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PRELIMINARY RESULTS ABOUT ROBUST LAGRANGE STABILITY IN ADAPTIVE NON-LINEAR REGULATION

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SUMMARY

We are concerned with the problem of regulating the equilibrium point of a non-linear system in the presence of both parametric and dynamic uncertainties. For the parametric uncertainty we propose a new adaptive controller based on a Lyapunov design and guaranteeing the global boundedness of the solution if a growth condition is satisfied. For the dynamic uncertainty we propose a new way of characterizing the unmodelled effects which encompasses some singular and regular perturbations as illustrated by our worked example. Finally we show how, by modifying the above controller, the boundedness property can be made robust to these unmodelled effects.

KEY WORDS Non-linear systems Lagrange stability Adaptive non-linear control Unmodelled effects Robustness

1. INTRODUCTION

Important progress has been made in the adaptive control of non-linear systems. The main difficulties are now well understood and some very sophisticated solutions are available. Most of the results are synthesized in Reference 1 and the references cited therein, while more recent developments are given in References 2 and 3. However, these studies concern the ideal case, i.e. the case where the system to be controlled is exactly modelled up to the knowledge of some constant parameters. We know from the linear case that robustness to unmodelled effects of the properties of adaptive systems is a very difficult issue. For the non-linear case, results are already available for particular systems about the robustness of Lagrange stability to some unmodelled effects: Taylor *et al.*[4] and Kanellakopoulos *et al.*[5] have studied feedback linearizable systems in the presence of singular perturbations; Campion and Bastin[6,7] and Reed and Ioannou[8] have considered manipulators under bounded disturbances and singular perturbations. The objective of this paper is to report some preliminary results for more general circumstances about the following two aspects:

(1) introducing a new way of characterizing unmodelled effects,
(2) studying the robustness of boundedness of solutions given by a new adaptive regulator based on a Lyapunov design.

In proposing a characterization of the unmodelled effects in Section 3, our goal is to study to what class of uncertainties the boundedness is robust. We look for an as general as possible description which could encompass as many types of effects as possible. However, to remain

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simple, we shall focus our attention on qualitative more than quantitative results. The idea to get this characterization is to generalize to non-linear systems what was proposed in the linear case in Reference 9, namely the so-called normalizing signal technique. It has been shown to be a very powerful concept and its ability to describe all the possible linear unmodelled effects has been established in References 10–12. With the examples we propose in Section 2 we shall see that it may also be very fruitful for non-linear systems.

On the basis of a Lyapunov design, we propose in Section 4 an adaptive controller which allows us to stabilize a larger class of ideal systems than the one considered in Reference 13, namely those which are stabilizable by a state feedback such that a particular growth condition on the non-linearities is satisfied. Moreover, for the purpose of robustness, specific modifications will be introduced—parameter update projection and signal normalization.

To remain as simple as possible in this preliminary study, we have considered only the global case. This will be the reason for some overly restrictive assumptions.

2. MOTIVATION: AN EXAMPLE

Before entering a more general framework, let us look at an example. This will allow us to show the need for and the interest in a new way of characterizing unmodelled effects as well as the need for a new adaptive regulator.

Consider the following family of models indexed by a parameter $p \in \mathbb{R}$:

$$\dot{x}_1 = x_2 + p x_1^2 - x_1^3, \qquad \dot{x}_2 = u \tag{1}$$

where the state (x_1, x_2) in $\mathbb{R}^2$ is assumed to be measured and u in $\mathbb{R}$ is the control. Choosing one element in this family may be an appropriate way of getting a model for a system whose dynamics are actually described by the state equation

$$\dot{x}_1 = x_2 + p^* x_1^2 - y, \qquad \dot{x}_2 = u, \qquad \mu \dot{y} = -y + x_1^3 \tag{2}$$

where y is unmeasured and μ, a small positive real number, and $p^* \in [-p_{\max}, p_{\max}]$ are unknown but $p_{\max} > 0$ is known. Such a model may also be adequate for a system described by

$$\dot{x}_1 = x_2 + p^* x_1^2 - x_1^3 + \mu y + d(t), \qquad \dot{x}_2 = u, \qquad \dot{y} = -y + (1 + u^2)^\sigma - 1 \tag{3}$$

with d an unmeasured bounded C^0 time function and σ a positive real number we introduce for the purpose of the discussion.

The problem we want to address is to design an adaptive regulator for the model (1) guaranteeing boundedness of all the closed-loop solutions as well as regulation, i.e.

$$\lim_{t \to \infty} (x_1(t), x_2(t)) = 0 \tag{4}$$

However, we also require that this same regulator gives at least boundedness of all the solutions when placed in feedback with systems 'close' to the model (1), as are for example the systems (2) and (3) when μ is small enough. Indeed, in this case, (1) is the reduced-order system of the singularly perturbed system (2) and the regularly perturbed system (3).

To show that such a design is possible, we first address the problem of regulating the model (1) when $p = p^*$ is known and with guaranteed solution boundedness for the system (2). This study will allow us to extract a sufficient property on the unmodelled effects implying closed-loop solution boundedness. In this way we shall get a definition of the notion of 'close' we mentioned above. To check that this definition is appropriate, we will show that it is satisfied by the system (3). However, before this, it is important to remark that the extra component

$y(t)$ of a solution in both systems (2) and (3) is bounded as soon as the other components $x_1(t)$ and $x_2(t)$ of this solution are bounded and the control u is given by a bounded input–bounded output system with $(x_1(t), x_2(t))$ as input. This key property of 'boundedness of part of the state vector implying boundedness of the complete state vector' will be called later *boundedness observability*.

Regulation of the model $p = p^*$ known

Let us design a family of controllers for the model (1) with each one solving our problem if p^* were known.

By applying the adding-an-integrator technique,[14,15] we design a controller in two steps.

Step 1. We start by stabilizing the system

$$\dot{x}_1 = u_1 + p^* x_1^2 - x_1^3 \tag{5}$$

Denote by $u_1(x_1, p^*)$ and V_1 the functions

$$u_1(x_1, p^*) = -x_1 - p^* x_1^2 + x_1^3 + v_1(x_1), \qquad V_1(x_1) = \tfrac{1}{2} x_1^2 \tag{6}$$

where v_1 is a function introduced as a degree of freedom and satisfying

$$x_1 v_1(x_1) \leqslant 0 \quad \forall x_1, \qquad v_1(0) = 0 \tag{7}$$

By applying u_1 to the system (5) and computing the time derivative of V_1, we get

$$\dot{V}_1 = -2V_1(x_1) + x_1 v_1(x_1) \stackrel{\text{def}}{=} -W_1(x_1) \tag{8}$$

Step 2. We add an integrator, i.e. we consider the system (1). We define

$$V(x_1, x_2, p) = \frac{1}{2j}(2V_1(x_1))^j + \tfrac{1}{2}\chi_2^2 \tag{9}$$

where j is a strictly positive interger left as a new degree of freedom and χ_2, introduce to simplify the notations, is given by

$$\chi_2 = x_2 - u_1(x_1, p) = x_2 + x_1 + p x_1^2 - x_1^3 - v_1(x_1) \tag{10}$$

Note that χ_2 is in fact a function of (x_1, x_2, p). Along the solutions of the system (1) the time derivative of $V(x_1, x_2, p^*)$ satisfies†

$$\begin{aligned} \dot{V} = & -(2V_1(x_1))^{j-1} W_1(x_1) \\ & + \chi_2[x_1(2V_1(x_1))^{j-1} + u + (1 + 2p^* x_1 - 3x_1^2 - v_1'(x_1))(x_2 + p^* x_1^2 - x_1^3)] \end{aligned} \tag{11}$$

Therefore we get

$$\begin{aligned} \dot{V} = & -2jV(x_1, x_2, p^*) + x_1 v_1(x_1)(2V_1(x_1))^{j-1} + (x_2 + x_1 + p^* x_1^2 - x_1^3 - v_1(x_1)) v_2(x_1, x_2, p^*) \\ \stackrel{\text{def}}{=} & -W(x_1, x_2, p^*) \end{aligned} \tag{12}$$

when we choose

$$\begin{aligned} u = & \, u_n(x_1, x_2, p^*) \\ \stackrel{\text{def}}{=} & -x_1(2V_1(x_1))^{j-1} - (1 + 2p^* x_1 - 3x_1^2 - v_1'(x_1))(x_2 + p^* x_1^2 - x_1^3) - j\chi_2 + v_2(x_1, x_2, p^*) \end{aligned} \tag{13}$$

† Throughout this paper we denote by f' the derivative of the real function f.

where v_2, a third degree of freedom, is a function satisfying

$$\begin{aligned}&(x_2 + x_1 + px_1^2 - x_1^3 - v_1(x_1))v_2(x_1, x_2, p) \leqslant 0 \quad \forall (x_1, x_2, p) \in \mathbb{R}^3 \\ &v_2(x_1, x_2, p) = 0 \quad \forall (x_1, x_2, p) \in \{(x_1, x_2, p) : x_2 + x_1 + px_1^2 - x_1^3 - v_1(x_1) = 0\}\end{aligned} \tag{14}$$

Clearly from (12) the state feedback u_n defined in (13) guarantees that all the solutions of (1) with $p = p^*$ are bounded and satisfy our objective (4). The property described by equation (12) about the existence of a feedback law u_n guaranteeing that the time derivative of a proper positive function V is negative along the solutions of the known model is called *stabilizability* in the following.

Boundedness of solutions when $\mu \neq 0$ but p^* is known

Let us investigate now if the feedback u_n in closed loop with our actual systems (2) and (3) provides at least boundedness of all the solutions. We start this study by considering system (2).

