &= \hat{x}_3 + g_2(y) + k_2(y - \hat{x}_1), \\ &\vdots \\ \dot{\hat{x}}_r &= u + g_r(y) + k_r(y - \hat{x}_1), \end{aligned} \tag{17}$$

where the gains k_i's are appropriately chosen so that the observation error

$$\tilde{x} = \hat{x} - x \tag{18}$$

satisfies

$$\dot{z} = h(z, y), \qquad \dot{\tilde{x}} = A\tilde{x} - \Phi(z, y), \tag{19}$$

where A is a strictly Hurwitz matrix. From assumption IIS, system (19) is ISS. Indeed, the matrix A being strictly Hurwitz, there exist strictly positive real numbers K_1, K_2 and λ such that, for any initial condition $(\tilde{x}_0, z_0)$ and any C^0 time function $y: [0, T) \to \mathbb{R}$, there exists a unique solution $(\tilde{x}(t), z(t))$ of (19) which is defined on $[0, T)$ and satisfies, for all t in $[0, T)$.

$$|\tilde{x}(t)| \leqslant K_1 \exp(-\lambda t)|\tilde{x}_0| + K_2 \int_0^t \exp(-\lambda(t-s))|\Phi(z(s), y(s))|\,ds, \tag{20}$$

$$|z(t)| \leq \beta(|z_0|, t) + \gamma\left(\sup_{0 \leq s \leq t} \{|y(s)|\}\right). \tag{21}$$

On the other hand, we have

$$|\Phi(z, y)| \leq \gamma_y(|y|) + \gamma_z(|z|), \tag{22}$$

where the functions γ_y and γ_z are defined in (14). These functions being nondecreasing, by using (22) with [17, equation (12)], (20) becomes

$$|\tilde{x}(t)| \leq \beta_x\left(\left|\begin{pmatrix}\tilde{x}_0 \\ z_0\end{pmatrix}\right|, t\right) + \frac{K_2}{\lambda}(\gamma_y + \gamma_z \circ 2\gamma)\left(\sup_{0 \leq s \leq t} \{|y(s)|\}\right), \tag{23}$$

where β_x, defined as follows, is of class $\mathcal{KL}$:

$$\begin{aligned} \beta_x\left(\left|\begin{pmatrix}\tilde{x}_0 \\ z_0\end{pmatrix}\right|, s\right) &= \exp(-\lambda s)\left[K_1\left|\begin{pmatrix}\tilde{x}_0 \\ z_0\end{pmatrix}\right| + \frac{K_2}{\lambda}\gamma_z\left(2\beta\left(\left|\begin{pmatrix}\tilde{x}_0 \\ z_0\end{pmatrix}\right|, 0\right)\right)\right] \\ &\quad + K_2 \int_0^s \exp(-\lambda(s-\tau))\gamma_z\left(2\beta\left(\left|\begin{pmatrix}\tilde{x}_0 \\ z_0\end{pmatrix}\right|, \tau\right)\right) d\tau. \end{aligned} \tag{24}$$

[3] We consider here a full-order observer. This will lead to an r-dimensional output feedback. But exactly the same procedure could be applied with a reduced-order observer leading to an $(r-1)$-dimensional output feedback as in [9, 12].

Note that the $\tilde{x}$ and z subsystems driven by y being time-invariant, inequalities (21) and (23) hold also for any initial condition at any starting time, i.e.

$$\left.\begin{aligned} |z(t)| &\le \beta(|z(s)|, (t-s)) + \gamma\left(\sup_{s\le\tau\le t}\{|y(\tau)|\}\right) \\ |\tilde{x}(t)| &\le \beta_x\left(\left|\begin{pmatrix}\tilde{x}(s)\\ z(s)\end{pmatrix}\right|, (t-s)\right) + \left(\frac{K_2}{\lambda}(\gamma_y + \gamma_z\circ 2\gamma)\right)\left(\sup_{s\le\tau\le t}\{|y(\tau)|\}\right)\end{aligned}\right\} \quad \forall s\in[0,t]. \tag{25}$$

Step 2: We design a state feedback for the "observer" (17) or, more precisely, for

$$\begin{aligned} \dot{y} &= \hat{x}_2 + g_1(y) + \delta_1(\tilde{x}, z, y), \\ \dot{\hat{x}}_2 &= \hat{x}_3 + g_2(y) + \delta_2(\tilde{x}, z, y), \\ &\ \vdots \\ \dot{\hat{x}}_r &= u + g_r(y) + \delta_r(\tilde{x}, z, y), \end{aligned} \tag{26}$$

where the δ_i's are given by

$$\delta_1(\tilde{x}, z, y) = x_2 - \hat{x}_2 + f_1(z, y), \qquad \delta_i(\tilde{x}, z, y) = k_i(y - \hat{x}_1), \quad 2 \le i \le r. \tag{27}$$

