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ADAPTIVE REGULATION: LYAPUNOV DESIGN WITH A GROWTH CONDITION

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SUMMARY

We propose a new Lyapunov design of an adaptive regulator under some restriction on the dependence of a Lyapunov function on the parameters.

This restriction has been introduced by Praly *et al.* Its interest is to involve only a Lyapunov function and not explicitly the system non-linearities. We show it is satisfied by strict pure feedback systems with polynomial growth non-linearities and some other non-feedback linearizable systems.

Our new Lyapunov design leads to an adaptive regulator where the adapted parameter vector is transformed before being used in the control law; namely, the so-called certainty equivalence principle is not applied. Unfortunately, the implementation of this regulator needs the explicit solution of a fixed point problem, so in a second stage we propose a more practical solution obtained by replacing the fixed point static equation by a dynamical system with this fixed point as equilibrium.

KEY WORDS Adaptive stabilization Non-linear systems Control Lyapunov function

1. INTRODUCTION

For linear systems it is now well established that parameterized controllers can be made adaptive.[1-3] In the non-linear case this is not true in general when we are concerned with global stability. As shown in Reference 4, this follows from the fact that in general the closed-loop system depends on the parameters. Two routes have been explored to overcome this difficulty.

The first route assumes that the parameters can be rejected when considered as disturbances with measured time derivatives. This is the so-called matching condition introduced by Taylor *et al.*,[5] extended by Kanellakopoulos *et al.*[6] (see also Reference 7) and generalized by Praly *et al.*[4] This generalized matching condition depends on the open-loop system and an assignable Lyapunov function (i.e. a control Lyapunov function). Kanellakopoulos *et al.*[8] have shown that, at least for systems in a pure feedback form, a simultaneous design of the control and the adaptation law allows us to satisfy systematically this generalized matching condition in a very specific but sufficient way. This technique has been generalized in Reference 9 and improved by Krstic *et al.*[10] in order to reduce the dynamics of the adaptive controller.

The second route has been followed by Nam and Arapostathis,[11] Sastry and Isidori,[12] Pomet and Praly[13,14] and Praly *et al.*[4] Robustness is used instead of disturbance rejection as before and the matching condition is replaced by some growth condition. This latter condition is such that we can design an adaptive controller making the closed-loop system Lagrange stability robust with respect to the effects of adaptation. Unfortunately, in contrast to the first

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route where sufficient geometric conditions on the open-loop system are known for the generalized matching condition to hold,[4,6] there is no precise characterization of the systems for which the various proposed growth conditions hold.

In this paper we follow the second route. In Section 2 we present our assumptions with, in particular, the same growth condition as the one introduced in Reference 4 for a least squares estimation scheme with initialized filters. In Section 3 we show that this condition is satisfied by some systems in a strict pure feedback form but also by some systems for which no other adaptive controller is known. In Section 4 we design a first basic adaptive regulator from a Lyapunov design and prove Lagrange stability. However, this controller involving the solution of a fixed point problem has reduced practical interest. Thus in Section 5 we propose a more practical regulator and prove again Lagrange stability. Section 6 is devoted to some extensions. Finally, conclusions are given in Section 7.

2. ASSUMPTIONS

Let the system to be controlled have a measured state x in $\mathbb{R}^n$ and an input u in $\mathbb{R}^m$. We assume:

Assumption LP (linear parameterization) (1)

There exist two known C^1-functions a and A and an unknown vector p^* in $\mathbb{R}^l$ such that the dynamics of the system to be controlled are globally described by

$$\dot{x} = a(x, u) + A(x, u)p^* \tag{2}$$

We shall restrict our attention to the case where p^* is in a known closed convex set Π^* whose boundary is a level set of a function $\mathscr{P}$. In fact, for technical reasons, we shall need to know that Π^* is the smallest set in a chain $\Pi^* \subsetneq \Pi_0 \subsetneq \Pi_1 \subsetneq \Pi_2 \subsetneq \Pi_3 \subsetneq \Pi$ where all the sets are also closed and convex with a level set of $\mathscr{P}$ as boundary. Precisely, we assume:

Assumption ICS (imbedded convex sets) (3)

There exists a known convex C^2-function $\mathscr{P}: \mathbb{R}^l \to \mathbb{R}$ such that:

1. $[-1, 4]$ is a subset of $\mathscr{P}(\mathbb{R}^l)$ and for each real number λ we define the set
$$\Pi_\lambda = \{p \mid \mathscr{P}(p) \leqslant \lambda\} \tag{4}$$
and we denote Π^* (respectively $\Pi_0, \Pi_1, \Pi_2, \Pi_3, \Pi$) the set obtained for $\lambda = -1$ (respectively $\lambda = 0$, $\lambda = 1$, $\lambda = 2$, $\lambda = 3$, $\lambda = 4$).
2. There exists a strictly positive constant N such that
$$\left\| \frac{\partial \mathscr{P}}{\partial p}(p) \right\| \geqslant N, \quad \forall p \in \{p \mid -1 \leqslant \mathscr{P}(p) \leqslant 3\} \tag{5}$$
3. The parameter vector p^* of the system to be controlled is in Π^*.
4. The minimum δ among the distances from Π_2 to the complement of Π, from Π_1 to the complement of Π_2 and from Π^* to the complement of Π_0 is strictly positive, i.e.

$$\Delta = \inf\{ \inf_{p_1 \in \Pi_2,\, p_2 \notin \Pi} \| p_1 - p_2 \|, \quad \inf_{p_1 \in \Pi_1,\, p_2 \notin \Pi_2} \| p_1 - p_2 \|, \quad \inf_{p_1 \in \Pi^*,\, p_2 \notin \Pi_0} \| p_1 - p_2 \| \} > 0 \tag{6}$$

To design our adaptive controller, we consider the system to be controlled as a particular element of $\{S_p\}_{p\in\Pi}$, the following family of systems indexed by $p\in\Pi$:

$$\dot{x} = a(x,u) + A(x,u)p \tag{7}$$

We assume that each element in this family is Lagrange stabilizable in the following sense:

Assumption LS (Lagrange stabilizability) (8)

There exist two known functions

$$u_{\rm n}:\mathbb{R}^n\times\Pi\to\mathbb{R}^m \text{ which is } C^1 \quad\text{and}\quad V:\mathbb{R}^n\times\Pi\to\mathbb{R}_+ \text{ which is } C^3$$

such that

1. For all positive real numbers K_v and all compact subsets Π_c of Π, the set $\{x \mid \exists p\in\Pi_c : V(x,p)\leqslant K_v\}$ is a compact subset of $\mathbb{R}^n$,
2. For all (x,p) in $\mathbb{R}^n\times\Pi$ we have

$$\frac{\partial V}{\partial x}(x,p)(a(x,u_{\rm n}(x,p)) + A(x,u_{\rm n}(x,p))p) \stackrel{\rm def}{=} -W(x,p)\leqslant 0 \tag{9}$$

Namely, if we apply the state feedback $u_{\rm n}(\cdot,p)$ to the system S_p (7), we get the so-called nominal closed-loop system

$$\dot{x} = a(x,u_{\rm n}(x,p)) + A(x,u_{\rm n}(x,p))p \tag{10}$$

It is Lagrange stable with Lyapunov function V whose time derivative is $-W$. For the more stronger asymptotic Lyapunov stability of a desired setpoint $\mathscr{E}^*$ we will invoke:

Assumption ALS (asymptotic Lyapunov stabilizability) (11)

For all C^1 time functions $\hat{p}:\mathbb{R}_+\to\Pi$ with bounded derivative the only bounded solution of

$$\dot{x} = a(x,u_{\rm n}(x,\hat{p}(t))) + A(x,u_{\rm n}(x,\hat{p}(t)))p^* \tag{12}$$

satisfying

$$W(x(t),\hat{p}(t)) = 0 \quad \text{for all } t\in\mathbb{R}_+ \tag{13}$$

is the trivial solution $x(t) = \mathscr{E}^*$.

Indeed, with this assumption, asymptotic Lyapunov stability will follow from LaSalle's theorem (Reference 15, 5.2.81).

The nominal closed-loop system (10) depends on the parameter vector p. More precisely, the Lyapunov function V depends on p. As mentioned in Section 1 and discussed in Reference 4, this is the origin of most of the difficulties in adaptive non-linear control. To specify this dependence in our approach, we assume:

Assumption GC (growth condition) (14)

There exists a known positive real number γ such that for all (x,p) in $\mathbb{R}^n\times\Pi$ we have

$$\left\|\frac{\partial V}{\partial p}(x,p)\right\| \leqslant \gamma(1+V(x,p)), \qquad \left\|\frac{\partial^2 V}{\partial p^2}(x,p)\right\| \leqslant \gamma^2(1+V(x,p)) \tag{15}$$

Assumption GC (14) differs from the following growth condition introduced by Pomet and Praly:[14]

$$\left\|\frac{\partial V}{\partial p}(x,p)\right\|\left\|\frac{\partial V}{\partial x}(x,p)A(x,u_n(x,p))\right\| \leqslant \gamma(1+V(x,p)^2) \tag{16}$$

However, it is the same as the one considered in Reference 4, Proposition (375) with an estimation design. The interest of the growth condition (15) is that the system non-linearities are not involved explicitly — compare with (16). It concerns only the parameter dependence of the control Lyapunov function V. For instance, it is satisfied if the set Π is compact and V is given by

$$V(x,p)=\Phi(x)^{\mathrm{T}}P(p)\Phi(x) \tag{17}$$

with a matrix $P(p)\geqslant \lambda I \geqslant 0$ for all p in the set Π.

3. EXAMPLES

Let us illustrate our assumptions by means of examples.

3.1. A three-dimensional system

Let us consider the non-feedback linearizable system

$$\dot{x}=p_1^*z+p_2^*z^2, \qquad \dot{y}=z+p_3^*y^3, \qquad \dot{z}=u \tag{18}$$

where the parameters p_1^*, p_2^* and p_3^* are unknown. We are interested in asymptotically stabilizing the setpoint $\mathscr{E}^*=(0,0,0)$. Since equations (18) are linear in the p_i^*, Assumption LP (1) is satisfied.