When we compute the time derivative of $V(x_1, x_2, p^*)$ defined in (9) along the solutions of the system (2) with $u = u_n$, we get

$$\dot{V} = -W + \frac{\partial V}{\partial x_1}(x_1^3 - y) \tag{15}$$

Compared with (12), we see that the effect of the mismodelling is the introduction of the term $(\partial V/\partial x_1)(x_1^3 - y)$. Our objective now is to specify the feedback law u_n by choosing j, v_1 and v_2 independent of (μ, y) in order to counteract these effects. To obtain such a choice, the only difficulty is to relate these effects to V and W. This is what we call *unmodelled effects characterization* in this paper. We have

$$\left|\frac{\partial V}{\partial x_1}(x_1^3 - y)\right| \leqslant |x_1^{2j-1} + \chi_2(1 + 2p^*x_1 - 3x_1^2 - v_1'(x_1))||x_1^3 - y| \tag{16}$$

$$\begin{aligned}&\leqslant 2x_1^{2j+2} + \frac{2j-1}{j+1}|\chi_2|^{(2j+2)/(2j-1)}|1 + 2p^*x_1 - 3x_1^2 - v_1'(x)|^{(2j+2)/(2j+1)} \\ &\quad + \frac{3}{j+1}y^{(2j+2)/3}\end{aligned} \tag{17}$$

where we have used Young's inequality,[4] i.e. $\forall n > 1$ and $\forall (x, y) \in \mathbb{R}_+^2$,†

$$xy \leqslant \frac{1}{n}x^n + \frac{n-1}{n}y^{n/(n-1)}$$

Therefore if we choose v_1 and v_2 so that W defined in (12) satisfies

$$2x_1^{2j+2} + \frac{2j+1}{j+1}|\chi_2|^{(2j+2)/(2j-1)}|1 + 2p^*x_1 - 3x_1^2 - v_1'(x_1)|^{(2j+2)/(2j-1)} \leqslant \tfrac{1}{2}W(x_1, x_2, p^*) \tag{18}$$

then (15) and the equation of $\dot{y}$ give the system

$$\dot{V} \leqslant -\tfrac{1}{2}W + \frac{3}{j+1}|y|^{(2j+2)/3}, \qquad \mu\dot{\overline{y^{(2j+2)/3}}} = \frac{2j+2}{3}(-y^{(2j+2)/3} + y^{(2j-1)/3}x_1^3) \tag{19}$$

† Concavity of logarithms gives: $\forall n \geqslant 1$ and $\forall (x, y) \in \mathbb{R}_+^2$,

$$(1/n)\log x + ((n-1)/n)\log y \leqslant \log((1/n)x + ((n-1)/n)y)$$

or, again with Young's inequality,

$$\dot{V} \leqslant -\tfrac{1}{2} W + \frac{3}{j+1} |y|^{(2j+2)/3}, \qquad \mu \dot{\overbrace{y^{(2j+2)/3}}} \leqslant -y^{(2j+2)/3} + x_1^{2j+2} \tag{20}$$

The following expressions for v_1 and v_2 satisfy the constraint (18) we have proposed:

$$\begin{aligned} v_1(x_1) &= -c_1 x_1^3 \\ v_2(x_1, x_2, p) &= -c_2(x_2 + x_1 + p x_1^2 - x_1^3 - v_1(x_1))^{3/(2j-1)} |1 + 2p x_1 \\ &\quad - 3(1 - c_1) x_1^2|^{(2j+2)/(2j-1)} - c_1 j^{(j+1)/j}(x_2 + x_1 + p x_1^2 - x_1^3 - v_1(x_1))^{(j+2)/j} \end{aligned} \tag{21}$$

with

$$c_1 \geqslant 4, \qquad c_2 \geqslant \frac{4j-2}{j+1} \tag{22}$$

Note that the last term in the definition of v_2 is not motivated by (18) but by inequality (30) to follow. This yields in particular

$$x_1^{2j+2} \leqslant \frac{1}{c_1} W(x_1, x_2, p) \quad \forall (x_1, x_2, p) \in \mathbb{R}^3 \tag{23}$$

It follows that for any strictly positive real number ε we have

$$\dot{\overbrace{V + \mu \varepsilon y^{(2j+2)/3}}} \leqslant -\left(\frac{1}{2} - \frac{\varepsilon}{c_1}\right) W - \left(\varepsilon - \frac{3}{j+1}\right) y^{(2j+2)/3} \tag{24}$$

Therefore by choosing j and c_1 in the control and ε for the analysis such that

$$\frac{c_1}{2} > \varepsilon > \frac{3}{j+1} \tag{25}$$

we have achieved our objective. Namely, we are guaranteed that, whatever the positive value of μ is, along the solutions of the closed-loop system (2), (13) the function $V + \mu\varepsilon y^{(2j+2)/3}$ is decaying, these solutions are bounded on $[0, \infty)$ and satisfy

$$\lim_{t \to \infty} W(x_1(t), x_2(t), p^*) = \lim_{t \to \infty} y(t) = 0 \tag{26}$$

Let us remark that for proving this result we have not followed the standard route[17] of approximating the slow manifold by $y = x_1^3$, the equilibrium manifold at $\mu = 0$. Indeed, the deviation $\eta = y - x_1^3$ from this latter manifold satisfies

$$\mu \dot{\eta} = -(1 - 3\mu x_1^2)\eta - 3\mu x_1^2(x_2 + p^* x_1^2 - x_1^3) \tag{27}$$

The term $1 - 3\mu x_1^2$ not being positive uniformly in x_1, difficulties will follow when wishing to establish global regulation.

In conclusion, to get bounded solutions for both the model (1) and the system (2), we can choose the control (13) with the expressions (21) for v_1 and v_2. In fact, this is a family of controllers, since j, c_1 and c_2 are still to be chosen. Note, however, that we have been taking no care of smoothness of the control. In particular, for $j > 2$, v_2 may be only Hölder continuous. Also, we have shown in our analysis that an appropriate Lyapunov function for studying the closed-loop system is

$$U(x_1, x_2, y, p^*) = V(x_1, x_2, p^*) + \mu \varepsilon y^{(2j+2)/3} \tag{28}$$

Unfortunately, an important drawback of this Lyapunov function is that it involved explicitly the unknown constant μ and the unmeasured state component y. A consequence of this dependence is that we shall not be able to use it to design an adaptive regulator by applying the standard Lyapunov design,[1] since in this case we may expect the parameter update law to involve μ and y. Thus, before studying the other system (3), we have to find another Lyapunov function in terms of x_1, x_2 and any other signal which can be effectively obtained from the measurements of x_1 and x_2.

To attack this problem, we shall try to understand the relation between y and x_1 and x_2. Let $\hat{p}(t)$ be an arbitary bounded C^1 time function and $u(t)$ be another arbitrary C_0 time function. Let $(x_1(t), x_2(t), y(t))$ be a solution defined on $[0, T)$ of the system (2) with $u(t)$ as input. In (20) the equation satisfied by $\widehat{y^{(2j+2)/3}}$ gives for all $t \in [0, T)$

$$y^{(2j+2)/3}(t) \leqslant \exp\left(-\frac{t}{\mu}\right) y^{(2j+2)/3}(0) + \frac{1}{\mu}\int_0^t \exp\left(-\frac{t-s}{\mu}\right) x_1^{2j+2}(s)\,\mathrm{d}s \tag{29}$$

However, with (23) and (9) we notice that we can find a function Υ such that

$$c_1 x_1^{2j+2} \leqslant \Upsilon(V(x_1, x_2, p)) \leqslant 2^{1/j} W(x_1, x_2, p) \quad \forall (x_1, x_2, p) \in \mathbb{R}^3 \tag{30}$$

Indeed, we may choose

$$\Upsilon(V) = c_1 (2jV)^{(j+1)/j} \tag{31}$$

Noticing that p can be chosen arbitrarily in (30), this inequality and (29) yield for all $t \in [0, T)$

$$y^{(2j+2)/3}(t) \leqslant \exp\left(-\frac{t}{\mu}\right) y^{(2j+2)/3}(0) + \frac{1}{c_1\mu}\int_0^t \exp\left(-\frac{t-s}{\mu}\right) \Upsilon(V(x_1(s), x_2(s), \hat{p}(s)))\,\mathrm{d}s \tag{32}$$

Now, with Hölder's inequality, for any (finite) integer number $q > \mu$ we have

$$\int_0^t \exp\left(-\frac{t-s}{\mu}\right)\Upsilon(V(s))\,\mathrm{d}s \leqslant \left(\frac{\mu(q-1)}{q-\mu}\right)^{(q-1)/q} \left(\int_0^t \exp[-(t-s)]\,\Upsilon(V(s))^q\,\mathrm{d}s\right)^{1/q} \tag{33}$$

with $V(s)$ standing for $V(x_1(s), x_2(s), \hat{p}(s))$ to abbreviate the notation. Finally we observe that the integral on the right-hand side can be obtained as follows.

Given a positive real number r_0, we define what we call the normalizing signal r by

$$\dot{r} = -r + \Upsilon(V(t))^q, \qquad r(0) = r_0 \tag{34}$$

We have

$$\left(\int_0^t \exp[-(t-s)]\,\Upsilon(V(s))^q\,\mathrm{d}s\right)^{1/q} \leqslant 2^{(q-1)/q} r^{1/q} - \exp\left(-\frac{t}{q}\right) r_0^{1/q} \tag{35}$$

Therefore we have obtained

$$y^{(2j+2)/3}(t) \leqslant \exp\left(-\frac{t}{\mu}\right) y^{(2j+2)/3}(0) - \frac{1}{c_1\mu}\left(\frac{\mu(q-1)}{q-\mu}\right)^{(q-1)/q} \exp\left(-\frac{t}{q}\right) r_0^{1/q} + \frac{1}{c_1\mu}\left(\frac{2\mu(q-1)}{q-\mu}\right)^{(q-1)/q} r(t)^{1/q} \tag{36}$$

With (17) and (18) this yields the following key property for the unmodelled effects for all $t \in [0, T)$:

$$\left|\frac{\partial V}{\partial x_1}(x_1(t), x_2(t), \hat{p}(t))(x_1^3(t) - y(t))\right| \leqslant \mu_1 W(x_1(t), x_2(t), \hat{p}(t)) + \mu_2 r(t)^{1/q} + D(t) \tag{37}$$

where the constants μ_1 and μ_2 are given by

$$\mu_1 = \tfrac{1}{2}, \qquad \mu_2 = \frac{3}{j+1}\frac{1}{c_1\mu}\left(\frac{2\mu(q-1)}{q-\mu}\right)^{(q-1)/q} \tag{38}$$

and the time function D is defined by

$$D(t) = \frac{3}{j+1}\exp\left(-\frac{t}{\mu}\right)y^{(2j+2)/3}(0) - \frac{3}{c_1\mu(j+1)}\left(\frac{\mu(q-1)}{q-\mu}\right)^{(q-1)/q}\exp\left(-\frac{t}{q}\right)r_0^{1/q} \tag{39}$$

Recall that inequality (37) holds for any input $u(t)$ and any time function $\hat{p}(t)$. We note also that since $q > \mu$, $D(t)$ is negative for all t sufficiently large. Nevertheless, the fact that $\sup_t D(t)$ depends on the initial conditions may be the cause of a non-global result. To emphasize this aspect, it is opportune to define for any strictly positive real number η the open set

$$\mathscr{X}_\eta = \{(x_1, x_2, y) \in \mathbb{R}^3 \,\|\, y \,| < \eta\} \tag{40}$$

and the function

$$\mathscr{D}(\eta, r_0) = \sup_{t\in\mathbb{R}_+,\, |y(0)|<\eta} D(t) = \max\left\{0, \frac{3}{j+1}\eta^{(2j+2)/3} - \frac{3}{c_1\mu(j+1)}\left(\frac{\mu(q-1)}{q-\mu}\right)^{(q-1)/q} r_0^{1/q}\right\} \tag{41}$$

This function is increasing in η but decreasing in r_0. Then, if our controller involves the normalizing signal r and therefore fixes r_0 and our analysis requires a bound Δ on $\sup_t D(t)$, our result will hold only for some initial conditions defined as follows. Let η^* be given by

$$\mathscr{D}(\eta^*, r_0) = \Delta \tag{42}$$

The constraint on $\sup_t D(t)$ is satisfied if the initial conditions $(x_1(0), x_2(0), y(0))$ are in the set $\mathscr{X}_{\eta^*}$. Such a fact will motivate in our general framework the introduction of the set $\mathscr{X}$ and the function $\mathscr{D}(\mathscr{X})$.