If the f_i's were zero, according to [18] it would be sufficient to design u to make (26) with input δ_i's ISS. But when the f_i's are not zero, a more specific result is needed. For this, we remark on the existence of a positive real number K_3 such that, for all i in $\{1, \ldots, r\}$,

$$|\delta_i(\tilde{x}, z, y)| \le K_3|\tilde{x}| + \gamma_y(|y|) + \gamma_z(|z|). \tag{28}$$

Then it follows from (25) that, for all i in $\{1, \ldots, r\}$, we have

$$|\delta_i(\tilde{x}(t), z(t), y(t))| \le \beta_\delta\left(\left|\begin{pmatrix}\tilde{x}(s)\\ z(s)\end{pmatrix}\right|, (t-s)\right) + \gamma_\delta\left(\sup_{s\le\tau\le t}\{|y(\tau)|\}\right) \quad \forall s\in[0,t], \tag{29}$$

where β_δ of class $\mathcal{KL}$ and γ_δ of class $\mathcal{K}$ are defined as follows:

$$\beta_\delta(s,t) = K_3\beta_x(s,t) + \gamma_z(2\beta(s,t)), \qquad \gamma_\delta(s) = \left[1 + \frac{K_2K_3}{\lambda}\right](\gamma_y + \gamma_z\circ 2\gamma)(s). \tag{30}$$

Inequality (29) can be viewed as showing the existence of a "nonlinear L^∞-gain" for the operator $y \mapsto (\delta_i)_{i\in\{1,\ldots,r\}}$. This motivates our interest for the following lemma about the stability of composite systems which is in the spirit of the small-gain theorem [10] (see [7] for a more general statement).

Lemma 3.1. *Consider the following composite system*:

$$\begin{aligned} \dot{\zeta} &= \psi_1(\zeta, \varpi), & \omega &= \psi_2(\zeta, \varpi), \\ \dot{\chi} &= \psi_3(\omega, \chi), & \varpi &= \psi_4(\chi), \end{aligned} \tag{31}$$

where ζ is in $\mathbb{R}^{n_1}$, χ is in $\mathbb{R}^{n_2}$, ψ_1, ψ_2 and ψ_4 are C^1 functions, with $\psi_4(0) = 0$, and ψ_3 is a C^0 function. Assume the following:

(1) *For the ζ-subsystem with input ϖ and output ω, there exist functions β_1 and β_2* of class *$\mathcal{KL}$, γ_1 and γ_2 of class $\mathcal{K}$ such that, for any initial condition $\zeta(0)$ and any C^0 time function $\varpi: [0, T) \to \mathbb{R}^{n_2}$, there exists a unique solution. It is defined on $[0, T)$ and satisfies, for all $t \in [0, T)$*,

$$\left.\begin{aligned} |\zeta(t)| &\le \beta_1(|\zeta(s)|, (t-s)) + \gamma_1\left(\sup_{s\le\tau\le t}\{|\varpi(\tau)|\}\right) \\ \|\omega(t)\| &\le \beta_2(|\zeta(s)|, (t-s)) + \gamma_2\left(\sup_{s\le\tau\le t}\{|\varpi(\tau)|\}\right)\end{aligned}\right\} \quad \forall s\in[0,t]. \tag{32}$$

(2) *For the χ-subsystem with input ω and output ϖ, there exist a positive-definite proper C^1 function $\mho$ and a convex function ϕ of class $\mathcal{K}$ such that*

$$\frac{\partial \mho}{\partial \chi}(\chi)\psi_3(\omega, \chi) \le -\mho(\chi) + \phi(\|\omega\|) \quad \forall(\omega, \chi) \tag{33}$$

and, for some positive real number η,

$$\phi(2\gamma_2(|\psi_4(\chi)|)) \le \mho(\chi) + \eta \quad \forall \chi. \tag{34}$$

Under these conditions, system (31) *has well-defined solutions for any initial conditions. They are bounded and converge to a compact set Γ_η and, in particular, we have*

$$\limsup_{t \to +\infty} \mho(\chi(t)) \le \eta. \tag{35}$$

Moreover, if $\eta = 0$ in (34) *then Γ_η is reduced to the origin.*

In the following, the role of the ζ-subsystem will be played by (19) with $(\tilde{x}, z)$ standing for ζ and y for ϖ, whereas the role of the χ-subsystem will be played by (26) with the vector of the δ_i's standing for ω. Having already established (2) with (25) and (29), this lemma indicates that the state feedback for (26) should be designed so that the "nonlinear gain" ϕ of the operator $\omega \mapsto \chi$ (see 33) be "small enough" to satisfy (34). We shall come back to this point after proving this lemma.