For Assumption ICS (3) to hold, it is sufficient to know that the vector $(p_1^*,p_2^*,p_3^*)^{\mathrm{T}}$ satisfies

$$(p_1^*-p_{01})^2+(p_2^*-p_{02})^2+(p_3^*-p_{03})^2 \leqslant R^2-\delta^2 \tag{19}$$

where the p_{0i} and $R>\delta>0$ are arbitrary but known. Indeed, in this case we can define the function $\mathscr{P}$ by

$$\mathscr{P}(p_1,p_2,p_3)=\frac{1}{\delta^2}\left[(p_1-p_{01})^2+(p_2-p_{02})^2+(p_3-p_{03})^2-R^2\right] \tag{20}$$

Clearly we have

$$\mathscr{P}(\mathbb{R}^3)=[-R^2/\delta^2,\infty)\supset[-1,4] \tag{21}$$

Also, $\mathscr{P}(p_1,p_2,p_3)\geqslant -1$ implies

$$\left\|\frac{\partial\mathscr{P}}{\partial p}(p_1,p_2,p_3)\right\|=\frac{2}{\delta^2}\left\|\begin{pmatrix}p_1-p_{01}\\ p_2-p_{02}\\ p_3-p_{03}\end{pmatrix}\right\|\geqslant\frac{2}{\delta^2}\sqrt{(R^2-\delta^2)} \tag{22}$$

Finally we get readily

$$\mathscr{P}(p_1^*,p_2^*,p_3^*)\leqslant -1 \tag{23}$$

To meet Assumption LS (8), we choose the control Lyapunov function

$$V(x,y,z,p_1,p_2,p_3)=y^8/8+\tfrac{1}{2}\left[z+(p_3+1)y^3\right]^2+\tfrac{1}{2}(x-h(y,p_1,p_2,p_3))^2 \tag{24}$$

where, to simplify the notation, we denote

$$h(y, p_1, p_2, p_3) = p_1(p_3+1)y - \frac{p_2(p_3+1)^2}{4}\,y^4 \tag{25}$$

Then a Lyapunov design gives the control law

$$\begin{aligned} u_n(x, y, z, p_1, p_2, p_3) = & -[z+(p_3+1)y^3] - y^7 - 3(p_3+1)y^2(z+p_3y^3) \\ & -(x-h)(p_2z - p_3[p_1 - p_2(p_3+1)y^3)] \end{aligned} \tag{26}$$

It follows that (9) in Assumption LS (8) is satisfied with

$$W(x, y, z, p_1, p_2, p_3) = y^{10} + [z+(p_3+1)y^3]^2 \tag{27}$$

Assumptions ALS (11) holds also if the set Π defined by (20) is such that

$$(p_1, p_2, p_3) \in \Pi \quad \Rightarrow \quad p_1p_3 \neq 0 \tag{28}$$

Indeed, for any C_1 time function $(\hat{p}_1(t), \hat{p}_2(t), \hat{p}_3(t)) \in \Pi$, any solution $(x(t), y(t), z(t))$ of (18) with

$$u = u_n(x, y, z, \hat{p}_1(t), \hat{p}_2(t), \hat{p}_3(t)) \tag{29}$$

which satisfies

$$W(x, y, z, \hat{p}_1(t), \hat{p}_2(t), \hat{p}_3(t)) = 0, \quad \forall t \tag{30}$$

is necessarily such that

$$y(t) = z(t) = 0, \quad \forall t \tag{31}$$

However, from (18) and (25), (26) this implies

$$\hat{p}_1(t)\hat{p}_3(t)x(t) = 0, \quad \forall t \tag{32}$$

The conclusion follows from (28).

It remains to check that Assumption GC (14) holds. A straightforward computation gives

$$\frac{\partial V}{\partial p} = [z+(p_3+1)y^3]\begin{pmatrix} 0 \\ 0 \\ y^3 \end{pmatrix}^{\mathrm{T}} - (x-h)\begin{pmatrix} (p_3+1)y \\ -[(p_3+1)^2/4]\,y^4 \\ p_1y - [p_2(p_3+1)/2]\,y^4 \end{pmatrix}^{\mathrm{T}} \tag{33}$$

$$\begin{aligned} \frac{\partial^2 V}{\partial p^2} = & \begin{pmatrix} 0 & 0 & (x-h)y \\ 0 & 0 & -(x-h)[(p_3+1)/2]\,y^4 \\ (x-h)y & -(x-h)[(p_3+1)/2]\,y^4 & y^6 - (x-h)(p_2/2)y^4 \end{pmatrix} \\ & + \begin{pmatrix} (p_3+1)y \\ -[(p_3+1)^2/2]\,y^4 \\ p_1y - [p_2(p_3+1)/4]\,y^4 \end{pmatrix}\begin{pmatrix} (p_3+1)y \\ -[(p_3+1)^2/2]\,y^4 \\ p_1y - [p_2(p_3+1)/4]\,y^4 \end{pmatrix}^{\mathrm{T}} \end{aligned} \tag{34}$$

Therefore there exists positive continuous functions Γ_1 and Γ_2 such that for all (x, y, z, p_1, p_2, p_3)

$$\left\| \frac{\partial V}{\partial p}(x, y, z, p_1, p_2, p_3) \right\| \leqslant \Gamma_1(p_1, p_2, p_3)(1 + V(x, y, z, p_1, p_2, p_3)) \tag{35}$$

$$\left\| \frac{\partial^2 V}{\partial p^2}(x, y, z, p_1, p_2, p_3) \right\| \leqslant \Gamma_2(p_1, p_2, p_3)(1 + V(x, y, z, p_1, p_2, p_3)) \tag{36}$$

With (20) this implies that (15) holds with

$$\gamma = \sup_{(p_1, p_2, p_3) \in \Pi} \{\Gamma_1(p_1, p_2, p_3), \sqrt{(\Gamma_2(p_1, p_2, p_3))}\} \tag{37}$$

Finally we remark that for the globally stabilizable system (18) we do not know any functions V and u_n such that Assumption LS (8) and the growth condition (16) hold.

3.2. *Strict pure feedback systems*

Let us consider now a system which maybe after parameter-dependent diffeomorphism and feedback can be written in the following form, called the strict pure feedback form in Reference 8:

$$\begin{aligned} \dot{x}_1 &= x_2 + f_1(x_1)p_1^* \\ &\vdots \\ \dot{x}_i &= x_{i+1} + f_i(x_1, \ldots, x_i)p_i^* \\ &\vdots \\ \dot{x}_n &= u \end{aligned} \tag{38}$$

where the x_i in $\mathbb{R}$ are measured, the f_i are known C^∞-function row vectors and the p_i are unknown parameter vectors in known convex compact sets Π_i. For this system we are interested in asymptotically stabilizing the setpoint $\mathscr{E}^* = (0, e_2^*, \ldots, e_n{}^*)$ which is uniquely defined by

$$e_{i+1}^* = -f_i(0, \ldots, e_i^*)p_i^*, \quad \forall i \geqslant 1 \tag{39}$$

Clearly Assumption LP (1) is satisfied. For Assumption ICS (3) we need to state precisely what was meant above by 'known convex compact sets Π_i'. For instance, if these sets are defined from proper convex positive functions $\mathscr{P}_i$ by

$$\Pi_i = \{p \mid \mathscr{P}_i(p) \leqslant 1\} \tag{40}$$

then the function $\mathscr{P}$ involved in Assumption ICS (3) can be chosen as

$$\mathscr{P}(p_1, p_2, \ldots, p_{n-1}) = \frac{5}{\varepsilon}\left(\sum_{i=1}^{n-1} \mathscr{P}_i(p_i)^l - 1 + \frac{4\varepsilon}{5}\right) \tag{41}$$

where $l \geqslant 2$ and $\varepsilon \in (0, 1)$ are two real numbers.

To show that functions u_n and V can be found to satisfy Assumptions LS (8), ALS (11) and GC (14), we apply the iterative Lyapunov design procedure suggested in Reference 16, Theorem 3.c (see also Reference 17). For this we introduce the following more compact notation:

$$\begin{aligned} X_i &= (X_{i-1}^{\mathrm{T}}, x_i)^{\mathrm{T}}, & X_1 &= x_1 \\ P_i &= (P_{i-1}^{\mathrm{T}}, p_i^{\mathrm{T}})^{\mathrm{T}}, & P_1 &= p_1 \end{aligned}$$

$$F_i(X_i, x_{i+1}, P_i) = \begin{pmatrix} x_2 + f_1(x_1)p_1 \\ \vdots \\ x_{i+1} + f_i(x_1, \ldots, x_i)p_i \end{pmatrix} \tag{42}$$

Step 2. Initialization. Consider the two-dimensional system

$$\dot{X}_l = x_2 - f_1(X_1)p_1, \qquad \dot{x}_2 = u_2 \tag{43}$$

In this case we propose the C^∞ control Lyapunov function

$$V_2(X_1, x_2, p_1) = \tfrac{1}{2}(x_2 + X_1 + f_1(X_1)p_1)^2 + \frac{|X_1|^{m_1}}{m_1} \tag{44}$$

where $m_1 \geqslant 2$ is an integer number. Its time derivative is

$$\dot{V}_2 = (x_2 + X_1 + f_1 p_1)\left[u_2 + \left(1 + \frac{\partial f_1}{\partial X_1} p_1\right)(x_2 + f_1 p_1) + X_1 |X_1|^{m_1 - 2}\right] - |X_1|^{m_1} \tag{45}$$

Therefore, by choosing the C^∞ control

$$u_2(X_1, x_2, p_1) = -(x_2 + X_1 + f_1(X_1)p_1) - \left(1 + \frac{\partial f_1}{\partial X_1} p_1\right)(x_2 + f_1 p_1) - X_1 |X_1|^{m_1 - 2} \tag{46}$$

we get

$$\dot{V}_2 = -(x_2 + X_1 + f_1 p_1)^2 - |X_1|^{m_1} \stackrel{\text{def}}{=} -W_2 \tag{47}$$

This implies that Assumption LS (8) is satisfied for the system (43). About Assumption GC (14) we have:

$$\frac{\partial V_2}{\partial p_1} = (x_2 + X_1 + f_1 p_1) f_1, \qquad \frac{\partial^2 V_2}{\partial p_1^2} = f_1^{\mathrm{T}} f_1 \tag{48}$$

Therefore, since p_1 is in a compact set, Assumption GC (14) is satisfied if we can choose m_1 such that with some positive real number $\mu_1 \geqslant 0$ we have

$$\| f_1(X_1) \| \leqslant \mu_1 (1 + |X_1|^{m_1/2}) \tag{49}$$

Namely, f_1 has a polynomial growth.

