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Adding an integrator for the stabilization problem

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Abstract: We study the relationship between the following two properties: P1: The system $\dot{x} = f(x, y)$, $\dot{y} = v$ is locally asymptotically stabilizable; and P2: The system $\dot{x} = f(x, u)$ is locally asymptotically stabilizable; where $x \in \mathbb{R}^n$, $y \in \mathbb{R}$. Dayawansa, Martin and Knowles have proved that these properties are equivalent if the dimension $n = 1$. Here, using the so called Control Lyapunov function approach, (a) we propose another more constructive and somewhat simpler proof of Dayawansa, Martin and Knowles's result; (b) we show that, in general, P1 does not imply P2 for dimensions n larger than 1; (c) we prove that P2 implies P1 if some extra assumptions are added like homogeneity of the system. By using the latter result recursively, we obtain a sufficient condition for the local asymptotic stabilizability of systems in a triangular form.

Keywords: Continuous stabilization; dynamic feedback; control Lyapunov function.

1. Introduction and Results

The goal of this paper is to study the relation between the following two properties:

P1: The system $\dot{x} = f(x, y)$, $\dot{y} = v$ is locally asymptotically stabilizable (LAS). Namely, there exist a neighborhood U of (0, 0) in $\mathbb{R}^n \times \mathbb{R}$ and a continuous function $v : U \to \mathbb{R}$ such that $v(0, 0) = 0$ and (0, 0) is an asymptotically stable equilibrium point of $\dot{x} = f(x, y)$, $\dot{y} = v$.

P2: The system $\dot{x} = f(x, u)$ is LAS.

Our starting point in this study will be to provide a constructive and somewhat simpler proof and to generalize a nice result recently obtained in [5,8] by Dayawansa, Martin and Knowles on the asymptotic stabilization of two dimensional real analytic systems. This result concerns systems which, may be after a diffeomorphism and a preliminary feedback, can be written as

$$\dot{x} = f(x, y), \qquad \dot{y} = v, \tag{1}$$

where f is a function defined in an open neighborhood Ω of (0, 0) in $\mathbb{R}^2$.

Theorem 1 (Dayawansa, Martin and Knowles [5,8]). *If the dimension* $n = 1$ *and* f *is a real analytic function, property* P1 *is equivalent to*:

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P3. *For all $\varepsilon > 0$ there exists two points (x_0, y_0) and (x_1, y_1) in $\mathbb{R}^2$ such that*

$$x_0 > 0, \quad x_1 < 0, \quad f(x_0, y_0) < 0, \quad f(x_1, y_1) > 0, \quad x_0 < \varepsilon, \quad |y_0| < \varepsilon, \quad |x_1| < \varepsilon \text{ and } |y_1| < \varepsilon. \tag{2}$$

Remarks. (a) In [5,8] it is also proved that if P1 or P3 holds then v in P1 can be chosen such that $v \in C^\infty(U \setminus \{(0, 0)\})$. In fact, this is a consequence of a general result proved by Sontag in [17, Section 7] and saying that, for n dimensional systems, condition P1 is equivalent to the same condition with the extra property $v \in C^\infty(U \setminus \{(0, 0)\})$. In this paper, the function v we propose is in $C^1(U \setminus \{(0, 0)\})$. Note also that it follows from [17, Section 7] that one does not have to specify in the definition of *locally asymptotically stabilizable* if one requires uniqueness of the trajectory for a given initial data.

(b) In [5,8] it is moreover proved that P3 implies P1 with a function v such that, for some constant C,

$$|v(x, y)| \leq C(|x| + |y|) \quad \forall (x, y) \in U. \tag{3}$$

Here, instead of (3), we shall get that, for some constants α in $(0, 1]$ and C,

$$|v(x, y) - v(x', y')| \leq C(|x - x'|^\alpha + |y - y'|) \quad \forall (x, y) \in U, \forall (x', y') \in U. \tag{4}$$

This generalizes the result of Kawski [12] that two dimensional systems which are locally controllable at (0, 0) are stabilizable by Hölder continuous state feedback. This result is in some sense optimal since Dayawansa and Martin have proved in [6], that v in general cannot be smoother than Hölder continuous.

As noticed by Dayawansa and Martin in [5,8] the implication P1 $\Rightarrow$ P3 is obvious. Our proof of P3 $\Rightarrow$ P1 is divided into two lemmas.

Lemma 1. *Assume that* P1 *holds. There exist $\delta > 0$, two real analytic functions u_+ and u_- in $(-\delta, \delta)$ which do not vanish at zero and four positive integers p_+, q_+, p_- and q_- such that*:

1. *By defining the control law*

$$u(x) = \begin{cases} x^{p_+/q_+} u_+(x^{1/q_+}) & \text{if } x \geq 0, \\ |x|^{p_-/q_-} u_-(|x|^{1/q_-}) & \text{if } x < 0, \end{cases} \tag{5}$$

we have

$$x f(x, u(x)) < 0 \quad \forall x \in (-\delta, \delta) \setminus \{0\}, \tag{6}$$

i.e. the origin is an asymptotically stable equilibrium point of the system $\dot{x} = f(x, u(x))$.

2. *For (x, y) in a neighborhood of* (0, 0), *we have*

$$|f(x, y)| \leq C(|x| + |y|^{1/\alpha}) \tag{7}$$

where

$$\alpha = \min\left\{1, \frac{p_+}{q_+}, \frac{p_-}{q_-}\right\}. \tag{8}$$

With the function u defined in Lemma 1, we can now follow [15, Section 3]. For the desingularizing function $\phi(x, y)$, we take $y^p - u(x)^p$ and, for $h_0(x)$, we take $x^{m+1}/(m+1)$ where p and m are odd integers. This provides the control Lyapunov function

$$h(x, y) = \frac{y^{p+1}}{p+1} - u(x)^p y + \frac{p}{p+1} u(x)^{p+1} + \frac{x^{m+1}}{m+1} \tag{9}$$

and the control law

$$v(x, y) = \frac{y - u(x)}{y^p - u(x)^p} f(x, y) \frac{\mathrm{d}u^p}{\mathrm{d}x}(x) - \left(y^p - u(x)^p\right) - x^m \frac{f(x, y) - f(x, u(x))}{y^p - u(x)^p}. \tag{10}$$

Our second lemma proves that such a control law v is well defined and studies its smoothness.