Proof of Lemma 3.1. We first remark that $\mho$ being positive-definite and proper, there exists an invertible proper function γ_v of class $\mathcal{K}$ such that [14, Theorem 7.13]

$$\gamma_v(|\chi|) \le \mho(\chi). \tag{36}$$

Moreover, ψ_4 being a C^1 function with $\psi_4(0) = 0$, there exists a function γ_3 of class $\mathcal{K}$ such that

$$|\psi_4(\chi)| \le \gamma_3(|\chi|) \quad \forall \chi. \tag{37}$$

Then, the system (31) having a continuous right-hand side, for each initial condition, there exist (may be nonunique) solutions right maximally defined on $[0, T)$, for some strictly positive real number T. Along each solution, we have from (33), for all t in $[0, T)$,

$$\dot{\mho} \le -\mho(\chi(t)) + \phi(\|\omega(t)\|). \tag{38}$$

But the function ϕ being convex, the function $\phi(2\gamma_2)$ being nondecreasing and the function $\psi_4(\chi(t))$ being C^0 on $[0, T)$, (32) and (34) yield, for $0 \le s \le t < T$,

$$\phi(\|\omega(t)\|) \le \tfrac{1}{2}\phi(2\beta_2(|\zeta(s)|, (t-s))) + \tfrac{1}{2}\phi\left(2\gamma_2\left(\sup_{s \le \tau \le t}\{|\psi_4(\chi(\tau))|\}\right)\right) \tag{39}$$

$$\le \tfrac{1}{2}\phi(2\beta_2(|\zeta(s)|, (t-s))) + \tfrac{1}{2}\sup_{s \le \tau \le t}\{\phi(2\gamma_2(|\psi_4(\chi(\tau))|))\} \tag{40}$$

$$\le \tfrac{1}{2}\phi(2\beta_2(|\zeta(s)|, (t-s))) + \tfrac{1}{2}\sup_{s \le \tau \le t}\{\mho(\chi(\tau))\} + \tfrac{1}{2}\eta. \tag{41}$$

Therefore, we have for $0 \le s \le t < T$,

$$\dot{\mho} \le -\mho(\chi(t)) + \tfrac{1}{2}\sup_{s \le \tau \le t}\{\mho(\chi(\tau))\} + \tfrac{1}{2}\phi(2\beta_2(|\zeta(s)|, (t-s))) + \tfrac{1}{2}\eta. \tag{42}$$

From this inequality, boundedness of $\mho$ and consequently of χ and ζ can be proved by invoking the small-gain theorem [10]. Here is a more specific proof.

By letting

$$v_0 = \max\{\mho(\chi(0)), \phi(2\beta_2(|\zeta(0)|, 0)) + \eta\}, \tag{43}$$

we get with Gronwall inequality

$$\mho(\chi(t)) - v_0 \leq \tfrac{1}{2} \sup_{0\leq s\leq t} \{\mho(\chi(s)) - v_0\}. \tag{44}$$

It follows readily that $\mho(\chi)$ is bounded by v_0 on $[0, T)$. But, with (36) and (32), this yields

$$|(\zeta(t), \chi(t))^{\mathrm{T}}| \leq \beta_1(|\zeta(0)|, t) + (\mathrm{Id} + \gamma_1 \circ \gamma_3)(\gamma_v^{-1}(v_0)) \quad \forall t \in [0, T), \tag{45}$$

with Id denoting the identity function. This implies with a contradiction argument that $T = +\infty$ and that the corresponding solution is defined and bounded on $[0, +\infty)$. And, in particular, we have

$$|\zeta(t)| \leq \zeta_0 \overset{\text{def}}{=} \beta_1(|\zeta(0)|, 0) + \gamma_1 \circ \gamma_3 \circ \gamma_v^{-1}(v_0) \quad \forall t \in [0, +\infty). \tag{46}$$

Let us now prove that

$$\limsup_{t\to+\infty} \mho(\chi(t)) \leq \eta. \tag{47}$$

For this, with v_0 defined above, let us define $t_0 = 0$. If $v_0 \leq \eta$, we augment it so that $v_0 > \eta$. Then, by induction, we assume the existence of a positive real number v_n and a time instant t_n such that

$$v_n > \eta, \quad \mho(\chi(t)) \leq v_n \quad \forall t \geq t_n. \tag{48}$$

In this condition, (42) yields, for all t larger than t_n,

$$\dot{\mho} \leq -\mho(\chi(t)) + \tfrac{1}{2} v_n + \tfrac{1}{2}\phi(2\beta_2(|\zeta(t_n)|, (t - t_n))) + \tfrac{1}{2}\eta. \tag{49}$$