Assumption ALS (11) holds also since $W_2 = 0$ implies $X_1 = 0$. In addition, the first equation of (43) implies $x_2 = e_2^*$.

Step i + 1. The induction assumption is: for the system

$$\dot{X}_{i-1} = F_{i-1}(X_{i-1}, x_i, P_{i-1}), \qquad \dot{x}_i = u_i \tag{50}$$

we know C^∞-functions $V_i(X_{i-1}, x_i, P_{i-1})$ and $u_i(X_{i-1}, x_i, P_{i-1})$ such that Assumptions LS (8), ALS (11) and GC (14) hold. Precisely, there exist three positive continuous functions W_i, ν_i and ξ_i such that for all (X_{i-1}, x_i, P_{i-i})

$$\begin{gathered} \dot{V}_i = \frac{\partial V_i}{\partial X_{i-1}} F_{i-1} + \frac{\partial V_i}{\partial x_i} u_i \stackrel{\text{def}}{=} -W_i \leqslant 0 \\ \left\| \frac{\partial V_i}{\partial P_{i-1}} \right\| \leqslant \nu_i(P_{i-1})(1 + V_i), \qquad \left\| \frac{\partial^2 V_i}{\partial P_{i-1}^2} \right\| \leqslant \xi_i(P_{i-1})(1 + V_i) \end{gathered} \tag{51}$$

Assume also the following extra condition: there exist a real number $a_i \geqslant 1$ and positive continuous functions μ_i and λ_i such that for all (X_{i-1}, x_i, P_{i-i})

$$\| f_i \| + \left\| \frac{\partial u_i}{\partial P_{i-1}} \right\| \leqslant \mu_i(P_{i-1})(1 + V_i^{a_i}), \qquad \left\| \frac{\partial^2 u_i}{\partial P_{i-1}^2} \right\| \leqslant \lambda_i(P_{i-1})(1 + V_i^{a_i}) \tag{52}$$

Under these conditions we shall show that Assumptions LS (8), ALS (11) and GC (14) are satisfied by C^∞-functions V_{i+1} and u_{i+1} for the system

$$\begin{aligned} \dot{X}_i &= F_i(X_i, x_{i+1}, P_i) \\ \dot{x}_{i+1} &= u_{i+1} \end{aligned} \tag{53}$$

Indeed, we consider the following C^∞ control Lyapunov function with $m_i \geqslant 2a_i$ a strictly positive integer number:

$$V_{i+1}(X_i, x_{i+1}, P_i) = \tfrac{1}{2}(x_{i+1} + f_i(X_i)p_i - u_i(X_i, P_{i-1}))^2 + \frac{1}{m_i} V_i(X_i, P_{i-1})^{m_i} \tag{54}$$

We have

$$\dot{V}_{i+1} = (x_{i+1} + f_i p_i - u_i)\left[u_{i+1} + \left(\frac{\partial f_i}{\partial X_i} p_i - \frac{\partial u_i}{\partial X_i}\right)F_i\right] + V_i^{m_i - 1}\left(\frac{\partial V_i}{\partial X_{i-1}} F_{i-1} + \frac{\partial V_i}{\partial x_i}(x_{i+1} + f_i p_i)\right) \tag{55}$$

In view of (51), by choosing the C^∞ control

$$u_{i+1} = -\left(\frac{\partial f_i}{\partial X_i} p_i - \frac{\partial u_i}{\partial X_i}\right)F_i - V_i^{m_i - 1}\frac{\partial V_i}{\partial x_i} - (x_{i+1} + f_i p_i - u_i) \tag{56}$$

we obtain

$$\dot{V}_{i+1} \leqslant -(x_{i+1} + f_i p_i - u_i)^2 - V_i^{m_i - 1} W_i \stackrel{\text{def}}{=} -W_{i+1} \leqslant 0 \tag{57}$$

This is (9) in Assumption LS (8). Also, Assumption ALS (11) follows by induction. On the other hand, we have

$$\frac{\partial V_{i+1}}{\partial P_i} = (x_{i+1} + f_i p_i - u_i)\left(-\frac{\partial u_i}{\partial P_{i-1}}, f_i\right) + V_i^{m_i - 1}\left(\frac{\partial V_i}{\partial P_{i-1}}, 0\right) \tag{58}$$

$$\begin{aligned}\frac{\partial^2 V_{i+1}}{\partial P_i^2} = &\begin{pmatrix} -(\partial u_i/\partial P_{i-1})^{\mathrm{T}} \\ f_i \end{pmatrix}\left(-\frac{\partial u_i}{\partial P_{i-1}}, f_i\right) + (x_{i+1} + f_i p_i - u_i)\begin{pmatrix} -\partial^2 u_i/\partial P_{i-1}^2 & 0 \\ 0 & 0\end{pmatrix} \\ &+ V_i^{m_i - 2}\begin{pmatrix} V_i \partial^2 V_i/\partial P_{i-1}^2 + (m_i - 1)(\partial V_i/\partial P_{i-1})^{\mathrm{T}} \partial V_i/\partial P_{i-1} & 0 \\ 0 & 0\end{pmatrix}\end{aligned} \tag{59}$$

Therefore from (51), (52) and (54) we get

$$\begin{aligned}\left\|\frac{\partial V_{i+1}}{\partial P_i}\right\| &\leqslant (2V_{i+1})^{1/2}\mu_i(P_{i-1})(1 + V_i^{a_i}) + V_i^{m_i - 1}\nu_i(P_{i-1})(1 + V_i) \\ &\leqslant (2V_{i+1})^{1/2}\mu_i(P_{i-1})[1 + (m_i V_{i+1})^{a_i/m_i}] + (m_i V_{i+1})^{(m_i - 1)/m_i}\nu_i(P_{i-1}) \\ &\quad \times [1 + (m_i V_{i+1})^{1/m_i}] \\ &\leqslant \nu_{i+1}(P_i)(1 + V_{i+1})\end{aligned} \tag{60}$$

with

$$\nu_{i+1}(P_i) \stackrel{\text{def}}{=} 2[\sqrt{(2m_i)}\mu_i(P_{i-1}) + m_i \nu_i(P_{i-1})] \tag{61}$$

Also, we have

$$\begin{aligned}\left\|\frac{\partial^2 V_{i+1}}{\partial P_i^2}\right\| &\leqslant \mu_i^2(1 + V_i^{a_i})^2 + (2V_{i+1})^{1/2}\lambda_i(1 + V_i^{a_i}) + V_i^{m_i - 2}(1 + V_i)[\xi_i V_i + (m_i - 1) \\ &\quad \times \nu_i^2(1 + V_i)] \\ &\leqslant \mu_i^2[1 + (m_i V_{i+1})^{a_i/m_i}]^2 + (2V_{i+1})^{1/2}\lambda_i[1 + (m_i V_{i+1})^{a_i/m_i}] \\ &\quad + (m_i V_{i+1})^{(m_i - 2)/m_i}[1 + (m_i V_{i+1})^{1/m_i}]^2[\xi_i + (m_i - 1)\nu_i^2] \\ &\leqslant \lambda_{i+1}(P_i)(1 + V_{i+1})\end{aligned} \tag{62}$$

with

$$\lambda_{i+1}(P_i) \stackrel{\text{def}}{=} 2\{2m_i\mu_i(P_{i-1})^2 + \sqrt{(2m_i)}\lambda_i(P_{-1}) + 2m_i[\xi_i(P_{i-1}) + (m_i - 1)\nu_i(P_{i-1})^2]\} \tag{63}$$

Since ν_{i+1} and λ_{i+1} are continuous functions, when P_i is in a compact set, (60) and (62) are exactly (15) of Assumption GC (14). Finally, note that (57), (60) and (62) are nothing but (51) for Step $i+2$.

Following this iterative procedure, we can design explicitly C_∞-functions V and u_n such that Assumptions LS (8), ALS (11) and GC (14) hold for the system (38). Unfortunately, this procedure is not systematic. At each step we have to check that the extra assumption (52) holds. However, as observed for (49), this extra assumption can always be satisfied if the f_i and their successive derivatives have a polynomial growth.

We conclude that Assumptions LP (1), LS (8), ALS (11) and GC (14) are satisfied for linearly parametrized systems in a strict pure feedback form with polynomial growth nonlinearities and a parameter vector in a known convex compact set — a subclass of the family of systems considered by Kanellakopoulos *et al.*[8]

4. A THEORETICAL ADAPTIVE CONTROLLER

To design a controller guaranteeing at least Lagrange stability for the system (2) with p^* unknown, we follow the standard adaptive control procedure and propose the dynamic state feedback

$$\dot{\hat{p}} = v(x,\hat{p}), \qquad u = u_n(x,\hat{p}) \tag{64}$$

where v, the control of the extended system, is to be designed.

Let $L:\mathbb{R}_+ \to \mathbb{R}_+$ be some positive proper C^1-function with a strictly positive derivative denoted L'. This function L will be chosen later. For the time being we use it to define the function

$$S(x,\hat{p}) \stackrel{\text{def}}{=} L(V(x,\hat{p})) \tag{65}$$

The time derivative of this function along the solutions of (2), (64) is

$$\dot{S} = L'(V(x,\hat{p}))\left(\frac{\partial V}{\partial x}(x,\hat{p})(a(x,u_n(x,\hat{p})) + A(x,u_n(x,\hat{p}))p^*) + \frac{\partial V}{\partial p}(x,\hat{p})v(x,\hat{p})\right) \tag{66}$$

Then, using (9) in Assumption LS (8) and our dynamic controller (64), we get the differential inequalities system

$$\dot{S} \leqslant -L'W - L'\frac{\partial V}{\partial x}A(\hat{p}-p^*) + L'\frac{\partial V}{\partial p}v, \qquad \dot{\widehat{(\hat{p}-p^*)}} = v \tag{67}$$

We remark that if V were independent of p, i.e. $\partial V/\partial p$ were identically zero, we would have, by using the positivity of L' and W,

$$\dot{S} \leqslant -L'\frac{\partial V}{\partial x}A(\hat{p}-p^*), \qquad \dot{\widehat{(\hat{p}-p^*)}} = v \tag{68}$$

In such a case it would be very easy to find v. Indeed, by applying the very standard Lyapunov design (see e.g. Reference 18), we would choose a control Lyapunov function

$$U = S(x,\hat{p}) + \frac{1}{2\alpha}\|\hat{p}-p^*\|^2 \tag{69}$$

with α some strictly positive real number. Its time derivative would satisfy

$$\dot{U} \leqslant \left(\frac{1}{\alpha}v^{\mathrm{T}} - L'\frac{\partial V}{\partial x}A\right)(\hat{p}-p^*) \tag{70}$$

Thus the control v could be determined so that this time derivative be negative, i.e.

$$v = \alpha\left(L' \frac{\partial V}{\partial x} A\right)^{\mathrm{T}} \tag{71}$$

This would imply that S and $\hat{p}$ remain bounded. Then, in view of point 1 of Assumption LS (8), Lagrange stability would hold.