Lemma 2. *Let Ω be an open subset of $\mathbb{R}^2$ containing $(0, 0)$ and let δ and r be two strictly positive real numbers such that*

$$U := \{(x, y) \mid |x| < r,\ |y| < r\} \subset \Omega \cap (-\delta, \delta) \times \mathbb{R}. \tag{11}$$

Consider three functions $f: \Omega \to \mathbb{R}$, $u: (-\delta, \delta) \to \mathbb{R}$ and $v: U \to \mathbb{R}$ such that, for some constants C, $\alpha \in (0, 1]$ and $\beta \geq \alpha$, we have

$$u \in C^0((-\delta, \delta)) \cap C^2((-\delta, \delta) \setminus \{0\}), \quad u(0) = 0, \tag{12}$$

$$\left|\frac{\mathrm{d}^2 u}{\mathrm{d}x^2}(x)\right| \leq C|x|^{-2+\alpha} \text{ and } |x|\left|\frac{\mathrm{d}u}{\mathrm{d}x}(x)\right| \leq C|u(x)| \quad \forall x \in (-\delta, \delta) \setminus \{0\}, \tag{13}$$

$$|u(x)| \geq \frac{1}{C}|x|^\beta \quad \forall x \in (-\delta, \delta), \tag{14}$$

$$f \in C^2(\Omega), \quad f(0, 0) = 0 \text{ and } \left|\frac{\partial f}{\partial y}(x, y)\right| \leq C(|x| + |y|^{(1/\alpha)-1}) \ \forall (x, y) \in \Omega, \tag{15}$$

and

$$v(0, y) = -y^p,$$

$$v(x, u(x)) = \frac{f(x, u(x)) \dfrac{\mathrm{d}u^p}{\mathrm{d}x}(x) - x^m \dfrac{\partial f}{\partial y}(x, u(x))}{p u(x)^{p-1}} \quad \text{if } x \neq 0,$$

$$v(x, y) = \frac{y - u}{y^p - u^p} f \frac{\mathrm{d}u^p}{\mathrm{d}x} - (y^p - u^p) - x^m \frac{f(x, y) - f(x, u)}{y^p - u^p} \quad \text{if } y \neq u \text{ and } x \neq 0, \tag{16}$$

where m and p are odd integers, chosen to satisfy

$$\alpha p > 2, \tag{17}$$

$$m \geq \beta(p - 1) + \alpha. \tag{18}$$

Under these conditions, $v \in C^0(U) \cap C^1(U \setminus \{(0, 0)\})$ and satisfies for some constant C_1,

$$|v(x, y) - v(x', y')| \leq C_1(|x - x'|^\alpha + |y - y'|) \quad \forall (x, y) \in U,\ \forall (x', y') \in U. \tag{19}$$

The implication P3 $\Rightarrow$ P1 now follows from Lemma 1 and Lemma 2. Indeed, with u and α defined by Lemma 1 and with $\beta = \max\{p_+/q_+, p_-/q_-\}$, one easily sees that (12), (13), (14) and (15) are satisfied. Moreover if one computes the time derivative of the Lyapunov function h, defined in (9), one gets

$$\dot{h} = f(x, y) \frac{\partial h}{\partial x}(x, y) + v \frac{\partial h}{\partial y}(x, y) = -\left(y^p - u(x)^p\right)^2 - x^m f(x, u(x)). \tag{20}$$

Hence by (6), $\dot{h} < 0$ in $U \setminus \{(0, 0)\}$ which proves that $(0, 0)$ is an asymptotically stable equilibrium point of (1).

A consequence of Theorem 1 and of our construction is that, when the dimension $n = 1$, if $\dot{x} = f(x, y)$, $\dot{y} = v$ is LAS and f is real analytic then this system can be locally asymptotically stabilized with an Hölder continuous feedback law. It would be interesting to know if this holds for any analytic systems in higher dimensions.

Let us remark also that Lemma 1 shows that, when f is a real analytic function and the dimension $n=1$, property P3 implies property P2. Hence another consequence of Theorem 1 is the implication:

$$\text{if } \dot{x}=f(x, y),\ \dot{y}=v \text{ is LAS then } \dot{x}=f(x, u) \text{ is LAS.}$$

This is particular to the dimension $n=1$. Indeed we will prove:

Proposition 1. *Let, for a constant C and $n \geq 2$, a function $f : \mathbb{R}^n \times \mathbb{R} \simeq \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^n$ be defined by*

$$f(x, y)=f(x_1, x_2, y)=-\left[\left(|x_1|^2+x_2^2\right)^3-C^2\left(y^3-|x_1|^2 y+x_2^3\right)^2\right] x. \tag{21}$$

For C large enough, the system $\dot{x}=f(x, u)$ is not LAS *but the system $\dot{x}=f(x, y)$, $\dot{y}=v$ is* LAS.

Remarks. (a) This phenomenon is purely nonlinear: if f is linear and $\dot{x}=f(x, y)$, $\dot{y}=v$ is LAS, then $\dot{x}=f(x, u)$ is LAS.

(b) This proposition shows that it may be helpful to add integrators to locally asymptotically stabilize a system. It has been previously proved that integrators can also be useful for global stabilization (Sontag and Sussmann [18]) and for smoothing feedback laws (Boothby and Marino [3]).

(c) A consequence of our proof of Proposition 1 is that for C large enough and n odd, there exists no function u such that, for some neighborhood W of 0,

$$u \in C^0(W), \quad u(0)=0, \quad f(x, u(x)) \neq 0 \ \forall x \in W \setminus \{0\} \text{ and } \operatorname{index}(f(\cdot, u(\cdot)), 0)=(-1)^n. \tag{22}$$

Hence the existence of a control law u satisfying (22) which is necessary for the system $\dot{x}=f(x, u)$ to be LAS (see [11, Theorem 5.2.1]) is not necessary for that system to be dynamically locally asymptotically stabilizable.

These considerations lead naturally to the following open question:

Question. Let f be a real analytic map defined in a neighborhood Ω of (0, 0) in $\mathbb{R}^n \times \mathbb{R}$ with values into $\mathbb{R}^n$. Assume that $f(0, 0)=0$ and that the system $\dot{x}=f(x, u)$ is LAS. Is the system $\dot{x}=f(x, y)$, $\dot{y}=v$ LAS?

It has been proved by Tsinias in [19, Theorem 3.c] that the answer to that question is yes if one assumes moreover that $\dot{x}=f(x, u)$ is locally asymptotically stabilizable with a C^1 feedback law $u(x)$. Our next proposition is an improvement of this result which, with Lemma 1, also gives a proof of P3 $\Rightarrow$ P1 for $n=1$.

Proposition 2. *Let Ω be an open subset of $\mathbb{R}^n \times \mathbb{R}$ containing* (0, 0) *and $f : \Omega \to \mathbb{R}^n$ be a C^2 map such that* $f(0, 0)=0$. *Assume the existence of an open neighborhood U of 0 in $\mathbb{R}^n$ and of a continuous map $u : U \to \mathbb{R}$ such that, for some constant α in* (0, 1] *and some constant C in* $(0, +\infty)$, *we have*

$$u(0)=0, \tag{23}$$

$$u \in C^1(U \setminus \{0\}), \tag{24}$$

$$\left|\frac{\partial u}{\partial x}(x)\right| \leq C|x|^{\alpha-1} \quad \forall x \in U \setminus \{0\}, \tag{25}$$

$$|f(x, y)| \leq C\left(|x|+|y|^{1/\alpha}\right) \quad \forall (x, y) \in \Omega, \tag{26}$$

$$0 \textit{ is an asymptotically stable equilibrium point of } \dot{x}=f(x, u). \tag{27}$$

Under these conditions, the system $\dot{x}=f(x, y)$, $\dot{y}=v$ is LAS.

