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Topological Orbital Equivalence with Asymptotic Phase for a Two Time-Scales Discrete-Time System*

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Abstract. Existence, smoothness, approximation, and attractiveness of a locally integral manifold are established for a two time-scales discrete-time system. This manifold contains all the solutions remaining in a specific compact subset. It allows us to define locally a triangular system which is topologically orbitally equivalent with asymptotic phase. It follows that (in)stability properties and existence of solutions of the original system, remaining after some time instant in the above-mentioned compact subset, can be established from the study of a reduced-order system. We study this reduced-order system for a weakly nonstationary case, applying the stroboscopic method to approximate it by a practically meaningful, slowly time-varying system.

Key words. Discrete-time system, Two time-scales system, Topological equivalence, Integral manifold, Averaging.

1. Introduction

The objective of this paper is to provide a tool for local analysis of the following discrete-time system:

$$\begin{aligned} X_{k+1} &= A(\theta_k)X_k + B(\theta_k)u_k, \\ \theta_k &= \theta_{k-1} + \varepsilon C(X_k, \theta_{k-1}, k). \end{aligned} \tag{S_ε}$$

For ε small enough and when $A(\theta)$ has no eigenvalue on the unit circle, this system exhibits a two time-scales property. The state of the fast subsystem is X in $\mathbb{R}^n$ and the state of the slow subsystem is θ in $\mathbb{R}^p$. The u_k represents a (closed-loop) forcing term.

To determine conditions for which S_ε has solutions bounded on $\mathbb{Z}$ or a semi-infinite time interval and when these solutions are stable, we consider S_ε as a small pertubation of the "frozen" system

$$\begin{aligned} X_{k+1} &= A(\theta_k)X_k + B(\theta_k)u_k, \\ \theta_k &= \theta_{k-1}, \end{aligned} \tag{S_0}$$

which can be viewed as a family of linear systems indexed by θ_k.

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System S_ε arises from the study of linear time-varying systems where the time variations, although small, depend on the state itself. Also, as noticed by Ljung and Söderström [LS] for example, S_ε describes most adaptive linear systems studied in the literature.

When the sequence u_k satisfies the so-called test input assumption (see [RPK] and [PR]), i.e., when there exists a solution of S_ε on $\mathbb{Z}$ or on a semi-infinite time interval whose θ-component is constant, Riedle and Kokotovic [RK1] have performed stability analysis by linearization and by invoking the Krylov–Bogoliubov–Mitropolski averaging theory (see [H1]). This theory has long been used in the case when u_k is a stationary stochastic process. In particular, it has been used by Ljung to derive the "ordinary differential equation" technique for the case when $C(X, \theta, k)$ decreases to zero as k tends to infinity [L1], [LS], [KC]. This technique has also been justified for the case where $C(X, \theta, k)$ is small but not decreasing [BMP], [DF]. In the deterministic case and for the specific system S_ε, its use as a heuristic in the nondecreasing case has been considered by Åström [A1], [A2]. Based on a linear averaging technique, but incorporating total stability arguments, some relaxation of the test input assumption has been obtained [ABJ].

When u_k has only l_2-stationary properties (periodic, almost periodic, ...), existence of a particular $\mathbb{Z}$-bounded solution has been established using nonlinear averaging theory [BSA] or the Poincaré expansion method [P4], [PP]. It relies on the existence of a solution of a bifurcation equation. Again stability properties are established by linearization.

In this paper, up to Section 4, no other assumption, besides boundedness, are needed on u_k. Moreover, we are interested in a complete description of all the bounded solutions. Such a description is easily obtained for S_0 for any θ-set where $A(\theta)$ is noncritical with respect to u_k. For example, all the $\mathbb{Z}$-bounded solutions of S_0 whose θ-component lies in S (subset of $\mathbb{R}^p$ for which the eigenvalues of $A(\theta)$ are in the open unit disk), are given by the graph of the function M_0: $\mathbb{R}^p \times \mathbb{Z} \to \mathbb{R}^n$:

$$M_0(\theta, k) = \sum_{i=-\infty}^{k-1} A(\theta)^{k-1-i} B(\theta) u_i. \tag{1}$$

This graph is an integral manifold which, restricted to S for example, is normally attractive. A general theory is available proving the persistence of normally hyperbolic invariant manifolds under small perturbations (see [F2], [H1], [HPS], and [O]). Therefore we expect the existence of a function M_ε whose graph is an integral manifold of S_ε. And like M_0 for S_0, M_ε should describe (at least locally) all the $\mathbb{Z}$-bounded solutions of S_ε.

After Section 2 where we introduce our assumptions and notations, in Section 3 we apply this general theory to establish existence and regularity of this function M_ε and we prove that any solution (X_{k+1}, θ_k) of S_ε which remains for ever in $B(O, x) \times$ S lies in the graph of M_ε ($B(O, x)$ is the closed ball of center O, radius x).

Our main result is given in Section 4. We prove that the so-called "reduction principle" applies with an asymptotic phase. This "principle" has been introduced by Pliss [P1] and generalized by Aulbach [A3], Henry [H2], and Kelley [K]. It is of practical importance since it shows that stability properties and existence of solutions of S_ε remaining in $B(O, x) \times$ S after time k_0 can be established from the reduced-order system given by the restriction of S_ε to the graph of M_ε, i.e., from the system SM_ε^θ obtained from the θ-equation of S_ε by replacing X_k by $M_\varepsilon(\theta_{k-1}, k)$. We will see that this "principle" or more precisely the existence of a topological orbital equivalence is a consequence of Bowen's Shadowing Lemma (see [HPS] or Proposition 8.19 of [S]) which is again an aspect of normal hyperbolicity.

In Section 5 we study the system SM_ε^θ for the case where u_k is weakly nonstationary. We apply Minorsky's stroboscopic method ideas [M]. This allows us to define a subset of $\mathbb{R}^p$ where the solutions of SM_ε^θ are hyperbollically attractive (and therefore, the same property holds for solutions of S_ε). This proves in particular that Åström's heuristic approach is technically sound when restricted to this subset.

Some of the results presented here have been established for the continuous-time case by Riedle and Kokotovic [RK2]. For the discrete-time case, they have been announced in [P3].

2. Assumptions and Notations

In the following, $\|\cdot\|$ denotes the Euclidian norm. The case when u_k and $C(X, \theta, k)$ are periodic in k is called the periodic case. Throughout this paper, ε is positive.

The following assumptions are used:

A1. The sequence u_k is bounded: $\|u_k\| \le u$ for all $k \in \mathbb{Z}$.

Let S be a compact set in $\mathbb{R}^p$, with a nonempty interior.

A2. If θ lies in S, the eigenvalues of $A(\theta)$ are strictly inside the unit circle: $|\lambda\{A(\theta)\}| \le \lambda_0$ for all $\theta \in$ S, $\lambda_0 < 1$.

A3. The functions $A(\theta)$, $B(\theta)$ are Lipschitz continuous on S with a_1, b_1 the respective Lipschitz constants.

A direct consequence of assumptions A2 and A3 is the existence of a, b, λ such that (see [F3]) $\|B(\theta)\| \le b$ and $\|A(\theta)^i\| \le a\lambda^i$, where $\lambda_0 < \lambda < 1$, for all $\theta \in$ S, for all $i \in \mathbb{N}$.

Let $B(O, x)$ be the closed ball of $\mathbb{R}^n$ with center O and radius x.

A4. The function C is Lipschitz continuous uniformly on $B(O, x) \times \mathrm{S} \times \mathbb{Z}$:

$$\|C(X, \theta, k)\| \le c(x),$$

$$\|C(X^0, \theta^0, k) - C(X^1, \theta^1, k)\| \le c_1(x)(\|X^0 - X^1\| + \|\theta^0 - \theta^1\|),$$

where c, c_1 are positive nondecreasing functions whose argument is omitted when no confusion is possible. In the following, the function C could also depend on ε.

To study the smoothness properties of $M_\varepsilon(\theta, k)$, we need:

A5. The functions A, B, C are Lipschitz continuously differentiable: There exist linear maps $\partial A/\partial\theta$, $\partial B/\partial\theta$, $\partial C/\partial X$, and $\partial C/\partial\theta$ such that, with (X, θ) an interior point of $B(O, x) \times$ S and h_x, h_θ two (sufficiently small) vectors in $\mathbb{R}^n$, $\mathbb{R}^p$ respectively, the functions δ_A, δ_B, δ_C defined as (similarly for δ_A, δ_B)

$$\delta_C(X, \theta, k, h_x, h_\theta)$$
$$= C(X + h_x, \theta + h_\theta, k) - C(X\ \theta, k) - \frac{\partial C}{\partial X}(X, \theta, k)h_x - \frac{\partial C}{\partial \theta}(X, \theta, k)h_\theta$$

satisfy (similarly for δ_A, δ_B)

$$\limsup_{h_\theta \to 0,\, h_x \to 0} \frac{\|\delta_C(X, \theta, k, h_x, h_\theta)\|}{\|h_x\| + \|h_\theta\|} = 0.$$

Moreover, the functions $\partial A/\partial\theta(\theta)$, $\partial B/\partial\theta(\theta)$, $\partial C/\partial X(X, \theta, k)$, and $\partial C/\partial\theta(X, \theta, k)$ are Lipschitz continuous uniformly on $B(O, x) \times \mathrm{S} \times \mathbb{Z}$ with a_2, b_2, $c_2(x)$, and $c_2(x)$ the respective Lipschitz constants.

Assumptions A1 and A3–A5 are smoothness conditions. Assumption A2 concerns the possibility of finding a constant θ such that the (linear) X-subsystem is exponentially stable. These assumptions are very similar to conditions C1 and C2 on p. 158 of [LS], introduced to derive the associated differential equation technique. One of the consequence of the results presented in this paper is to provide a geometric justification of the substitution of X_{k+1} by $M_0(\theta_k, k)$ in the θ-equation of S_ε, used to derive this associated differential equation. However, notice that for the time being no stationarity or decaying ε is needed compared with C3–C6 used in the above reference. These extra assumptions are discussed in Section 5, when studying a reduced-order system.

To obtain this geometric insight, we consider S_ε as a map from $\mathbb{R}^n \times \mathbb{R}^p \times \mathbb{Z}$ to $\mathbb{R}^n \times \mathbb{R}^p \times \mathbb{Z}$ taking

$$S_\varepsilon\colon \quad \begin{bmatrix} X \\ \theta \\ k \end{bmatrix} \mapsto \begin{bmatrix} Y = A(\psi)X + B(\psi)u_k \\ \psi = \theta + \varepsilon C(X, \theta, k) \\ k+1 \end{bmatrix}.$$

Similarly, given the function $M_\varepsilon\colon \mathrm{S} \times \mathbb{Z} \to \mathbb{R}^n$, we consider the system SM_ε defined by

$$SM_\varepsilon\colon \quad \begin{bmatrix} X^M \\ \theta^M \\ k \end{bmatrix} \mapsto \begin{bmatrix} Y^M = A(\psi^M)X^M + B(\psi^M)u_k \\ \psi^M = \theta^M + \varepsilon C(M_\varepsilon(\theta^M, k), \theta^M, k) \\ k+1 \end{bmatrix}.$$

The difference between S_ε and SM_ε is that, in the θ-equation, X is replaced by M_ε. Consequently, SM_ε, being "lower triangular," is structurally much simpler than S_ε.

To study boundedness and stability properties of these systems, we consider the following sets: Let Θ be a compact subset of S, we define $\mathbf{B}([k_0, k_1), \Theta)$ (resp. $\mathbf{B}^M([k_0, k_1), \Theta))$ as the set of sequences (X_{k+1}, θ_k) (resp. (X_{k+1}^M, θ_k^M)) solutions of S_ε (resp. SM_ε) for k in $[k_0, k_1)$ and satisfying

$$X_{k_0+1} - M_\varepsilon(\theta_{k_0}, k_0 + 1) \in B\left(O, \frac{x - m}{\alpha}\right)$$
$$\left(\text{resp. } X_{k_0+1}^M - M_\varepsilon(\theta_{k_0}^M, k_0 + 1) \in B\left(O, \frac{x \pm m}{\alpha}\right)\right),$$
$$\theta_k \in \Theta \qquad (\text{resp. } \theta_k^M \in \Theta) \qquad \text{for all} \quad k \in [k_0, k_1),$$

where m, α are specified later. For sequences in these sets we define the elevation above the manifold as (similarly for E^M) $E_{k+1} = X_{k+1} - M_\varepsilon(\theta_k, k+1)$.

In the following, we say that the graph of M_ε is an integral manifold of S_ε locally on Θ if, for any (θ_k^M) solution of

$$\theta_{k+1}^M = \theta_k^M + \varepsilon C(M_\varepsilon(\theta_k^M, k+1), \theta_k^M, k+1) \qquad (SM_\varepsilon^\theta)$$

such that θ_k^M lies in Θ for all k in $[k_0, k_1)$, the sequence (X_{k+1}, θ_k) defined by $X_{k+1} = M_\varepsilon(\theta_k^M, k+1)$, $\theta_k = \theta_k^M$, is a solution of S_ε on $[k_0, k_1)$. Note that this implies that the graph of M_ε is an integral manifold of SM_ε locally on Θ.