Thus our idea is to transform S in such a way that a system similar to (68) is obtained. For this we notice that the bad term $L'(\partial V/\partial p)\, v$ in (67) is part of the time derivative of $L'(\partial V/\partial p)(\hat{p} - p^*)$, i.e.

$$\overbrace{L' \frac{\partial V}{\partial p}(\hat{p} - p^*)}^{\cdot} - L' \frac{\partial V}{\partial p} v = \overbrace{L' \frac{\partial V}{\partial p}}^{\cdot}(\hat{p} - p^*) \tag{72}$$

It follows that by replacing S in (69) by $S - L'(V)(\partial V/\partial p)(\hat{p} - p^*)$, we may readily apply the Lyapunov design described above. Indeed, with this substitution U is defined now by

$$U = L(V) - L'(V)\frac{\partial V}{\partial p}(\hat{p} - p^*) + \frac{1}{2\alpha}\|\hat{p} - p^*\|^2 \tag{73}$$

$$= L(V) - \frac{\alpha}{2}\left\| L'(V)\frac{\partial V}{\partial p}\right\|^2 + \frac{1}{2\alpha}\left\|\hat{p} - p^* - \alpha L'(V)\left(\frac{\partial V}{\partial p}\right)\right\|^2 \tag{74}$$

Its time derivative satisfies

$$\dot{U} \leqslant \left(\frac{1}{\alpha} v^{\mathrm{T}} - L' \frac{\partial V}{\partial x} A - \overbrace{L' \frac{\partial V}{\partial p}}^{\cdot}\right)(\hat{p} - p^*) \tag{75}$$

It is made negative by choosing the control

$$v = \alpha\left(L' \frac{\partial V}{\partial x} A + \overbrace{L' \frac{\partial V}{\partial p}}^{\cdot}\right)^{\mathrm{T}} \tag{76}$$

Two questions arise about the definitions (76) and (74) we have obtained.

1. Is the equation for $\hat{p}$ realizable — a time derivative is involved on the right-hand side of (76)?
2. Is U in (74) a non-negative and proper function of the state vector $(x, \hat{p})$ of the closed-loop system?

The answer to the first question is simple. A state space realization of (64) with (76) is

$$\dot{q} = \alpha L'(V(x,\hat{p}))A(x, u_{\mathrm{n}}(x,\hat{p}))^{\mathrm{T}} \frac{\partial V}{\partial x}(x,\hat{p})^{\mathrm{T}}, \qquad \hat{p} = q + \alpha L'(V(x,\ \hat{p}))\frac{\partial V}{\partial p}(x,\hat{p})^{\mathrm{T}} \tag{77}$$

Unfortunately, a difficulty remains since the last equation is implicit in $\hat{p}$. We shall address this point after the following answer to the second question.

Since the function L is to be designed as a positive proper function and V is positive and satisfies point 1 of Assumption LS (8), U is a positive proper function of $(x, \hat{p})$ if there exists a strictly positive real number $\varepsilon < 1$ such that

$$\frac{\alpha}{2}\left\| L'(V)\frac{\partial V}{\partial p}\right\|^2 \leqslant (1 - \varepsilon)L(V) \tag{78}$$

However, from Assumption GC (14) this inequality is satisfied if $\hat{p}$ is in the set Π and

$$\frac{\alpha}{2} L'(V)^2\gamma^2(1 + V)^2 \leqslant (1 - \varepsilon)L(V) \tag{79}$$

Therefore it is sufficient to choose

$$L(V) = 1 + \log(1 + V), \qquad \alpha < 2/\gamma^2 \tag{80}$$

Now, coming back to the problem of implicit definition of $\hat{p}$, we note that, not only should a solution exist, but also, for (79) to hold with L defined by (80), this solution $\hat{p}$ should be in Π. The second equation of (77) is

$$\hat{p} = \hat{q} + \alpha \frac{(\partial V/\partial p)(x, \hat{p})^{\mathrm{T}}}{1 + V(x, \hat{p})} \tag{81}$$

It follows from an elementary fixed point argument (see Lemma 1 in Appendix) that Assumption GC (14) and

$$\alpha < \min\{\delta/\gamma, 1/2\gamma^2\} \tag{82}$$

imply the existence of a C^1-function $\rho : \mathbb{R}^n \times \Pi_2 \to \Pi$ such that, for all $(x, \hat{q})$ in $\mathbb{R}^n \times \Pi_2$,

$$\rho(x, \hat{q}) = \hat{q} + \alpha \frac{(\partial V/\partial p)(x, \rho(x, \hat{q}))^{\mathrm{T}}}{1 + V(x, \rho(x, \hat{q}))} \tag{83}$$

To use this function ρ in (77), it remains to guarantee that $\hat{q}$ is in Π_2. This is achieved by using the standard projection trick.

As a result we have designed the adaptive controller

$$\dot{\hat{q}} = \alpha \operatorname{Proj}\left(\hat{q}, \frac{A(x, u_n(x, \rho(x, \hat{q})))^{\mathrm{T}}(\partial V/\partial x)(x, \rho(x, \hat{q}))^{\mathrm{T}}}{1 + V(x, \rho(x, \hat{q}))}, 0, I\right), \qquad u = u_n(x, \rho(x, \hat{q})) \tag{84}$$

where the C^1-function $\rho : \mathbb{R}^n \times \Pi_2 \to \mathbb{R}^l$ is defined by (83) and, with $\mathcal{M}$ the set of symmetric positive definite $l \times l$ matrices, the locally Lipschitz continuous function $\operatorname{Proj} : \Pi \times \mathbb{R}^l \times \mathbb{R} \times \mathcal{M} \to \mathbb{R}^l$ is defined by (see Reference 4, Lemma (103))

$$\operatorname{Proj}(q, y, \lambda, M) = \begin{cases} y & \text{if } \mathscr{P}(q) \leqslant \lambda \text{ or } \dfrac{\partial \mathscr{P}}{\partial p}(q) y > 0 \\ y - \dfrac{(\mathscr{P}(q) - \lambda)(\partial \mathscr{P}/\partial p)(q) y}{(\partial \mathscr{P}/\partial p)(q) M (\partial \mathscr{P}/\partial p)(q)^{\mathrm{T}}} M \dfrac{\partial \mathscr{P}}{\partial p}(q)^{\mathrm{T}} & \text{if } \mathscr{P}(q) > \lambda \text{ and } \dfrac{\partial \mathscr{P}}{\partial p}(q) y > 0 \end{cases} \tag{85}$$

and satisfies

$$(q - p)^{\mathrm{T}} M^{-1} \operatorname{Proj}(q, y, \lambda, M) \leqslant (q - p)^{\mathrm{T}} M^{-1} y, \quad \forall (q, p, y) \in \mathbb{R}^l \times \Pi_\lambda \times \mathbb{R}^l \tag{86}$$

Moreover, an appropriate Lyapunov function to study the dynamics of the closed-loop system should be

$$U(x, \hat{q}) = 1 + \log(1 + V(x, \rho(x, \hat{q}))) - \frac{\alpha}{2} \left\| \frac{(\partial V/\partial p)(x, \rho(x, \hat{q}))}{1 + V(x, \rho(x, \hat{q}))} \right\|^2 + \frac{1}{2\alpha} \| \hat{q} - p^* \|^2 \tag{87}$$

The arguments used up to now are given only to motivate for the controller (84) and the Lyapunov function (87). By no means are they rigorous enough to substantiate any result. Nevertheless, we shall prove:

Proposition 1

Let Assumptions LP (1), ICS (3), LS (8) and GC (14) hold and α be chosen such that

$$0 < \alpha < \min\{\delta/\gamma, 1/2\gamma^2\} \tag{88}$$

Under these conditions, for any initial condition $(x(0), \hat{q}(0))$ in $\mathbb{R}^n \times \Pi_0$ there exists a unique solution $(x(t), \hat{q}(t))$ of (2), (84). This solution is bounded on $[0, \infty)$ and, if Assumption ALS (11) holds, it satisfies

$$\lim_{t\to\infty} x(t) = \mathcal{E}^* \tag{89}$$

Proof. The closed-loop system we consider is

$$\begin{aligned} \dot{x} &= a(x, u_n(x, \rho(x, \hat{q}))) + A(x, u_n(x, \rho(x, \hat{q})))p^* \\ \dot{\hat{q}} &= \alpha \operatorname{Proj}\left(\hat{q}, \frac{A(x, u_n(x, \rho(x, \hat{q})))^{\mathrm{T}} (\partial V/\partial x)(x, \rho(x, \hat{q}))^{\mathrm{T}}}{1 + V(x, \rho(x, \hat{q}))}, 0, I\right) \end{aligned} \tag{90}$$

From our smoothness assumptions on the functions a, A, u_n and V, Lemma 1 and Reference 4, Lemma (103) this system has a locally Lipschitz continuous right-hand side in the open set $\mathbb{R}^n \times \mathring{\Pi}_2$. It follows that for any initial condition $(x(0), \hat{q}(0))$ in this open set and therefore in particular in $\mathbb{R}^n \times \Pi_0$ there exists a unique solution $(x(t), \hat{q}(t))$ defined on a right maximal interval $[0, T)$, with T maybe infinite. Moreover, from Reference 4, Lemma (103), point 5 we know that $\hat{q}(t) \in \Pi_1 \subset \mathring{\Pi}_2$ for all t in $[0, T)$.