We prove also that for any dimension n, but for homogeneous systems and homogeneous feedback laws, P2 implies P1. More precisely, we prove:

Proposition 3. *Let* $f=(f_i)_{i=1,n}:\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}^n$ *be a* C^1 *map such that*

$$\forall i\in\{1,\dots,n\},\ \forall x=(x_i)_{i=1,n}\in\mathbb{R}^n,\ \forall\varepsilon\geq 0,\ \forall u\in\mathbb{R}$$
$$f_i(\varepsilon^{r_1}x_1,\dots,\varepsilon^{r_n}x_n,\ \varepsilon^{r_{n+1}}u)=\varepsilon^{\tau+r_i}f_i(x_1,\dots,x_n,\ u) \tag{28}$$

for some $r_i>0$ *and some* $\tau\in(-\min_j\{r_j\},\ +\infty)$. *Assume that the system* $\dot{x}=f(x,\ u)$ *is globally asymptotically stabilizable with a continuous feedback law* $u:\mathbb{R}^n\to\mathbb{R}$ *such that*

$$u(\varepsilon^{r_1}x_1,\dots,\varepsilon^{r_n}x_n)=\varepsilon^{r_{n+1}}u(x_1,\dots,x_n)\quad\forall x=(x_i)_{i=1,n}\in\mathbb{R}^n,\ \forall\varepsilon\geq 0. \tag{29}$$

Under these conditions, the system $\dot{x}=f(x,\ y)$, $\dot{y}=v$ *is globally asymptotically stabilizable with a continuous feedback law* $v:\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}$ *such that*

$$\forall x=(x_i)_{i=1,n}\in\mathbb{R}^n,\ \forall y\in\mathbb{R},\ \forall\varepsilon\geq 0\quad v(\varepsilon^{r_1}x_1,\dots,\varepsilon^{r_n}x_n,\ \varepsilon^{r_{n+1}}y)=\varepsilon^{\tau+r_{n+1}}v(x_1,\dots,x_n,\ y). \tag{30}$$

Remark. (a) Stabilization of homogeneous systems has recently received a lot of attention – see for example [1,7,8,10,13]. In particular, Proposition 3 has been proved previously by Dayawansa and Martin in [7] for the special case $n=2$, $r_1=r_2$ and by Andreini, Bacciotti and Stefani in [1] for the special case where u can be chosen C^1.

(b) In the definition of *global asymptotic stabilizability* we are using here, there is no need to precise if uniqueness of the solutions is required. This follows from the following lemma which is a key technical step in the proof of Proposition 3:

Lemma 3. *Under the assumptions of Proposition* 3, *there exist*:

- *a stabilizing control law* $\bar{u}:\mathbb{R}^n\to\mathbb{R}$ *in* $C^1(\mathbb{R}^n\setminus\{0\})\cap C^0(\mathbb{R}^n)$ *which satisfies*

$$\bar{u}(\varepsilon^{r_1}x_1,\dots,\varepsilon^{r_n}x_n)=\varepsilon^{r_{n+1}}\bar{u}(x_1,\dots,x_n)\quad\forall x=(x_i)_{i=1,n}\in\mathbb{R}^n,\ \forall\varepsilon\geq 0; \tag{31}$$

- *a* C^1 *function* $\bar{V}:\mathbb{R}^n\to\mathbb{R}$ *which is positive definite* (*see* [9, *Def.* 24.3]), *radially unbounded* (*see* [9, *Def.* 24.5]) *and satisfies*

$$\bar{V}(\varepsilon^{r_1}x_1,\dots,\varepsilon^{r_n}x_n)=\varepsilon^{k}\bar{V}(x_1,\dots,x_n)\quad\forall x=(x_i)_{i=1,n}\in\mathbb{R}^n,\ \forall\varepsilon\geq 0, \tag{32}$$

where k is a real number satisfying

$$k>r_i,\quad\forall i\in\{1,\dots,n\}; \tag{33}$$

such that

$$\frac{\partial\bar{V}}{\partial x}(x)f(x,\ \bar{u}(x))<0\quad\forall x\in\mathbb{R}^n\setminus\{0\}. \tag{34}$$

A consequence of Proposition 3 and of a theorem on robust stability for homogeneous systems, proved by Hermes in [10, Theorem 1], is:

Corollary 1. *Let* $f=(f_i)_{i=1,n}:\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}^n$ *be a* C^∞ *map such that*:

$$f(0,0)=0; \tag{35}$$

the function $f_i(x,\ u)$ *does not depend on* $(x_{i+2},\dots,u)$ *for* $i\in\{1,\dots,n-1\}$, (36) *for any* $i \in \{1, \ldots, n\}$, *there exists an odd integer* p_{i+1} *such that, with* $x_{n+1} = u$,

$$\frac{\partial^{p_{i+1}} f_i}{\partial x_{i+1}^{p_{i+1}}}(0) \neq 0 \quad \text{and} \quad \frac{\partial^l f_i}{\partial x_{i+1}^l}(0) = 0 \quad \forall l \in \{0, \ldots, p_{i+1} - 1\}. \tag{37}$$

Under these conditions, the system $\dot{x} = f(x, u)$ *is* LAS.

According to this corollary, the following system for example is LAS:

$$\dot{x}_1 = f_1(x_1) + x_2^{p_2}, \tag{38a}$$

$$\dot{x}_i = f_i(x_1, \ldots, x_i) + x_{i+1}^{p_{i+1}}, \tag{38b}$$

$$\dot{x}_n = f_n(x_1, \ldots, x_n) + u^{p_{i+1}}, \tag{38c}$$

when the p_i's are odd integer numbers.

The following sections are devoted to the proof of our various statements.

2. Proof of Lemma 1

We prove this lemma for $x > 0$. The proof for $x < 0$ is similar. Let A be the following set:

$$A = \{(x, y) \in \Omega \mid f(x, y) < 0 \text{ and } x > 0\}. \tag{39}$$

It is semianalytic and, from P3, (0, 0) is in its closure $\bar{A}$. Hence, by a theorem due to Whitney and Bruhat [20], we know there exists an analytic function $\phi : (-1, 1) \to \Omega$, $s \mapsto (x(s), y(s))$ such that

$$x(0) = y(0) = 0 \quad \text{and} \quad (x(s), y(s)) \in A \neq \forall s \in (0, 1). \tag{40}$$

From (40) and since the functions f and x are analytic, one gets the existence of an integer l such that, for all sufficiently small strictly positive real number s, we have

$$f(x(s), y(s)) + x(s)^l < 0. \tag{41}$$

Now, let $\eta \in (0, 1]$ be such that

$$f(t, 0) + \eta t^l \not\equiv 0 \tag{42}$$

and let $g : \Omega \to \mathbb{R}$ be the real analytic function defined by

$$g(x, y) = f(x, y) + \eta x^l. \tag{43}$$

It follows from (40) and (41) that

$$(0, 0) \in \overline{\{(x, y) \in \Omega \mid g(x, y) < 0 \text{ and } x > 0\}}. \tag{44}$$