Our focus in this paper concerns the properties of solutions of S_ε remaining in $B(O, x) \times \mathrm{S}$. To facilitate our study, we use the classical trick consisting of modifying the system S_ε into $\bar{S}_\varepsilon$, such that S_ε and $\bar{S}_\varepsilon$ coincide on a compact subset of $B(O, x) \times \mathrm{S}$. For a compact set Θ in $\mathbb{R}^p$ we denote by $\Theta + \eta$, the "η-augmented" compact set of Θ:

$$\Theta + \eta = \{\theta \in \mathbb{R}^p | \exists \psi \in \Theta: \|\psi - \theta\| \leq \eta\}.$$

Since S has a nonempty interior we can find η and compact sets S_0, S_1 with nonempty interiors, such that $\mathrm{S}_0 + \eta \subset \mathrm{S}_1$ and $\mathrm{S}_1 + \eta \subset \mathrm{S}$. We call a stopping function s: $\mathbb{R}^p \to [0, 1]$, a C^r-function $(0 \leq r \leq \infty)$ given by the Urysohn theorem satisfying $s(\theta) = 1$ if and only if $\theta \in \mathrm{S}_0$, and $\theta \notin \mathrm{S}_1$ implies $s(\theta) = 0$.

Let s_1, s_2 denote the Lipschitz constants of s and its differential, respectively. We can take $s_1 = 1/\eta$ if s is only Lipschitz continuous. We define the function $\bar{C}$: $\mathbb{R}^n \times \mathbb{R}^p \times \mathbb{Z} \to \mathbb{R}^p$ by $\bar{C}(X, \theta, k) = s(\theta)C(X, \theta, k)$. The function $\bar{C}$ has the same properties as C, with, in particular, $\bar{c} \leq c$, $\bar{c}_1 \leq s_1 c + c_1$, and $\bar{c}_2 \leq s_2 c + 2s_1 c_1 + c_2$.

In this paper we always assume $\varepsilon \leq \varepsilon_0$ with $\varepsilon_0 \leq \mathrm{Min}(\eta/c, 1/\bar{c}_1)$. In this condition, if ψ is defined as $\psi = \theta + \varepsilon\bar{C}(X, \theta, k)$, we have (i) $\theta \in \mathrm{S}_1$ implies $\psi \in \mathrm{S}$, (ii) $\psi \in \mathrm{S} - \mathrm{S}_1$ implies $\theta = \psi$, and (iii) $\theta \in \mathrm{S}$ implies the segment $[\theta, \psi] \subset \mathrm{S}$. For $\bar{C}$, we define the modified system $\bar{S}_\varepsilon$ (similarly for $\bar{S}M_\varepsilon$):

$$\bar{S}_\varepsilon: \quad \begin{bmatrix} X \\ \theta \\ k \end{bmatrix} \to \begin{bmatrix} Y = A(\psi)X + B(\psi)u_k \\ \psi = \theta + \varepsilon\bar{C}(X, \theta, k) \\ k+1 \end{bmatrix}.$$

In $\bar{S}_\varepsilon$ we smoothly stop the θ-component of any solution trying to leave S_1. Clearly, a solution of $\bar{S}_\varepsilon$ is a solution of S_ε on $[k_0, k_1)$ if, for all k in $[k_0, k_1)$, θ_k lies in S_0. The idea of preventing the θ-vector from leaving an admissible region is also used in practice [E], [LS].

To end this section, let us mention that the nonexplicit form of S_ε is chosen to simplify the forthcoming derivations. However, this is done with no loss of generality since the system

$$Y_{k+1} = A(\theta_k)Y_k + B(\theta_k)u_k,$$

$$\theta_{k+1} = \theta_k + \varepsilon C(Y_k, \theta_k, k+1)$$

can also be written as

$$\begin{pmatrix} Y_{k+1} \\ Y_k \end{pmatrix} = \begin{pmatrix} A(\theta_k) & 0 \\ I & 0 \end{pmatrix} \begin{pmatrix} Y_k \\ Y_{k-1} \end{pmatrix} + \begin{pmatrix} B(\theta_k) \\ 0 \end{pmatrix} u_k,$$

$$\theta_k = \theta_{k-1} + \varepsilon C(Y_{k-1}, \theta_{k-1}, k).$$

3. Existence of a Normally Attractive Locally Integral Manifold

Theorem 1. *Under assumptions* A1–A4, *for any compact set* S_0 *strictly contained in* S, *there exists* ε^* *such that for any* ε, $\varepsilon \le \varepsilon^*$, *we can find constants* α, f^0, f^1, f_1^0, *a function* $\tau_0(\varepsilon, x)$, $0 \le \tau_0 < 1$, *and a (possibly nonunique) function* $M_\varepsilon\colon \mathrm{S} \times \mathbb{Z} \to \mathbb{R}^n$ *which is periodic in k in the periodic case, such that:*

(i) *The graph of* M_ε *is an integral manifold of* S_ε *locally on* S_0.

(ii) *Smoothness*: M_ε *is bounded and Lipschitz continuous uniformly on* $\mathrm{S} \times \mathbb{Z}$ *with bounds* m *and* m_1, *respectively. Moreover, if assumption* A5 *holds,* M_ε *is Lipschitz continuously differentiable in* θ *with* m_2 *as a Lipschitz constant of its differential.*

(iii) *Approximation: let* M_0 *be the function defined in* (1), *then, uniformly on* $S_0 \times \mathbb{Z}$, $\|M_\varepsilon(\theta, k) - M_0(\theta, k)\| \le \varepsilon f^0$. *Moreover, if assumption* A5 *holds,* M_0 *is continuously differentiable in* θ *and letting* $(M_0^1(\theta, k))$ *be the unique* $\mathbb{Z}$*-bounded solution of*

$$X_{k+1} = A(\theta)X_k + B(\theta)u_k - \varepsilon A(\theta)\frac{\partial M_0}{\partial \theta}(\theta, k)C(M_0(\theta, k), \theta, k),$$

we have, uniformly on $S_0 \times \mathbb{Z}$,

$$\|M_\varepsilon(\theta, k) - M_0^1(\theta, k)\| \le \varepsilon^2 f^1, \qquad \left\|\frac{\partial M_\varepsilon}{\partial \theta}(\theta, k) - \frac{\partial M_0}{\partial \theta}(\theta, k)\right\| \le \varepsilon f_1^0.$$

(iv) *Attractiveness: let* (X_{k+1}, θ_k) *be a sequence of* $\mathbf{B}([k_0, k_1), S_0)$, *with* (E_k) *its corresponding sequence of elevations above the manifold; we have for all* k', k, $k_0 \le k' \le k < k_1$, $\|E_{k+1}\| \le \alpha \tau_0(\varepsilon, x)^{k-k'}\|E_{k'+1}\|$. *Moreover, if* (X_{k+1}, θ_k) *is a solution of* S_ε *which lies in* $B(O, x) \times S_0$ *for all* k *in* $\mathbb{Z}$, *then this solution lies in the graph of* M_ε, *namely, for all* k, $X_{k+1} = M_\varepsilon(\theta_k, k + 1)$.

(v) *All these properties hold for* SM_ε.

All the constants appearing in this statement are clarified in the proof.

Remarks. For the continuous-time case, existence of M_ε, approximation by M_0, and exponential decaying of the elevation above the manifold are established in [RK2].

This theorem is a technical step toward our main result of Section 4. However, it gives us a first important geometric property of S_ε:

Any solution of S_ε *remaining in the compact set* $B(O, x) \times S_0$ *lies in the graph of* M_ε.

The end of this section is devoted to the proof of this theorem. It is sufficient to establish this result for the modified system $\bar{S}_\varepsilon$. One possible proof would call upon general theorems on persistence of normally hyperbolic integral manifolds. We prefer a less technical direct proof. It is an adaptation of the proof of Theorem 5.2 of [S] and is based on the graph transform technique.

Consider the image by $\bar{S}_\varepsilon$ of the graph $\{(X = M_\varepsilon(\theta, k), \theta, k) | \theta \in \mathrm{S}, k \in \mathbb{Z}\}$. We obtain a set of $(Y, \psi, k + 1)$ contained in $\mathbb{R}^n \times \mathrm{S} \times \mathbb{Z}$. If the graph is an integral manifold, this set is contained in the graph itself, i.e., $Y = M_\varepsilon(\psi, k+1)$. This means that the following diagram commutes:

$$\begin{array}{ccc}
\begin{bmatrix} X = M_\varepsilon(\theta, k) \\ \theta \\ k \end{bmatrix} & \xrightarrow{\bar{S}_\varepsilon} & \begin{bmatrix} Y = M_\varepsilon(\psi, k+1) \\ \psi \\ k+1 \end{bmatrix} \\
\uparrow{\scriptstyle M_\varepsilon} & & \uparrow{\scriptstyle M_\varepsilon} \\
(\theta, k) & \xrightarrow{\bar{S}M_\varepsilon^\theta} & (\psi, k+1)
\end{array}$$

This also means that M_ε is a fixed point of the operator T, called the graph transform and defined as $T(M) = \bar{S}_\varepsilon \circ M \circ (\bar{S}M_\varepsilon^\theta)^{-1}$. Our problem is reduced to studying the properties of this operator. Let us first introduce an

Adapted Metric. Given θ in S, for any vector X in $\mathbb{R}^n$, we define its norm $|||X|||_\theta$ by $|||X|||_\theta = \sum_{i=0}^{\infty} \mu^{-i}\|A(\theta)^i X\|$, $\lambda < \mu < 1$. From assumptions A2 and A3 it can be seen that:

(i) $\|X\| \le |||X|||_\theta \le \alpha\|X\|$, $\alpha = a\mu/(\mu - \lambda)$.
(ii) $|||A(\theta)X|||_\theta \le \gamma|||X|||_\theta$, $\gamma = \mu[1 - (\mu - \lambda)/a\mu]$.
(iii) $|||X|||_\psi \le (1 + \beta\|\theta - \psi\|)|||X|||_\theta$, $\beta = a_1 a/(\mu - \lambda)$.

Remark. This metric allows us to exhibit the normal hyperbolicity property. Let (X^0, θ^0, k), (X^1, θ^1, k) be two points in $B(O, x) \times \mathrm{S} \times \mathbb{Z}$ and let $(Y^0, \psi^0, k+1)$, $(Y^1, \psi^1, k+1)$ be their respective images by $\bar{S}_\varepsilon$. We have the following inequalities (compare with (2.21) and (2.22) of [P2]):

$$|||Y^0 - Y^1|||_{\psi^0} \le \gamma(1 + \varepsilon\beta\bar{c})|||X^0 - X^1|||_{\theta^0} + \alpha(a_1 x + b_1 u)\|\psi^0 - \psi^1\|,$$
$$\|\psi^0 - \psi^1\| \ge -\varepsilon\bar{c}_1|||X^0 - X^1|||_{\theta^0} + (1 - \varepsilon\bar{c}_1)\|\theta^0 - \theta^1\|.$$

Introducing the positive function $l(x)$ satisfying

$$\frac{\gamma(1 + \varepsilon\beta\bar{c}(x))l(x)}{1 - \varepsilon\bar{c}_1(x)(1 + l(x))} + \alpha(a_1 x + b_1 u) \le l(x),$$

we obtain the following key technical triangular system:

$$\begin{aligned}
|||Y^0 - Y^1|||_{\psi^0} - l\|\psi^0 - \psi^1\| &\le \tau_0(\varepsilon, x)(|||X^0 - X^1|||_\theta - l\|\theta^0 - \theta^1\|), \\
\|\psi^0 - \psi^1\| &\ge -\varepsilon\bar{c}_1(|||X^0 - X^1|||_\theta - l\|\theta^0 - \theta^1\|) \\
&\quad + (1 - \varepsilon\bar{c}_1(1 + l))\|\theta^0 - \theta^1\|
\end{aligned} \tag{2}$$

with

$$\tau_0(\varepsilon, x) = \gamma(1 + \varepsilon\beta\bar{c}(x))\left(1 + \varepsilon\frac{\bar{c}_1(x)l(x)}{1 - \varepsilon\bar{c}_1(x)(1 + l(x))}\right).$$

The normal attractivity property appears here. In particular, $\tau_0(\varepsilon, x)$ characterizes the contraction property of S_ε in the direction normal to the integral manifold (see attractiveness). The term $(1 - \varepsilon\bar{c}_1(1 + l))^{-1}$ characterizes the possible expansion along the manifold (see Lemma 1). From these characterizations and following [F2] or

[HPS] we expect the existence of an integral manifold which can be up to r times continuously differentiable if r is the largest integer such that

$$\tau_0(\varepsilon, x)\left(\frac{1}{1-\varepsilon\bar{c}_1(x)(1+l(x))}\right)^r < 1. \tag{3}$$

This means that existence and smoothness of this integral manifold depend directly on how much more sharply S_ε contracts in the normal direction than expands in the tangent direction or else on how much faster the X-component converges to its "steady-state" value than the θ-component is changed.

Let $\mathbf{M}$ be the set of functions $M: \mathbf{S} \times \mathbb{Z} \to \mathbb{R}^n$ satisfying uniformly

$$|||M(\theta, k)|||_\theta \leq m < x,$$

$$|||M(\theta^0, k) - M(\theta^1, k)|||_{\theta^i} \leq m_1\|\theta^0 - \theta^1\|, \qquad i = 0, 1,$$

where m, m_1 are as specified later. In the periodic case, $M(\theta, k)$ is chosen periodic in k. Equipped with the distance associated with the norm $|M| = \mathrm{Sup}_{\theta \in \mathbf{S}, k \in \mathbb{Z}} |||M(\theta, k)|||_\theta$, $\mathbf{M}$ is a complete metric space.

In our above definition of the graph transform, we used the inverse function $(\bar{S}M_\varepsilon^\theta)^{-1}$. Let us prove that this makes sense.

Lemma 1. *There exists ε^* such that for any ε, $\varepsilon \leq \varepsilon^*$, M in $\mathbf{M}$, and k in $\mathbb{Z}$ we can find a unique function $D(M, k)$: $\mathbf{S} \to \mathbb{R}^p$ such that, uniformly on $\mathbf{M} \times \mathbb{Z} \times \mathbf{S}$:*

(i) $\|D(M, k)(\psi)\| \leq \bar{c}$.
(ii) $\|D(M, k)(\psi^0) - D(M, k)(\psi^1)\| \leq d_1\|\psi^0 - \psi^1\|$.
(iii) $\|D(M^0, k)(\psi) - D(M^1, k)(\psi)\| \leq d_1^m|M^0 - M^1|$.
(iv) $D(M, k)(\psi) = \bar{C}(M(\psi - \varepsilon D(M, k)(\psi), k), \psi - \varepsilon D(M, k)(\psi), k)$.