But, the function β_2 being of class $\mathcal{KL}$ and the function ϕ of class $\mathcal{K}$, there exists a time instant $s_n \geq t_n$ such that

$$\phi(2\beta_2(|\zeta(t_n)|, (t - t_n))) \leq \tfrac{1}{2}(v_n - \eta) \quad \forall t \geq s_n. \tag{50}$$

This implies that

$$\dot{\mho} \leq -\mho(\chi(t)) + \tfrac{3}{4} v_n + \tfrac{1}{4}\eta \quad \forall t \geq s_n. \tag{51}$$

Therefore, let

$$t_{n+1} = s_n + \log 2, \qquad v_{n+1} = \tfrac{7}{8} v_n + \tfrac{1}{8}\eta. \tag{52}$$

We have established that

$$v_{n+1} > \eta, \quad \mho(\chi(t)) \leq v_{n+1} \quad \forall t \geq t_{n+1}. \tag{53}$$

Property (47) follows since the sequence v_n converges to η.

We have established, for each positive integer number n,

$$|(\zeta(t), \chi(t))^{\mathrm{T}}| \leq \beta_1(|\zeta(t_n)|, (t - t_n)) + (\mathrm{Id} + \gamma_1 \circ \gamma_3)(\gamma_v^{-1}(v_n)) \quad \forall t \geq t_n. \tag{54}$$

But β_1 being of class $\mathcal{KL}$, with (46), we have

$$\beta_1(|\zeta(t_n)|, (t - t_n)) \leq \beta_1(\zeta_0, t_n) \quad \forall t \geq 2t_n. \tag{55}$$

Therefore, we have finally, for each positive integer number n,

$$|(\zeta(t), \chi(t))^{\mathrm{T}}| \leq \beta_1(\zeta_0, t_n) + (\mathrm{Id} + \gamma_1 \circ \gamma_3)(\gamma_v^{-1}(v_n)) \quad \forall t \geq 2t_n, \tag{56}$$

where the sequence $\beta_1(\zeta_0, t_n)$ converges to 0 since the sequence t_n goes to infinity.

The conclusion of the lemma follows by defining the compact set Γ_η as the closed ball centered at the origin with radius $\rho(\eta)$, where the function ρ of class $\mathcal{K}$ is given by

$$\rho(\eta) = (\mathrm{Id} + \gamma_1 \circ \gamma_3) \circ \gamma_v^{-1}(\eta). \quad \square \tag{57}$$

As mentioned above our proofs of Lemma 2.1 and Theorem 2.2 will be complete if we can design a control law for the system (26) so that (33) and (34) are satisfied. To achieve this objective, we note that the system (26) is in feedback form, a form which has received a lot of attention (see [3] and the references therein for the unperturbed case and [4, 13] for the perturbed case). We follow here a mixture with some extension of the ideas contained in these studies. Indeed, our interpretation of the results we are aware of is that they are based on the following two elementary technical lemmas.

Lemma 3.2 (adding a perturbation). *Assume that for the system*

$$\dot{x} = f(x) + \Delta + gu, \tag{58}$$

there exist a C^l control law w and a C^{l+1} function V, such that $w(0) = 0$, $L_gV(0) = 0$[4] *and, for any solution $x(t)$ with $u = w(x)$, we have*

$$\dot{V} \le -V(x) + \alpha \tag{59}$$

for some time function α. Consider the system

$$\dot{x} = f(x) + \Delta + g(u + \delta). \tag{60}$$

Then, for any function φ of class $\mathscr{L}\mathscr{F}^l$, the control law

$$\bar{w}(x) = w(x) - \frac{\ell\varphi(|L_gV(x)|)}{L_gV(x)} \tag{61}$$

is C^l, $\bar{w}(0) = 0$ and, along the solutions of (61) with $u = \bar{w}$, we get

$$\dot{V} \le -V(x) + \alpha + \varphi(|\delta|). \tag{62}$$

Proof. Let us compute the time derivative of V along the solutions of (60) with

$$u = v + w(x). \tag{63}$$

We get, with obvious omitted arguments,

$$\dot{V} \le -V + \alpha + L_gV(v + \delta). \tag{64}$$

But with (3), we get

$$L_gV(v + \delta) \le L_gV\left(v + \frac{\ell\varphi(|L_gV|)}{L_gV}\right) + \varphi(|\delta|). \tag{65}$$

The result follows readily. □

Lemma 3.3 (adding one integrator). *Assume that for the system*

$$\dot{x} = f(x) + \Delta + gu, \tag{66}$$

with $f(0) = 0$, there exist a C^{l+1} control law w and a C^{l+1} function V such that $w(0) = 0$, $L_gV(0) = 0$ and, for any solution $x(t)$ with $u = w(x)$, we have

$$\dot{V} \le -V(x) + \alpha. \tag{67}$$

Then, for the system

$$\begin{aligned} \dot{x} &= f(x) + \Delta + gy, \\ \dot{y} &= h(x, y) + u, \end{aligned} \tag{68}$$