Then we compute the time derivative of $U(x(t), \hat{q}(t))$ defined in (87). With Assumption LS (8) and (83) we get

$$\dot{U} = \frac{1}{1+V}\left(\frac{\partial V}{\partial x}(a + Ap^*) + \frac{\partial V}{\partial p}\dot{\rho}\right) - \alpha \frac{\partial V/\partial p}{1+V}\,\widehat{\frac{(\partial V/\partial p)^{\mathrm{T}}}{1+V}}^{\cdot} + \frac{1}{\alpha}(\hat{q} - p^*)^{\mathrm{T}}\dot{\hat{q}} \tag{91}$$

$$\leqslant -\frac{W}{1+V} + \left(\hat{q} + \alpha\frac{(\partial V/\partial p)^{\mathrm{T}}}{1+V} - p^*\right)^{\mathrm{T}}\left(\frac{\dot{\hat{q}}}{\alpha} - \frac{A^{\mathrm{T}}(\partial V/\partial x)^{\mathrm{T}}}{1+V}\right) \tag{92}$$

However, since $\hat{q}(t)$ is in $\Pi_1 \subset \Pi_2$, ρ is in Π. With ρ in Π, Assumption GC (14) and (88) imply

$$\left\| \alpha \frac{(\partial V/\partial p)(x, \rho)^{\mathrm{T}}}{1 + V(x, \rho)} \right\| \leqslant \alpha\gamma < \delta \tag{93}$$

Since, from Assumption ICS (3), p^* is in Π^* and δ is defined by (6), this inequality implies that $p^* - \alpha(\partial V/\partial p)^{\mathrm{T}}/(1+V)$ is in Π_0. Therefore with the expression of $\dot{\hat{q}}$ and (86) we get finally

$$\dot{U} \leqslant -\frac{W(x(t), \rho(x(t), \hat{q}(t)))}{1 + V(x(t), \rho(x(t), \hat{q}(t)))} \tag{94}$$

This implies that $U(x(t), \hat{q}(t))$ is a non-increasing time function. Moreover, we have with Assumption GC (14)

$$U \geqslant \left(1 - \frac{\alpha\gamma^2}{2}\right)[1 + \log(1 + V)] + \frac{\alpha}{2}\left(\gamma^2 - \left\|\frac{\partial V/\partial p}{1+V}\right\|^2\right) + \frac{1}{2\alpha}\|\hat{q} - p^*\|^2 \tag{95}$$

$$\geqslant \left(1 - \frac{\alpha\gamma^2}{2}\right)[1 + \log(1 + V)] + \frac{1}{2\alpha}\|\hat{q} - p^*\|^2 \geqslant 0 \tag{96}$$

It follows from (88), Lemma 1, Assumption GC (14) and point 1 of Assumption LS (8) that, for $\hat{q}$ in Π_2, U is a positive function proper in x and $\hat{q}$. Therefore by contradiction T is infinite and the solution $(x(t), \hat{q}(t))$ is bounded on $[0, \infty)$.

Finally, it follows from Reference 15, Lemma 5.2.81 and (94) that any solution of the autonomous system (90) converges to the largest invariant set of points $(x, \hat{q})$ satisfying

$W(x, \rho(x, \hat{q})) = 0$. However, any bounded solution in this invariant set is also a solution of (12), (13) in Assumption ALS (11) with

$$\hat{p}(t) = \rho(x(t), \hat{q}(t)) \tag{97}$$

The conclusion follows readily. □

Unfortunately, even though from a theoretical point of view the adaptive controller (84) may be satisfactory, it is not yet a practical solution. Indeed, its implementation requires an explicit expression for the function ρ. It is clear from our examples of Section 3 that an analytical expression for this function is inaccessible in general.

5. A MORE PRACTICAL ADAPTIVE CONTROLLER

An idea to obtain a more practical controller is to replace the fixed point equation (81) by a dynamical system whose equilibrium point, given x and $\hat{q}$, is this fixed point, i.e. we consider the dynamic state feedback

$$\dot{\hat{q}} = v_q(x, \hat{p}, \hat{q}), \qquad \dot{\hat{p}} = v_p(x, \hat{p}, \hat{q}), \qquad u = u_n(x, \hat{p}) \tag{98}$$

To obtain the new controls v_p and v_q, we will again apply a Lyapunov design inspired by Artstein's theorem.[19,20] The closed-loop system (2), (98) can be rewritten in a more compact form as

$$\dot{X} = F(X, p^*, \hat{p}), \qquad \dot{\hat{p}} = v_p \tag{99}$$

with $X = (x, \hat{q})$. We know from Section 4 that the implication

$$\hat{p} = \rho(x, \hat{q}) = \rho(X) \quad \Rightarrow \quad v_q = \alpha \operatorname{Proj}\left(\hat{q}, \frac{A^{\mathrm{T}}(\partial V/\partial x)^{\mathrm{T}}}{1+V}, 0, I\right) \tag{100}$$

guarantees that the time derivative of U, defined in (87), along the solutions of

$$\dot{X} = F(X, p^*, \rho(X)) \tag{101}$$

is negative, i.e.

$$\frac{\partial U}{\partial X}(X) F(X, p^*, \rho(X)) \leqslant 0 \tag{102}$$

On the other hand, Artstein's theorem states that if a positive real function $\mathcal{U}(X, \hat{p})$ satisfies

$$\frac{\partial \mathcal{U}}{\partial \hat{p}}(X, \hat{p}) = 0 \quad \Rightarrow \quad \frac{\partial \mathcal{U}}{\partial X}(X, \hat{p}) F(X, p^*, \hat{p}) \leqslant 0 \tag{103}$$

then it is possible to find a control v_p such that the time derivative of $\mathcal{U}$ along the solutions of (99) is negative. However, clearly with (102), a function $\mathcal{U}$ satisfying

$$\frac{\partial \mathcal{U}}{\partial \hat{p}}(X, \hat{p}) = 0 \quad \Rightarrow \quad \hat{p} = \rho(X), \qquad \mathcal{U}(X, \rho(X)) = U(X) \tag{104}$$

satisfies (103) necessarily. Now we observe from (87) and the fixed point equation (81) that, with a strictly positive real number β to be chosen later sufficiently large, $\mathcal{U}$ defined by

$$\begin{aligned} \mathcal{U}(X, \hat{p}) &= \mathcal{U}(x, \hat{q}, \hat{p}) \\ &= [1 + \log(1 + V(x, \hat{p}))] - \frac{1}{2\alpha}\| \hat{p} - \hat{q} \|^2 + \frac{1}{2\alpha}\| \hat{q} - p^* \|^2 \\ &\quad \times \frac{\beta}{2}\left\| \hat{p} - \hat{q} - \alpha \frac{(\partial V/\partial p)(x, \hat{p})^{\mathrm{T}}}{1 + V(x, \hat{p})} \right\|^2 \end{aligned} \tag{105}$$

satisfies (104) (use (113) and (112) below) and therefore Artstein's theorem applies.

To obtain explicit expressions for the controls v_p and v_q, we compute the time derivative of $\mathscr{U}$. For this we note that

$$\frac{\dot{\overbrace{(\partial V/\partial p)^{\mathrm{T}}}}}{1+V} = b + Bp^* + H\dot{\hat{p}} \tag{106}$$

where the l-vector b and the $l\times l$ matrices B and H are defined by

$$b(x,\hat{p}) = \frac{\partial}{\partial x}\left(\frac{(\partial V/\partial p)(x,\hat{p})^{\mathrm{T}}}{1+V(x,\hat{p})}\right) a(x, u_{\mathrm{n}}(x,\hat{p}))$$

$$B(x,\hat{p}) = \frac{\partial}{\partial x}\left(\frac{(\partial V/\partial p)(x,\hat{p})^{\mathrm{T}}}{1+V(x,\hat{p})}\right) A(x, u_{\mathrm{n}}(x,\hat{p})) \tag{107}$$

$$H(x,\hat{p}) = \frac{\partial}{\partial p}\left(\frac{(\partial V/\partial p)(x,\hat{p})^{\mathrm{T}}}{1+V(x,\hat{p})}\right)$$

Now, by using point 2 of Assumptions LS (8), we get

$$\dot{\mathscr{U}} \leqslant -\frac{W}{1+V} + \frac{(\partial V/\partial x)A}{1+V}(p^*-\hat{p}) + \frac{\partial V/\partial p}{1+V} v_p - \frac{1}{\alpha}(\hat{p}-\hat{q})^{\mathrm{T}}(v_p - v_q) + \frac{1}{\alpha}(\hat{q}-p^*)^{\mathrm{T}} v_q$$
$$+ \beta\left[\hat{p}-\hat{q}-\alpha\left(\frac{\partial V/\partial p}{1+V}\right)^{\mathrm{T}}\right]^{\mathrm{T}} [v_p - v_q - \alpha(b + Bp^* + Hv_p)] \tag{108}$$

or

$$\dot{\mathscr{U}} \leqslant -\frac{W}{1+V} - \sigma\left\|\hat{p}-\hat{q}-\alpha\left(\frac{\partial V/\partial p}{1+V}\right)^{\mathrm{T}}\right\|^2 + \left[p^*-\hat{q}-\alpha\left(\frac{\partial V/\partial p}{1+V}\right)^{\mathrm{T}}\right]^{\mathrm{T}}\left(Z_q - \frac{1}{\alpha}v_q\right)$$
$$-\left[\hat{p}-\hat{q}-\alpha\left(\frac{\partial V/\partial p}{1+V}\right)^{\mathrm{T}}\right]^{\mathrm{T}}\left(\beta(I-\alpha H) - \frac{1}{\alpha}I\right)(Z_p - v_p) \tag{109}$$

where σ is an arbitrarily chosen strictly positive real number and $Z_q(x,\hat{q},\hat{p})$ and $Z_p(x,\hat{q},\hat{p},v_q)$ are defined by