Proof. Given M, k, ψ in $\mathbf{M} \times \mathbb{Z} \times \mathbf{S}_1$, we consider the complete metric space of vectors $\mathbf{D}$ of $\mathbb{R}^p$ such that $\|D\| \leq \bar{c}$. With our assumption on ε, if ψ is in $\mathbf{S}_1$, $\psi - \varepsilon D$ is in $\mathbf{S}$. On $\mathbf{D}$, we define an operator $T(M, \psi, k)$ as $T(M, \psi, k)(D) = \bar{C}(M(\psi - \varepsilon D, k), \psi - \varepsilon D, k)$. We have

$$\begin{aligned}&\|T(M^0, \psi^0, k)(D^0) - T(M^1, \psi^1, k)(D^1)\| \\ &\qquad \leq \bar{c}_1[(1+m_1)\|\psi^0 - \psi^1\| + \varepsilon(1+m_1)\|D^0 - D^1\| + |M^0 - M^1|].\end{aligned}$$

From the Uniform Contraction Mapping Theorem [H1], the result follows with

$$d_1 = \frac{\bar{c}_1(1+m_1)}{1-\varepsilon\bar{c}_1(1+m_1)}, \qquad d_1^m = \frac{1}{1-\varepsilon\bar{c}_1(1+m_1)}.$$

However, up to now ψ was restricted to lie in $\mathbf{S}_1$. The extension to $\mathbf{S}$ is obtained by taking $D(M, k)(\psi) = 0$ for all $\psi \in \mathbf{S} - \mathbf{S}_1$. Using the properties of $\bar{C}$, we can easily check that this is a valid extension satisfying (i)–(iv). ■

This lemma proves that $\psi = \theta + \varepsilon\bar{C}(M(\theta, k), \theta, k)$ if and only if $\theta = \psi - \varepsilon D(M, k)(\psi)$ and $\psi \in \mathbf{S} - \mathbf{S}_1$ implies $D(M, k)(\psi) = 0$. We also remark that $D(M, k)(\psi)$ is periodic in the periodic case.

Having established that the graph transform is well defined, we study its properties in a slightly more general context. Let us consider a graph transform T defined on $\mathbf{M}$ by

$$T(M)(\psi, k+1) = A(\psi)M(\theta, k) + B'(\psi, M(\theta, k), k),$$
$$\theta = \psi - \varepsilon D(M, k)(\psi),$$

where $B'\colon \mathbf{S} \times B(O, m) \times \mathbb{Z} \to \mathbb{R}^n$ satisfies uniformly:

(i) $|||B'(\psi, X, k)|||_\psi \le b'$.
(ii) $|||B'(\psi^0, X^0, k) - B'(\psi^1, X^1, k)|||_{\psi^i} \le b_1^\psi \|\psi^0 - \psi^1\| + \varepsilon b_1^x |||X^0 - X^1|||_{\psi^i}, i = 0, 1.$

Lemma 2. *Function $T(M)$ is contained in* $\mathbf{M}$ *and T is a contraction.*

Proof. (a) Function $T(M)$ is in $\mathbf{M}$:

(i) With our adapted metric, we have

$$|||T(M)(\psi, k+1)|||_\psi \le \gamma(1 + \varepsilon\beta\bar{c})|||M(\theta, k)|||_\theta + b'.$$

Hence m should satisfy, with ε sufficiently small, $\gamma(1 + \varepsilon\beta\bar{c})m + b' \le m$.

(ii) Using the properties of M, D, for ψ^0, ψ^1 in $\mathbf{S}$, we have

$$\begin{aligned}&|||T(M)(\psi^0, k+1) - T(M)(\psi^1, k+1)|||_{\psi^i}\\ &\quad\le [(\gamma + \varepsilon b_1^x)(1 + \varepsilon\beta\bar{c})m_1(1 + \varepsilon d_1) + \alpha a_1 m + b_1^\psi]\|\psi^0 - \psi^1\|,\\ &\quad i = 0, 1.\end{aligned}$$

Hence m_1 should satisfy, with ε sufficiently small,

$$(1 + \varepsilon d_1)(\gamma + \varepsilon b_1^x)(1 + \varepsilon\beta\bar{c})m_1 + \alpha a_1 m + b_1^\psi \le m_1.$$

(iii) Function $T(M)$ is periodic in k in the periodic case.

(b) The graph transform T is a contraction: Let M^0, M^1 be two elements of $\mathbf{M}$, we have

$$\begin{aligned}&|||T(M^0)(\psi, k+1) - T(M^1)(\psi, k+1)|||_\psi\\ &\quad\le (\gamma + \varepsilon b_1^x)(1 + \varepsilon\beta\bar{c})|||M^0(\theta^0, k) - M^1(\theta^1, k)|||_{\theta^0}\\ &\quad\le (\gamma + \varepsilon b_1^x)(1 + \varepsilon\beta\bar{c})[|M^0 - M^1| + \varepsilon m_1\|D(M^0, k)(\psi) - D(M^1, k)(\psi)\|]\\ &\quad\le (\gamma + \varepsilon b_1^x)(1 + \varepsilon\beta\bar{c})(1 + \varepsilon m_1 d_1^m)|M^0 - M^1|.\end{aligned}$$

Since $\gamma < 1$, taking the supremum on $\mathbf{S} \times \mathbb{Z}$ gives the result for ε sufficiently small. ■

To prove existence of $M_\varepsilon(\theta, k)$, we apply this lemma with $B'(\psi, X, k) = B(\psi)u_k$. It follows that the graph transform has a fixed point in $\mathbf{M}$ if

$$\gamma(1 + \varepsilon\beta\bar{c}(m))m + bu\alpha \le m \le x,$$

$$\frac{m_1}{1 - \varepsilon\bar{c}_1(m)(1 + m_1)}\gamma(1 + \varepsilon\beta\bar{c}(m)) + \alpha(b_1 u + a_1 m) \le m_1,$$

$$\tau_0(\varepsilon, m) = \gamma(1 + \varepsilon\beta\bar{c}(m))\left(1 + \varepsilon\frac{\bar{c}_1(m)m_1}{1 - \varepsilon\bar{c}_1(m)(1 + m_1)}\right) < 1.$$

In particular, this confirms condition (3) of the previous Remark.

We have proved the existence of a Lipschitz continuous function M_ε: $\mathrm{S} \times \mathbb{Z} \to \mathbb{R}^n$ whose graph is an integral manifold of S_ε locally on S_0 and which is periodic in the periodic case. We now establish that, under assumption A5, M_ε is Lipschitz continuously differentiable in θ. Assuming for the time being that this property holds and denoting by $L_\varepsilon(\theta, k)$ what the differential of $M_\varepsilon(\theta, k)$ should be, formal derivations show that $L_\varepsilon(\theta, k)$ satisfies

$$L_\varepsilon(\psi, k+1) = A(\psi)L_\varepsilon(\theta, k) + B'(\psi, L_\varepsilon(\theta, k), k),$$
$$\psi = \theta + \varepsilon\bar{C}(M_\varepsilon(\theta, k), \theta, k),$$

where, with $\otimes$ denoting the tensor product,

$$B'(\psi, Y, k) = -\varepsilon A(\psi) Y \nabla\bar{C}(Y, \theta, k)(I + \varepsilon\nabla\bar{C}(Y, \theta, k))^{-1}$$
$$+ \frac{\partial A}{\partial\theta}(\psi) \otimes M_\varepsilon(\theta, k) + \frac{\partial B}{\partial\psi}(\psi) \otimes u_k,$$

$$\nabla\bar{C}(Y, \theta, k) = \frac{\partial\bar{C}}{\partial\theta}(M_\varepsilon(\theta, k), \theta, k) + \frac{\partial\bar{C}}{\partial X}(M_\varepsilon(\theta, k), \theta, k)Y.$$

Hence L_ε should be the fixed point of the following operator:

$$T(L)(\psi, k+1) = A(\psi)L(\theta, k) + B'(\psi, L(\theta, k), k),$$
$$\theta = \psi - \varepsilon D(M_\varepsilon, k)(\psi).$$

Let us prove that this operator has a fixed point. From its definition and assumption A5, B' satisfies the assumptions of Lemma 2 with, in particular,

$$|||X|||_\theta \leq m_1 \;\Rightarrow\; |||B'(\psi, X, k)|||_\psi \leq \varepsilon\gamma(1 + \varepsilon\beta\bar{c})m_1 \frac{\bar{c}_1(1 + m_1)}{1 - \varepsilon\bar{c}_1(1 + m_1)} + \alpha(a_1 m + b_1 u).$$

Noting that m_1 plays the role of m and m_2 (the Lipschitz constant of L_ε) the role of m_1, Lemma 2 applies if

$$\gamma(1 + \varepsilon\beta\bar{c}(m))\frac{m_1}{1 - \varepsilon\bar{c}_1(m)(1 + m_1)} + \alpha(a_1 m + b_1 u) \leq m_1,$$

$$(1 + \varepsilon\beta\bar{c}(m))\frac{(\gamma + \varepsilon b_1^x)m_2}{1 - \varepsilon\bar{c}_1(m)(1 + m_1)} + \alpha a_1 m_1 + b_1^\psi \leq m_2,$$

$$\tau_1'(\varepsilon, m) = (1 + \varepsilon\beta\bar{c}(m))(\gamma + \varepsilon b_1^x)\left(1 + \varepsilon\frac{m_2\bar{c}_1(m)}{1 - \varepsilon\bar{c}_1(m)(1 + m_1)}\right) < 1.$$

Note that the condition $\tau_1' < 1$ guarantees not only existence but also Lipschitz continuity of L_ε (compare with (3)).

To complete our proof let us show that the candidate L_ε is indeed the differential of M_ε, i.e., for any θ interior point of S we have, uniformly in k,

$$\limsup_{h\to 0} \frac{\|M_\varepsilon(\theta + h, k) - M_\varepsilon(\theta, k) - L_\varepsilon(\theta, k)h\|}{\|h\|} = 0.$$

To prove this equality, we consider ψ an interior point of S and $\bar{h}$ a (sufficiently small) vector in $\mathbb{R}^p$ and we introduce the following notations (see assumption A5):

$$\theta = \psi - \varepsilon D(M_\varepsilon, k)(\psi),$$
$$h = \bar{h} + \varepsilon[D(M_\varepsilon, k)(\psi) - D(M_\varepsilon, k)(\psi + \bar{h})],$$
$$\delta_M(\theta, k, h) = M_\varepsilon(\theta + h, k) - M_\varepsilon(\theta, k) - L_\varepsilon(\theta, k)h,$$
$$\bar{\delta}_C(\theta, k, h) = \delta_{\bar{C}}(M_\varepsilon(\theta, k), \theta, k, M_\varepsilon(\theta + h) - M_\varepsilon(\theta, k), h),$$
$$\Delta(\theta, k) = \limsup_{h\to 0} \frac{|||\delta_M(\theta, k, h)|||_\theta}{\|h\|}.$$

Note that θ is also an interior point of S. From assumption A5 and since M_ε is Lipschitz continuous, we have

$$\limsup_{h\to 0} \frac{\|\bar{\delta}_C(\theta, k, h)\|}{\|h\|} = 0.$$

Let us find a recurrence satisfied by δ_M. The definition of M_ε yields

$$\delta_M(\psi, k + 1, \bar{h}) = A(\psi + \bar{h})\delta_M(\theta, k, h) + A(\psi + \bar{h})L_\varepsilon(\theta, k)h - L_\varepsilon(\psi, k + 1)\bar{h}$$
$$+ (A(\psi + \bar{h}) - A(\psi))M_\varepsilon(\theta, k) + (B(\psi + \bar{h}) - B(\psi))u_k.$$

We also have

$$\bar{h} = [I + \varepsilon\nabla\bar{C}(L_\varepsilon(\theta, k), \theta, k)]h + \varepsilon\frac{\partial\bar{C}}{\partial X}(M_\varepsilon(\theta, k), \theta, k)\delta_M(\theta, k, h)$$
$$+ \varepsilon\bar{\delta}_C(\theta, k, h),$$
$$L_\varepsilon(\psi, k + 1) = A(\psi)L_\varepsilon(\theta, k)[I + \varepsilon\nabla\bar{C}(L_\varepsilon(\theta, k), \theta, k)]^{-1}$$
$$+ \frac{\partial A}{\partial\psi}(\psi) \otimes M_\varepsilon(\theta, k) + \frac{\partial B}{\partial\psi}(\psi) \otimes u_k.$$

This yields

$$\delta_M(\psi, k + 1, \bar{h})$$
$$= A(\psi + \bar{h})\delta_M(\theta, k, h) + \delta_A(\psi, k, \bar{h})M_\varepsilon(\theta, k) + \delta_B(\psi, k, \bar{h})u_k$$
$$+ (A(\psi + \bar{h}) - A(\psi))L_\varepsilon(\theta, k)h - \varepsilon A(\psi)L_\varepsilon(\theta, k)$$
$$\times (I + \varepsilon\nabla\bar{C}(M_\varepsilon(\theta, k), \theta, k))^{-1}\left[\frac{\partial\bar{C}}{\partial X}(M_\varepsilon(\theta, k), \theta, k)\delta_M(\theta, k, h) + \bar{\delta}_C(\theta, k, h)\right].$$

Noticing that $\|h\|/(1 + \varepsilon d_1) \le \|\bar{h}\| \le (1 + \varepsilon\bar{c}_1(1 + m_1))\|h\|$, taking the $|||\cdot|||$-norm, dividing by $\|h\|$, and taking the lim sup for h going to zero, we obtain

$$\Delta(\psi, k + 1)\frac{1}{1 + \varepsilon d_1} \le \gamma(1 + \varepsilon\beta\bar{c})\left(1 + \varepsilon\frac{m_1\bar{c}_1}{1 - \varepsilon\bar{c}_1(1 + m_1)}\right)\Delta(\theta, k).$$

On the other hand, for any interior point θ of S, the properties of M_ε, L_ε give, for all k, $\Delta(\theta, k) \le 2m_1$. Given k and θ, an interior point of S, we construct a sequence

(θ_j), $j \in \mathbb{N}$, as $\theta_{j+1} = \theta_j - \varepsilon D(M_\varepsilon, k - j - 1)(\theta_j)$, $\theta_0 = \theta$. By induction, θ_j is an interior point of S and, from the previous inequality, we have $\Delta(\theta, k) \leq \tau_1(\varepsilon, m)^j \Delta(\theta_j, k-j) \leq \tau_1(\varepsilon, m)^j 2m_1$ with

$$\tau_1(\varepsilon, m) = \gamma(1 + \varepsilon\beta\bar{c}(m))\left(1 + \varepsilon\frac{m_1\bar{c}_1(m)}{1 - \varepsilon\bar{c}_1(m)(1 + m_1)}\right)\frac{1}{1 - \varepsilon\bar{c}_1(m)(1 + m_1)}$$

$$= \tau_0(\varepsilon, m)\frac{1}{1 - \varepsilon\bar{c}_1(m)(1 + m_1)}.$$

Our relation holds for all positive j, and, as expected from the previous Remark, we have established that L_ε is the differential of M_ε if $\tau_1(\varepsilon, m) < 1$.