[4] L_gV denotes as usual $(\partial V/\partial x)g$.

with $h(0,0)=0$ *and for any function* φ *of class* $\mathscr{L}\mathscr{F}^{l}$, *the control law*

$$\bar{w}(x,y) = -\tfrac{1}{2}(y-w(x)) - L_g V(x) + \frac{\partial w}{\partial x}(x)(f(x)+gy) - \frac{\ell\varphi\left(|y-w(x)|\left|\frac{\partial w}{\partial x}(x)\right|\right)}{y-w(x)} - h(x,y) \tag{69}$$

is C^l, $\bar{w}(0,0)=0$, *the function*[5]

$$U(x,y) = V(x) + \tfrac{1}{2}(y-w(x))^2 \tag{70}$$

is C^{l+1} *and we have along the solutions of* (68), *with* $u=\bar{w}$,

$$\dot{U} \le -U(x,y) + \alpha + \varphi(|\Delta|). \tag{71}$$

Note that one degree of smoothness may be lost while going from w and V to $\bar{w}$.

Proof. Let us compute the time derivative of U along the solution of (68). With (67) and

$$u = v - \tfrac{1}{2}(y-w(x)) - L_g V(x) + \frac{\partial w}{\partial x}(f(x)+gy) - h(x,y), \tag{72}$$

we get

$$\dot{U} \le -U + \alpha + (y-w)\left(v - \frac{\partial w}{\partial x}\Delta\right). \tag{73}$$

But with (3), we get

$$(y-w)\left(v-\frac{\partial w}{\partial x}\Delta\right) \le (y-w)\left(v + \frac{\ell\varphi\left(|y-w|\left|\frac{\partial w}{\partial x}\right|\right)}{y-w}\right) + \varphi(|\Delta|). \tag{74}$$

The result follows readily. □

The main point of these lemmas is that, when going from system (58) to system (61) or from system (66) to system (68), one can find control laws so that the effect of the uncertainties is only to increment α by "φ (uncertainties)" (see (62) and (71)).

We are now ready to solve our design problem for system (26) by following the technique of induction on adding integrators. Precisely, we apply Lemmas 3.2 and 3.3 recursively while adding integrators.

Let φ be a function of class $\mathscr{L}\mathscr{F}^{r-1}$, $V_1(y)$ a proper positive-definite C^r function and u_0 a C^{r-1} function such that $u_0(0)=0$ and

$$V_1'(y)u_0(y) \le -V_1(y). \tag{75}$$

These three functions will be specified later.

We consider the system

$$\dot{y} = u_{11} + g_1(y). \tag{76}$$

With (5), the control law

$$u_{11}(y) = u_0(y) - g_1(y) \tag{77}$$

[5] More generally, one can take $U(x,y)=h_1(V(x)) + h_2(y-w(x))$, where h_1 and h_2 are any smooth, positive, increasing and proper functions with h_2 quadratic on a neighborhood of 0.

is $C^{r-1}, u_{11}(0) = 0$ and along, the solutions of (76), we have

$$\dot{V}_1 \leq -V_1. \tag{78}$$

Next we consider the system

$$\dot{y} = u_{12} + g_1(y) + \delta_1(\tilde{x}, z, y). \tag{79}$$

The assumptions of Lemma 3.2 are satisfied with

$$l = r - 1, \qquad V = V_1, \qquad \alpha = 0. \tag{80}$$

Therefore, (61) gives us appropriate C^{r-1} control law $u_{12}(y)$ and Lyapunov function V_1 for the system (79) in the sense that the assumptions of Lemma 3.3 are satisfied with

$$\Delta = \delta_1, \quad l = r - 2, \quad V = V_1, \quad \alpha = \varphi(|\delta_1|), \quad u_{12}(0) = 0, \quad V_1'(0) = 0. \tag{81}$$

We can now consider the system

$$\dot{y} = \hat{x}_2 + g_1(y) + \delta_1(\tilde{x}, z, y), \qquad \dot{\hat{x}}_2 = g_2(y) + u_{21}. \tag{82}$$

By applying Lemma 3.3, we get an appropriate C^{r-2} control law $u_{21}(y, \hat{x}_2)$ and C^{r-1} Lyapunov function V_2 for system (83) in the sense that the assumptions of Lemma 3.2 are satisfied with

$$\delta = \delta_2, \quad l = r - 2, \quad V_2(y, \hat{x}_2) = V_1(y) + \tfrac{1}{2}(\hat{x}_2 - u_{12}(y))^2, \quad \alpha = 2\varphi(|\delta_1|), \quad u_{21}(0, 0) = 0. \tag{83}$$