$$Z_q(x,\hat{q},\hat{p})^{\mathrm{T}} = \frac{(\partial V/\partial x)(x,\hat{p})}{1+V(x,\hat{p})} A(x,u_{\mathrm{n}}(x,\hat{p})) - \beta\alpha\left(\hat{p}^{\mathrm{T}} - \hat{q}^{\mathrm{T}} - \alpha\frac{(\partial V/\partial p)(x,\hat{p})}{1+V(x,\hat{p})}\right) B(x,\hat{p}) \tag{110}$$

$$Z_p(x,\hat{q},\hat{p},v_q) = \left(\frac{1}{\alpha}I - \beta(I-\alpha H)\right)^{-1}\left\{\left(\frac{1}{\alpha}-\beta\right)v_q - \beta\alpha B(x,\hat{p})\left(\hat{q} + \alpha\frac{(\partial V/\partial p)(x,\hat{p})^{\mathrm{T}}}{1+V(x,\hat{p})}\right)\right.$$
$$\left. - \beta\alpha b(x,\hat{p}) + \sigma\left[\hat{p}-\hat{q}-\alpha\left(\frac{\partial V/\partial p}{1+V}\right)^{\mathrm{T}}\right] - \left(\frac{(\partial V/\partial x)(x,\hat{p})}{1+V(x,\hat{p})}A(x,u_{\mathrm{n}}(x,\hat{p}))\right)^{\mathrm{T}}\right\} \tag{111}$$

where the matrix $(1/\alpha)I - \beta(I - \alpha H(x,p))$ is symmetric negative definite for all (x,p) in $\mathbb{R}^n \times \Pi$ if

$$\frac{1}{\alpha} - \beta + 2\beta\alpha\gamma^2 \leqslant 0 \tag{112}$$

This follows from Assumption GC (14) and (see (165))

$$\sup_{(x,p)\in\mathbb{R}^n\times\Pi} \|H(x,p)\| \leqslant 2\gamma^2 \tag{113}$$

Now, in the proof of Proposition 1 (see after (93)) we noticed that $p^* - \alpha(\partial V/\partial p)^{\mathrm{T}}/(1+V)$ is in Π_0 if $\alpha\gamma < \delta$. However, similarly, if $\hat{q}$ is in Π_1, $\hat{q} + \alpha(\partial V/\partial p)^{\mathrm{T}}/(1+V)$ is in Π_2. With (86) this implies

$$\left[p^* - \hat{q} - \alpha\left(\frac{\partial V/\partial p}{1+V}\right)^{\mathrm{T}}\right]^{\mathrm{T}} Z_q \leqslant \left[p^* - \hat{q} - \alpha\left(\frac{\partial V/\partial p}{1+V}\right)^{\mathrm{T}}\right]^{\mathrm{T}} \operatorname{Proj}(\hat{q}, Z_q, 0, I)$$

$$- \left[\hat{p} - \hat{q} - \alpha\left(\frac{\partial V/\partial p}{1+V}\right)^{\mathrm{T}}\right]^{\mathrm{T}}\left(\beta(I - \alpha H) - \frac{1}{\alpha} I\right) Z_p \leqslant - \left[\hat{p} - \hat{q} - \alpha\left(\frac{\partial V/\partial p}{1+V}\right)^{\mathrm{T}}\right]^{\mathrm{T}}\left(\beta(I - \alpha H) - \frac{1}{\alpha} I\right)$$

$$\times \operatorname{Proj}\left(\hat{p}, Z_p, 2, \left(\beta(I - \alpha H) - \frac{1}{\alpha} I\right)^{-1}\right) \qquad (114)$$

This yields

$$\dot{\mathscr{U}} \leqslant -\frac{W}{1+V} - \sigma \left\| \hat{p} - \hat{q} - \alpha\left(\frac{\partial V/\partial p}{1+V}\right)^{\mathrm{T}} \right\|^2$$
$$+ \left[p^* - \hat{q} - \alpha\left(\frac{\partial V/\partial p}{1+V}\right)^{\mathrm{T}}\right]^{\mathrm{T}} \times \left(\operatorname{Proj}(\hat{q}, Z_q, 0, I) - \frac{1}{\alpha} v_q\right)$$
$$- \left[\hat{p} - \hat{q} - \alpha\left(\frac{\partial V/\partial p}{1+V}\right)^{\mathrm{T}}\right]^{\mathrm{T}}$$
$$\times \left(\beta(I - \alpha H) - \frac{1}{\alpha}\right)\left(\operatorname{Proj}\left(\hat{p}, Z_p, 2, \left(\beta(I - \alpha H) - \frac{1}{\alpha}\right)^{-1}\right) - v_p\right) \qquad (115)$$

It follows that, by choosing

$$\begin{aligned} v_q(x, \hat{q}, \hat{p}) &= \alpha \operatorname{Proj}(\hat{q}, Z_q(x, \hat{q}, \hat{p}), 0, I) \\ v_p(x, \hat{q}, \hat{p}, v_q) &= \operatorname{Proj}(\hat{p}, Z_p(x, \hat{q}, \hat{p}, v_q), 2, [\beta(I - \alpha H(x, \hat{p})) - I/\alpha]^{-1}) \end{aligned} \qquad (116)$$

we get

$$\dot{\mathscr{U}} \leqslant -\frac{W}{1+V} - \sigma \left\| \hat{p} - \hat{q} - \alpha\left(\frac{\partial V/\partial p}{1+V}\right)^{\mathrm{T}} \right\|^2 \qquad (117)$$

As a result we have designed the adaptive controller

$$\begin{aligned} \dot{\hat{q}} &= \alpha \operatorname{Proj}(\hat{q}, Z_q(x, \hat{q}, \hat{p}), 0, I) \\ \dot{\hat{p}} &= \operatorname{Proj}(\hat{p}, Z_p(x, \hat{q}, \hat{p}, \dot{\hat{p}}), 2, [\beta(I - \alpha H(x, \hat{p})) - I/\alpha]^{-1}) \\ u &= u_{\mathrm{n}}(x, \hat{p}) \end{aligned} \qquad (118)$$

where the functions Z_p and Z_q are defined in (110) and (111). Note that (100) is satisfied. We have:

Proposition 2

Let Assumptions LP (1), ICS (3), LS (8) and GC (14) hold and α and β be chosen such that

$$0 < \alpha < \min\{\delta/\gamma, 1/2\gamma^2\}, \qquad \frac{1}{\alpha(1 - 2\alpha\gamma^2) < \beta} \qquad (119)$$

Under these conditions, for any initial condition $(x(0), \hat{q}(0), \hat{p}(0))$ in $\mathbb{R}^n \times \Pi_0 \times \Pi_1$ there exists a unique solution $(x(t), \hat{q}(t), \hat{p}(t))$ of (2), (118). This solution is bounded on $[0, \infty)$ and satisfies

$$\lim_{t\to\infty} \left\| \hat{p}(t) - \hat{q}(t) - \alpha\left(\frac{(\partial V/\partial p)(x(t), \hat{p}(t))}{1 + V(x(t), \hat{p}(t))}\right)^{\mathrm{T}} \right\| = 0 \tag{120}$$

Moreover, if Assumption ALS (11) holds, we also have

$$\lim_{t\to\infty} x(t) = \mathscr{E}^* \tag{121}$$

Proof. The closed-loop system we consider is:

$$\begin{aligned} \dot{x} &= a(x, u_{\mathrm{n}}(x, \hat{p})) + A(x, u_{\mathrm{n}}(x, \hat{p}))p^* \\ \dot{\hat{q}} &= \alpha \operatorname{Proj}(\hat{q}, Z_q(x, \hat{q}, \hat{p}), 0, I) \\ \dot{\hat{p}} &= \operatorname{Proj}(\hat{p}, Z_p(x, \hat{q}, \hat{p}, \dot{\hat{q}}), 2, [\beta(I - \alpha H(x, \hat{p})) - I/\alpha]^{-1}) \end{aligned} \tag{122}$$

with the definitions (107), (110) and (111). From our smoothness assumptions on the functions a, A, u_{n} and V and Reference 4, Lemma (103) this system has a locally Lipschitz continuous right-hand side in the open set $\mathbb{R}^n \times \mathring{\Pi}_2 \times \mathring{\Pi}$. It follows that for any initial condition $(x(0), \hat{q}(0), \hat{p}(0))$ in this open set and therefore in particular in $\mathbb{R}^n \times \Pi_0 \times \Pi_1$ there exists a unique solution $(x(t), \hat{q}(t), \hat{p}(t))$ defined on a right maximal interval $[0, T)$, with T maybe infinite. Moreover, from Reference 4, Lemma (103), point 5 we know that $\hat{q}(t) \in \Pi_1 \subset \mathring{\Pi}_2$ and $\hat{p}(t) \in \Pi_3 \subset \mathring{\Pi}$ for all t in $[0, T)$.

Then from our design we know that the time derivative of $\mathscr{U}(x(t), \hat{q}(t), \hat{p}(t))$ defined in (105) satisfies (see (117))

$$\dot{\mathscr{U}} \leqslant -\frac{W(x(t), \hat{p}(t))}{1 + V(x(t), \hat{p}(t))} - \sigma \left\| \hat{p}(t) - \hat{q}(t) - \alpha\left(\frac{(\partial V/\partial p)(x(t), \hat{p}(t))}{1 + V(x(t), \hat{p}(t))}\right)^{\mathrm{T}} \right\|^2 \tag{123}$$

This implies that $\mathscr{U}(x(t), \hat{q}(t), \hat{p}(t))$ is a non-increasing time function. On the other hand, we remark that for all $\varepsilon \in (0, \beta - 1/\alpha)$ the inequality

$$-\frac{1}{2\alpha}\|\hat{p} - \hat{q}\|^2 + \frac{\beta}{2}\left\|\hat{p} - \hat{q} - \alpha\frac{(\partial V/\partial p)^{\mathrm{T}}}{1 + V}\right\|^2 \geqslant -\frac{(1 + \alpha\varepsilon)\beta\alpha^2}{2[\alpha\beta - (1 - \alpha\varepsilon)]}\left\|\frac{(\partial V/\partial p)^{\mathrm{T}}}{1 + V}\right\|^2 + \frac{\varepsilon}{2}\|\hat{p} - \hat{q}\|^2 \tag{124}$$

yields with Assumption GC (14) (as in (95))

$$\mathscr{U} \geqslant \left(1 - \frac{(1 + \alpha\varepsilon)\beta\alpha^2\gamma^2}{2[\alpha\beta - (1 + \alpha\varepsilon)]}\right)[1 + \log(1 + V)] + \frac{1}{2\alpha}\|\hat{q} - p^*\|^2 + \frac{\varepsilon}{2}\|\hat{p} - \hat{q}\| \tag{125}$$