Approximation. We have the existence of M_ε, the solution of

$$M_\varepsilon(\psi, k + 1) = A(\psi)M_\varepsilon(\psi - \varepsilon D(M_\varepsilon, k)(\psi), k) + B(\psi)u_k.$$

Getting a solution of this system for each ψ, would be equivalent to getting all the solutions of $\bar{S}_\varepsilon$ bounded on $\mathbb{Z}$. However, we notice that this equation can be rewritten as

$$M_\varepsilon(\psi, k + 1) = A(\psi)M_\varepsilon(\psi, k) + B(\psi)u_k + A(\psi)[M_\varepsilon(\psi - \varepsilon D(M_\varepsilon, k)(\psi), k) - M_\varepsilon(\psi, k)].$$

This is a linear system with a nonlinear forcing term which disappears for $\varepsilon = 0$. Using the Poincaré method of expansion with respect to a small parameter, we can obtain a family of approximations:

"0"-order approximation: taking $\varepsilon = 0$ gives

$$M_0(\psi, k + 1) = A(\psi)M_0(\psi, k) + B(\psi)u_k.$$

This is the solution of S_0.

"1"-order approximation: retaining the first-order term in ε gives

$$M_1(\psi, k + 1) = A(\psi)M_1(\psi, k) + B(\psi)u_k - \varepsilon A(\psi)L_0(\psi, k)\bar{C}(M_0(\psi, k), \psi, k),$$

again a linear system.

The quality of these approximations can be characterized as follows:

(i) We notice that M_0 belongs to the set **M**, hence the graph transform properties imply

$$|M_0 - M_\varepsilon| \leq \frac{|M_0 - T(M_0)|}{1 - \tau_0(\varepsilon, m)}.$$

But

$$|||M_0(\psi, k + 1) - T(M_0)(\psi, k + 1)|||_\psi \leq \gamma|||M_0(\psi - \varepsilon D(M_0, k)(\psi), k) - M_0(\psi, k)|||_\psi.$$

It follows that

$$|M_0 - M_\varepsilon| \leq \varepsilon\frac{\gamma m_1\bar{c}}{1 - \tau_0(\varepsilon, m)} = \varepsilon f^0. \quad \blacklozenge$$

Similarly, for the gradient matrix, we get

$$|L_0 - L_\varepsilon| \le \frac{|L_0 - T(L_0)|}{1 - \tau_1'(\varepsilon, m)}$$

and

$$\begin{aligned} L_0(\psi, k+1) &- T(L_0)(\psi, k+1) \\ &= A(\psi)[L_0(\psi, k) - L_0(\theta, k)(I + \varepsilon\nabla\bar{C}(L_0(\theta, k), \theta, k))^{-1}] \\ &\quad + \frac{\partial A}{\partial\psi}(\psi) \otimes (M_0(\psi, k) - M_\varepsilon(\theta, k)) \end{aligned}$$

with $\theta = \psi - \varepsilon D(M_0, k)(\psi)$. Hence

$$|L_0 - L_\varepsilon| \le \frac{\gamma\left(\varepsilon m_2\bar{c} + \varepsilon\dfrac{m_1\bar{c}_1(1+m_1)(1+\varepsilon\beta\bar{c})}{1-\varepsilon\bar{c}_1(1+m_1)}\right) + \alpha a_1 m_1(\varepsilon m_1\bar{c} + \varepsilon f^0)}{1 - \tau_1'(\varepsilon, m)} = \varepsilon f_1^0.$$

(ii) The function M_0^1, defined in (iii) of Theorem 1, is given by a noncritical linear system with bounded input. Hence, it is uniquely defined and bounded. Denoting $\Delta(\psi, k) = M_\varepsilon(\psi, k) - M_0^1(\psi, k)$, we have

$$\Delta(\psi, k+1) = A(\psi)[\Delta(\psi, k) + M_\varepsilon(\theta, k) - M_\varepsilon(\psi, k) + \varepsilon L_0(\psi, k)\bar{C}(M_0(\psi, k), \psi, k)],$$
$$\theta = \psi - \varepsilon D(M_\varepsilon, k)(\psi).$$

We know that the segment $[\theta, \psi]$ is contained in S. The Mean Value Theorem gives for some ξ, $0 \le \xi \le 1$,

$$M_\varepsilon(\theta, k) - M_\varepsilon(\psi, k) = -\varepsilon L_\varepsilon(\psi + \xi(\theta - \psi), k)\bar{C}(M_\varepsilon(\theta, k), \theta, k).$$

Hence, using the properties of L_ε, $L_0 - L_\varepsilon$, $M_0 - M_\varepsilon$, D, we get

$$\begin{aligned} &|||M_\varepsilon(\theta, k) - M_\varepsilon(\psi, k) + \varepsilon L_0(\psi, k)\bar{C}(M_0(\psi, k), \psi, k)|||_\psi \\ &\qquad \le \varepsilon^2 m_2\xi\bar{c}^2 + \varepsilon^2 f_1^0\bar{c} + \varepsilon m_1\bar{c}_1(\varepsilon(1+m_1)\bar{c} + \varepsilon f^0). \end{aligned}$$

This yields

$$|M_\varepsilon - M_1| \le \varepsilon^2\gamma\frac{m_2\bar{c}^2 + f_1^0\bar{c} + m_1\bar{c}_1(\bar{c}(1+m_1) + f^0)}{1-\gamma} = \varepsilon^2 f^1.$$

Attractiveness. Let (X, θ, k) be an element of $B(O, x) \times S_0 \times \mathbb{Z}$, for $x \ge m$, and let $(Y, \psi, k+1)$ be its image by S_ε. Paying attention to the fact that we have X only in $B(O, x)$ and not in $B(O, m)$, we can however proceed, as in Lemma 1, to get

$$\|D(M_\varepsilon, k)(\psi) - C(X, \theta, k)\| \le d_1^m(x)|||X - M_\varepsilon(\theta, k)|||_\theta,$$

$$d_1^m(x) = \frac{\bar{c}_1(x)}{1 - \varepsilon\bar{c}_1(x)(1+m_1)}.$$

Then, as in Lemma 2, we obtain $|||Y - M_\varepsilon(\psi, k+1)|||_\psi \le \tau_0(\varepsilon, x)|||X - M_\varepsilon(\theta, k)|||_\theta$.

Applying this inequality to a sequence (X_{k+1}, θ_k) of $\mathbf{B}([k_0, k_1), \mathbf{S}_0)$, we obtain for all $k, k', k_0 \le k' \le k < k_1$,

$$\|X_{k+1} - M_\varepsilon(\theta_k, k+1)\| \le \tau_0(\varepsilon, x)^{k-k'}\alpha\|X_{k'+1} - M_\varepsilon(\theta_{k'}, k'+1)\|.$$

Moreover, if X_{k+1}, θ_k lies in $B(O, x) \times \mathbf{S}_0$ for all k, then this inequality holds for all $k', k' \le k$. This implies $X_{k+1} = M_\varepsilon(\theta_k, k+1)$.

This completes the proof of Theorem 1.

In the following we are interested in characterizing the set $\mathbf{B}([k_0, \infty), \mathbf{S}_0)$. The attractiveness property implies that this set is a subset of $l_\infty(\mathbb{R}^n \times \mathbb{R}^p)$ from which it inherits a complete metric space property. We also see that the corresponding sequences of elevation above the manifold have a natural exponentially weighted supremum norm:

$$|E| = \operatorname*{Sup}_{k \ge k_0} \{\rho^{k_0-k}\|E_k\|\}, \qquad \rho < 1.$$

For $\rho = \tau_0(\varepsilon, x)$, we know that the map from $\mathbf{B}([k_0, \infty), \mathbf{S}_0)$ to $\mathbb{R}^n$, giving $(X, \theta) \mapsto E$ is bounded with this norm. We even have

Lemma 3. *Under assumptions* A1–A5, *there exists ε^* such that for any ε, $\varepsilon \le \varepsilon^*$, we can find $\rho < 1$ such that the above map is Lipschitz continuous.*

A key technical lemma to prove this statement is the following consequence of Hadamard's lemma [AE]:

Lemma 4. *Let f be a Lipschitz continuously differentiable function $f: \mathbf{C} \subset \mathbb{R}^n \to \mathbb{R}^n$, with f_1, f_2 as Lipschitz constants. If the segments $[x^0, x^0 + \delta^0]$, $[x^1, x^1 + \delta^1]$ are contained in* **C**, *then*

$$\begin{aligned}&\|f(x^0 + \delta^0) - f(x^0) + f(x^1) - f(x^1 + \delta^1)\| \\ &\qquad \le (f_1 + \tfrac{1}{2}f_2\|\delta^0\|)\|\delta^0 - \delta^1\| + f_2\|\delta^0\|\,\|x^0 - x^1\|.\end{aligned}$$

Proof of Lemma 3. Let (X^0, θ^0, k), (X^1, θ^1, k) be two points in $B(O, x) \times \mathbf{S}_0 \times \mathbb{Z}$ and (Y^0, ψ^0, k), (Y^1, ψ^1, k) be the respective images by S_ε. For $i = 0, 1$, we define

$$E^i = X^i - M_\varepsilon(\theta^i, k), \qquad F^i = Y^i - M_\varepsilon(\psi^i, k), \qquad G^i = X^i - M_\varepsilon(\theta^i + \Delta^i, k),$$
$$\Delta^i = \psi^i - \theta^i - \varepsilon D(M_\varepsilon, k)(\psi^i) = \varepsilon[C(X^i, \theta^i, k) - C(M_\varepsilon(\theta^i + \Delta^i, k), \theta^i + \Delta^i, k)].$$

We have $\|\Delta^0\| \le \varepsilon c_1(\|G^0\| + \|\Delta^0\|)$ and $\|G^0\| \le \|E^0\| + m_1\|\Delta^0\|$ which implies

$$\|\Delta^0\| \le \varepsilon\frac{c_1}{1 - \varepsilon c_1(1 + m_1)}\|E^0\|, \qquad \|G^0\| \le \frac{1 - \varepsilon c_1}{1 - \varepsilon c_1(1 + m_1)}\|E^0\|.$$

On the other hand, by definition, $F^0 - F^1 = A(\psi^0)(G^0 - G^1) + (A(\psi^0) - A(\psi^1))G^1$. But, noticing that, for $i = 0, 1$, the segments $[X^i, M_\varepsilon(\theta^i + \Delta^i, k)]$ and $[\theta^i, \theta^i + \Delta^i]$ are contained in $B(O, x)$ and S_1, respectively, we apply Lemma 4 to M_ε and C to obtain

$$\begin{aligned}
|||G^0 - G^1|||_{\psi^0} &\leq |||E^0 - E^1|||_{\psi^0} + (m_1 + \tfrac{1}{2}m_2\|\Delta^0\|)\|\Delta^0 - \Delta^1\| \\
&\quad + m_2\|\Delta^0\|\,\|\theta^0 - \theta^1\|, \\
\|\Delta^0 - \Delta^1\| &\leq \varepsilon[(c_1 + \tfrac{1}{2}c_2(\|\Delta^0\| + \|G^0\|))(\|G^0 - G^1\| + \|\Delta^0 - \Delta^1\|) \\
&\quad + c_2(\|\Delta^0\| + \|G^0\|)(\|X^0 - X^1\| + \|\theta^0 - \theta^1\|)].
\end{aligned}$$

It follows that

$$\begin{aligned}
&[1 - \varepsilon(c_1 + \tfrac{1}{2}c_2(\|\Delta^0\| + \|G^0\|))(1 + m_1 + \tfrac{1}{2}m_2\|\Delta^0\|)]\,|||G^0 - G^1|||_{\psi^0} \\
&\quad \leq [1 - \varepsilon(c_1 + \tfrac{1}{2}c_2(\|\Delta^0\| + \|G^0\|))]\,|||E^0 - E^1|||_{\psi^0} \\
&\qquad + [(1 - \varepsilon c_1)m_2\|\Delta^0\| + \varepsilon c_2 m_1(\|\Delta^0\| + \|G^0\|)]\,\|\theta^0 - \theta^1\| \\
&\qquad + \varepsilon c_2(\|\Delta^0\| + \|G^0\|)(m_1 + \tfrac{1}{2}m_2\|\Delta^0\|)\|X^0 - X^1\|.
\end{aligned}$$