Now, coming back to the key inequality (37) satisfied by the unmodelled effect, we can try to prove boundedness via the study of the following system instead of (20) as we did above:

$$\dot{V} \leqslant -(1-\mu_1)W + \mu_2 r^{1/q} + D(t), \qquad \dot{r} = -r + \Upsilon(V)^q \tag{43}$$

with (see (30))

$$\Upsilon(V) \leqslant 2^{1/j} W \tag{44}$$

The inequalities (43) are satisfied when $u = u_n(x_1, x_2, p^*)$ and $\hat{p}(t) = p^*$. For such a system of inequalities it turns out that an appropriate Lyapunov function is

$$U(x_1, x_2, r, p^*) = \int_0^{V(x_1, x_2, p^*)} \Upsilon(v)^{2q-1}\,\mathrm{d}v + \frac{\varepsilon}{2}r^2 \tag{45}$$

Indeed, it allows us to prove that V and r are bounded if (see the proof of Proposition 2)

$$0 < 1 - \mu_1 - 2^{1+1/j}\mu_2 = \tfrac{1}{2} - 2^{1/j}\mu^{-1/q}\frac{6}{c_1(j+1)}\left(\frac{2(q-1)}{q-\mu}\right)^{(q-1)/q} \tag{46}$$

This means that by following this approach, boundedness can be established without any bound on $\sup_t D(t)$ (see the discussion above), but only if, for some real number c_3 in $(0, 1)$,

$$\left(\frac{12}{2^{-1/j}c_1(j+1)}\right)^q \left(\frac{2(q-1)}{(1-c_3)q}\right)^{q-1} < \mu \leqslant c_3 q \tag{47}$$

This is more conservative than what we obtained above, since no constraint on μ was needed in our previous analysis. Note, however, that if in the control u_n, c_1 is chosen such that

$$\frac{24}{2^{-1/j}(j+1)(1-c_3)} < c_1 \tag{48}$$

then by choosing q sufficiently large, we can enlarge arbitarily the domain of admissible μ's for our second approach.

Though more restrictive, our second approach is more general. Indeed, the only assumption needed about the unmodelled effects is that (37) holds. To check that such a characterization of the unmodelled effects allows us to encompass a wide variety of unmodelled effects and not only the singular perturbation of the system (2), let us see if it is satisfied by the system (3) with regular perturbation.

As in the previous case, to know what the unmodelled effects are for the system (3), we compute the time derivative of $V(x_1, x_2, p^*)$ defined in (9) along the solutions of this system. For this we need to choose a control. Let $\hat{p}(t)$ be an arbitrary bounded C^1 time function. Let $(x_1(t), x_2(t), y(t))$ be a solution defined on $[0, T)$ of the system (3) with $u = u_n(x_1, x_2, \hat{p}(t))$ as input. Note that u is not arbitrary as it was in the previous case. Along this solution we get

$$\dot{V} = -W + \frac{\partial V}{\partial x_1}(\mu y + d(t)) \tag{49}$$

Thus in this case the effect of the mismodelling is the introduction of the term $(\partial V/\partial x_1)(\mu y + d(t))$. Our task now is to check if an inequality in the form (37) is satisfied, with the same definition (34) for the normalizing signal r. As in the previous case with (30), such an inequality holds if we can find constants δ_1 and D_1 such that for all $t \in [0, T)$ we have

$$[(1 + u_n^2(x_1(t), x_2(t), \hat{p}(t)))^\sigma - 1]^{(2j+2)/3} \leqslant \delta_1 \Upsilon(V(x_1(t), x_2(t), \hat{p}(t)) + D_1 \tag{50}$$

Tedious computations with application of Young's inequality show that such constants exist provided that

$$\sigma \leqslant \tfrac{3}{10}, \qquad \frac{3 + 8\sigma}{6 - 14\sigma} \leqslant j \leqslant \frac{3 + 2\sigma}{4\sigma} \tag{51}$$

These very strong constraints illustrate the trouble that may be caused by the direct action of the input on the unmodelled dynamics. In particular, by increasing j, which may be interpreted as augmenting the gain of the controller, we may excite these unmodelled dynamics up to such a point that stability is no longer guaranteed.

Boundedness of solutions when $\mu \neq 0$ and p^* is unknown

When p^* is unknown, many adaptive controllers proposed in the literature can be applied to our simplistic model (1). For example, following the terminology of Reference 1, since the matching condition is satisfied, we may apply a Lyapunov design. Namely, we choose an adaptation law in order to make the time derivative of a Lyapunov function negative along the solutions of the closed-loop system. Precisely, using the definitions (9), (12) and (13), we look for two functions v_p and v_u such that the time derivative of

$$U(x_1, x_2, \hat{p}) = V(x_1, x_2, \hat{p}) + \tfrac{1}{2}|\hat{p} - p^*|^2 \tag{52}$$

along the solutions of

$$\dot{x}_1 = x_2 + p^* x_1^2 - x_1^3, \qquad \dot{x}_2 = u_n(x_1, x_2, \hat{p}) + v_u(x_1, x_2, \hat{p}), \qquad \dot{\hat{p}} = v_p(x_1, x_2, \hat{p}) \tag{53}$$

is negative. By using (12), we get

$$\dot{U} = -W + (\hat{p} - p^*)\left(v_p - \frac{\partial V}{\partial x_1} x_1^2\right) + \frac{\partial V}{\partial p} v_p + \frac{\partial V}{\partial x_2} v_u \tag{54}$$

However, we notice that

$$\frac{\partial V}{\partial p} = \chi_2 x_1^2, \qquad \frac{\partial V}{\partial x_2} = \chi_2 \tag{55}$$

imply that the matching condition is satisfied (see Reference 1, P. 371). Therefore, by choosing

$$v_p = \frac{\partial V}{\partial x_1} x_1^2, \qquad v_u = v_p x_1^2 = -\frac{\partial V}{\partial x_1} x_1^4 \tag{56}$$

we get

$$\dot{U} = -W \tag{57}$$

Having designed the adaptive controller $u = u_n + v_u, \dot{\hat{p}} = v_p$ for the model (1), let us investigate now the properties it gives when applied to the actual system (2). We compute the time derivative of U defined in (52) along the solutions of

$$\begin{aligned} \dot{x}_1 &= x_2 + p^* x_1^2 - y, \qquad & \dot{x}_2 &= u_n(x_1, x_2, \hat{p}) + v_u(x_1, x_2, \hat{p}), \\ \mu \dot{y} &= -y + x_1^3, \qquad & \dot{\hat{p}} &= v_p(x_1, x_2, \hat{p}) \end{aligned} \tag{58}$$

By using (57), we get

$$\dot{U} = -W + \frac{\partial V}{\partial x_1}(x_1^3 - y) \tag{59}$$

We observe that the mismodelling has the same effect $(\partial V/\partial x_1)(x_1^3 - y)$ as in the case where p^* was known. Since no constraint was imposed on the control u to obtain (37), this inequality holds again. By using this characterization of the unmodelled effects, the analysis can be pursued as will be shown in remark 5 following Proposition 2. Unfortunately, if we analyse the properties given by the same adaptive controller based on the matching condition when applied now to the system (3), we encounter difficulties. Indeed, the unmodelled effects characterization (37) has been obtained for system (3) only when $u = u_n$, whereas we have now $u = u_n + v_u$. Precisely, the closed-loop system is

$$\begin{aligned} \dot{x}_1 &= x_2 + p^* x_1^2 - x_1^3 + \mu y + d(t), \qquad & \dot{x}_2 &= u_n(x_1, x_2, \hat{p}) + v_u(x_1, x_2, \hat{p}) \\ \dot{y} &= -y + [1 + (u_n(x_1, x_2, \hat{p}) + v_u(x_1, x_2, \hat{p}))^2]^\sigma - 1, \qquad & \dot{\hat{p}} &= v_p(x_1, x_2, \hat{p}) \end{aligned} \tag{60}$$

As we have already observed, in the case where p^* is known, inequality (37) holds if we can find constants δ_2 and D_2 such that for all $t \in [0, T)$ we have

$$\{[1 + (u_n(x_1(t), x_2(t), \hat{p}(t)) + v_u(x_1(t), x_2(t), \hat{p}(t)))^2]^\sigma - 1\}^{(2j+2)/3} \leqslant \delta_2 \Upsilon(V(x_1(t), x_2(t), \hat{p}(t)) + D_2 \tag{61}$$

After lengthy calculations with repeated use of Young's inequality, we show that such constants exist provided that either

$$\sigma \leqslant \tfrac{1}{6}, \qquad \frac{3 + 8\sigma}{6 - 14\sigma} \leqslant j \leqslant \frac{3 - 6\sigma}{4\sigma} \tag{62}$$

or

$$\tfrac{1}{6} < \sigma \leqslant \frac{15 - 3\sqrt{6}}{38}, \qquad \frac{3 + 18\sigma}{6 - 14\sigma} \leqslant j \leqslant \frac{3 - 12\sigma}{32\sigma} \tag{63}$$

It is clear that the set of admissible σ's and the corresponding set of acceptable j's are smaller than those defined by (51) for the case where p^* is known. We note that this loss is due to the presence of v_p in the control. In other words, the adaptive controller obtained by the matching condition cannot guarantee the property of boundedness of the solutions robust with respect to some unmodelled effects. This leads us to find another controller design.