Then, there are only two possibilities: either

$$\text{in a neighborhood of } (0, 0), \text{ “}x > 0\text{” implies “}g(x, y) < 0\text{”}, \tag{45}$$

or

$$(0, 0) \in \overline{\{(x, y) \in \Omega \mid g(x, y) = 0 \text{ and } x > 0\}} =: B. \tag{46}$$

In case (45) one can take for example $u_+(x) = x$, $p_+ = 1$, $q_+ = 1$. In case (46), by applying Whitney and Bruhat Theorem [20] to the semianalytic set B, by expressing s in terms of x^{1/q_+} and by substituting in y, we obtain the existence of two positive integer numbers p_+ and q_+, of a strictly positive real number δ and of an analytic function u_+ defined in $(-\delta, \delta)$ such that, using (42), i.e. $g(x, 0) \not\equiv 0$,

$$g(x, x^{p_+/q_+} u_+(x^{1/q_+})) = f(x, x^{p_+/q_+} u_+(x^{1/q_+})) + \eta x^l = 0 \quad \forall x \in (0, \delta), \tag{47}$$

$$u_+(0) \neq 0. \tag{48}$$

Equation (47) gives

$$f\left(x,\ x^{p_+/q_+}u_+\left(x^{1/q_+}\right)\right) < 0 \quad \forall x \in (0,\ \delta) \tag{49}$$

Finally, using (47) and (48) in the expansion of f in powers of x and y, we get

$$|f(x,\ y)| \le C\left(|x| + |y|^{\max\{1, q_+/p_+\}}\right), \tag{50}$$

for some constant C and in a neighborhood of (0, 0).

3. Proof of Lemma 2

We denote by C the various constants independent of $(x,\ y)$ in U. First note:
From (12) and (13), we have

$$\left|\frac{\mathrm{d}u}{\mathrm{d}x}(x)\right| \le C|x|^{\alpha-1} \quad \text{and} \quad |u(x)| \le C|x|^{\alpha}; \tag{51}$$

the function $(y^p - u^p)/(y-u)$ is a non-negative C^∞ function of $(y,\ u)$ which is zero iff $y = u = 0$; (52)

$$u(x) = 0 \text{ implies } x = 0. \tag{53}$$

It follows that v, defined in (16), is in $C^1(U \setminus \{(0, 0)\})$. In fact, if u is given by Lemma 1 then, for any k we can find p and m – large enough – so that v is in $C^k(U \setminus \{(0, 0)\})$. Now, let us establish the continuity of v at (0, 0). We shall prove the following inequality:

$$|v(x,\ y)| \le C(|x|^{\alpha} + |y|). \tag{54}$$

It implies the continuity of v at (0, 0). Inequality (54) clearly holds for $x = 0$. Hence we consider the case $x \neq 0$. From the definition (16) of v and (15), we get

$$|v| \le C\left\{\frac{u^{p-1}}{y^{p-1}+u^{p-1}}\left|\frac{\mathrm{d}u}{\mathrm{d}x}\right||f| + |y|^p + |u|^p + |x|^m \frac{|x| + |u|^{(1/\alpha)-1} + |y|^{(1/\alpha)-1}}{y^{p-1}+u^{p-1}}\right\}. \tag{55}$$

Since (15) implies

$$|f(x,\ y)| \le C\left(|x| + |y|^{1/\alpha}\right), \tag{56}$$

with (55), (56), (18) and (14), we obtain:

$$|v| \le C\left(|x|^{\alpha} + |y|^p + \frac{|y|^{1/\alpha}|x|^{p\alpha-1}}{y^{p-1} + |x|^{(p-1)\alpha}} + \frac{|x|^m |y|^{(1/\alpha)-1}}{y^{p-1} + |x|^{(p-1)\beta}}\right). \tag{57}$$

But, with Young's inequality – see also (17),

$$|y|^{1/\alpha}|x|^{p\alpha-1-\alpha} \le \frac{1}{\alpha(p-1)}y^{p-1} + \left(1 - \frac{1}{\alpha(p-1)}\right)|x|^{(p-1)\alpha}, \tag{58}$$

$$|y|^{(1/\alpha)-1}|x|^{m-\alpha} \le \left(\frac{1-\alpha}{\alpha(p-1)}\right)y^{p-1} + \left(1 - \frac{1-\alpha}{\alpha(p-1)}\right)|x|^{(m-\alpha)(p-1)\alpha/(\alpha p-1)}, \tag{59}$$

inequality (54) follows from (57), (58), (59), (17) and (18).

It remains only to prove (19). We are going to show that, in the set $U \setminus (\{0\} \times \mathbb{R})$, we have

$$\left|\frac{\partial v}{\partial x}(x,\ y)\right| \le C|x|^{\alpha-1}, \qquad \left|\frac{\partial v}{\partial y}(x,\ y)\right| \le C. \tag{60}$$

Inequality (19) is a consequence of (60). Therefore, let (x, y) be a point in $U \backslash (\{0\} \times \mathbb{R})$. We write v as the sum $v = v_1 + v_2 + v_3$ with:

$$v_1(x, y) = p \frac{y - u(x)}{y^p - u(x)^p} f(x, y) u(x)^{p-1} \frac{\mathrm{d}u}{\mathrm{d}x}(x), \tag{61a}$$

$$v_2(x, y) = -y^p + u(x)^p, \tag{61b}$$

$$v_3(x, y) = -x^m \frac{f(x, y) - f(x, u(x))}{y^p - u(x)^p}. \tag{61c}$$

Using (17), (62) and (51), we have

$$\left| \frac{\partial v_2}{\partial x} \right| \le C. \tag{62}$$

On the other hand, we have

$$\left| \frac{\partial v_1}{\partial x} \right| \le C \left[\frac{|u|^{p-2}}{y^{p-1} + u^{p-1}} |f| \left| \frac{\mathrm{d}u}{\mathrm{d}x} \right|^2 + \left| \frac{\mathrm{d}u}{\mathrm{d}x} \right| + \frac{u^{p-1}}{y^{p-1} + u^{p-1}} |f| \left| \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \right| \right]. \tag{63}$$

Therefore, using (13), (51) and (56), we get

$$\left| \frac{\partial v_1}{\partial x} \right| \le C \left(|x|^{\alpha - 1} + \frac{|x|^{p\alpha - 2} |y|^{1/\alpha}}{y^{p-1} + |x|^{(p-1)\alpha}} \right). \tag{64}$$

Using (58) and (64), we obtain

$$\left| \frac{\partial v_1}{\partial x} \right| \le C |x|^{\alpha - 1}. \tag{65}$$

Similar computations give

$$\left| \frac{\partial v_3}{\partial x} \right| \le C |x|^{\alpha - 1}. \tag{66}$$

From (62), (65) and (66), we get

$$\left| \frac{\partial v}{\partial x} \right| \le C |x|^{\alpha - 1}. \tag{67}$$