Continuing this procedure, we finally get a C^0 control u_{r2} and a C^1 function

$$V_r(y, \hat{x}_2, \ldots, \hat{x}_r) = V_1(y) + \frac{1}{2}\sum_{i=1}^{r-1} (\hat{x}_{i+1} - u_{i2}(y, \hat{x}_2, \ldots, \hat{x}_i))^2, \tag{84}$$

where each of the u_{i2}'s is C^{r-i} and is zero at zero. Moreover, along the solutions of (26), with the perturbations δ_i's generated by (19) and satisfying (29), and

$$u = u_{r2}(y, \hat{x}_2, \ldots, \hat{x}_r), \tag{85}$$

we have

$$\dot{V}_r \leq -V_r(y(t), \hat{x}_2(t), \ldots, \hat{x}_r(t)) + (2r - 1)\varphi\left(\sup_{i \in \{1, \ldots, r\}} \{|\delta_i|\}\right). \tag{86}$$

On the other hand, we know that the function V_1 is proper and positive-definite. It follows that the function $V_r(y, \hat{x}_2, \ldots, \hat{x}_r)$ is a positive-definite proper function.

Then, let us check now if we can apply Lemma 3.1 with the ζ-subsystem given by (19) and the χ-subsystem given by (26) with $u = u_{r2}$. From (86), assumption (33) is sastisfied with V_r playing the role of U, $(\delta_i)_{i \in \{1, \ldots, r\}}$ the role of ω and

$$\phi(s) = (2r - 1)\varphi(s). \tag{87}$$

We have already mentioned that (2) holds from (25) and (29) with, in particular,

$$\gamma_2(s) = \left[1 + \frac{K_2 K_3}{\lambda}\right](\gamma_y + \gamma_z \circ 2\gamma)(s). \tag{88}$$

It remains to check condition (35). To meet this condition, we adjust our three functions φ, V_1 and u_0. We have the following technical lemma (see the appendix).

Lemma 3.4. *For any function γ_2 of class $\mathscr{K}$ and any strictly positive real number η, there exist a function φ of class $\mathscr{L}\mathscr{F}^{r-1}$, a proper positive-definite C^r function $V_1(y)$ and a C^{r-1} function u_0 such that $u_0(0) = 0$ and, for all $y \in \mathbb{R}$, we have*

$$-V_1'(y)u_0(y) \geq V_1(y) \geq (2r - 1)\varphi(2\gamma_2(|y|)) - \eta. \tag{89}$$

Moreover if, for some strictly positive real numbers κ, χ and C, with $\kappa \le 1$, we have, for all $x \in [0, \chi]$,

$$\gamma_2(x) \le \begin{cases} Cx & \text{if } r > 1 \\ Cx^{\kappa} & \text{if } r = 1, \end{cases} \tag{90}$$

then the same result is satisfied with $\eta = 0$.

Consequently, by choosing η strictly positive if (13) does not hold or zero if not, assumption (34) of Lemma 3.1 is satisfied from (87)–(89) with γ_2 given in (88) and $\psi_4(\chi) = y$. Lemma 2.1 and Theorem 2.2 follow from the conclusion of Lemma 3.1 and, in particular, we have

$$\limsup_{t \to +\infty} V_r(y(t), \hat{x}_2(t), \ldots, \hat{x}_r(t)) \le \eta . \tag{91}$$

4. Example

Consider the system

$$\begin{aligned} \dot{z} &= -z^3 + x_1 , \\ \dot{x}_1 &= x_2 + z x_1 , \\ \dot{x}_2 &= u , \\ y &= x_1 . \end{aligned} \tag{92}$$

Following [17, Proof of Theorem 1], we can see that the system

$$\dot{z} = -z^3 + y \tag{93}$$

is ISS with

$$\beta(s,t) = \sqrt{\frac{4s^2}{4 + 7s^2 t}}, \quad \gamma(s) = 2 s^{1/3} . \tag{94}$$

Also, by choosing $\theta(x) = x^2$ in (15), we get, for $x \in \mathbb{R}_+$,

$$\gamma_y(x) = x^{3/2}, \qquad \gamma_z(x) = x^3 . \tag{95}$$

It follows, for $x \in \mathbb{R}_+$ that

$$\gamma_y(x) + \gamma_z(2\gamma(x)) = x^{3/2} + 64x . \tag{96}$$

We conclude that the assumptions of Theorem 2.2 are satisfied. Therefore, there exists a continuous dynamic output feedback which makes the origin a globally stable equilibrium point.