To conclude, the study of this example introduced three assumptions: *boundedness observability*, *stabilizability* and *unmodelled effects characterization*. These assumptions will be presented in the general framework in the next section. This study showed us also that an adaptive controller appropriate for the ideal case, i.e. applied to the model, may lead to problems in the non-ideal case, i.e. applied to systems only close to the model. This emphasizes the need for modification of this adaptive controller. This problem will be addressed in Section 4.

3. UNMODELLED EFFECTS

Let the system to be controlled admit a finite state representation on $\mathbb{R}^N$ and its dynamics, maybe augmented by input and output filters, be described globally by.

$$\dot{X} = F(X, t, u), \qquad x = H(X, t) \tag{64}$$

where the vector X is the state in $\mathbb{R}^N$ which is not measured and the dimension N is unknown, u is the input vector in $\mathbb{R}^m$, x is a measured output in $\mathbb{R}^n$ and finally:

Assumption R (regularity) (65)

F and H are C^1 unknown functions with $(\partial H/\partial X)\ (X, t)$, $(\partial H/\partial t)\ (X, t)$ and $F(X, t, u)$ bounded for all (X, u) in compact sets and $t \geqslant 0$.

We assume also:

Assumption BO (boundedness observability) (66)

For all compact subsets $\mathscr{K}_x$ in $\mathbb{R}^n$ and $\mathscr{K}_u$ in $\mathbb{R}^m$ and for all initial conditions $X(0)$ in $\mathbb{R}^N$, we can find a compact subset $\mathscr{K}_X$ in $\mathbb{R}^N$ such that for the corresponding solution $X(t)$ of (64) defined on $[0, T)$,

$$x(t) \in \mathscr{K}_x \text{ and } u(t) \in \mathscr{K}_u \quad \forall t \in [0, T) \quad \text{implies } X(t) \in \mathscr{K}_X \quad \forall t \in [0, T)$$

Namely, to know that the trajectory $\{X(t)\}_{t \in [0, T)}$ is bounded, it is sufficient to observe that the trajectories $\{x(t)\}_{t \in [0, T)}$ and $\{u(t)\}_{x \in [0, T)}$ are bounded.

Our problem is: design a controller such that the solutions $X(t)$ of (64) are bounded and $\lim_{t \to \infty} H(X(t), t) = \mathscr{E}$, a desired set point for the measurement x.

Since the system to be controlled is only partially known, we shall work from a design model whose state is x. This is why it may be interesting to augment the dynamics of the system to be controlled by filters (see Reference 1, Example (24)). The dynamics of this model are chosen as being described by an equation involving an unknown constant parameter vector p^*:

$$\dot{x} = a(x,u) + A(x, u)p^* \tag{67}$$

where the functions a and A are known and continuously differentiable and p^* is an unknown parameter vector in a known compact convex subset Π of $\mathbb{R}^l$. This model is said to be linearly parameterized in explicit form. This is more restrictive than the case of linear parametrization in implicit form

$$(b(x) + B(x)p^*)\dot{x} = a(x, u) + A(x, u)p^* \tag{68}$$

as obtained with manipulators for example and considered in References 6–8. Our model is supposed to be stabilizable for all p:

Assumption S (stabilizabilty) (69)

There exist three known functions u_n, W and V such that

(1) $u_n : \mathbb{R}^n \times \Pi \to \mathbb{R}^m$ is of class C^1
(2) W is positive, of class C^0, $W(x, p) = 0$ if $x = \mathscr{E}$
(3) V is of class C^2, positive, $V(x, p) = 0$ iff $x = \mathscr{E}$ and for any positive real number K_v the set $\{x \mid V(x, p) \leqslant K_v,\ p \in \Pi\}$ is a compact subset of $\mathbb{R}^n$
(4) for all (x, p) in $\mathbb{R}^n \times \Pi$, we have

$$\frac{\partial V}{\partial x}(a(\cdot, u_n) + A(\cdot, u_n)p) \leqslant -W(x, p) \tag{70}$$

Namely, for any p in Π, the point $\mathscr{E}$, which may depend on p, is a globally asymptotically stable equilibrium point of the system

$$\dot{x} = a(x, u_n(x, p)) + A(x, u_n(x, p))p \tag{71}$$

and V is a corresponding Lyapunov function for this closed-loop model with time derivative less than $-W$.

Knowing that the model can be stabilized whatever the (constant) value of the parameter vector is, we need now to characterize the discrepancy between our model (67) and the actual system (64). For this we need to choose two strictly positive real numbers α and r_0, a strictly increasing C^1 and convex function $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\Psi(0) = 0$ and a positive C^1 real function $\Upsilon : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying, for some strictly positive real number c,

$$\begin{aligned} &\Upsilon(V(x, p)) \leqslant cW(x, p) \quad \forall (x, p) \in \mathbb{R}^n \times \Pi \\ &\Upsilon(v) = 0 \ \text{ iff } v = 0, \qquad \liminf_{v \to \infty} \Upsilon(v) > 0 \end{aligned} \tag{72}$$

Then we assume:

Assumption UEC ($\alpha, r_0, \Psi, \Upsilon$) (unmodelled effects characterization) (73)

Let $\mathscr{X}$ be an open subset of $\mathbb{R}^N$ with $H(\mathscr{X}; 0) = \mathbb{R}^n$. There exist a positive real function $\mathscr{D}(\mathscr{X})$ and positive real numbers μ_1, μ_2 and ρ such that for any C^1 time function $\hat{p} : \mathbb{R}_+ \to \Pi$ and any solution $X(t)$ of

$$\dot{X} = F(X, t, u_n(x, \hat{p})), \qquad x = H(X, t), \qquad X(0) \in \mathscr{X} \tag{74}$$

defined on $[0, T)$ there exist a C^1 time function $p^* : [0, T) \to \Pi$ with $\|\dot{p}^*\| \leqslant \rho$ and a C^0 time function $D : [0, T) \to [0, \mathscr{D}(\mathscr{X})]$ satisfying, for all $t \in [0, T)$,

$$\left| \frac{\partial V}{\partial x}(x, \hat{p})[\dot{x} - a(x, u_n(x, \hat{p})) - A(x, u_n(x, \hat{p}))p^*] \right| \leqslant \mu_1 W(x, \hat{p}) + \mu_2 \Psi^{-1}(\alpha, r) + D(t) \tag{75}$$

where r, called the normalizing signal, is defined by

$$\dot{r} = -\alpha r + \Psi(\Upsilon(V(x,\hat{p}))), \qquad r(0) = r_0 \tag{76}$$

Note that p^* is allowed to be time-dependent and to depend on $\hat{p}$. The condition $H(\mathscr{X}; 0) = \mathbb{R}^n$ means that we are looking for results which are global with respect to the model state initial condition $x(0)$. On the other hand, we may have restriction on the initial condition $X(0)$ if, as discussed in the previous section, constraints are imposed on $\mathscr{X}$ by the fact that $\mathscr{D}(\mathscr{X})$ should satisfy some inequality.

This assumption is the unmodelled effects characterization we mentioned previously. Note the presence of the positive real number α as a pole defining the normalizing signal r. It should be chosen in general depending on the neglected fast time constants. This implies some knowledge about these time constants. Nevertheless, as seen in our example with (47), by introducing the function Ψ, we can make this dependence much weaker. Also, in the case of linear systems this dependence can be overcome by input filtering as shown by Ioannou and Tsakalis.[10] Another motivation for introducing an integrator is to prevent the input from having a direct action on the unmodelled dynamics.[10,18] Such a direct action is no trouble when dealing only with a local stability analysis,[4,5,7] but it may be a real problem for a global analysis as we have seen above by studying the system (3). Adding integrators can be done easily in the non-adaptive case as follows from Reference 14, Theorem 3.c or Reference 15. However, in the context of adaptive control, adding integrators may cause difficulties, the parameter dependence of the closed-looped system being reinforced (see the discussion about the matching condition in Reference 1). In particular, for the case of manipulators as considered by Reed and Ioannou[8] and Campion and Bastin,[7] we do not see how integrators could be added in the adaptive case owing to the fact that the model is linearly parametrized only in the implicit form (68).

The characterization (75), (76) is in some sense a closed-loop one. Instead of asking for inequality (75) to hold for all possible input functions, u, which would be a very stringent requirement, we require only that it holds for the particular class of input functions $u_n(\cdot, \hat{p})$, among which will be the one actually used. However, one open-loop aspect remains, since not knowing *a priori* what will be the time function $\hat{p}$, we are led to ask for (75) to hold for all possible time functions $\hat{p}$. Also, the closed-loop model Lyapunov function V is involved in inequality (75). One way to understand this is: the control law u_n should be designed in such a way that the corresponding V satisfies (75), i.e. the unmodelled effects should be taken into account in the control design. This aspect was illustrated in the previous section when we chose v_1, v_2 and j.

One of the difficulties encountered when dealing with Assumption UEC $(\alpha, r_0, \Psi, \Upsilon)$ (73) is its checkability. This aspect is the topic of current research. For the time being, our objective in this paper is only to get a qualitative result. Namely, we intend to introduce a notion of distance between model and system—quantified later by $\mu_1 + 2c\mu_2$—and to establish if possible that Lagrange stability is an open property with respect to the induced topology. In the linear case this programme has succeeded completely. Indeed, it is proved in Reference 12, Property 5 that an assumption similar to Assumption UEC$(\alpha, r_0, \Psi, \Upsilon)$ (73) is related to the graph topology introduced by Vidyasagar.[19] Another way to realize the interest of Assumption UEC$(\alpha, r_0, \Psi, \Upsilon)$ (73) is to figure out what kind of unmodelled dynamics can be captured by inequality (75). This was the motivation in Section 2 for considering examples with singular and regular perturbations.

To help the designer in choosing the constant α and the function Ψ which are involved in Assumption UEC$(\alpha, r_0, \Psi, \Upsilon)$ (73) and will be explicitly used in the controller we shall propose in Section 4.3, we have the following lemma whose proof is straightforward from the arguments of Reference 16, Chap. 3.

Lemma 1

Let $v(t)$ be a C^1 function defined on $[0, T)$. We have the following properties.