It remains to prove that $|\partial v / \partial y| \le C$. Clearly,

$$\left| \frac{\partial v_2}{\partial y} \right| \le C. \tag{68}$$

A straightforward computation leads to

$$\left| \frac{\partial v_1}{\partial y} \right| \le C \left\{ \frac{|u|^{p-2}}{|y|^{p-1} + |u|^{p-1}} |f| \left| \frac{\mathrm{d}u}{\mathrm{d}x} \right| + \frac{u^{p-1}}{y^{p-1} + u^{p-1}} \left| \frac{\partial f}{\partial y} \right| \left| \frac{\mathrm{d}u}{\mathrm{d}x} \right| \right\}. \tag{69}$$

Hence, using (13), (15), (51) and (56) we obtain

$$\left| \frac{\partial v_1}{\partial y} \right| \le C \left\{ 1 + \frac{|y|^{1/\alpha} |x|^{\alpha p - 1 - \alpha} + |y|^{(1/\alpha) - 1} |x|^{\alpha p - 1}}{|y|^{p-1} + |x|^{\alpha(p-1)}} \right\}. \tag{70}$$

By Young's inequality, we have

$$|y|^{(1/\alpha)-1}|x|^{\alpha p-1} \le \left(\frac{1-\alpha}{\alpha(p-1)}\right) y^{p-1} + \left(\frac{\alpha p-1}{\alpha(p-1)}\right)|x|^{\alpha(p-1)}. \tag{71}$$

With (58), (59) and (70), this leads to

$$\left|\frac{\partial v_1}{\partial y}\right| \le C. \tag{72}$$

The proof of

$$\left|\frac{\partial v_3}{\partial y}\right| \le C \tag{73}$$

is similar and is omitted. Inequality $|\partial v/\partial y| \le C$ follows from (68), (72) and (73).

4. Proof of Proposition 1

Let us first prove that if C is large enough, the system $\dot{x} = f(x, u)$ is not LAS. Assume this is not the case. Then there exist a neighborhood U of 0 in $\mathbb{R}^n$ and a continuous map $u: U \to \mathbb{R}$ such that 0 is an asymptotically stable equilibrium point of $\dot{x} = f(x, u(x))$. With (21), it follows that, in a neighborhood of 0, $|x|^5 - C^2(u(x)^3 - |x_1|^2 u(x) + x_2^3)^2$ cannot be 0 except at 0. Hence this expression has a constant sign which, for stability, has to be positive. Hence, there exists $\delta > 0$ such that

$$|x|^2 = |x_1|^2 + x_2^2 \le \delta^2 \;\Rightarrow\; \left|u(x_1, x_2)^3 - |x_1|^2 u(x_1, x_2) + x_2^3\right| \le \frac{1}{C}\left(|x_1|^2 + x_2^2\right)^{3/2}. \tag{74}$$

Fix $\bar{x}_1 \in \mathbb{R}^{n-1}$ with $|\bar{x}_1| = 1$ and, for $\nu \in (0, \delta/\sqrt{2})$, define the function $v: [-1, 1] \to \mathbb{R}$ by

$$v(s) = \nu^{-1} u(\nu \bar{x}_1, \nu s). \tag{75}$$

From (74) and (75), we get

$$|v^3(s) - v(s) - s^3| \le \frac{1}{C}(1 + s^2)^{3/2} \le \frac{1}{C} 2^{3/2}. \tag{76}$$

Also, if u is continuous, v is continuous. But there is no continuous function $y(x)$ whose graph is close to the locus in $\mathbb{R}^2$ of the zeros of $y^3 - y + x^3 = 0$. Hence, if C is large enough there exists no continuous map $v: [-1, 1] \to \mathbb{R}$ which satisfies (76).

Let us now prove that

$$\dot{x} = f(x, y), \qquad \dot{y} = v \tag{77}$$

is LAS – in fact globally asymptotically stabilizable. Let $V: \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the function defined by

$$V(x_1, x_2, y) = \tfrac{1}{4} y^4 - \tfrac{1}{2}|x_1|^2 y^2 + x_2^3 y + \tfrac{1}{4} M\left(|x_1|^4 + x_2^4\right) \tag{78}$$

where M is large enough so that V is positive definite and radially unbounded in $\mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}$. For system (77), we have

$$\dot{V} = -|x_1|^2 |x|^6 \left(M|x_1|^2 - y^2\right) - x_2^2 |x|^6 \left(M x_2^2 + 3 x_2 y\right) + \tilde{v}\left(y^3 - |x_1|^2 y + x_2^3\right) \tag{79}$$

where

$$\tilde{v} = v + C^2\left(y^3 - |x_1|^2 y + x_2^3\right)\left(|x_2|^2\left(M|x_1|^2 - y^2\right) + x_2^2\left(M x_2^2 + 3 x_2 y\right)\right). \tag{80}$$

Note

$$y^3 - |x_1|^2 y + x_2^3 = 0 \;\Rightarrow\; |y| \le 2\sqrt{|x_1|^2 + x_2^2}\,. \tag{81}$$

Hence, for M large enough, V is a control Lyapunov function, i.e. (see [16, Section 1])

$$y^3 - |x_1|^2 y + x_2^3 = 0 \;\Rightarrow\; \dot{V} < 0 \text{ or } |x_1| = x_2 = y = 0. \tag{82}$$

Moreover the 'small control property' (in the sense of [16, Section 2]) follows by homogeneity. The fact that $\dot{x} = f(x, y)$, $\dot{y} = v$ is LAS and, in fact, globally asymptotically stabilizable follows from Artstein's theorem on nonlinear stabilization [2] (see also [16]).

5. Proof of Proposition 2

By a generalization due to Kurzweil [14] of a classical Lyapunov's theorem we may assume that for some $\delta > 0$ such that

$$(-\delta, \delta)^n \subset U \quad \text{and} \quad (-\delta, \delta)^{n+1} \subset \Omega, \tag{83}$$

there exists $V : (-\delta, \delta)^n \to [0, +\infty)$, a positive definite function, such that

$$V \in C^\infty\big((-\delta, \delta)^n\big) \quad \text{and} \quad \frac{\partial V}{\partial x}(x) f(x, u(x)) < 0 \;\; \forall x \in (-\delta, \delta)^n \setminus \{0\}. \tag{84}$$

Let F be the closed set of $\mathbb{R}^n \times \mathbb{R}$ defined by

$$F = \big\{(x, u(x)) \mid x \in \big[-\tfrac{1}{2}\delta, \tfrac{1}{2}\delta\big]^n\big\} \tag{85}$$

and let $d : \mathbb{R}^{n+1} \to [0, \infty)$ be the distance to F, i.e.

$$d(x, y) = \min_{\bar{x} \in [-\delta/2, \delta/2]^n} \Big\{\sqrt{|\bar{x} - x|^2 + |u(\bar{x}) - y|^2}\Big\}. \tag{86}$$

By a theorem due to Calderon and Zygmund [4], there exists a continuous map $\Delta : \mathbb{R}^{n+1} \to [0, +\infty)$ such that, for some positive constant C,

$$\frac{1}{C} d(x, y) \le \Delta(x, y) \le C d(x, y) \quad \forall (x, y) \in \mathbb{R}^n \times \mathbb{R}, \tag{87}$$

$$\Delta \in C^\infty(\mathbb{R}^{n+1} \setminus F) \quad \text{and} \quad \left|\frac{\partial \Delta}{\partial x}(x, y), \frac{\partial \Delta}{\partial y}(x, y)\right| \le C \;\; \forall (x, y) \in \mathbb{R}^n \times \mathbb{R} \setminus F. \tag{88}$$