Therefore with (119), point 1 of Assumption LS (8) and $\hat{p} \in \Pi$, $\mathscr{U}(x, \hat{q}, \hat{p})$ is a positive proper function in $(x, \hat{q}, \hat{p})$. Therefore by contradiction T is infinite and the solution $(x(t), \hat{q}(t), \hat{p}(t))$ is bounded on $[0, \infty)$. The conclusion follows from (123) as in Proposition 1. □

From (123) and LaSalle's theorem we know that all the solutions converge to the largest invariant set contained in

$$\mathscr{I} = \left\{(x, p, q) \,\middle|\, W(x, p) = 0,\ p = q + \alpha\frac{(\partial V/\partial x)(x, p)}{1 + V(x, p)}\right\} \tag{126}$$

In particular, in this set we have

$$\dot{\hat{p}} = \dot{\hat{q}} + \alpha \overbrace{\frac{(\partial V/\partial x)(x,\hat{p})^{\mathrm{T}}}{1+V(x,\hat{p})}}^{\cdot} \tag{127}$$

or, with (106),

$$(I - \alpha H)\dot{\hat{p}} = \dot{\hat{q}} + \alpha(b + Bp^*) \tag{128}$$

On the other hand, if $\hat{q} \in \Pi_0$, then $\hat{p} = \hat{q} + \alpha(\partial V/\partial x)(1+V) \in \Pi_1$ and (110), (111), (118) and (134) give

$$\left(\beta(I - \alpha H(x,\hat{p})) - \frac{1}{\alpha} I\right)\dot{\hat{p}} = \beta\dot{\hat{q}} + \beta\alpha(b + B\hat{p}) \tag{129}$$

We conclude that, asymptotically, for each time t such that $\hat{q}(t) \in \Pi_0$ we have

$$\dot{\hat{p}}(t) = -\beta\alpha^2 B(x(t))(\hat{p}(t) - p^*) \tag{130}$$

It follows that if B is a stable matrix — in an appropriate sense — then $\hat{p}(t)$ converges to p^*.

Note also that the fixed point equation is satisfied asymptotically, since with (83) and (167) we have

$$(1 - 2\alpha\gamma^2)\,\|\hat{p} - \rho(x,\hat{q})\| \leqslant \left\|\hat{p} - \hat{q} - \alpha\left(\frac{(\partial V/\partial p)(x,\hat{p})}{1+V(x,\hat{p})}\right)^{\mathrm{T}}\right\| \tag{131}$$

6. EXTENSIONS

6.1. About point 1 of Assumption LS

In order to facilitate the satisfaction of the growth condition (14), it may be useful to relax the constraint of V being proper partially in x (see point 1 in Assumption LS (8)). For our Propositions 1 and 2 it is sufficient that the following so-called boundedness observability condition holds.[4]

Assumption BO (boundedness observability)

For all positive real numbers v, all compact subsets Π_c of Π and all vectors $x_0 \in \mathbb{R}^n$ there exists a compact set Γ such that for any C^1-function $\hat{p}: \mathbb{R}_+ \to \Pi_c$ and for the corresponding solution $x(t)$ of

$$\dot{x} = a(x, u_n(x,\hat{p}(t))) + A(x, u_n(x,\hat{p}(t)))p^*, \qquad x(0) = z_0 \tag{133}$$

defined on $[0, T)$ we have the implication

$$V(x(t), p^*) \leqslant v, \quad \forall t \in [0, T) \quad \Rightarrow \quad x(t) \in \Gamma, \quad \forall t \in [0, T) \tag{134}$$

Indeed, our adaptive controllers guarantee for each solution the existence of a positive constant l and a compact set Π_c, depending on its initial condition, such that for all t where it exists we have

$$\log(1 + V(x(t), \hat{p}(t))) \leqslant l, \quad \hat{p}(t) \in \Pi_c \tag{135}$$

However, with Assumption GC (14) and convexity of Π we get for all $(x, p^*, p) \in \mathbb{R}^n \times \Pi \times \Pi$

$$\log(1 + V(x, p^*)) \leqslant \log(1 + V(x,p)) + \gamma\,\|p - p^*\| \tag{136}$$

It follows that the right-hand side of (134) is satisfied by any solution of the closed-loop system.

What Assumption BO (132) actually means is that V is the output function of a minimum phase system; namely, if we can find a feedback to keep this output bounded, then necessarily this feedback guarantees boundedness of the full state vector.

6.2. *About the set* Π^*

One of the reasons for the complexity of the adaptive controller (118) is the involvement of the function Proj. This function is introduced to take into account that the set Π^*, where the parameter vector is supposed to lie, may not be equal to the complete set $\mathbb{R}^l$. There are typically two motivations for Π^* to be a strict subset of $\mathbb{R}^l$. The first one is that there may not exist any functions V and u_n satisfying Assumption LS (8) and depending smoothly on p globally for p in $\mathbb{R}^l$. The second motivation is, as can be seen in our examples of Section 3, that Assumption GC (14) holds only in the form

$$\left\| \frac{\partial V}{\partial p}(x,p) \right\| \leqslant \gamma(p)(1+V(x,p)), \qquad \left\| \frac{\partial^2 V}{\partial p^2}(x,p) \right\| \leqslant \gamma(p)^2(1+V(x,p)) \tag{137}$$

with γ a continuous function of p. This leads us to restrict the set Π^* to be a compact subset of $\mathbb{R}^l$. Fortunately, if both V and u_n are defined globally for p in $\mathbb{R}^l$, this restriction can be overcome when, for all p in $\mathbb{R}^l$,

$$\gamma(p) \leqslant \gamma_1 \exp[\gamma_2 \surd(1+\|p-p_0\|^2)] + \gamma_3 \tag{138}$$

with γ_1, γ_2 and γ_3 some strictly positive real numbers and p_0 a vector in $\mathbb{R}^l$. This is satisfied for example by Γ_1 and Γ_2 in (35) and (36). Indeed, in this case we may replace the given function $V(x,p)$ by

$$\log(1+V(x,p)) + \exp[2\gamma_2 \surd(1+\|p-p_0\|^2)] \tag{139}$$

since this yields

$$\left\| \frac{\partial}{\partial p}\{\log(1+V(x,p)) + \exp[2\gamma_2\surd(1+\|p-p_0\|^2)]\} \right\|$$
$$\leqslant (\gamma_1 + 2\gamma_2 + \gamma_3\{1+\log(1+V(x,p)) + \exp[2\gamma_2\surd(1+\|p-p_0\|^2)]\} \tag{140}$$

and

$$\left\| \frac{\partial^2}{\partial p^2}\{\log(1+V(x,p)) + \exp[2\gamma_2\surd(1+\|p-p_0\|^2)]\} \right\| \leqslant 2(\gamma_1^2 + 2\gamma_2^2 + \gamma_3^2)$$
$$\times \{1+\log(1+V(x,p)) + \exp[2\gamma_2\surd(1+\|p-p_0\|^2)]\} \tag{141}$$

Therefore, by modifying V as indicated in (139), we are allowed to take $\Pi^* = \mathbb{R}^l$. In this case Assumption ICS (3) is not needed and Proposition 2 holds with the simpler adaptive controller

$$\dot{\hat{q}} = \alpha Z_q(x,\hat{q},\hat{p}), \qquad \dot{\hat{p}} = Z_p(x,\hat{q},\hat{p},\dot{\hat{q}}), \qquad u = u_n(x,\hat{p}) \tag{142}$$

with the functions Z_p and Z_q still defined by (110) and (111). That such a controller is sufficient can be readily seen from (109).

6.3. Our design as a step in a more complex design

In Reference 9 it has been emphasized that several kinds of Lyapunov designs can be mixed together to get designs of adaptive controllers for more complex cases. This remark applies here since the Lyapunov design we have proposed can be used in combination with others.

For example, let us consider the system

$$\dot{x}_1 = x_2 + p^* x_1 u, \qquad \dot{x}_2 = u \tag{143}$$

where the parameter p^* is unknown. We are interested in asymptotically stabilizing the setpoint $\mathscr{E}^* = (0, 0)$.

Clearly Assumption LP (1) is satisfied. Also, by choosing the control Lyapunov function

$$V(x_1, x_2, p) = \tfrac{1}{2} \log[1 + x_1^2 \exp(-2px_2)] + \frac{1}{2}\left(x_2 + \frac{x_1}{1 + p^2 + x_1^2}\right)^2 \tag{144}$$

and the control law

$$u_n(x_1, x_2, p) = -\frac{\dfrac{x_1 \exp(-2px_2)}{1 + x_1^2 \exp(-2px_2)} + \dfrac{x_2(1 + p^2 - x_1^2)}{(1 + p^2 + x_1^2)^2} + x_2 + \dfrac{x_1}{1 + p^2 + x_1^2}}{1 + \dfrac{px_1(1 + p^2 - x_1^2)}{(1 + p^2 + x_1^2)^2}} \tag{145}$$

Assumption LS (8) is met. The interest of the choice (144) is that Assumption GC (14) holds with $\Pi = \mathbb{R}^l$. Finally, Assumption ALS (11) holds also since (144) and (145) give (9) with

$$W(x_1, x_2, p) = \frac{1}{1 + p^2 + x_1^2} \frac{x_1^2 \exp(-2px_2)}{1 + x_1^2 \exp(-2px_2)} + \left(x_2 + \frac{x_1}{1 + p^2 + x_1^2}\right)^2 \tag{146}$$

According to Section 6.2, Proposition 2 applies with an adaptive controller of the following form given in (142):

$$\dot{\hat{q}} = v_q(x_1, x_2, \hat{q}, \hat{p}), \qquad \dot{\hat{p}} = v_p(x_1, x_2, \hat{q}, \hat{p}), \qquad u = u_n(x_1, x_2, \hat{p}) \tag{147}$$