1. If Ψ is a strictly increasing convex function with $\Psi(0)=0$, then $\Psi(x)/x$ (resp. $\Psi^{-1}(x)/x$) is non-decreasing (resp. increasing) and for all positive x, y and $k \geqslant 1$,

$$\Psi^{-1}(kx) \leqslant k\Psi^{-1}(x), \qquad \Psi^{-1}(x-y) \leqslant \Psi^{-1}(x)+\Psi^{-1}(y) \tag{77}$$

2. Let r_1 and r_2 be positive and such that

$$\dot{r}_1 \leqslant -\alpha_1 r_1 + \Psi_1(v), \qquad \dot{r}_2 = -\alpha_2 r_2 + \Psi_2(v) \tag{78}$$

where $\alpha_1 \geqslant \alpha_2 > 0$ are real numbers and Ψ_1 and Ψ_2 are strictly increasing functions with $\Psi_1(0)=\Psi_2(0)=0$ and Ψ_2 and $\Psi_2\Psi_1^{-1}$ are convex. For all $t \in [0, T)$ we have

$$\Psi_1^{-1}(\alpha_1 r_1(t)) \leqslant \frac{\alpha_2}{\alpha_2}\Psi_2^{-1}(\alpha_2 r_2(t)) + \Psi_2^{-1}(\mathrm{Max}\{0, \Psi_2\Psi_1^{-1}(\alpha_1 r_1(0))\mathrm{e}^{-\alpha_1 t} - \alpha_1 r_2(0)\mathrm{e}^{-\alpha_2 t}\}) \tag{79}$$

3. Let r_3 and r_4 be positive and such that

$$\dot{r}_3 \leqslant -\alpha r_3 + \beta\Psi(v) + \gamma, \qquad \dot{r}_4 = -\alpha r_4 + \Psi(v) \tag{80}$$

where $\alpha > 0$, $\beta \geqslant 0$ and $\gamma \geqslant 0$ are real numbers and Ψ is a strictly increasing and convex function with $\Psi(0)=0$. For all $t \in [0, T)$ we have

$$\Psi^{-1}(\alpha r_3(t)) \leqslant \mathrm{Max}\{1,\beta\}\Psi^{-1}(\alpha r_4(t)) + \Psi^{-1}(\mathrm{Max}\{0, \gamma + (\alpha r_3(0) - \gamma - \alpha\beta r_4(0))\mathrm{e}^{-\alpha t}\}) \tag{81}$$

Proof. Point 1 is straightforward. For point 2, according to Jensen's inequality with $\Psi_2\Psi_1^{-1}$ convex, we have

$$\Psi_1^{-1}(\alpha_1 r_1) \leqslant \Psi_1^{-1}\left(\alpha_1 \mathrm{e}^{-\alpha_1 t} r_1(0) + \alpha_1 \int_0^t \mathrm{e}^{-\alpha_1(t-\tau)}\Psi_1(v(\tau))\,\mathrm{d}\tau\right) \tag{82}$$

$$\leqslant \Psi_2^{-1}(a) \tag{83}$$

where a is defined by

$$a = \mathrm{e}^{-\alpha_1 t}\Psi_2\Psi_1^{-1}(\alpha_1 r_1(0)) + \alpha_1 \int_0^t \mathrm{e}^{-\alpha_1(t-\tau)}\Psi_2(v(\tau))\,\mathrm{d}\tau \tag{84}$$

With point 1 we obtain

$$\Psi_1^{-1}(\alpha_1 r_1) \leqslant \frac{\alpha_1}{\alpha_2}\Psi_2^{-1}\left(\alpha_2 \mathrm{e}^{-\alpha_2 t} r_2(0) + \alpha_2 \int_0^t \mathrm{e}^{-\alpha_2(t-\tau)}\Psi_2(v(\tau))\,\mathrm{d}\tau\right) + \Psi_2^{-1}(\mathrm{Max}\{0, \Psi_2\Psi_1^{-1}(\alpha_1 r_1(0))\mathrm{e}^{-\alpha_1 t} - \alpha_1 r_2(0)\mathrm{e}^{-\alpha_2 t}\}) \tag{85}$$

This is point 2.

It remains to prove point 3. From the definition of r_3 we have

$$\Psi^{-1}(\alpha r_3) \leqslant \Psi^{-1}\left(\alpha \mathrm{e}^{-\alpha t} r_3(0) + \alpha\beta \int_0^t \mathrm{e}^{-\alpha(t-s)}\Psi(v(s))\,\mathrm{d}s + \gamma(1-\mathrm{e}^{-\alpha t})\right) \tag{86}$$

With point 1 the conclusion follows as in the proof of point 2. □

4. ROBUST BOUNDEDNESS

Our objective now is to study whether Assumptions R (65), BO (66), S (68) and UEC($\alpha, r_0, \Psi, \Upsilon$) (73) are sufficient to guarantee the existence of a controller solving our problem. We proceed in increasing order of difficulty.

4.1. p^* given and constant

When the vector p^* in (75) is given and constant, the value of p^* is available for computation. Therefore we may propose the control law

$$u = u_{\mathrm{n}}(x, p^*) \tag{87}$$

we get:

Proposition 1

Let Assumptions R (65), BO (66) and S (69) hold. Under these conditions, if for some α, r_0, Ψ and Υ, Assumption UEC($\alpha, r_0, \Psi, \Upsilon$) (73) holds with p^* known and constant, i.e. $\rho = 0$, and

$$1 - \mu_1 - 2c\mu_2 - c \limsup_{v \to \infty} \frac{\mathscr{D}(\mathscr{X})}{\Upsilon(v)} > 0 \tag{88}$$

then all the solutions of (64), (87) with $X(0) \in \mathscr{X}$ are well defined on $[0, \infty)$, unique and bounded. Moreover, if for some $T_0 > 0, D(t) \equiv 0 \quad \forall t \geqslant T_0$ in (75), then

$$\lim_{t \to \infty} x(t) = \mathscr{E} \tag{89}$$

The proof of this proposition is based on the Lyapunov function

$$\mathscr{V}(x, r) = I(V(x, p^*)) + \frac{\varepsilon}{2} r^2 \tag{90}$$

with ε a strictly positive real number and $I(V)$ the function defined by

$$I(V) = \int_0^V \frac{\Psi(\Upsilon(v))^2}{\Upsilon(v)} \, \mathrm{d}v \tag{91}$$

A complete proof can be found in the Appendix.

We remark that if $\liminf_{v \to \infty} \Upsilon(v) = \infty$, there is no constraint on $\mathscr{D}(\mathscr{X})$. It follows that if the function $\mathscr{D}$ is such that $\mathscr{D}(\mathscr{X})$ is well defined for any open set $\mathscr{X}$ of $\mathbb{R}^N$ satisfying $H(\mathscr{X}; 0) = \mathbb{R}^n$, then our result holds globally, i.e. for any initial condition $X(0)$. Also, α, r_0, Ψ and Υ are free to be chosen since they are not fixed by our controller. In particular, α can be chosen depending on the singular perturbation parameter if needed (see the discussion after (48)). Also, we can adjust r_0 to each initial value $X(0)$ in such a way that $D(t)$ can meet the assumption $D(t) \equiv 0 \ \forall t > T_0$ for some T_0 (see (41) for an illustration).

4.2. p^* unknown and time-varying; V does not depend on p

When the vector p^* is unknown and time-varying, the control law (87) cannot be implemented. Instead we use a dynamic controller with state $\hat{p}$,

$$\dot{\hat{p}} = \mathscr{F}(x, \hat{p}), \qquad u = u_{\mathrm{n}}(x, \hat{p}) \tag{92}$$

which is obtained by applying Lyapunov design (see e.g. Reference 20). We have just mentioned that $\mathcal{V}$ defined in (90) is an appropriate Lyapunov function for the case where p^* is known. Since the model equation is affine in the parameter vector, we may try the control Lyapunov function

$$U(x,r,\hat{p}) = I(V(x,\hat{p})) + \frac{\varepsilon}{2} r^2 + \frac{\gamma}{2} \| \hat{p} - p^* \|^2 \tag{93}$$

where γ is a strictly positive real number. Namely, let us design the function $\mathcal{F}$ in (92) so that the time derivative of U along the solutions of the closed-loop model (67), (76), (92)—which is not (64), (92)—is negative assuming (but for the design only) that p^* is constant. If such solutions exist, we get with (70) in Assumptions S (69) and (72)

$$\dot{U} \leqslant -\frac{1}{c} \Psi^2 - \varepsilon\alpha r^2 + \varepsilon r\Psi + \left(\gamma \dot{\hat{p}}^{\mathrm{T}} - \frac{\Psi^2}{\Upsilon} \frac{\partial V}{\partial x} A \right)(\hat{p} - p^*) + \frac{\Psi^2}{\Upsilon} \frac{\partial V}{\partial p} \dot{\hat{p}} \tag{94}$$

Hence in the case where V does not depend on p, this leads us to choose

$$\mathcal{F}(x,\hat{p}) = \frac{1}{\gamma} \frac{\Psi(\Upsilon(V(x)))^2}{\Upsilon(V(x))} A(x, u_{\mathrm{n}}(x,\hat{p}))^{\mathrm{T}} \frac{\partial V}{\partial x}(x)^{\mathrm{T}} \tag{95}$$

Note that as a consequence the values of ε and r are not needed to implement the controller. Moreover, we know that p^* is in the known convex compact subset Π. We use this *a priori* knowledge by projecting $\mathcal{F}$ onto the boundary of Π whenever $\hat{p}$ is on this boundary and $\mathcal{F}$ is pointing outside Π. The following controller results:

$$\dot{\hat{p}} = \mathrm{Proj}\left(\hat{p}, \frac{\Psi(\Upsilon(V(x)))^2}{\gamma \Upsilon(V(x))} A(x, u_{\mathrm{n}}(x,\hat{p}))^{\mathrm{T}} \frac{\partial V}{\partial x}(x)^{\mathrm{T}} \right), \qquad u = u_{\mathrm{n}}(x,\hat{p}) \tag{96}$$

with $\hat{p}(0)$ an interior of Π, i.e. $\hat{p}(0) \in \mathring{\Pi}$, and with some extra but weak restrictions on the set Π, the function Proj can be made locally Lipschitz continuous and to have the property[1,13]

$$(p - p^*)^{\mathrm{T}} \mathrm{Proj}(p, y) \leqslant (p - p^*)^{\mathrm{T}} y, \qquad \| \mathrm{Proj}(p, y) \| \leqslant \| y \| \tag{97}$$

for all (p, p^*, y) in $\mathbb{R}^l \times \mathring{\Pi} \times \mathbb{R}^l$. It remains to study the properties that this controller provides to the actual closed-loop system (64), (96). We have:

Proposition 2

Let Assumptions R (65), BO (66) and S (69) hold with V independent of p. Under these conditions, if for some α, r_0, Ψ and Υ Assumption UEC($\alpha, r_0, \Psi, \Upsilon$) (73) is satisfied with

$$1 - \mu_1 - 2c\mu_2 - c \limsup_{v \to \infty} \frac{\mathscr{D}(\mathscr{X})}{\Upsilon(v)} > 0 \tag{98}$$

and if γ is chosen sufficiently small so that

$$\gamma\rho \sup_{(p_1, p_2) \in \Pi^2} \| p_1 - p_2 \| < \frac{1}{c} \left(1 - \mu_1 - 2c\mu_2 - c \limsup_{v \to \infty} \frac{\mathscr{D}(\mathscr{X})}{\Upsilon(v)} \right) \liminf_{v \to \infty} \Psi(\Upsilon(v))^2 \tag{99}$$

then all the solutions of (64), (96) with $X(0) \in \mathscr{X}$ are well defined on $[0, \infty)$, unique and bounded. Moreover, if $\rho = 0$ and for some $T_0 > 0$, $D(t) \equiv 0 \quad \forall t \geqslant T_0$ in (75), then

$$\lim_{t \to \infty} x(t) = \mathscr{E} \tag{100}$$

Proof. See Appendix. □

We remark:

1. Inequality (99) implies that the larger the speed ρ of the unknown vector p^* or the larger the parametric uncertainty $\sup_{(p_1, p_2) \in \Pi^2} \| p_1 - p_2 \|$, the faster the adaptation should be.
2. Proposition 2 confirms one of the conclusions that can be drawn from the work of Reed and Ioannou[8] and Campion and Bastin[7] for manipulators. In this case where we can choose V independent of the updated parameter vector $\hat{p}$, the only modification which is needed compared with the known parameter vector case is a mechanism guaranteeing boundedness of the updated parameter vector $\hat{p}$. Instead of the projection used here and in Reference 7, Reed and Ioannou proposed the so-called σ-modification.
3. The normalizing signal r is not explicitly used in the controller (96). It follows that α and r_0 are free (see the comments after Proposition 1).
4. In this case where singular perturbations are present, only a local result is obtained in References 7 and 8. This follows from the fact that in these two cases the control appears in the fast subsystem. In our framework this implies that we do not know any quadruple $(\alpha, r_0, \Psi, \Upsilon)$ such that Assumption UEC$(\alpha, r_0, \Psi, \Upsilon)$ (73) holds.
5. Robustness of Lagrange stability has also been established locally by Taylor *et al.*[4] for feedback linearizable models with a parameter-independent linearizing diffeomorphism. This independence implies our parameter-independent V assumption. There is, however, a possibility to extend those results to the case where V depends on the updated parameter vector if a so-called matching condition is satisfied.[1,5] Indeed, if this condition holds, it is possible by augmenting the control in

$$u = u_n(x, \hat{p}) + v(x, \hat{p}, \dot{\hat{p}}) \tag{101}$$

to annihilate by v the term $(\Psi^2/\Upsilon)\ (\partial V/\partial p)\ \dot{\hat{p}}$ in (94). Unfortunately, in this case Assumption UEC$(\alpha, r_0, \Psi, \Upsilon)$ (73) is not sufficient since u is no longer in the class $u_n(\cdot, \hat{p})$. We need to make Assumption UEC$(\alpha, r_0, \Psi, \Upsilon)$ (73) more restrictive by replacing

for any C^1 time function $\hat{p}: \mathbb{R}_+ \to \Pi$ and any solution $X(t)$ of

$$\dot{X} = F(X, t, u_n(H(X, t), \hat{p})), \qquad X(0) \in \mathscr{X} \tag{102}$$

by

for any C^1 time function $\hat{p}: \mathbb{R}_+ \to \Pi$ and $u: \mathbb{R}_+ \to \mathbb{R}^m$ and any solution $X(t)$ of

$$\dot{X} = F(X, t, u(t)), \qquad X(0) \in \mathscr{X} \tag{103}$$

4.3. p^* *unknown but constant,* V *depends on* p

When V depends on p, we have the extra term $(\Psi^2/\Upsilon)\ (\partial V/\partial p)\ \dot{\hat{p}}$ in (94). If such a term cannot be annihilated via the control, we have to consider it as a disturbance and to design a controller which will guarantee robustness and boundedness of the solutions with respect to it. For this design we propose to replace the control Lyapunov function U in (93) by

$$U(x, r, \hat{p}) = L\left(I(V(x, \hat{p})) + \frac{\varepsilon}{2} r^2\right) + \frac{\gamma}{2} \| \hat{p} - p^* \|^2 \tag{104}$$

where the function L is to be designed. For this new function U, the same Lyapunov design as in Section 4.2—without projection—leads to the following inequality, replacing (94):

$$\dot{U} \leqslant \left[\frac{\Psi^2}{c} + \varepsilon\alpha r^2 - \varepsilon\alpha r\Psi - \frac{1}{\gamma}\left(\frac{\Psi^2}{\Upsilon}\right)^2 \frac{\partial V}{\partial p} A^{\mathrm{T}} \frac{\partial V^{\mathrm{T}}}{\partial x} L'\right] L' \tag{105}$$

where L' is the derivative of L. We conclude that, to make the last term in the brackets smaller, this derivative should be positive but as small as possible while guaranteeing radial unboundedness and positive definiteness of L. This leads us to choose $L(x) = \log(1+x)$ and to propose the controller

$$\dot{r} = -\alpha r + \Psi(\Upsilon(V(x,\hat{p}))), \qquad r(0) = r_0$$

$$\dot{\hat{p}} = \mathrm{Proj}\left(\hat{p}, \frac{\dfrac{\Psi(\Upsilon(V(x,\hat{p})))^2}{\Upsilon(V(x,\hat{p}))} A(x, u_{\mathrm{n}}(x,\hat{p}))^{\mathrm{T}} \dfrac{\partial V}{\partial x}(x,\hat{p})^{\mathrm{T}}}{(1 + I(V(x,\hat{p})) + \dfrac{\varepsilon}{2} r^2)\gamma}\right), \qquad u = u_{\mathrm{n}}(x,\hat{p}) \tag{106}$$

with $\hat{p}(0) \in \mathring{\Pi}$ and I defined in (91). Note that α, r_0, Ψ and Υ are involved in this controller. In this case we have:

Proposition 3

Let Assumptions R (65), BO (66) and S (69) hold with, for all $(x,p) \in \mathbb{R}^n \times \Pi$,

$$\left\|\frac{\partial V}{\partial p}\right\| \left\|\frac{\partial V}{\partial x} A(\cdot, u_{\mathrm{n}})\right\| \leqslant d\left(1 + \int_0^V \Upsilon(v)\,\mathrm{d}v\right) \tag{107}$$

where d is a positive real number. In the controller we choose $\alpha, r_0, \Psi, \Upsilon, \varepsilon$ and γ such that

(1) For some positive real number k,

$$\frac{\left(1 + \displaystyle\int_0^v \Upsilon(t)\,\mathrm{d}t\right)^k}{1 + \displaystyle\int_0^v (\Psi(\Upsilon(t))^2/\Upsilon(t))\,\mathrm{d}t}$$

is a non-decreasing function for $v \geqslant 0$

(2)

$$c\varepsilon < 2\alpha\left(1 - \frac{cdk}{\gamma}\right) \tag{108}$$

Under these conditions, if Assumption UEC($\alpha, r_0, \Psi, \Upsilon$) (73) holds with the above given α, r_0, Ψ and Υ and, moreover, if $\mu_1, \mu_2, \mathscr{D}(\mathscr{X})$ and ρ satisfy $\rho = 0$ and

$$c(\varepsilon + \alpha\mu_2)^2 < 2\alpha\varepsilon\left(1 - \mu_1 - c\mu_2 - \frac{cdk}{\gamma} - c \limsup_{v\to\infty} \frac{\mathscr{D}(\mathscr{X})}{\Upsilon(v)}\right) \tag{109}$$

then all the solutions of (64), (106) with $X(0) \in \mathscr{X}$ are well defined on $[0, \infty)$, unique and bounded. Moreover, if for some $T_0 > 0, D(t) \equiv 0 \quad \forall t \geqslant T_0$ in (75), then

$$\lim_{t\to\infty} x(t) = \mathscr{E} \tag{110}$$

Proof. See Appendix. □

We remark:

1. In this case where V depends on the updated parameter vector $\hat{p}$, together with the parameter update projection another modification is used: a normalization. Namely, compared with (96), we have introduced in (106) the denominator $1 + I(V) + (\varepsilon/2)r^2$. Consequently, the normalizing signal r appears explicitly in the controller. This implies in particular that α, r_0, Ψ and Υ are no longer the free parameters we can adjust to show that Assumption UEC($\alpha, r_0, \Psi, \Upsilon$) (73) is satisfied. This is the opposite to the case of Proposition 2 (see remark 3 following Proposition 2).
2. Inequality (107) generalizes the growth condition introduced in Reference 13 for the case $\Upsilon(v) = v$.
3. Monotonicity of
$$\frac{\left(1 + \int_0^v \Upsilon(t)\,\mathrm{d}t\right)^k}{1 + \int_0^v (\Psi(\Upsilon(t))^2/\Upsilon(t))\,\mathrm{d}t}$$
is a weak (technical) growth condition on the functions Ψ and Υ. For example, it is satisfied by
$$\Upsilon(v) = v^n, \qquad \Psi(v) = v^m, \qquad k \geqslant (2m-2)n + \frac{(2m-1)n+1}{n+1}$$
or by
$$\Upsilon(v) = \frac{v}{v+1}, \qquad \Psi(v) = v^m, \qquad k \geqslant 2^{2m+1}$$
4. In contrast with remark 1 following Proposition 2, γ should be large enough for (108) to hold. This is the well-known robustness versus fast adaptation trade-off.
5. All our assumptions are satisfied if the system to be controlled is linear, V is quadratic in x, u_n is linear in x, $\Psi(\Upsilon) = \Upsilon$ and $\Upsilon(v) = cv$. In this case (106) is a new—as far as we know— robust adaptive linear controller which does not require any augmented error.