Let now $\phi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be a function defined by:

$$\phi(x, y) = \begin{cases} \Delta(x, y) & \text{if } y \ge u(x), \\ -\Delta(x, y) & \text{if } y < u(x). \end{cases} \tag{89}$$

Proceeding as in [15, Section 3], with ϕ as 'desingularizing' function, we define, in the set $Q := (-\frac{1}{2}\delta, \frac{1}{2}\delta)^{n+1}$,

$$\Phi(x, y) = \int_{u(x)}^{y} \phi(x, s)\, \mathrm{d}s \quad \text{and} \quad h(x, y) = \Phi(x, y) + V(x). \tag{90}$$

From [15, Lemma 1], h is positive definite in the set Q,

$$\frac{\partial \Phi}{\partial y} = \phi, \quad \frac{\partial \Phi}{\partial x} = \int_{u(x)}^{y} \frac{\partial \phi}{\partial x}(x, s)\, \mathrm{d}s \text{ and } h \in C^1(Q). \tag{91}$$

We have

$$\dot{h} = \phi v + \left(\frac{\partial \Phi}{\partial x} + \frac{\partial V}{\partial x}\right) f. \tag{92}$$

and, in Q, "$\phi = 0$" implies "$y = u(x)$". Hence, by (84), (91) and (92), we have

$$\dot{h} < 0 \quad \forall (x, y) \neq (0, 0). \tag{93}$$

It follows that h is a control Lyapunov function. So, to prove that $\dot{x} = f(x, y)$, $\dot{y} = v$ is LAS it is sufficient to check the small control property holds (see [2,16]), i.e.:

For all $\varepsilon > 0$, there exists $\eta < \frac{1}{2}\delta$ such that:

$$(x, y) \neq (0, 0) \text{ and } (x, y) \in (-\eta, \eta)^{n+1} \Rightarrow \exists v \in (-\varepsilon, \varepsilon) \text{ such that } \dot{h} < 0. \tag{94}$$

Since "$\phi(x, y) = 0$, $(x, y) \in Q \setminus \{(0, 0)\}$ and $v = 0$" imply "$\dot{h} < 0$", we will assume in the following that $\phi(x, y) \neq 0$ and $(x, y) \in Q$. We denote by C various constants independent of (x, y). Let us choose the following expression for v (see [15, Section 3]):

$$v = -(\phi + A_1 + A_2) \tag{95}$$

where

$$A_1(x, y) = \frac{\partial \Phi}{\partial x}(x, y) \frac{f(x, y)}{\phi(x, y)} \quad \text{and} \quad A_2 = \frac{\partial V}{\partial x}(x) \frac{f(x, y) - f(x, u)}{\phi(x, y)}. \tag{96}$$

From (84), such an expression for v yields

$$\dot{h} < 0. \tag{97}$$

Let us show that this v is also a good candidate for meeting (94). With (87) and (89), we note that

$$\frac{1}{C} d(x, y) \leq |\phi(x, y)| \leq C d(x, y) \leq C |y - u(x)|. \tag{98}$$

and (25) and (23) yield

$$|u(x)| \leq C |x|^{\alpha}. \tag{99}$$

Hence, from (96) and (97), we have

$$|\phi(x, y)| \leq C(|x|^{\alpha} + |y|). \tag{100}$$

We will check that A_i, $i = 1, 2$, satisfy the same inequality, i.e.

$$|A_1(x, y)| \leq C(|x|^{\alpha} + |y|) \quad \text{and} \quad |A_2(x, y)| \leq C(|x|^{\alpha} + |y|). \tag{101}$$

The small control property (94) follows from (95), (97), (100) and (101). Now, we verify (101) for A_1. Let $\bar{x}$ in $[-\frac{1}{2}\delta, \frac{1}{2}\delta]^n$ be such that

$$d(x, y) = \left(|\bar{x} - x|^2 + |u(\bar{x}) - y|^2\right)^{1/2}. \tag{102}$$

From (26), (88), (89), (91), (96) and (98), we get

$$|A_1(x, y)| \leq C(|x| + |y|^{1/\alpha}) \frac{|y - u(x)|}{|x - \bar{x}| + |y - u(\bar{x})|} \tag{103}$$

and

$$|A_1(x, y)| \leq C(|x| + |y|^{1/\alpha}) \left(1 + \frac{|u(\bar{x}) - u(x)|}{|x - \bar{x}| + |y - u(\bar{x})|}\right). \tag{104}$$

From (25), we get

$$|u(x)-u(\tilde{x})| \le C \min\left\{ |u(x)|+|u(x')|, \max_{x''\in[x,x']} \frac{|x-x'|}{|x''|^{1-\alpha}} \right\}. \tag{105}$$

Then, by comparing $|x-x'|$ with $\frac{1}{4}(|x|+|x'|)$ and by using (99), we can establish

$$|u(x)-u(x')| \le C\frac{|x-x'|}{|x|^{1-\alpha}+|x'|^{1-\alpha}}. \tag{106}$$

It follows from (104) and (106) that

$$|A_1(x, y)| \le C(|x|^\alpha + |y|^{1/\alpha}) + C|y|^{1/\alpha}\frac{|u(x)-u(\bar{x})|}{|x-\bar{x}|+|y-u(\bar{x})|}. \tag{107}$$

- If $|y-u(\bar{x})| \ge \frac{1}{2}|y|$ we get:

$$|A_1(x, y)| \le C(|x|^\alpha + |y|^{1/\alpha}) + C|u(x)-u(\bar{x})| \tag{108}$$

$$\le C(|x|^\alpha + |y|^{1/\alpha}) + C(|u(x)-y|+|u(\bar{x})-y|). \tag{109}$$

Since the definition of $\bar{x}$ implies

$$|u(\bar{x})-y| \le |u(x)-y| \le C(|x|^\alpha + |y|) \tag{110}$$

we finally obtain

$$|A_1(x, y)| \le C(|x|^\alpha + |y|). \tag{111}$$

- If $|y-u(\bar{x})| \le \frac{1}{2}|y|$ then we have

$$|y| \le 2|u(\bar{x})| \le C|\bar{x}|^\alpha. \tag{112}$$

Using (106), (107) and (112) we get again (111).