In particular, with (117) we know that the time derivative of $\mathscr{U}(x_1, x_2, \hat{q}, \hat{p})$ defined in (105) satisfies

$$\dot{\mathscr{U}} \leqslant -\frac{W}{1 + V} - \sigma \left\| \hat{p} - \hat{q} - \alpha\left(\frac{\partial V/\partial p}{1 + V}\right)^{\mathrm{T}} \right\|^2 \tag{148}$$

$$\stackrel{\text{def}}{=} -\mathscr{W}(x_1, x_2, \hat{q}, \hat{p}) \tag{149}$$

As observed in Reference 9, what we have done for the system (143) can be used as a first step towards the design of a controller for the system

$$\dot{x}_1 = x_2 + p^* x_1 y, \qquad \dot{x}_2 = y, \qquad \dot{y} = u \tag{150}$$

where the parameter p^* is unknown and the objective is to asymptotically stabilize $\mathscr{E}^* = (0, 0, 0)$. Indeed, (150) is a particular case of a system in the form

$$\dot{x} = a(x, y) + A(x, y)p^*, \qquad \dot{y} = u + b(x, y) + B(x, y)p^* \tag{151}$$

with $x \in \mathbb{R}^n$, $y \in \mathbb{R}$ and $u \in \mathbb{R}$. For such a system, where an integrator is added to the system

$$\dot{x} = a(x, u) + A(x, u)p^* \tag{152}$$

for which we know how to solve the problem, we can apply the design proposed by Krstic *et al.*[10]

Let us rewrite for convenience the function $\mathcal{U}$ defined in (105) as

$$\mathcal{U}(x,\hat{q},\hat{p}) = \mathcal{U}_0(x,\hat{q},\hat{p}) + \frac{1}{2\alpha}\|\hat{q}-p^*\|^2 \tag{153}$$

Following Reference 10, we evaluate the time derivative of

$$\mathcal{U}_1(x,y,\hat{q},\hat{p}) = \mathcal{U}_0(x,\hat{q},\hat{p}) + \frac{1}{2\alpha}\|\hat{q}-p^*\|^2 + \tfrac{1}{2}(y-u_{\rm n}(x,\hat{p}))^2 \tag{154}$$

along the solutions of (151) (as long as they exist). By using (148) (or (117)) obtained with (147), we get

$$\begin{aligned}\dot{\mathcal{U}}_1 \leqslant\ -\ \mathcal{W} + \frac{\partial \mathcal{U}_0}{\partial \hat{p}}(\dot{\hat{p}}-v_p) + \frac{\partial \mathcal{U}_0}{\partial \hat{q}}(\dot{\hat{q}}-v_q) + \frac{1}{\alpha}(\hat{q}-p^*)^{\rm T}(\dot{\hat{q}}-v_q)\\ + (y-u_{\rm n})\Bigg(u + b + Bp^* - \frac{\partial u_{\rm n}}{\partial x}(a+Ap^*) - \frac{\partial u_{\rm n}}{\partial p}\dot{\hat{p}}\\ + \frac{\partial \mathcal{U}_0}{\partial x}\frac{a(x,y)+A(x,y)p^* - a(x,u_{\rm n}) - A(x,u_{\rm n})p^*}{y-u_{\rm n}}\Bigg)\end{aligned} \tag{155}$$

It follows that, by choosing

$$\begin{aligned}\dot{\hat{q}} &= v_q + \alpha\left(B - \frac{\partial u_{\rm n}}{\partial x}A\right)^{\rm T}(y-u_{\rm n}) + \alpha\left(\frac{\partial \mathcal{U}_0}{\partial x}(A(x,y)-A(x,u_{\rm n}))\right)^{\rm T}\\ \dot{\hat{p}} &= v_p\\ u &= -(y-u_{\rm n}) - b - B\hat{q} + \frac{\partial u_{\rm n}}{\partial x}(a+A\hat{q}) + \frac{\partial u_{\rm n}}{\partial \hat{p}}v_p - \frac{\partial \mathcal{U}_0}{\partial x}\\ &\quad\times\frac{a(x,y)+A(x,y)\hat{q}-a(x,u_{\rm n})-A(x,u_{\rm n})\hat{q}}{y-u_{\rm n}}\\ &\quad-\alpha\frac{\partial \mathcal{U}_0}{\partial \hat{q}}\left(B - \frac{\partial u_{\rm n}}{\partial x}A + \frac{(\partial \mathcal{U}_0/\partial x)(A(x,y-A(x,u_{\rm n}))}{y-y_{\rm n}}\right)^{\rm T}\end{aligned} \tag{157}$$

we get

$$\dot{\mathcal{U}}_1 \leqslant\ -\ \mathcal{W} - (y-u_{\rm n})^2 \tag{157}$$

Specifying the general adaptive controller (156) to the system (150) allows us to give a positive answer to the stabilization problem we stated.

7. CONCLUSIONS

Our cornerstone in this paper is the growth condition (14). It does not involve explicitly the system non-linearities. It is satisfied by strict pure feedback systems with polynomial growth non-linearities and parameter vector in a known compact set, a subclass of the family of systems considered by Kanellakopoulos *et al.*[8] and Krstic *et al.*[10] It is also satisfied by some non-globally feedback linearizable systems (see (143)) and even some non-feedback linearizable systems (see (18)). This condition has been introduced by Praly *et al.* in Reference 4, Proposition (375). The have shown that it is sufficient to obtain an adaptive controller by an estimation design based on a least squares algorithm. However, for the regulation result to hold, the filters feeding this algorithm must be initialized to a particular value depending on the state initial condition. Here we have applied a Lyapunov design and obtained an adaptive regulator with a not necessarily vanishing adaptation gain and without requiring a specific initialization. This regulator is of a new type since the adapted parameter vector is transformed before being used in the control law. This means that it does not rely on the so-called certainty equivalence principle. Unfortunately, this transformation is given as the solution of a fixed point problem which cannot be computed explicitly in general. Thus in a second stage we proposed another adaptive regulator where the static fixed point equation is replaced by a dynamical system with this fixed point as equilibrium.

Our result has been extended in two directions.

1. We have shown that it is not necessary to use a proper function for the Lyapunov design. In fact, a function satisfying a so-called boundedness observability property (132) is sufficient. This remark is important since it allows us to make the growth condition less difficult to achieve.
2. We have observed that our design may be used as an ingredient in a more complex design. In particular, we have shown that it can be used in combination with the technique proposed by Krstic, *et al.*[10] and dealing with the case where integrators are added.

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APPENDIX: THE FIXED POINT $\rho(x, \hat{q})$

Lemma 1

Let $\Pi_2 \subsetneq \Pi$ be two closed convex subsets of $\mathbb{R}^l$ such that

$$\delta \overset{\text{def}}{=} \inf_{p_1 \in \Pi_2,\, p_2 \notin \Pi} \| p_1 - p_2 \| > 0 \tag{158}$$

Let also $V: \mathbb{R}^n \times \Pi \to \mathbb{R}_+$ be a C^3-function such that for all (x, p) in $\mathbb{R}^n \times \Pi$ we have

$$\left\| \frac{\partial V}{\partial p}(x, p) \right\| \leqslant \gamma(1 + V(x, p)), \qquad \left\| \frac{\partial^2 V}{\partial p^2}(x, p) \right\| \leqslant \gamma^2(1 + V(x, p)) \tag{159}$$

If α is a positive real number satisfying

$$\alpha < \min\{\delta/\gamma, 1/2\gamma^2\} \tag{160}$$

then there exists a C^1-function $\rho: \mathbb{R}^n \times \Pi_2 \to \Pi$ such that

$$\rho(x, q) = q + \alpha \frac{(\partial V/\partial p)(x, \rho(x, q))^{\mathrm{T}}}{1 + V(x, \rho(x, q))} \tag{161}$$

Proof. Let f be defined by

$$f(x, p, q) \overset{\text{def}}{=} q + \alpha \frac{(\partial V/\partial p)(x, p)^{\mathrm{T}}}{1 + V(x, p)} \tag{162}$$

For all (x, p) in $\mathbb{R}^n \times \Pi$ we have

$$\| f(x, p, q) - q \| \leqslant \alpha\gamma \tag{163}$$

Therefore from (158) and (160) $f(x, q, p)$ is an interior point of Π for all (x, q, p) in $\mathbb{R}^n \times \Pi_2 \times \Pi$. Now with assumption (159) we have

$$\sup_{(x,p)\in\mathbb{R}^n\times\Pi}\left\|\frac{\partial}{\partial p}\left(\frac{(\partial V/\partial p)^{\mathrm{T}}}{1+V}\right)\right\| = \sup_{(x,p)\in\mathbb{R}^n\times\Pi}\left\|\frac{\partial^2 V/\partial p^2}{1+V}-\left(\frac{\partial V/\partial p}{1+V}\right)\left(\frac{\partial V/\partial p}{1+V}\right)^{\mathrm{T}}\right\|^2 \quad (164)$$

$$\leqslant 2\gamma^2 \quad (165)$$

Then from this inequality, the mean value theorem and the convexity of Π we have for all (x, q) in $\mathbb{R}^n \times \Pi_2$ and p_1 and p_2 in Π.

$$\| f(x, p_1, q) - f(x, p_2, q)\| = \alpha \left\| \frac{(\partial V/\partial p)(x, p_1)}{1+V(x,p_1)} - \frac{(\partial V/\partial p)(x,p_2)}{1+V(x,p_2)}\right\| \quad (166)$$

$$\leqslant 2\alpha\gamma^2 \| p_1 - p_2 \| \quad (167)$$

Since Π is a complete metric space, it follows from the contraction mapping theorem that (160) implies the existence of a unique function $\rho : \mathbb{R}^n \times \Pi_2 \to \Pi$ such that

$$\rho(x,q) = f(x, \rho(x,q), q) \quad (168)$$

Moreover, from (165) and (160) the matrix

$$I - \frac{\partial f}{\partial p}(x,p,q) = I - \alpha\frac{\partial}{\partial p}\left(\frac{(\partial V/\partial p)(x,p)}{1+V(x,p)}\right) \quad (169)$$

is non-singular for all (x, p) in $\mathbb{R}^n \times \Pi$. It follows from the implicit function theorem and the uniqueness of ρ that this function ρ is C^1, as are V and $\partial V/\partial p$. □

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