Example (continued)

During the discussion and the analysis in Section 2 we have seen the Assumptions R (65), BO (66), S (69) and UEC($\alpha, r_0, \Psi, \Upsilon$) (73) are satisfied for the two systems (2) and (3) whose reduced-order models are (1). Now we want to show that the growth condition (107) is satisfied by V and Υ defined by (9) and (31) respectively. Indeed, we have
$$\left|\frac{\partial V}{\partial p}(x_1, x_2, p)\right| = |\chi_2 x_1^2| \leqslant (2V)^{1/2}(2jV)^{1/j} \tag{111}$$
and
$$\left|\frac{\partial V}{\partial x_1} x_1^2\right| = |x_1^{2j+1} + \chi_2[1 + 2px_1 - 3(1-c_1)x_1^2]x_1^2| \tag{112}$$
$$\leqslant (2jV)^{(2j+1)/(2j)} + (2V)^{1/2}(2jV)^{1/j}[1 + 2p_{\max}(2jV)^{1/(2j)} + 3(1+c_1)(2jV)^{1/j}] \tag{113}$$
Therefore there exist two positive real numbers c_4 and c_5 such that
$$\left|\frac{\partial V}{\partial p}\right| \left|\frac{\partial V}{\partial x_1} x_1^2\right| \leqslant c_4 V^{(3j+3)/(2j)} + c_5 \tag{114}$$

but since from (31) we have the equality

$$\int_0^V \Upsilon(v)\, \mathrm{d}v = \frac{c_1 j(2j)^{(j+1)/j}}{2j+1} V^{(2j+1)/j} \tag{115}$$

inequality (107) follows.

With remark 3 following Proposition 3, by using this proposition, our new adaptive regulator applied to (2) and (3) guarantees the global robust boundedness of the solutions (x_1, x_2).

5. CONCLUDING REMARKS

The interest of considering a normalizing signal in the study of the robustness of Lagrange stability of linear systems has been acknowledged by several authors. In particular, it is shown in Reference 12 that this technique allows all the qualitatively possible linear unmodelled dynamics to be captured. The generalization of this concept to non-linear systems is the main idea reported in this paper. We have shown with our worked example of Section 2 that it may also be fruitful in this context by introducing one new way of characterizing the unmodelled effects. Also, by combining this technique with a Lyapunov design and a growth condition, we have been able to establish some new Lagrange stability results for non-linear systems with both parametric and dynamic uncertainties. Nevertheless, we are reporting only preliminary results. More work remains to be done to get a better grip on the properties of this new unmodelled effects characterization we have introduced. It would also be interesting to identify the set of systems for which the growth condition is satisfied. Finally, all the results we have presented were aimed at providing global Lagrange stability. To be more realistic, the local case should be considered.

APPENDIX

Proof of Proposition 1

Since F, H and u_n are locally Lipschitz functions, for each initial condition $X(0) \in \mathscr{X}$ there exists a unique solution $X(t)$ of (64), (87) defined on a right maximal interval $[0, T)$.

To study the properties of this solution, we define a function $I: \mathbb{R}_+ \to \mathbb{R}_+$ by

$$I(V) = \int_0^V \frac{\Psi(\Upsilon(v))^2}{\Upsilon(v)}\, \mathrm{d}v \tag{116}$$

I is well defined. Indeed, from the properties of Ψ and point 1 of Lemma 1, $\Psi(x)/x$ can be defined by continuous extension as a continuous function from $\mathbb{R}_+$ to $\mathbb{R}_+$. The function Υ being continuous also, $\Psi(\Upsilon(v))^2/\Upsilon(v)$ is a continuous function from $\mathbb{R}_+$ to $\mathbb{R}_+$. Moreover, Ψ being strictly increasing and Υ satisfying (72), for all positive real numbers i there exists a positive real number v such that

$$I(V) \leqslant i \quad \Rightarrow \quad V \leqslant v \tag{117}$$

Now let $\mathscr{V}(x, r)$ be defined by

$$\mathscr{V}(x, r) = I(V(x, p^*)) + \frac{\varepsilon}{2} r^2 \tag{118}$$

with ε a strictly positive real number to be made precise later. The previous arguments prove that $\mathscr{V}$ is a C^1 positive definite and radially unbounded function. Let the unmodelled effect associated with the solution $X(t)$ be

$$\omega = \dot{x} - a(x, u) - A(x, u)p^* \tag{119}$$

with (70) in Assumption S (69) and (75), (76) in Assumption UEC$(\alpha, r_0, \Psi, \Upsilon)$ (73), according to the property (72) of Υ, the derivative of $\mathscr{V}$ along this solution satisfies

$$\dot{\mathscr{V}} = \frac{\Psi(\Upsilon)^2}{\Upsilon}\frac{\partial V}{\partial x}(a + Ap^* + \omega) - \varepsilon r(\alpha r - \Psi(\Upsilon)) \tag{120}$$

$$\leqslant -\left(1 - \mu_1 - \frac{c\mathscr{D}(\mathscr{X})}{\Upsilon(V)}\right)\frac{\Psi^2}{c} - \varepsilon\alpha r^2 + \mu_2\frac{\Psi(\Upsilon)^2\Psi^{-1}(\alpha r)}{\Upsilon} + \varepsilon r\Psi \tag{121}$$

Now we have

$$\Psi^{-1}(\alpha r) \leqslant \Upsilon \quad \Rightarrow \quad \frac{\Psi(\Upsilon)^2\Psi^{-1}(\alpha r)}{\Upsilon} \leqslant \Psi(\Upsilon)^2 \tag{122}$$

and, since $\Psi(x)/x$ is non-decreasing,

$$\Psi^{-1}(\alpha r) > \Upsilon \quad \Rightarrow \quad \frac{\Psi(\Upsilon)^2\Psi^{-1}(\alpha r)}{\Upsilon} \leqslant \alpha r\Psi(\Upsilon) \tag{123}$$

This yields

$$\dot{\mathscr{V}} \leqslant -\left(1 - \mu_1 - c\mu_2 - \frac{c\mathscr{D}(\mathscr{X})}{\Upsilon(V)}\right)\frac{\Psi^2}{c} - \varepsilon\alpha r^2 + (\mu_2\alpha + \varepsilon)r\Psi \tag{124}$$

Now we are going to prove the existence of positive real numbers v^*, r^* and $\lambda^* > 0$ such that at each time t where $V(t) \geqslant v^*$ or $r(t) \geqslant r^*$ we have

$$\dot{\mathscr{V}}(t) \leqslant -\frac{\lambda^*}{2}(\Psi(\Upsilon(V(t)))^2 + r(t)^2) \tag{125}$$

Indeed, by assumption we have

$$l \overset{\text{def}}{=} \limsup_{v\to\infty}\frac{\mathscr{D}(\mathscr{X})}{\Upsilon(c)} < \frac{1 - \mu_1 - 2c\mu_2}{c} \tag{126}$$

Then let us define l' by

$$l' \overset{\text{def}}{=} l + \frac{1 - \mu_1 - 2c\mu_2 - cl}{2c} \in \left(l, \frac{1 - \mu_1 - 2c\mu_2}{c}\right) \tag{127}$$

and choose

$$\varepsilon \overset{\text{def}}{=} \frac{\alpha(2 - 2\mu_1 - 3c\mu_2 - 2cl')}{c} \tag{128}$$

We have obtained that at each time t where $\mathscr{D}(\mathscr{X})/\Upsilon(V(t)) < l'$ we have

$$\dot{\mathscr{V}}(t) \leqslant -\lambda^*(\Psi(\Upsilon(V(t)))^2 + r(t)^2) \tag{129}$$

where λ^* is the smallest root of

$$c^2\lambda^2 - c[(1 + 2\alpha^2)(1 - \mu_1 - c\mu_2 - cl') - c\mu_2\alpha^2]\lambda + \alpha^2(1 - \mu_1 - c\mu_2 - cl')(1 - \mu_1 - 2c\mu_2 - cl') = 0 \tag{130}$$

which is strictly positive thanks to (88). On the other hand, from the definitions of l and l' there exists a positive real number v^* such that

$$\frac{\mathscr{D}(\mathscr{X})}{\Upsilon(v)} \geqslant l' \quad \Rightarrow \quad v \leqslant v^* \tag{131}$$

It follows that at each time t where $\mathscr{D}(\mathscr{X})/\Upsilon(V(t)) \geqslant l'$ we have

$$\dot{\mathscr{V}}(t) \leqslant -\frac{1 - \mu_1 - c\mu_2 - cl'}{c}\Psi(\Upsilon(V(t)))^2 - \varepsilon\alpha r(t)^2 + (\mu_2\alpha + \varepsilon)r(t)\Psi(\Upsilon(V(t)))$$

$$+ \left(\frac{\mathscr{D}(\mathscr{X})}{\Upsilon(V(t))} - l'\right)\Psi(\Upsilon(V(t)))^2 \tag{132}$$

$$\leqslant -\frac{\lambda^*}{2}(\Psi(\Upsilon(V(t)))^2 + r(t)^2) + \frac{\lambda^*}{2}(r^{*2} - r(t)^2) \tag{133}$$

where, thanks to the continuity of the function $\Psi(\Upsilon(v))^2/\Upsilon(v)$, the positive real number r^* is defined by

$$r^{*2} \overset{\text{def}}{=} \frac{2\mathscr{D}(\mathscr{X})}{\lambda^*} \sup_{0 \leqslant v \leqslant v^*} \frac{\Psi(\Upsilon(v))^2}{\Upsilon(v)} \tag{134}$$

Now assume that T is finite. This implies that

$$\lim_{t \to T} \| X(t) \| = \infty \tag{135}$$

However, since $\mathscr{V}(x, r)$ is radially unbounded, V satisfies point (3) in Assumption S (69) and $\dot{\mathscr{V}}$ is strictly negative whenever V or r is large, there exists a compact set K_x in $\mathbb{R}^n$ such that

$$x(t) \in K_x \quad \forall t \in [0, T) \tag{136}$$

With continuity of u_n and Assumption BO (66), this implies the existence of a compact set K_X such that

$$X(t) \in K_X \quad \forall t \in [0, T) \tag{137}$$

which contradicts (135). It follows that $T = \infty$ and the solution $X(t)$ is bounded.