This proves (101) for A_1. Inequality (101) for A_2 can be obtained with the same method since

$$|A_2(x, y)| \le C|x|\frac{|y-u(x)|}{|x-\bar{x}|+|y-u(\bar{x})|}. \tag{113}$$

6. Proof of Proposition 3

Postponing the proof of Lemma 3 to the end of this section, we use its conclusions. With (33), let Σ be the following 'sphere' in $\mathbb{R}^n \times \mathbb{R}$:

$$\Sigma = \left\{ (x, y) \in \mathbb{R}^n \times \mathbb{R} \mid \sum_{i=1}^{n} |x_i|^{k/r_i} + |y|^{k/r_{n+1}} = 1 \right\}. \tag{114}$$

This set Σ is a C^1 submanifold of $\mathbb{R}^{n+1}$ and the map $(x, y) \mapsto y - \bar{u}(x)$, from Σ to $\mathbb{R}$, is in $C^1(\Sigma \setminus \{(0, 1), (0, -1)\})$. Now, pick δ in $(0, 1)$ and let $\theta : \mathbb{R} \to \mathbb{R}$ be a C^1 function such that

$$s\theta(s) > 0 \quad \forall s \in \mathbb{R}\setminus\{0\}, \qquad \frac{\mathrm{d}\theta}{\mathrm{d}s}(s) = 0 \quad \forall s \in (-\infty, \delta] \cup [\delta, +\infty). \tag{115}$$

We define a desingularizing function ϕ as follows. First, we define a function $\phi_\Sigma : \Sigma \to \mathbb{R}$ by

$$\phi_\Sigma(x, y) = \theta(y - \bar{u}(x)). \tag{116}$$

It is in $C^1(\Sigma)$. Then, we extend it as a C^1 function $\phi:\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}$ by letting $\phi(0)=0$ and

$$\phi(x_1,\ldots,x_n,\ y)=\rho^l\phi_\Sigma\left(\frac{x_1}{\rho^{r_1}},\ldots,\frac{x_n}{\rho^{r_n}},\ \frac{y}{\rho^{r_{n+1}}}\right),\quad \rho=\left(\sum_{i=1}^n |x_i|^{k/r_i}+|y|^{k/r_{n+1}}\right)^{1/k}, \tag{117}$$

for some real number $l\geq k$. We have

$$\phi(\varepsilon^{r_1}x_1,\ldots,\varepsilon^{r_n}x_n,\ \varepsilon^{r_{n+1}}y)=\varepsilon^l\phi(x_1,\ldots,x_n,\ y) \tag{118}$$

and

$$\phi(x,\ y)=0\ \Leftrightarrow\ y=\bar{u}(x). \tag{119}$$

Now, as in [15, Section 3], we define the functions

$$\Phi(x,\ y)=\int_{\bar{u}(x)}^{y}\phi(x,\ s)\,\mathrm{d}s\quad\text{and}\quad h(x,\ y)=\Phi(x,\ y)^\beta+\bar{V}(x)^\alpha \tag{120}$$

where α and β are positive real numbers such that

$$\alpha>1,\quad \beta>1\ \text{ and }\ k\alpha=\beta(l+r_{n+1})=:\gamma>r_i\ \ \forall i\in\{1,\ldots,n+1\}. \tag{121}$$

The function h is positive definite, radially unbounded and in $C_1(\mathbb{R}^n\times\mathbb{R})$. We note that:

$$\Phi(\varepsilon^{r_1}x_1,\ldots,\varepsilon^{r_n}x_n,\ \varepsilon^{r_{n+1}}y)=\varepsilon^{l+r_{n+1}}\Phi(x_1,\ldots,x_n,\ y), \tag{122}$$

$$\frac{\partial\Phi}{\partial x_i}(\varepsilon^{r_1}x_1,\ldots,\varepsilon_n^{r}x_n,\ \varepsilon^{r_{n+1}}y)=\varepsilon^{l+r_{n-1}-r_i}\frac{\partial\Phi}{\partial x_i}(x_1,\ldots,x_n,\ y), \tag{123}$$

$$h(\varepsilon^{r_i}x_1,\ldots,\varepsilon^{r_n}x_n,\ \varepsilon^{r_{n+1}}y)=\varepsilon^\gamma h(x_1,\ldots,x_n,\ y). \tag{124}$$

Moreover, along the solutions of $\dot{x}_i=f_i(x_1,\ldots,x_n,\ y)$, $\dot{y}=v$, we have:

$$\dot{h}=\alpha\bar{V}^{\alpha-1}\frac{\partial\bar{V}}{\partial x}f+\beta\Phi^{\beta-1}\frac{\partial\Phi}{\partial x}f+\beta\Phi^{\beta-1}\phi v. \tag{125}$$

Since we have

$$\beta\Phi(x,\ y)^{\beta-1}\phi(x,\ y)=0\ \Rightarrow\ y=\bar{u}(x)\ \text{and}\ \frac{\partial\Phi}{\partial x_i}(x,\ y)=0, \tag{126}$$

with (34), we obtain

$$\beta\Phi(x,\ y)^{\beta-1}\phi(x,\ y)=0\ \text{and}\ (x,\ y)\neq(0,0)\ \Rightarrow\ \dot{h}<0. \tag{127}$$

This means that h is a control Lyapunov function. It follows from Artstein's theorem [2] (see also [16]) that there exists a function $\vartheta:\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}$ such that, by using $v=\vartheta$ in (125), we obtain

$$\dot{h}<0\quad\forall(x,\ y)\neq(0,0). \tag{128}$$

Moreover this function ϑ can be chosen at least continuous when restricted to the set Σ. Denoting ϑ_Σ this restriction, we define a continuous control law $v:\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}$ by $v(0)=0$ and

$$v(x_1,\ldots,x_n,\ y)=\rho^{\tau+r_{n+1}}\vartheta_\Sigma\left(\frac{x_1}{\rho^{r_1}},\ldots,\frac{x_n}{\rho^{r_n}},\ \frac{y}{\rho^{r_{n+1}}}\right),\quad \rho=\left(\sum_{i=1}^n |x_i|^{k/r_i}+|y|^{k/r_{n+1}}\right)^{1/k}. \tag{129}$$

We have

$$v(\varepsilon^{r_1}x_1,\ldots,\varepsilon^{r_n}x_n,\ \varepsilon^{r_{n+1}}y)=\varepsilon^{\tau+r_{n-1}}v(x_1,\ldots,x_n,\ y). \tag{130}$$

And, by homogeneity, (128) is again satisfied. This proves that (0, 0) is a globally asymptotically stable equilibrium point of $\dot{x} = f(x, y)$, $\dot{y} = v$.