This implies existence of a constant q_0 such that

$$|||F^0 - F^1|||_{\psi^0} \leq \rho|||E^0 - E^1|||_{\theta^0} + q_0\|E^0\|(\|X^0 - X^1\| + \|\theta^0 - \theta^1\|)$$

with

$$\rho = \frac{\gamma(1+\varepsilon\beta c)[1-\varepsilon(c_1+(\frac{1}{2}c_2(1-\varepsilon c_1(1+m_1)))\|E^0\|)]}{1-\varepsilon(c_1+(\frac{1}{2}c_2(1-\varepsilon c_1(1+m_1)))\|E^0\|)(1+m_1+\varepsilon\frac{1}{2}m_2c_1/(1-\varepsilon c_1(1+m_1))\|E^0\|)}.$$

Now we apply these inequalities to two sequences (X^0_{k+1}, θ^0_k), (X^1_{k+1}, θ^1_k) of $\mathbf{B}([k_0, \infty), \mathrm{S}_0)$: we have

$$\|E^0_k\| \leq (x - m)\tau_0^{k-k_0}, \quad |||E^0_{k_0+1} - E^1_{k_0+1}|||_{\theta^0_{k_0}} \leq \alpha\|X^0_{k_0+1} - X^1_{k_0+1}\| + m_1\|\theta^0_{k_0} - \theta^1_{k_0}\|,$$

hence

$$|||E^0_{k+1} - E^1_{k+1}|||_{\theta^0_k} \leq \rho|||E^0_k - E^1_k|||_{\theta^0_{k-1}} + q_0(x - m)\tau_0^{k-k_0}(\|X^0_k - X^1_k\| + \|\theta^0_k - \theta^1_k\|),$$

which yields

$$\|E^0_{k+1} - E^1_{k+1}\| \leq \rho^{k-k_0}\left(1 + \frac{\tau_0}{\rho - \tau_0}\right)q \operatorname*{Sup}_{k\geq k_0}(\|X^0_{k+1} - X^1_{k+1}\| + \|\theta^0_k - \theta^1_k\|)$$

with $q = \mathrm{Max}(\alpha, m_1, q_0(x - m))$. ∎

4. Topological Orbital Equivalence with Asymptotic Phase

In this section we establish the existence of a homeomorphism between subsets of $\mathbf{B}([k_0, \infty), \mathrm{S}_0)$ and $\mathbf{B}^M([k_0, \infty), \mathrm{S}_0)$. For such a strong property to hold, solutions of S_ε must be very close to solutions of SM_ε. As a consequence of attractiveness, we know that for any sequence (X_{k+1}, θ_k) (resp. (X^M_{k+1}, θ^M_k)) of $\mathbf{B}([k_0, \infty), \mathrm{S}_0)$ (resp. $(\mathbf{B}^M([k_0, \infty), \mathrm{S}_0))$ we have for all k, $k \geq k_0$,

$$\begin{aligned}
\rho^{k_0-k}\|(X_{k+1}, \theta_k, k + 1) - SM_\varepsilon(X_k, \theta_{k-1}, k)\| &\leq \varepsilon(1 + (a_1x + b_1u))c_1\alpha(x - m), \\
\rho &= \tau_0(\varepsilon, x),
\end{aligned}$$

and

$$\rho^{k_0-k}\|(X_{k+1}^M, \theta_k^M, k+1) - S_\varepsilon(X_k^M, \theta_{k-1}^M, k)\| \le \varepsilon(1 + (a_1 x + b_1 u))c_1\alpha(x - m),$$
$$\rho = \gamma(1 + \varepsilon\beta c).$$

This means that (X_{k+1}, θ_k) (resp. (X_{k+1}^M, θ_k^M)) is an ε-pseudo solution of SM_ε (resp. S_ε) in the neighborhood of a normally attractive integral manifold. Invoking the Shadowing Lemma (Proposition 8.19 of [S]) we may expect the existence of a unique solution $(\tilde{X}_{k+1}^M, \tilde{\theta}_k^M)$ (resp. $(\tilde{X}_{k+1}, \tilde{\theta}_k)$) of SM_ε (resp. S_ε) ε-close to the corresponding ε-pseudo solution for the exponentially weighted supremum norm. This allows us to define a map $\Phi(k_0)$ (resp. $\Phi^M(k_0)$) as $\Phi(k_0)(X, \theta)_k = (\tilde{X}_{k+1}^M, \tilde{\theta}_k^M)$ (resp. $\Phi^M(k_0)(X^M, \theta^M)_k = (\tilde{X}_{k+1}, \tilde{\theta}_k)$). Clearly, from the ε-closeness property, this map is a very good candidate for the homeomorphism we are looking for.

To obtain the Shadowing Lemma in our context, we have to shrink the set S_0 again: let S_0', S_1' be compact sets with nonempty interior such that $S_0' + \eta \subset S_1'$, $S_1' + \eta \subset S_0$.

Theorem 2. *Under assumptions* A1–A4, *for any compact set* S_1' *strictly contained in* S, *there exists* ε^* *such that for any* ε, $\varepsilon \le \varepsilon^*$, *we can find* g^x, g^θ, g_1, σ, $0 \le \sigma < 1$, *and, for any* k_0, *maps*

$$\Phi(k_0): \mathbf{B}([k_0, \infty), S_1') \to \mathbf{B}^M([k_0, \infty), S_0) \quad \textit{and}$$
$$\Phi^M(k_0): \mathbf{B}^M([k_0, \infty), S_1') \to \mathbf{B}([k_0, \infty), S_0)$$

such that (similarly for Φ^M*):*

(i) *For any sequence* (X_{k+1}, θ_k) *in* $\mathbf{B}([k_0, \infty), S_1')$, *its image* $(\tilde{X}_{k+1}^M, \tilde{\theta}_k^M)$ *by* $\Phi(k_0)$ *satisfies for all k,* $k \ge k_0$,

$$X_{k_0+1} = \tilde{X}_{k_0+1}^M, \qquad \sigma^{k_0-k}\|X_{k+1} - \tilde{X}_{k+1}^M\| \le \varepsilon g^x(\|E_{k_0+1}\|),$$
$$\sigma^{k_0-k}\|\theta_k - \tilde{\theta}_k^M\| \le \varepsilon g^\theta(\|E_{k_0+1}\|),$$

where g^x, g^θ *are positive nondecreasing functions of the norm of the elevation above the manifold at time* k_0.

(ii) $\Phi(k_0)(X, \theta)_k = \Phi(k_0')(X, \theta)_k$, $k_0 \le k_0' \le k$.

(iii) *If assumption* A5 *holds, then* $\Phi(k_0)$ *is Lipschitz continuous in the following strong sense: for all k,* $k \ge k_0$,

$$\|\tilde{X}_{k+1}^0 - \tilde{X}_{k+1}^1\| + \|\tilde{\theta}_k^0 - \tilde{\theta}_k^1\|$$
$$\le \|X_{k+1}^0 - X_{k+1}^1\| + \|\theta_k^0 - \theta_k^1\| + \sigma^{k-k_0} g_1 \sup_{\kappa \ge k_0} (\|X_{\kappa+1}^0 - X_{\kappa+1}^1\| + \|\theta_\kappa^0 - \theta_\kappa^1\|).$$

Moreover, $\Phi(k_0)$ *(resp.* $\Phi^M(k_0)$*) is injective and the restriction of* $\Phi^M(k_0) \circ \Phi(k_0)$ *(resp.* $\Phi(k_0) \circ \Phi^M(k_0)$*) to* $\mathbf{B}([k_0, \infty), S_0')$ *(resp.* $\mathbf{B}^M([k_0, \infty), S_0')$*) is the identity map.*

Remarks. For the continuous time case, Riedle and Kokotovic [RK2] have shown that if there exists a uniformly stable solution (θ_k^M) of SM_ε^θ, remaining in S_1' after time k_0, then $\mathbf{B}([k_0, \infty), S_0)$ is not empty and has a nonempty interior.

Since we have only $\sigma^{k_0-k}\|\theta_k - \tilde{\theta}_k\| \leq \eta$ we have to take θ_k in $\mathbf{S}'_0$ (resp. $\mathbf{S}'_1$) to guarantee that $\tilde{\theta}_k^M$ is in $\mathbf{S}'_1$ (resp. $\mathbf{S}_0$).

Since we have an exponentially decaying distance between (X_{k+1}, θ_k) and $(\tilde{X}_{k+1}^M, \tilde{\theta}_k^M)$, $\Phi(k_0)$ is an asymptotic phase.

We have established $\mathbf{B}([k_0, \infty), \mathbf{S}'_0) \subset \Phi^M(k_0)(\mathbf{B}^M([k_0, \infty), \mathbf{S}'_1))$. This means that any solution (X_{k+1}, θ_k) of S_ε which remains in $B(O, x) \times \mathbf{S}'_0$ on a semi-infinite time interval (i.e., belongs to $\mathbf{B}([k_0, \infty), \mathbf{S}'_0)$ for some k_0) can be approximated with an exponentially decaying distance by a solution of SM_ε satisfying the same property. Moreover, these two sequences have the same X-values X_{k_0+1} at time k_0 and their θ-values at this time are at an ε-distance, the magnitude of this distance increasing with the norm of the elevation above the manifold at time k_0. These two solutions have the same type of Lyapunov stability in each of the following cases: stability, uniform stability, asymptotic stability, uniform asymptotic stability, and instability. Unformity follows from (ii) and the independence of σ, g^x, g^θ, g_1 in k_0. Asymptotic or exponential property results from the σ^{k-k_0} term in (iii). (In)stability is a consequence of the continuity property (iii) as shown in the following lemma:

Lemma 5. *If the sequence (X_{k+1}, θ_k) (resp. (X_{k+1}^M, θ_k^M)) in the interior (in the $l_\infty(\mathbb{R}^n \times \mathbb{R}^p)$ sense) of $\mathbf{B}([k_0, \infty), \mathbf{S}'_0)$ (resp. $\mathbf{B}^M([k_0, \infty), \mathbf{S}'_0)$) is a stable solution of S_ε (resp. SM_ε), its image by $\Phi(k_0)$ (resp. $\Phi^M(k_0)$) is a stable solution of SM_ε (resp. S_ε).*

Proof. As above let $(\tilde{X}_{k+1}^M, \tilde{\theta}_k^M)$ denote the image of (X_{k+1}, θ_k) by $\Phi(k_0)$. Let (X_{k+1}^0, θ_k^0) be the stable solution of S_ε in the interior of $\mathbf{B}([k_0, \infty), \mathbf{S}'_0)$. Stability implies existence of an open neighborhood $\mathbf{V}(X_{k_0+1}^0, \theta_{k_0}^0)$ in $\mathbb{R}^n \times \mathbb{R}^p$ on which the injective map associating a solution of S_ε to its value at time k_0 is continuous. Composing by $\Phi(k_0)$, we obtain a continuous injective map:

$$\Psi\colon \mathbf{V}(X_{k_0+1}^0, \theta_{k_0}^0) \to \mathbf{B}^M([k_0, \infty), \mathbf{S}'_1),$$
$$(X_{k_0+1}, \theta_{k_0}) \mapsto (\tilde{X}_{k+1}^M, \tilde{\theta}_k^M).$$

Restricting Ψ to time $k = k_0$, we have a continuous injective map:

$$\Psi_{k_0}\colon \mathbf{V}(X_{k_0+1}^0, \theta_{k_0}^0) \subset \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n \times \mathbb{R}^p,$$
$$(X_{k_0+1}, \theta_{k_0}) \mapsto (\tilde{X}_{k_0+1}^M, \tilde{\theta}_{k_0}^M).$$

It follows from Brouwer's Domain Invariance Theorem (Theorem 3.3.2 of [L2]) that this application is a homeomorphism from the open set $\mathbf{V}(X_{k_0+1}^0, \theta_{k_0}^0)$ into its open image $\tilde{\mathbf{V}}(\tilde{X}_{k_0+1}^{M0}, \tilde{\theta}_{k_0}^{M0})$. We conclude that the following map is continuous and is defined on a nonempty open subset of $\mathbb{R}^n \times \mathbb{R}^p$:

$$\Psi \circ \Psi_{k_0}^{-1}\colon \mathbf{V}(\tilde{X}_{k_0+1}^{M0}, \tilde{\theta}_{k_0}^{M0}) \to \mathbf{B}^M([k_0, \infty), \mathbf{S}'_1).$$

This is nothing but the map associating the solution $(\tilde{X}_{k+1}^M, \tilde{\theta}_k^M)$ of SM_ε to its value $(\tilde{X}_{k_0+1}^M, \tilde{\theta}_{k_0}^M)$ at time k_0. Since this can be done for any k'_0, $k'_0 \geq k_0$, we have proved stability of $(\tilde{X}_{k+1}^{M0}, \tilde{\theta}_k^{M0})$. ■

From this lemma, instability is obtained by contradiction.