Now let us consider the case where $D(t) \equiv 0 \quad \forall t \geqslant T_0$. We know from the above analysis that all the solutions with $X(0) \in \mathscr{X}$ are well defined and bounded on $[0, \infty)$, so also in particular on $[0, T_0)$. Therefore we can reproduce this analysis but starting now at time T_0, allowing us in this case to replace $\mathscr{D}(\mathscr{X})$ by 0. This leads to $v^* = r^* = 0$. It follows that for each $t \geqslant T_0$ we have

$$\dot{\mathscr{V}}(t) \leqslant -\lambda^*(\Psi(\Upsilon(V(t)))^2 + r(t)^2) \tag{138}$$

From Assumption R (65) and Barbălat's lemma (Reference 21, p. 211) it follows that $\Psi(\Upsilon(V(x(t), p^*)))$ tends to zero as t tends to infinity. With the properties of Ψ and points (2) and (3) of Assumptions S (69) we conclude that

$$D(t) \equiv 0 \quad \forall t \geqslant T_0 \quad \Rightarrow \quad \lim_{t \to \infty} x(t) = \mathscr{E} \tag{139}$$

□

Proof of Proposition 2

Since the right-hand side of (64), (96) is locally Lipschitz, for each initial condition $\hat{p}(0) \in \mathring{\Pi}$ and $X(0) \in \mathscr{X}$ there exists a unique solution $(x(t), \hat{p}(t))$ of (64), (96) defined on a right maximal interval $[0, T)$. Along this solution we have, with (97),

$$\dot{U} \leqslant -\left(1 - \mu_1 - \frac{c\mathscr{D}(\mathscr{X})}{\Upsilon(V)}\right)\frac{\Psi^2}{c} - \varepsilon\alpha r^2 + \mu^2 \frac{\Psi(\Upsilon)^2 \Psi^{-1}(\alpha r)}{\Upsilon} + \varepsilon r \Psi - \gamma(\hat{p} - p^*)^{\mathrm{T}} \dot{p}^* \tag{140}$$

However, thanks to Proj, $\hat{p}$ remains in the compact set Π. Since by assumption the same holds for p^* and $\| \dot{p}^* \| \leqslant \rho$, there exists a positive real number k, depending only on Π, such that

$$\dot{U} \leqslant -\left(1 - \mu_1 \frac{c\mathscr{D}(\mathscr{X})}{\Upsilon} - \frac{c\gamma k \rho}{\Psi^2}\right)\frac{\Psi^2}{c} - \varepsilon\alpha r^2 + \mu_2 \frac{\Psi(\Upsilon)^2 \Psi^{-1}(\alpha r)}{\Upsilon} + \varepsilon r \Psi \tag{141}$$

Then, as in the proof of Proposition 1, we can show the existence of positive real numbers v^*, r^* and $\lambda^* > 0$ such that at each time t where $V(t) \geqslant v^*$ or $r(t) \geqslant r^*$ we have

$$\dot{U}(t) \leqslant \frac{\lambda^*}{2}(\Psi(\Upsilon(V(t)))^2 + r(t)^2) \tag{142}$$

Now assume that T is finite. Since $\hat{p}(t)$ remains in the compact set Π, this implies that

$$\lim_{t/T} \| X(t) \| = \infty \tag{143}$$

However, since $U(x, r, \hat{p})$ is radially unbounded, $\hat{p}$ remains in the compact set Π, V satisfies point (3) in Assumption S (69) and $\dot{U}$ is strictly negative whenever V or r is large, there exists a compact set K_x in $\mathbb{R}^n$ such that

$$x(t) \in K_x \quad \forall t \in [0, T) \tag{144}$$

With continuity of u_n and Assumption BO (66) this implies the existence of a compact set K_X in $\mathbb{R}^N$ such that

$$X(t) \in K_X \quad \forall t \in [0, T) \tag{145}$$

This contradicts (143). It follows that $T = \infty$ and the solution $X(t)$ is bounded. Finally we observe that if $\rho = 0$ and $D(t) \equiv 0 \quad \forall t \geqslant T_0$, then we can choose $v^* = r^* = 0$ for $t \geqslant T_0$. This implies that

$$\dot{U} \leqslant -\lambda^*(\Psi^2 + r^2) \tag{146}$$

From Assumption R (65) and Barbălat's lemma (References 21, p. 211) it follows that $\Psi(\Upsilon(V(x(t))))$ tends to zero as t tends to infinity. With the properties of Ψ and points (2) and (3) of Assumptions S (69) we conclude that

$$\rho = 0 \quad \text{and} \quad D(t) \equiv 0 \quad \forall t \geqslant T_0 \quad \Rightarrow \quad \lim_{t \to \infty} x(t) = \mathscr{E} \tag{147}$$

We remark that if $\mathscr{D}(\mathscr{X}) = \rho = 0$, the compactness of Π is not needed since in this case (142) implies the boundedness of U and therefore of $\hat{p} - p^*$. □

Proof of Proposition 3

Since the right-hand side of (64), (106) is locally Lipschitz, for each initial conditions $\hat{p}(0) \in \mathring{\Pi}$, $r(0) \geqslant 0$ and $X(0) \in \mathscr{X}$ there exists a unique solution $(X(t), \hat{p}(t), r(t))$ of (64), (106) defined on a right maximal interval $[0, T)$. Along this solution we have, with (97) and U defined in (104),

$$\dot{U} \leqslant \frac{1}{\Delta}\left[-\left(1 - \mu_1 - \frac{c\mathscr{D}(\mathscr{X})}{\Upsilon(V)}\right)\frac{\Psi^2}{c} - \varepsilon\alpha r^2 + \mu_2 \frac{\Psi(\Upsilon)^{2\Psi - 1}(\alpha r)}{\Upsilon} + \varepsilon r \Psi + \frac{\Psi^2}{\Upsilon}\frac{\partial V}{\partial p}\dot{\hat{p}}\right] \tag{148}$$

$$\Delta \overset{\text{def}}{=} 1 + \int_0^V \frac{\Psi(\Upsilon(v))^2}{\Upsilon(v)}\,\mathrm{d}v + \frac{\varepsilon}{2} r^2 \tag{149}$$

To go further, we observe that,

$$\frac{\left(1 + \int_0^v \Upsilon(t)\,\mathrm{d}t\right)^k}{1 + \int_0^v (\Psi(\Upsilon(t))^2/\Upsilon(t))\,\mathrm{d}t}$$

being a non-decreasing function, we have by taking its logarithm,

$$\frac{\Psi(\Upsilon(v))^2}{\Upsilon(v)^2}\,\frac{1 + \int_0^v \Upsilon(t)\,\mathrm{d}t}{1 + \int_0^v (\Psi(\Upsilon(t))^2/\Upsilon(t))\,\mathrm{d}t} \leqslant k \tag{150}$$

Then with (97), (106) and (107) we have

$$\left|\frac{\Psi^2}{\Upsilon}\frac{\partial V}{\partial p}\dot{\hat{p}}\right| \leqslant \frac{l}{\gamma\Delta}\frac{\Psi^4}{\Upsilon^2}\left\|\frac{\partial V}{\partial p}(x, \hat{p})\right\|\,\left\|A(x, u_n(x, \hat{p}))^{\mathrm{T}}\frac{\partial V}{\partial x}(x, \hat{p})^{\mathrm{T}}\right\| \tag{151}$$

$$\leqslant \Psi^2 \frac{dk}{\gamma} \tag{152}$$

Returning to (148) and following the same steps as after (121), we get

$$\dot{U} \leqslant \frac{1}{\Delta}\left[-\left(1 - \mu_1 - c\mu_2 - \frac{cdk}{\gamma} - \frac{c\mathscr{D}(\mathscr{X})}{\Upsilon}\right)\frac{\Psi^2}{c} - \varepsilon\alpha r^2 + (\varepsilon + \alpha\mu_2) r\Psi\right] \tag{153}$$

Then we define

$$l \overset{\text{def}}{=} \limsup_{v \to \infty} \frac{\mathscr{D}(\mathscr{X})}{\Upsilon(v)}, \qquad l' \overset{\text{def}}{=} l + \frac{1 - \mu_1 - c\mu_2 - cdk/\gamma - cl}{2c} > l \tag{154}$$

At each time t where $\mathscr{D}(\mathscr{X})/\Upsilon(V(t)) < l'$ we have

$$\dot{U} \leqslant \frac{1}{\Delta}\left[-\frac{1}{2c}\left(1 - \mu_1 - c\mu_2 - \frac{cdk}{\gamma} - cl\right)\Psi^2 - \varepsilon\alpha r^2 + (\varepsilon + \alpha\mu_2) r\psi\right] \tag{155}$$

$$\leqslant -\frac{\lambda^*}{\Delta}(\Psi^2 + r^2) \tag{156}$$

where λ^* is the smallest root of

$$4c\lambda^2 - 2\left(1 - \mu_1 - c\mu_2 - \frac{cdk}{\gamma} - cl + 2c\varepsilon\alpha\right)\lambda + \left[2\alpha\varepsilon\left(1 - \mu_1 - c\mu_2 - \frac{cdk}{\gamma} - cl\right) - c(\varepsilon + \alpha\mu_2)^2\right] = 0 \quad (157)$$

which is strictly positive thanks to (109). On the other hand, there exists a positive real number v^* such that

$$\frac{\mathscr{D}(\mathscr{X})}{\Upsilon(v)} \geqslant l' \quad \Rightarrow \quad v \leqslant v^* \qquad (158)$$

Therefore at each time t where $\mathscr{D}(\mathscr{X})/\Upsilon(V(t)) \geqslant l'$ we have

$$\begin{aligned} \dot{U} &\leqslant \frac{1}{\Delta}\left[-\frac{1}{2c}\left(1 - \mu_1 - c\mu_2 - \frac{cdk}{\gamma} - cl\right)\Psi^2 - \varepsilon\alpha r^2 + (\varepsilon + \alpha\mu_2)r\Psi + \left(\frac{\mathscr{D}(\mathscr{X})}{\Upsilon(V(t))} - l'\right)\Psi^2\right] \\ &\leqslant \frac{1}{\Delta}\left(-\lambda^*(\Psi^2 + r^2) + \sup_{0 \leqslant v \leqslant v^*} \frac{\mathscr{D}(\mathscr{X})\Psi(\Upsilon(v))^2}{\Upsilon(v)}\right) \end{aligned} \qquad (159)$$

The proof is concluded with the same arguments as in the proof of Proposition 2. □

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