6.1. Proof of Lemma 3

Lemma 3 is a consequence of [9, pp. 278–284] and – the proof of – a theorem due to Hermes [10, Theorem 1]. Let Σ' be the following 'sphere' in $\mathbb{R}^n$:

$$\Sigma' = \left\{ x \in \mathbb{R}^n \mid \sum_{i=1}^{n} |x_i|^{k/r_i} = 1 \right\}. \tag{131}$$

It is a C^1 compact submanifold of $\mathbb{R}^n$ and we may approach arbitrarily close the continuous function u restricted to Σ' by a function in $C^1(\Sigma')$, i.e. for any strictly positive real number η, we can find a C^1 function $\bar{u}_\Sigma : \Sigma' \to \mathbb{R}$ such that:

$$|u(x) - \bar{u}_\Sigma(x)| \le \eta \quad \forall x \in \Sigma'. \tag{132}$$

We may extend $\bar{u}_\Sigma$ as a function $\bar{u} : \mathbb{R}^n \to \mathbb{R}$ by letting $\bar{u}(0) = 0$ and

$$\bar{u}(x_1, \dots, x_n) = \rho^{r_{n+1}} \bar{u}_\Sigma \left(\frac{x_1}{\rho^{r_1}}, \dots, \frac{x_n}{\rho^{r_n}} \right), \quad \rho = \left(\sum_{i=1}^{n} |x_i|^{k/r_i} \right)^{1/k}. \tag{133}$$

This function is in $C^1(\mathbb{R}^n \setminus \{0\}) \cap C^0(\mathbb{R}^n)$ and satisfies

$$\bar{u}(\varepsilon^{r_1} x_1, \dots, \varepsilon^{r_n} x_n) = \varepsilon^{r_{n+1}} \bar{u}(x_1, \dots, x_n) \quad \forall \varepsilon \ge 0 \tag{134}$$

and, with (29),

$$|u(x) - \bar{u}(x)| \le \eta \left(\sum_{i=1}^{n} |x_i|^{k/r_i} \right)^{r_{n+1}/k} \qquad \forall x \in \mathbb{R}^n. \tag{135}$$

On the other hand, as in the proof of Proposition 2, from [14], we know the existence of a neighborhood U of 0 in $\mathbb{R}^n$ and of a function $V : U \to [0, +\infty)$ which is positive definite, in $C^\infty(U)$ and satisfies:

$$\frac{\partial V}{\partial x}(x) f(x, u(x)) < 0 \quad \forall x \in U \setminus \{0\}. \tag{136}$$

May be after multiplying this function V by a constant, we may assume

$$\{ x \in \mathbb{R}^n \mid V(x) \le 1 \} \subset U. \tag{137}$$

Then, let $\mathscr{V}$ be the following set:

$$\mathscr{V} = \{ x \in \mathbb{R}^n \mid V(x) = 1 \}. \tag{138}$$

The set $\mathscr{V}$ being compact, from (136), we may find $\eta > 0$ so that the associated function $\bar{u}$ satisfies (135) and

$$\frac{\partial V}{\partial x}(x) f(x, \bar{u}(x)) < 0 \quad \forall x \in \mathscr{V}. \tag{139}$$

From here, to prove that 0 is a globally asymptotically stable equilibrium point of $\dot{x} = f(x, \bar{u}(x))$ we note that by homogeneity, it is sufficient to prove a local asymptotic stability. And this follows from – the proof of – [10, Theorem 1].

It remains to prove the existence of $\bar{V}$. This follows from [9, Theorems 57.1, 57.2 and 57.4 and extension on pp. 283–284] (see also [21, Theorem 36]) applied to the homogeneous C^1 system

$$\dot{x} = \left(\sum_{i=1}^{n} |x_i|^{k/r_i} \right) f(x, \bar{u}(x)). \tag{140}$$

7. Proof of Corollary 1

Let $(r_i)_{i=1,\ldots,n+1}$ be defined recursively as follows:

$$r_1 = 1, \quad r_i = r_{i-1}/p_i. \tag{141}$$

The p_i's being odd integer numbers, we have $r_{i-1} \geq r_i$. But also $p_i = 1$ implies $r_i = r_{i-1}$. We denote by k_i the number of consecutive p_j's which are equal to 1 with $j \leq i$, namely, k_i satisfies

$$r_{i-k_i} > r_{i+1-k_i} = r_{i+1-k_i} = \cdots = r_{i-1} = r_i. \tag{142}$$

Then let, by denoting $x_{n+1} = u$,

$$a_{i+1} = \frac{\partial^{p_{i+1}} f_i}{\partial x_{i+1}^{p_{i+1}}}(0), \qquad b_{i,j} = \frac{\partial f_i}{\partial x_j}(0) \quad \forall j \in \{i+1-k_i, \ldots, i\}. \tag{143}$$

From Taylor expansion, there exist a constant C and a neighborhood of (0, 0) in $\mathbb{R}^n \times \mathbb{R}$ such that:

$$\left| f_i(x_1, \ldots, x_n, u) - a_{i+1} x_{i+1}^{p_{i+1}} - \sum_{j=i+1-k_i}^{i} b_{i,j} x_j \right| \leq C \left(\sum_{j=0}^{i-k_i} |x_j| + \sum_{j=i+1-k_i}^{i} |x_j|^2 + |x_{i+1}|^{1+p_{i+1}} \right) \tag{144}$$

for all $i \in \{1, \ldots, n\}$. Now we define new functions $\bar{f}_i : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ by

$$\bar{f}_i(x, u) = a_{i+1} x_{i+1}^{p_{i+1}} + \sum_{j=i+1-k_i}^{i} b_{i,j} x_j. \tag{145}$$

These functions are homogeneous, namely we have

$$\forall i \in \{1, \ldots, n\}, \ \forall x = (x_i)_{i=1,n} \in \mathbb{R}^n, \ \forall \varepsilon \geq 0, \ \forall u \in \mathbb{R}$$
$$\bar{f}_i(\varepsilon^{r_1} x_1, \ldots, \varepsilon^{r_n} x_n, \varepsilon^{r_{n+1}} u) = \varepsilon^{r_i} \bar{f}_i(x_1, \ldots, x_n, u). \tag{146}$$

Since the p_i's are odd, it follows, from Proposition 3, Lemma 3 and by induction on n, that there exists a feedback law $u : \mathbb{R}^n \to \mathbb{R}$ in $C^1(\mathbb{R}^n \setminus \{0\}) \cap C^0(\mathbb{R}^n)$ such that

$$0 \text{ is a globally asymptotically stable equilibrium point of } \dot{x}_i = \bar{f}_i(x, u(x)), \tag{147}$$

$$u(\varepsilon^{r_1} x_1, \ldots, \varepsilon^{r_n} x_n) = \varepsilon^{r_{n+1}} u(x_1, \ldots, x_n). \tag{148}$$

Then, by denoting

$$g_i(x) = f_i(x, u(x)) \text{ and } \bar{g}_i(x) = \bar{f}_i(x, u(x)) \quad \forall i \in \{1, \ldots, n\}, \tag{149}$$

we have for all i in $\{1, \ldots, n\}$, with (146) and (148),

$$\bar{g}_i(\varepsilon^{r_1} x_1, \ldots, \varepsilon^{r_n} x_n) = \varepsilon^{r_i} \bar{g}_i(x_1, \ldots, x_n), \tag{150}$$

and, with (144) and (148),

$$\lim_{\varepsilon \to 0_+} \frac{1}{\varepsilon^{r_i}} \left[g_i(\varepsilon^{r_1} x_1, \ldots, \varepsilon^{r_n} x_n) - \bar{g}_i(\varepsilon^{r_1} x_1, \ldots, \varepsilon^{r_n} x_n) \right] = 0, \tag{151}$$

uniformly in $\{x \in \mathbb{R}^n \mid |x| \leq 1\}$. Now, it follows from (147), (150) and (151) and – the proof of – [10, Theorem 1] that 0 is an asymptotically stable equilibrium point of $\dot{x} = f(x, u)$.

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