Noticing that if (X^M_{k+1}, θ^M_k) belongs to $\mathbf{B}^M([k_0, \infty), \mathrm{S}_0)$, then this solution of SM_ε and (θ^M_k) as a solution of SM^θ_ε have the same type of Lyapunov stability or instability, we have established the main result of this paper:

Main result. *Under assumptions* A1–A5, *for each* x, $x \geq m$, *there exists* ε^* *such that for any* ε, $\varepsilon \leq \varepsilon^*$, *the systems* S_ε *and* SM_ε, *when restricted to* $B(O, x) \times \mathrm{S}'_0$, *are topologically orbitally equivalent with asymptotic phase. More precisely, for any* k_0, *each solution* (X_{k+1}, θ_k) *of* S_ε *such that* $\theta_k \in \mathrm{S}'_0$ *for all* k, $k \geq k_0$, *and* $\|X_{k_0+1} - M_\varepsilon(\theta_{k_0}, k_0 + 1)\| \leq (x - m)/\alpha$, *can be obtained from a solution* (θ^M_k) *of* SM^θ_ε *satisfying* $\theta^M_k \in \mathrm{S}'_1$ *for all* k, $k \geq k_0$, *and* $X_{k+1} - M_\varepsilon(\theta^M_k, k + 1)$ *and* $\theta^M_k - \theta_k$ *decay exponentially. Moreover,* (θ^M_k) *and* (X_{k+1}, θ_k) *have the same type of Lyapunov stability or instability.*

Proof of Theorem 2. We study an auxiliary system defined as follows. Let (ϕ_k) be a sequence whose elements lie in S'_1 and satisfy, uniformly on $\mathbb{N}$, the set of positive integers, $\|\phi_{k+1} - \phi_k\| \leq \varepsilon c$. We consider the following auxiliary system on $\mathbb{R}^n \times \mathbb{R}^p$:

$$\begin{aligned} \partial X_{k+1} &= A(\phi_k + \partial\theta_k)\partial X_k + B''(\partial\theta_k, k), \qquad \partial X_1 \in B\left(O, \frac{x - m}{\alpha}\right), \\ \partial\theta_k &= \partial\theta_{k-1} + \varepsilon C''(\partial X_k, \partial\theta_{k-1}, k), \end{aligned} \tag{∂S}$$

where B'', C'' are defined and Lipschitz continuous on $B(O, \eta) \times \mathbb{N}$ and $B(O, x - m) \times B(O, \eta) \times \mathbb{N}$, respectively, with b''_1, c''_1 as respective Lipschitz constants and such that, uniformly on $\mathbb{N}$, $B''(0, k) = 0$, $\|C''(0, 0, k)\| \leq c''_1 x_1 \rho^{k-1}$, $\rho < 1$. For two sequences (ϕ^0_k), (ϕ^1_k) satisfying, uniformly on $\mathbb{N}$, $\|\phi^0_k - \phi^1_k\| \leq \nu$ we are interested in the relation between the solutions of the corresponding auxiliary systems ∂S^0, ∂S^1, assuming that

$$\begin{aligned} &\|B''^0(\partial\theta^0, k) - B''^1(\partial\theta^1, k)\| \leq b''_2[\mathrm{Max}\{\|\partial\theta^0\|, \|\partial\theta^1\|\}\nu + \|\partial\theta^0 - \partial\theta^1\|], \\ &\quad \|C''^0(\partial X^0, \partial\theta^0, k) - C''^1(\partial X^1, \partial\theta^1, k)\| \\ &\qquad \leq c''_2[(\mathrm{Max}\{\|\partial X^0\| + \|\partial\theta^0\|, \|\partial X^1\| + \|\partial\theta^1\|\} + \rho^{k-1})\nu \\ &\qquad\qquad + \|\partial X^0 - \partial X^1\| + \|\partial\theta^0 - \partial\theta^1\|]. \end{aligned}$$

As for describing the ε-pseudo solution property, we use an exponentially weighted supremum norm to study this system:

$$|\partial X| = \sup_{k \geq 0} (\sigma^{-k}\|\partial X_{k+1}\|), \qquad |\partial\theta| = \sup_{k \geq 0} (\sigma^{-k}\|\partial\theta_k\|), \qquad \rho \leq \sigma < 1.$$

The following lemma states the existence of a unique solution of ∂S which is bounded for this norm.

Lemma 6. *Under the above assumptions and assumptions* A2 *and* A3, *there exists* ε^* *such that for any* ε, $\varepsilon \leq \varepsilon^*$, *we can find constants* ∂^z, g_1, σ *such that for any initial* ∂X*-condition* ∂X_1 *in* $B(O, (x - m)/\alpha)$:

(i) *The system ∂S has a unique bounded solution satisfying*

$$|\partial X| \le x_1' = \mathrm{Max}(\alpha\|\partial X_1\|, \varepsilon\partial^x), \qquad |\partial\theta| \le \varepsilon\frac{c_1''(x_1 + x_1')}{1 - \sigma - \varepsilon c_1''} \le \eta.$$

(ii) *If $(\partial X_{k+1}^0, \partial\theta_k^0)$, $(\partial X_{k+1}^1, \partial\theta_k^1)$ are these solutions for the systems ∂S^0, ∂S^1, respectively, we have $|\partial X^0 - \partial X^1| + |\partial\theta^0 - \partial\theta^1| \le g_1(\|\partial X_1^0 - \partial X_1^1\| + \nu)$. The σ, ∂^x, and g_1 are specified in the proof.*

Let us explain the interest of this lemma for proving Theorem 2:

If (X_{k+1}, θ_k) is a sequence of $\mathbf{B}([k_0, \infty), \mathbf{S}_1')$, we define its auxiliary system ∂S by choosing

$$\phi_k = \theta_{k+k_0}, \qquad \partial X_1 = 0,$$
$$B''(\partial\theta, k) = (A(\phi_k + \partial\theta) - (A(\phi_k)))X_{k+k_0} + (B(\phi_k + \partial\theta) - B(\phi_k))u_{k+k_0},$$
$$C''(\partial X, \partial\theta, k) = C(M_\varepsilon(\phi_{k-1} + \partial\theta, k + k_0), \phi_{k-1} + \partial\theta, k + k_0) - C(X_{k+k_0}, \phi_{k-1}, k + k_0).$$

Using the properties of the elevation above the manifold and Lemma 4, we can check that B'', C'' satisfy our assumptions with

$$b_1'' = a_1 x + b_1 u,$$
$$b_2'' = \mathrm{Max}\left\{x\left(a_1 + \frac{a_2}{2}\eta\right) + u\left(b_1 + \frac{b_2}{2}\eta\right), a_2 x + b_2 u, a_1\right\},$$
$$c_1'' = c_1(1 + m_1),$$
$$\begin{aligned} c_2'' = \mathrm{Max}\Bigg\{&\left(c_1 + \frac{c_2}{2}(x - m) + \eta(1 + m_1)\right)\left(m_1 + \frac{m_2}{2}\eta\right), \\ &\left(c_1 + \frac{c_2}{2}(x - m) + \eta(1 + m_1)\right)m_2 + c_2(1 + m_1), \\ &\left(c_1 + \frac{c_2}{2}(x - m) + \eta(1 + m_1)\right)q\left(1 + \frac{\tau_0}{\rho - \tau_0}\right) + c_2(x - m)\Bigg\}, \end{aligned}$$

where η is the "distance" between the imbeded compact sets, q and ρ are given by Lemma 3, $x_1 = \alpha\|E_{k_0+1}\|$ and $\nu = \mathrm{Sup}_{k \ge k_0}(\|X_{k+1}^0 - X_{k+1}^1\| + \|\theta_k^0 - \theta_k^1\|)$. Then if $(\partial X_{k+1}, \partial\theta_k)$ is the particular solution of ∂S given by Lemma 6, we define $\Phi(k_0)(X, \theta)_k = (X_{k+1+k_0} + \partial X_{k+1}, \theta_{k+k_0} + \partial\theta_k)$. From the properties of $(\partial X_{k+1}, \partial\theta_k)$, it follows that $\Phi(k_0)$ satisfies (i)–(iii) of Theorem 2.

Similarly, if (X_{k+1}^M, θ_k^M) is a sequence of $\mathbf{B}^M([k_0, \infty), \mathbf{S}_1')$, we define ∂S by

$$\phi_k = \theta_{k+k_0}^M, \qquad \partial X_1 = X_{k_0+1}^M - M_\varepsilon(\theta_{k_0}^M, k_0 + 1),$$
$$\begin{aligned} B''(\partial\theta, k) &= (A(\phi_k + \partial\theta) - A(\phi_k))M_\varepsilon(\phi_{k-1}, k + k_0) \\ &\quad + (B(\phi_k + \partial\theta) - B(\phi_k))u_{k+k_0}, \\ C''(\partial X, \partial\theta, k) &= C(M_\varepsilon(\phi_{k-1}, k + k_0) + \partial X, \phi_{k-1} + \partial\theta, k + k_0) \\ &\quad - C(M_\varepsilon(\phi_{k-1}, k + k_0), \phi_{k-1}, k + k_0). \end{aligned}$$

Again our assumptions are satisfied with

$$b_1'' = a_1 m + b_1 u,$$

$$b_2'' = \mathrm{Max}\left\{m\left(a_1 + \frac{a_2}{2}\eta\right) + u\left(b_1 + \frac{b_2}{2}\eta\right), a_2 m + b_2 u + a_1 m_1\right\},$$

$$c_1'' = c_1,$$

$$c_2'' = \mathrm{Max}\left\{\left(c_1 + \frac{c_2}{2}(x - m + \eta)\right), c_2(1 + m_1)\right\},$$

and

$$\rho = 0, \qquad \nu = \operatorname*{Sup}_{k \geq k_0} \|\theta_k^{M0} - \theta_k^{M1}\|, \qquad x_1 = 0.$$

With $(\partial X_{k+1}, \partial\theta_k)$ the corresponding solution of ∂S, we define

$$\Phi^M(k_0)(X^M, \theta^M)_k = (M_\varepsilon(\theta_{k+k_0}^M, k + 1 + k_0) + \partial X_{k+1}, \theta_{k+k_0}^M + \partial\theta_k).$$

Then $\Phi^M(k_0)$ satisfies (i)–(iii) of Theorem 2. Note that, in this case, we have

$$g^x(e) = \frac{\alpha(a_1 x + b_1 u)}{\sigma - \gamma(1 + \varepsilon\beta c)} \frac{\sigma\alpha c_1''}{1 - \sigma - \varepsilon c_1''} e.$$

This is obtained by studying the sequence $(\tilde{X}_{k+1} - X_{k+1}^M)$, knowing that (for e larger than $\varepsilon\partial^x$)

$$\tilde{X}_{k_0+1} = X_{k_0+1}^M, \qquad \|\tilde{\theta}_k - \theta_k^M\| \leq \varepsilon\frac{c_1''}{1 - \sigma - \varepsilon c_1''}\alpha e\sigma^{k-k_0}.$$

Proof of Lemma 6. (i) Let $\mathbf{D}$ be the set of sequences $(\partial\theta_k)$ in $\mathbb{R}^p$ such that, for all positive k, we have $|\partial\theta| \leq \varepsilon\partial^\theta \leq \eta$. Here, $\mathbf{D}$ is a complete metric space for the distance associated with the norm $|\partial\theta|$.

For any sequence $(\partial\theta_k)$ in $\mathbf{D}$, we consider the sequence (∂X_k) uniquely defined as the solution of $\partial X_{k+1} = A(\tilde{\phi}_k)\partial X_k + B''(\partial\theta_k, k)$ with ∂X_1 as the initial condition and $\tilde{\phi}_k = \phi_k + \partial\theta_k$. Since $\tilde{\phi}_k$ is in S, we have

$$|||\partial X_{k+1}|||_{\tilde{\phi}_k} \leq \gamma(1 + \beta\|\tilde{\phi}_k - \tilde{\phi}_{k-1}\|)|||\partial X_k|||_{\tilde{\phi}_{k-1}} + \alpha b_1''\|\partial\theta_k\|.$$

It follows that $|\partial X| \leq x_1' = \mathrm{Max}(\alpha\|\partial X_1\|, \varepsilon\partial^x)$ if ε, σ, ∂^θ, ∂^x satisfy $\gamma(1 + \beta(\varepsilon c + \varepsilon\partial^\theta\sigma^{k-1}(1 + \sigma)))\partial^x + \alpha b_1''\partial^\theta\sigma \leq \sigma\partial^x$. Also for two sequences $(\partial\theta_k^0)$, $(\partial\theta_k^1)$ in $\mathbf{D}$, we have

$$|||\partial X_{k+1}^0 - \partial X_{k+1}^1|||_{\tilde{\phi}_k^0} \leq \gamma(1 + \beta\|\tilde{\phi}_k^0 - \tilde{\phi}_{k-1}^0\|)|||\partial X_k^0 - \partial X_k^1|||_{\tilde{\phi}_{k-1}^0}$$
$$+ \alpha(a_1 x_1'\sigma^{k-1} + b_1'')\|\partial\theta_k^0 - \partial\theta_k^1\|.$$

This implies $|\partial X^0 - \partial X^1| \leq \partial_1^x|\partial\theta^0 - \partial\theta^1|$ if

$$\gamma(1 + \beta(\varepsilon c + \varepsilon\partial^\theta\sigma^{k-1}(1 + \sigma)))\partial_1^x + \alpha(a_1 x_1'\sigma^k + b_1''\sigma) \leq \sigma\partial_1^x.$$

Let us now define an operator T on $\mathbf{D}$ by $T(\partial\theta)_k = \partial\theta_{k+1} - \varepsilon C''(\partial X_{k+1}, \partial\theta_k, k + 1)$. Clearly, from the above inequalities $(\partial X_{k+1}, \partial\theta_k)$, defined this way, is the solution mentioned in the lemma if and only if $(\partial\theta_k)$ is a unique fixed point of T in $\mathbf{D}$.

(a) The operator $T(\partial\theta)$ is in **D**. We have

$$\|T(\partial\theta)_k\| \leq |\partial\theta|\sigma^{k+1} + \varepsilon c_1''(|\partial X| + |\partial\theta| + x_1)\sigma^k$$
$$\leq \sigma^k\varepsilon(\sigma\partial^\theta + c_1''(x_1' + \varepsilon\partial^\theta + x_1))$$

which means that ε, x_1', ∂^θ, σ should satisfy $\rho \leq \sigma < 1$ and $\sigma\partial^\theta + c_1''(x_1 + x_1' + \varepsilon\partial^\theta) \leq \partial^\theta$.