Then, let us exhibit this dynamic output feedback. Following our design, we first consider the following observer:

$$\dot{\hat{x}}_1 + \hat{x}_2 + 3(y - \hat{x}_1), \qquad \dot{\hat{x}}_2 = u + 2(y - \hat{x}_1) . \tag{97}$$

In this case the error system is

$$\dot{\tilde{x}}_1 = -3\tilde{x}_1 + \tilde{x}_2 - zy, \qquad \dot{\tilde{x}}_2 = -2\tilde{x}_1 . \tag{98}$$

We have

$$\delta_1 = x_2 - \hat{x}_2 + zy, \qquad \delta_2 = 2(y - \hat{x}_1) \tag{99}$$

and (29) is satisfied with

$$K_2 = 7, \qquad \lambda = 1, \qquad K_3 = 2 . \tag{100}$$

It follows that by choosing

$$\varphi(x) = \tfrac{1}{2}x^2, \qquad V_1(y) = 1350\, y^2(\sqrt{|y|} + 64)^2, \qquad u_0(y) = -\frac{y(\sqrt{|y|} + 64)}{3\sqrt{|y|} + 128}, \tag{101}$$

inequality (89) of Lemma 3.4 is sastisfied with $\eta = 0$ and

$$\ell\varphi(x) = \tfrac{1}{2}x^2. \tag{102}$$

Then, by applying Lemmas 3.2 and 3.3 successively, we get

$$\begin{aligned}
u_{11}(y) &= -\frac{y(\sqrt{|y|} + 64)}{3\sqrt{|y|} + 128},\\
u_{12}(y) &= u_{11}(y) - 675y(\sqrt{|y|} + 64)(3\sqrt{|y|} + 128),\\
u_{21}(y, \hat{x}_2) &= -\tfrac{1}{2}(\hat{x}_2 - u_{12}(y)) - 675y(\sqrt{|y|} + 64)(3\sqrt{|y|} + 128)\\
&\quad + u'_{12}(y)\hat{x}_2 - \tfrac{1}{2}(\hat{x}_2 - u_{12}(y))u'_{12}(y)^2,\\
u_{22}(y, \hat{x}_2) &= u_{21}(y, \hat{x}_2) - \tfrac{1}{2}(\hat{x}_2 - u_{12}(y)).
\end{aligned} \tag{103}$$

Therefore, a globally stabilizing dynamic output feedback is

$$\dot{\hat{x}}_1 = \hat{x}_2 + 3(y - \hat{x}_1), \qquad \dot{\hat{x}}_2 = u + 2(y - \hat{x}_1), \qquad u = u_{22}(y, \hat{x}_2). \tag{104}$$

5. Concluding remarks

We have established that a subclass[6] of the set of systems which admit a global normal form (4) and whose inverse dynamics are ISS can be stabilized by dynamic output feedback.

This result has been obtained by noting that it was already potentially contained in [8, 12], where systems in the form (12) are considered. The main point was to realize that what is effectively used in [8, 12] about the z-subsystem in (12) is the fact that it is BIBO system with y as input. Then, for the system (4), the natural extension is the property, for its z-subsystem, to be ISS.

The instrumental result which allowed us to do this extension is Lemma 3.1. It concerns the stability of composite systems with conditions à la small-gain theorem. Since it was very helpful here, we decided to devote the companion paper [7] to generalize it.

However, this extension asks for a knowledge of the function γ characterizing the property of being ISS. In the linear context of [8, 12], this function can easily be derived from the data of the system. For our more nonlinear context, we know only the algorithm proposed in [17, proof of Theorem 1]. It is not very practical. So, for the time being, our result is more an existence result than a practical result. Another problem we have raised, but not solved, in this paper is the small-gain condition close to the equilibrium. The consequence is that only the asymptotic smallness of the output is always guaranteed.

Here we have concentrated our attention on the stabilization problem for a system already written in appropriate coordinates. The next step is to find a coordinate-free condition guaranteeing that the assumptions of Theorem 2.2 are satisfied. This has been done in [8, 12] for the special form (12).

[6] See our first remark after (6).

Acknowledgment

We are very grateful to Andrew Teel for many enlightening discussions, to Jean-Michel Coron for his very efficient help for Lemma 3.5, to Eduardo Sontag for his very helpful comments about ISS systems and to Ioannis Kanellakopoulos for showing us some subtleties of nonlinear output feedback.

Appendix

A.1. Proof of Lemma 3.4

The result of this lemma is very intuitive. We provide here a proof for the sake of completeness. It has no practical interest. It is based on the following lemma established by Jean-Michel Coron.