(b) The operator T is a contraction (uniformly in ∂X_1):

$$\|T(\partial\theta^0)_k - T(\partial\theta^1)_k\| \leq \sigma^{k+1}|\partial\theta^0 - \partial\theta^1| + \varepsilon c_1''(1 + \partial_1^x)\sigma^k|\partial\theta^0 - \partial\theta^1|$$
$$\leq \sigma^k(\sigma + \varepsilon c_1''(1 + \partial_1^x))|\partial\theta^0 - \partial\theta^1|.$$

Therefore the first part of our lemma holds if

$$\varepsilon\partial^\theta \leq \eta, \qquad \rho \leq \sigma < 1, \qquad x_1' = \mathrm{Max}(\alpha\|\partial X_1\|, \varepsilon\partial^x),$$
$$\alpha b_1''\partial^\theta \leq [\sigma - \gamma(1 + \varepsilon\beta(c + \partial^\theta(1 + \sigma)))]\partial^x,$$
$$\alpha(a_1 x_1' + b_1''\sigma) \leq [\sigma - \gamma(1 + \varepsilon\beta(c + \partial^\theta(1 + \sigma)))]\partial_1^x,$$
$$c_1''(x_1 + x_1') \leq (1 - \sigma - \varepsilon c_1'')\partial^\theta,$$
$$0 \leq 1 - \sigma - \varepsilon c_1''(1 + \partial_1^x).$$

(ii) Proceeding as for (i), we obtain

$$|||\partial X_{k+1}^0 - \partial X_{k+1}^1|||_{\tilde\phi_k^0}$$
$$\leq \gamma(1 + \varepsilon\beta(c + \partial^\theta\sigma^{k-1}(1 + \sigma)))|||\partial X_k^0 - \partial X_k^1|||_{\tilde\phi_{k-1}^0}$$
$$+ \alpha[a_1 x_1'\sigma^{k-1}(v + \sigma^k|\partial\theta^0 - \partial\theta^1|) + b_2''(\varepsilon\partial^\theta\sigma^k v + \sigma^k|\partial\theta^0 - \partial\theta^1|)].$$

It follows that $|\partial X^0 - \partial X^1| \leq \alpha\|\partial X_1^0 - \partial X_1^1\| + g_1^v v + g_1^\theta|\partial\theta^0 - \partial\theta^1|$ with

$$\gamma(1 + \varepsilon\beta(c + \partial^\theta\sigma^{k-1}(1 + \sigma)))g_1^v + \alpha(a_1 x_1' + \varepsilon\partial^\theta b_2''\sigma) \leq \sigma g_1^v,$$
$$\gamma(1 + \varepsilon\beta(c + \partial^\theta\sigma^{k-1}(1 + \sigma)))g_1^\theta + \alpha(a_1 x_1'\sigma^k + b_2''\sigma) \leq \sigma g_1^\theta,$$

Now since $(\partial\theta_k^0)$, $(\partial\theta_k^1)$ are fixed points of the operator T defined in (i), we have

$$\|\partial\theta_k^0 - \partial\theta_k^1\|$$
$$\leq \sigma^k[\sigma|\partial\theta^0 - \partial\theta^1| + \varepsilon c_2''((x_1' + \varepsilon\partial^\theta + 1)v + |\partial X^0 - \partial X^1| + |\partial\theta^0 - \partial\theta^1|)].$$

Our conclusion follows with

$$\alpha(a_1 x_1' + \varepsilon b_2''\partial^\theta\sigma) \leq [\sigma - \gamma(1 + \varepsilon\beta(c + \partial^\theta(1 + \sigma)))]g_1^v,$$
$$\alpha(a_1 x_1' + b_2''\sigma) \leq [\sigma - \gamma(1 + \varepsilon\beta(c + \partial^\theta(1 + \sigma)))]g_1^\theta,$$
$$\mathrm{Max}\left(\alpha, g_1^v + \varepsilon\frac{c_2''(1 + g_1^\theta)(x_1' + \varepsilon\partial^\theta + 1)}{1 - \sigma}\right) \leq \left(1 - \varepsilon\frac{c_2''(1 + g_1^\theta)}{1 - \sigma}\right)g_1. \qquad \blacksquare$$

To complete the proof of Theorem 2, we have to study the relation between Φ and Φ^M. Let (X_{k+1}, θ_k) be a sequence in $\mathbf{B}([k_0, \infty), \mathbf{S}_0')$. From part (i) of the theorem, its image $(\tilde{X}_{k+1}^M, \tilde{\theta}_k^M)$ by $\Phi(k_0)$ lies in $\mathbf{B}^M([k_0, \infty), \mathrm{S}_1')$. Similarly, the image $(\tilde{\tilde{X}}_{k+1}, \tilde{\tilde{\theta}}_k)$ by $\Phi^M(k_0)$ of $(\tilde{X}_{k+1}^M, \tilde{\theta}_k^M)$ lies in $\mathbf{B}([k_0, \infty), \mathrm{S}_0)$. Moreover, we have $\tilde{\tilde{X}}_{k_0+1} = \tilde{X}_{k_0+1} = X_{k_0+1}$. By uniqueness, $\Phi^M(k_0) \circ \Phi(k_0)$ is the identity map if $\tilde{\tilde{\theta}}_{k_0} = \theta_{k_0}$. From (i), we have $\|\theta_k - \tilde{\theta}_k\| \le \sigma^{k-k_0}\varepsilon g^\theta$ and $\|\tilde{\theta}_k - \tilde{\tilde{\theta}}_k\| \le \sigma^{k-k_0}\varepsilon g^\theta$. Hence $\|\tilde{\tilde{\theta}}_k - \theta_k\| \le 2\sigma^{k-k_0}\varepsilon g^\theta$. But since (X_{k+1}, θ_k) and $(\tilde{\tilde{X}}_{k+1}, \tilde{\tilde{\theta}}_k)$ are in $B(O, x) \times \mathrm{S}_0$ for all k, $k \ge k_0$, we apply inequalities (2) iteratively to get for all k, $k \ge k_0$,

$$\begin{aligned} |||X_{k+1} - \tilde{\tilde{X}}_{k+1}|||_{\theta_k} - l\|\theta_k - \tilde{\tilde{\theta}}_k\| &\le -\tau_0(\varepsilon, x)^{k-k_0} l \|\theta_{k_0} - \tilde{\tilde{\theta}}_{k_0}\|, \\ \|\theta_k - \tilde{\tilde{\theta}}_k\| &\ge (1 - \varepsilon c_1(1 + l))^{k-k_0}\|\theta_{k_0} - \tilde{\tilde{\theta}}_{k_0}\|. \end{aligned}$$

Therefore, for all k, $k \ge k_0$,

$$\|\theta_{k_0} - \tilde{\tilde{\theta}}_{k_0}\| \le 2\varepsilon g^\theta \left(\frac{\sigma}{1 - \varepsilon c_1(1 + l)}\right)^{k-k_0}.$$

Hence our result holds if $0 < 1 - \sigma - \varepsilon c_1(1 + l)$. Proofs of $\Phi(k_0) \circ \Phi^M(k_0) = I_d$ and injectivity properties follow the same lines. ■

5. Study of the Reduced-Order System SM_ε^θ

With Theorem 2, we have established that stability and existence of solutions of S_ε remaining in $B(O, x) \times \mathrm{S}_0'$ after time k_0 can be obtained from similar properties of solutions of SM_ε^θ. Therefore, we can concentrate our attention on this system. It is a nonlinear nonautonomous system and many approaches can be used to study stability and existence of bounded solutions.

In the l_2-stationary (stochastic process) case, averaging theory, leading to the associated differential equation technique, turns out to be a very appropriate tool to deal with the difficulty due to time dependence. This has been demonstrated by Ljung in the $C(X, \theta, k)$-vanishing case [L1], [LS], [KC] and by Anderson *et al.* [ABJ] and Benveniste *et al.* [BMP] in the nonvanishing case. Noticing that, as ε is made smaller, SM_ε^θ becomes closer to a first-order approximation of an ordinary differential equation, our result allows us to decompose the associated differential equation technique into three steps:

(i) Application of the topological orbital equivalence to replace S_ε by SM_ε.
(ii) Application of the averaging theory to SM_ε^θ.
(iii) Approximation of a difference equation by a differential equation.

In the nonstationary case, one possibility to simplify the time dependence is to extend to the discrete-time case the stroboscopic method ideas according to Minorsky [M] (see also [F1]): if u_k and $C(X, \theta, k)$ have a "slowly varying period" $p(K)$, then by flash illumination at times $t(K)$, $t(K) + p(K)$, $t(K) + p(K) + p(K + 1)$, ... we see a weakly nonstationary advance map, i.e., the solutions of SM_ε^θ observed on the time interval $[t(K), t(K) + p(K))$ are very similar to the solutions observed on the time interval $[t(K) + p(K), t(K) + p(K) + p(K + 1))$. The idea is then to approximate this advance map. Let $(p(K))$ be a sequence of bounded positive integers, $p(K) \le p$. Given k_0, we define flash illumination times after k_0 by

$t(K+1) = t(K) + p(K)$, $t(0) = k_0$. The advance map $SM^\theta_{\varepsilon K}$ from $t(K)$ to $t(K+1)$ of SM^θ_ε, written as

$$\theta^M_{t(K+1)} = \theta^M_{t(K)} + \varepsilon C^\Sigma_\varepsilon(\theta^M_{t(K)}, K+1) \qquad (SM^\theta_{\varepsilon_K})$$

is obtained from

$$\theta^M_k = \theta^M_{t(K)} + \varepsilon \sum_{i=t(K)}^{k-1} C(M_\varepsilon(\theta^M_i, i+1), \theta^M_i, i+1), \qquad t(K)+1 \le k \le t(K+1).$$

To approximate $SM^\theta_{\varepsilon_K}$, we introduce the system $SM^\theta_{0_K}$:

$$\psi_{K+1} = \psi_K + \varepsilon C^\Sigma_0(\psi_K, K+1) \qquad (SM^\theta_{0_K})$$

with

$$C^\Sigma_0(\psi, K+1) = \sum_{i=t(K)}^{t(K+1)-1} C(M_0(\psi, i+1), \psi, i+1).$$

Under assumptions A1–A5, C^Σ_0 is Lipschitz continuously differentiable on $\mathrm{S} \times \mathbb{N}$ with c^σ_1, c^σ_2 the respective Lipschitz constants: $c^\sigma_1 \le pc_1(1+m_1)$ and $c^\sigma_2 \le p(c_2(1+m_1)^2 + c_1 m_2)$.

System SM^θ_{0K} is simpler than SM^θ_ε. In particular, for adaptive systems, the C-function is given by the controller designer and is typically the product of a gain vector times an adaptation error. In this case, C^Σ_0 and $\partial C^\Sigma_0/\partial\psi$ are correlations on the time interval $[t(K), t(K+1))$ of components of M_0 or $\partial M_0/\partial\theta$. And, from their definitions, M_0 and $\partial M_0/\partial\theta$ can be obtained by implementation of sensitivity filters and observation of the feedback system using a constant parameter vector θ, i.e., in a classical linear feedback context. This latter aspect makes the assumptions on C^Σ_0 interpretable in terms of signals properties.

Although simpler, SM^θ_{0K} is very helpful for understanding the behavior of solutions of SM^θ_ε. This is possible since SM^θ_{0K} is an ε^2-approximation of $SM^\theta_{\varepsilon K}$. Indeed, from Theorem 1, under assumptions A1–A5, we have, uniformly on $\mathrm{S}_0 \times \mathbb{N}$,

$$\|C^\Sigma_0(\psi, K) - C^\Sigma_\varepsilon(\psi, K)\| \le \varepsilon v_0, \qquad \left\| \frac{\partial C^\Sigma_0}{\partial\psi}(\psi, K) - \frac{\partial C^\Sigma_\varepsilon}{\partial\psi}(\psi, K) \right\| \le \varepsilon v_1,$$

where v_0, v_1 can be obtained by induction on p as

$$v_0 \le pc_1 f^0 + c^\sigma_1 c \frac{p-1}{2},$$

$$v_1 \le p(c_2(1+m_1)f^0 + c_1 f^0_1) + c^\sigma_2 c \frac{p-1}{2} + ((1+\varepsilon c_1(1+m_1))^{p-1} - 1)c_1(1+m_1).$$

Again invoking hyperbolicity properties, namely conservation of stable and unstable manifolds of hyperbolic solutions under small perturbations (see [H1] and [S]), we may expect that, to any hyperbolic solution of $SM^\theta_{0_K}$, there corresponds an ε-close solution of SM^θ_ε with the same hyperbolicity property.

However, a difficulty remaining in the study of stability of solutions of $SM^\theta_{0_K}$ is the time variations. They have two causes: the time variations of the system itself and the motion of the solutions studied.

To take care of the system time variations, we consider the case where, uniformly on $S_0 \times \mathbb{N}$, $\|(\partial C_0^\Sigma/\partial\psi)(\psi, K) - (\partial C_0^\Sigma/\partial\psi)(\psi, K+1)\|$ is "small." This assumption concerns essentially the k-dependence of $C(X, \theta, k)$ and u_k. In particular, it is trivially satisfied in the periodic case by choosing $p(K)$ constant, equal to the period. It holds also in the almost-periodic case, i.e., if there exists a Lipschitz continuously differentiable function $\hat{C}_0^\Sigma(\psi)$ such that, for any ε, we can find p for which we have, uniformly on $S \times \mathbb{N}$,

$$\|C_0^\Sigma(\psi, K) - \hat{C}_0^\Sigma(\psi)\| \le \varepsilon\hat{v}_0, \qquad \left\|\frac{\partial C_0^\Sigma}{\partial\psi}(\psi, K) - \frac{\partial\hat{C}_0^\Sigma}{\partial\psi}(\psi)\right\| \le \varepsilon\hat{v}_1.$$

In this case, we replace $C_0^\Sigma(\psi, K)$ by $\hat{C}_0^\Sigma(\psi)$ in the definition of SM_{0K}^θ.