Lemma A.1. *Let $k:[0, +\infty)\to\mathbb{R}$ be a continuous function such that, for some strictly positive real number χ, we have, for all $x\in[0,\chi]$,*

$$k(x) \le 0. \tag{105}$$

Under this condition, there exists a proper strictly increasing C^∞ function $W:[0, +\infty)\to[0, +\infty)$ and a strictly positive real number μ satisfying:

$$\begin{aligned} &\text{(i)}\ W(x) = \tfrac{1}{2}x^2 \quad \forall x\in[0,\mu],\\ &\text{(ii)}\ W(x) \ge k(x) \quad \forall x \ge 0,\\ &\text{(iii)}\ W'(x) > 0 \quad \forall x > 0. \end{aligned} \tag{106}$$

Proof. For any i in $\mathbb{Z}$, let $\mathscr{S}_i$ be the following open set:

$$\mathscr{S}_i = \{x\in\mathbb{R} \mid 2^{i-1} < x < 2^{i+1}\}. \tag{107}$$

We have

$$\bigcup_{i\in\mathbb{Z}} \mathscr{S}_i = (0, +\infty). \tag{108}$$

Then let $\{\alpha_i\}_{i\in\mathbb{Z}}$ be the partition of unity subordinate to this open covering [1, Theorem V.4.4]. Let also the real numbers c_i's be defined by

$$c_i = \sup\{\{k(x) \mid x\in\mathscr{S}_i\}\bigcup\{0\}\}. \tag{109}$$

Then $\bar{W}_1$, defined by

$$\bar{W}_1(x) = \sum_{i\in\mathbb{Z}} c_i\alpha_i(x), \tag{110}$$

is a C^∞ function satisfying

$$\begin{aligned} &\text{(i)}\ \bar{W}_1(x) = 0 \quad \forall x\in[0,\mu_1],\\ &\text{(ii)}\ \bar{W}_1(x) \ge k(x) \quad \forall x \ge 0, \end{aligned} \tag{111}$$

with μ_1 some strictly positive real number. With the same procedure but with k replaced by $-\bar{W}'_1$, we get a second function $\bar{W}_2$. Then the conclusion of the lemma follows by taking

$$W(x) = \tfrac{1}{2}x^2 + \int_0^x \bar{W}_2(s)\,ds + \bar{W}_1(x). \qquad \square \tag{112}$$

With this result, the first point of Lemma 3.4 is obtained by choosing

$$\varphi(x) = \tfrac{1}{2}x^2, \tag{113}$$

$$u_0(y) = -\frac{V_1(y)}{V_1'(y)}. \tag{114}$$

and

$$V_1(y) = V_1(-y) = W(y) \quad \forall y \geq 0, \tag{115}$$

where W is the function given by lemma A.1 with

$$k(x) = (2r-1)\varphi(2\gamma_2(x)) - \eta. \tag{116}$$

Note that if, for some positive real numbers C and $\chi > 0$, we have

$$2\gamma_2(x) \leq Cx \quad \forall x \in [0, \chi], \tag{117}$$

then, with φ as defined in (113), we get

$$\varphi(2\gamma_2(x)) \leq \frac{C^2}{2}x^2 \quad \forall x \in [0, \chi]. \tag{118}$$

In this case, we define u_0 as in (114), but with V_1 given by

$$V_1(y) = V_1(-y) = W(y) + (2r-1)C^2y^2 \quad \forall y \geq 0, \tag{119}$$

where now W is obtained from

$$k(x) = (2r-1)\varphi(2\gamma_2(x)) - (2r-1)C^2x^2. \tag{120}$$

Similarly, when $r = 1$, if, for some positive real numbers C, $\chi > 0$ and $\kappa \leq 1$, we have

$$2\gamma_2(x) \leq Cx^\kappa \quad \forall x \in [0, \chi], \tag{121}$$

then we choose φ as

$$\varphi(x) = \frac{\kappa}{1+\nu}|x|^{(1+\nu)/\kappa}, \tag{122}$$

with ν some strictly positive real number. In this case, we get

$$\ell\varphi(x) = \frac{1+\nu}{1+\nu-\kappa}|x|^{(1+\nu)/(1+\nu-\kappa)}. \tag{123}$$

So, clearly $\ell\varphi(|x|)/x$ is continuous, i.e. φ is of class $\mathscr{LF}^0$. Moreover, we get

$$\varphi(2\gamma_2(x)) \leq \frac{\kappa C^{(1+\nu)/\kappa}}{1+\nu}|x|^{1+\nu} \quad \forall x \in [0, \chi]. \tag{124}$$

In this case, we define again u_0 by (115), with V_1 given by

$$V_1(y) = V_1(-y) = W(y) + C^{(1+\nu)/\kappa}|y|^{1+\nu} \quad \forall y \geq 0, \tag{125}$$

where now W is obtained from

$$k(x) = \varphi(2\gamma_2(x)) - C^{(1+\nu)/\kappa}|x|^{1+\nu}. \tag{126}$$

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