To take care of the motion of the solutions, we consider those evolving in a set where $C_0^\Sigma(\psi, K)$ is "small" uniformly in K. This condition is trivially satisfied if $SM_{0_K}^\theta$ has a fixed point, i.e., if there exits ψ^* such that, for all K, $C_0^\Sigma(\psi^*, K) = 0$. This equation is precisely the bifurcation equation obtained by averaging theory [BSA], [RK2] or critical systems theory [PP]. Existence of solutions for this equation has been studied for model reference adaptive systems with a fixed-point argument (Section 4.5 of [R]) or applying degree theory [PCP].

From this discussion, we introduce the following definition:

Definition. Given strictly positive constants ζ, ε, we define the set $\mathbf{P}(\zeta, \varepsilon)$ as

$$\mathbf{P}(\zeta,\varepsilon) = \left\{ \psi \in S_0' \;\middle|\; \begin{array}{ll} \text{A6.1:} & \left\|\dfrac{\partial C_0^\Sigma}{\partial\psi}(\psi, K) - \dfrac{\partial C_0^\Sigma}{\partial\psi}(\psi, K+1)\right\| \\ & \quad + \varepsilon c_2^\sigma \|C_0^\Sigma(\psi, K)\| \le \varepsilon w < \varepsilon\zeta^2 \quad \text{for all } K, \\ \text{A6.2:} & \displaystyle\max_{i\in[1,p]} \operatorname{Re}\left(\lambda_i\left\{\frac{\partial C_0^\Sigma(\psi, K)}{\partial\psi}\right\}\right) \le -\zeta \quad \text{for all } K, \end{array} \right\}$$

where $\operatorname{Re}(\cdot)$ and $\lambda_i\{\cdot\}$ denote the real part and the ith eigenvalue, respectively.

Remark. In the adaptive linear systems context, the inequality A6.2, involved in the definition of $\mathbf{P}(\zeta, \varepsilon)$, is related to the so-called "signal dependent positivity condition." In the test input assumption case (see the Introduction), it can be interpreted as the positivity of an operator restricted to act on specific signals (see [ABJ], [RK1], and [RPK]).

Theorem 3. *Under assumptions* A1–A5, *there exists* ε^* *such that if we can find* ζ *and* ε, $0 < \varepsilon \le \varepsilon^*$, *for which* $\mathbf{P}(\zeta, \varepsilon)$ *contains a solution* (ψ_K) *of* SM_{0K}^θ, *then* SM_ε^θ *has an exponentially stable solution remaining in* S_1' *after time* k_0 *and* ε*-close to* (ψ_K) *at times* $t(K)$, *i.e., for all* K, $\|\theta_{t(K)}^M - \psi_K\| \le \varepsilon\delta$ *with* $v_0/\zeta < \delta$.

The constant δ, appearing in this statement, is clarified in the proof.

Remarks. This stroboscopic method approach extends, in more general situations, the local averaging technique proposed by Kosut *et al.* [KAM] (see also [ABJ]) for studying SM_0^θ after linearization under a relaxed test input assumption.

With Theorems 2 and 3, we have established the following result:

For ε sufficiently small, if SM_{0K}^{θ} has a solution remaining in a set $\mathbf{P}(\zeta, \varepsilon)$, then S_ε has exponentially stable solutions remaining in $B(O, x) \times \mathbf{S}_0$ after time k_0.

In other words, this proves that, as far as stability is concerned, the heuristic technique proposed by Åström [A1], [A2] is theoretically sound when restricted to the set $\mathbf{P}(\zeta, \varepsilon)$.

Finally, (ψ_K) is an ε^2-approximation of a solution of $SM_{\varepsilon_K}^{\theta}$ if C_0^{Σ} is replaced by C_1^{Σ}, an ε^2-approximation of C_0^{Σ} defined with M_1, the ε^2-approximation of M_ε.

As a key step to proving Theorem 3, we establish the following result:

Lemma 7. *Let $F(K_1, K_0)$ be the transition matrix of*

$$\Delta_{K+1} = \left(I + \varepsilon \frac{\partial C_0^{\Sigma}}{\partial \psi}(\psi_K, K+1)\right)\Delta_K$$

with (ψ_K), the solution in $\mathbf{P}(\zeta, \varepsilon)$ given by Theorem 3. For any ζ', $0 < \zeta' < \zeta - \varepsilon(c_1^{\sigma})^2/2$, there exists v (independent of ψ_K) such that for any K_0, K_1, $0 \le K_0 < K_1$,

$$\|F(K_1, K_0)\| \le \frac{1 + \varepsilon v'}{1 - \varepsilon\zeta''}(1 - \varepsilon\zeta'')^{K_1 - K_0}$$

with $\zeta'' = \zeta' - [(1 + \varepsilon v)(1 - \varepsilon\zeta')w]^{1/2}$ and $v' = v(1 - \varepsilon\zeta') - \zeta''$, where w is obtained from A6.1.

Proof. Given K_1, the transition matrix satisfies for any N, $K_0 \le N < K_1$,

$$F(N+1, K_0) = \left(I + \varepsilon \frac{\partial C_0^{\Sigma}}{\partial \psi}(\psi_{K_1 - 1}, K_1)\right) F(N, K_0) + \varepsilon\left[\frac{\partial C_0^{\Sigma}}{\partial \psi}(\psi_N, N+1) - \frac{\partial C_0^{\Sigma}}{\partial \psi}(\psi_{K_1-1}, K_1)\right] F(N, K_0).$$

To derive the property of $F(K_1, K_0)$, we use the following three inequalities:

(i) From assumption A6.2 in the definition of the set $\mathbf{P}(\zeta, \varepsilon)$, we have, uniformly on $\mathbf{P}(\zeta, \varepsilon) \times \mathbb{N}$,

$$\left\| I + \varepsilon \frac{\partial C_0^{\Sigma}}{\partial \psi}(\psi, K) \right\| \le 1 + \varepsilon c_1^{\sigma},$$

$$\operatorname*{Max}_{i \in [1,p]} \left| \lambda_i \left\{ I + \varepsilon \frac{\partial C_0^{\Sigma}}{\partial \psi}(\psi, K) \right\} \right| \le \left[1 - 2\varepsilon\zeta + \varepsilon^2 \left\| \frac{\partial C_0^{\Sigma}}{\partial \psi}(\psi, K) \right\|^2 \right]^{1/2} \le 1 - \varepsilon\zeta + \varepsilon^2 \frac{(c_1^{\sigma})^2}{2}.$$

Then it follows from Theorem 5 of [F3] that for any ζ', $0 < \zeta' < \zeta -$

$\varepsilon(c_1^\sigma)^2/2$, there exists v ($v \geq c_1^\sigma$) such that, uniformly on $\mathbf{P}(\zeta, \varepsilon) \times \mathbb{N}$,

$$\left\| \left(I + \varepsilon \frac{\partial C_0^\Sigma}{\partial \psi}(\psi, K) \right)^i \right\| \leq (1 + \varepsilon v)(1 - \varepsilon \zeta')^i \qquad \text{for all} \quad i \in \mathbb{N}.$$

(ii) From assumption A6.1 and the fact that (ψ_K) is a solution of $SM_{0_K}^\theta$, we obtain

$$\begin{aligned}
&\left\| \frac{\partial C_0^\Sigma}{\partial \psi}(\psi_N, N + 1) - \frac{\partial C_0^\Sigma}{\partial \psi}(\psi_{K_1}, K_1) \right\| \\
&\quad \leq \sum_{K=N+1}^{K_1 - 1} \left\| \frac{\partial C_0^\Sigma}{\partial \psi}(\psi_{K-1}, K) - \frac{\partial C_0^\Sigma}{\partial \psi}(\psi_K, K + 1) \right\| \\
&\quad \leq \varepsilon w(K_1 - N - 1).
\end{aligned}$$

(iii) Let u_k be a sequence of positive real numbers satisfying, for any k, $k \geq 1$, $u_k \leq a\lambda^k u_0 + b \sum_{i=0}^{k-1} \lambda^{k-1-i}(k - 1 - i)u_i$, $a \geq 0$, $b \geq 0$, $\lambda \geq 0$, then we can check by induction that

$$u_k \leq \left[\frac{\lambda a + \sqrt{\lambda b}}{2}(\lambda + \sqrt{\lambda b})^{k-1} + \frac{\lambda a - \sqrt{\lambda b}}{2}(\lambda - \sqrt{\lambda b})^{k-1} \right] u_0.$$

Now use the variation of constants formula; take the Euclidian norm and use the two first inequalities to obtain

$$\begin{aligned}
\|F(K_1, K_0)\| &\leq (1 + \varepsilon v)(1 - \varepsilon\zeta')^{K_1 - K_0} \\
&\quad + \varepsilon^2 w(1 + \varepsilon v) \sum_{N=K_0}^{K_1 - 1} (1 - \varepsilon\zeta')^{K_1 - 1 - N}(K_1 - 1 - N)\|F(N, K_0)\|.
\end{aligned}$$

The result follows from the third inequality ■

Proof of Theorem 3. (a) Existence: The idea is to find a solution $(\theta_{t(K)}^M)$ of $SM_{\varepsilon K}^\theta$ satisfying $\|\theta_{t(K)}^M - \psi_K\| \leq \varepsilon\delta$. This will solve our problem, since the corresponding solution (θ_k^M) of SM_ε^θ satisfies, for all k, $t(K) - p/2 \leq k \leq t(K) + p/2$,

$$\|\theta_{t(K)}^M - \theta_k^M\| \leq \varepsilon c \frac{p}{2}.$$

In particular, knowing that ψ_K lies in $\mathbf{S}_0'$, θ_k^M lies in $\mathbf{S}_1'$ if δ and ε satisfy $\delta \leq \eta/\varepsilon - c(p/2)$. Hence, let Δ_K be defined recursively by

$$\begin{aligned}
\Delta_0 &= 0, \\
\Delta_{K+1} &= \left(I + \varepsilon \frac{\partial C_0^\Sigma}{\partial \psi}(\psi_K, K + 1) \right) \Delta_K \\
&\quad + \varepsilon \left[C_\varepsilon^\Sigma(\psi_K + \Delta_K, K + 1) - C_0^\Sigma(\psi_K, K + 1) - \frac{\partial C_0^\Sigma}{\partial \psi}(\psi_K, K + 1)\Delta_K \right],
\end{aligned}$$

then $(\psi_K + \Delta_K)$ is a solution of $SM_{\varepsilon K}^\theta$. We remark that for any Δ such that the segment $[\psi_K, \psi_K + \Delta]$ is contained in $\mathbf{S}_0$, application of the Mean Value Theorem yields for some ξ, $0 \le \xi \le 1$,

$$\left\| C_\varepsilon^\Sigma(\psi_K + \Delta, K+1) - C_0^\Sigma(\psi_K, K+1) - \frac{\partial C_0^\Sigma}{\partial \psi}(\psi_K, K+1)\Delta) \right\|$$

$$\le \varepsilon v_0 + \left\| \frac{\partial C_0^\Sigma}{\partial \psi}(\psi_K + \xi\Delta, K+1) - \frac{\partial C_0^\Sigma}{\partial \psi}(\psi_K, K+1) \right\| \|\Delta\| \le \varepsilon v_0 + c_2^\sigma \|\Delta\|^2.$$

Now apply the variation of constants formula and Lemma 7. Then

$$\|\Delta_{K+1}\| \le \varepsilon \frac{1+\varepsilon v'}{1-\varepsilon\zeta''} \sum_{N=0}^{K} (1-\varepsilon\zeta'')^{K-N}(\varepsilon v_0 + c_2^\sigma \|\Delta_N\|^2)$$

Hence, we have established $\|\Delta_K\| \le \varepsilon\delta$ and, therefore, the existence of a solution (θ_k^M) of SM_ε^θ remaining in S_1' after time k_0, if ζ', δ satisfy

$$\frac{1+\varepsilon v'}{1-\varepsilon\zeta''}\frac{v_0 + \varepsilon c_2^\sigma \delta^2}{\zeta''} \le \delta \le \frac{\eta}{\varepsilon} - c\frac{p}{2}.$$

(b) Exponential stability: For any two solutions (θ_k^{M0}), (θ_k^{M1}) of SM_ε^θ remaining in S_0, we have, uniformly in K,

$$\|\theta_k^{M0} - \theta_k^{M1}\| \le (1 + \varepsilon c_1(1+m_1))^p \|\theta_{t(K)}^{M0} - \theta_{t(K)}^{M1}\|, \qquad t(K) \le k \le T(K+1).$$

Hence exponential stability of the solution (θ_k^M) obtained in part (a) follows from exponential stability of $(\theta_{t(K)}^M)$, the solution of $SM_{\varepsilon K}^\theta$. But, from Lyapunov's theorem, for this property to hold it is sufficient that the origin be an exponentially stable solution of the following linear system:

$$\Delta_{K+1} = \left(I + \varepsilon \frac{\partial C_\varepsilon^\Sigma}{\partial \psi}(\theta_{t(K)}^M, K+1) \right) \Delta_K.$$

From part (a), we know

$$\left\| \frac{\partial C_\varepsilon^\Sigma}{\partial \psi}(\theta_{t(K)}^M, K+1) - \frac{\partial C_0^\Sigma}{\partial \psi}(\psi_K, K+1) \right\| \le \varepsilon v_1 + \varepsilon c_1^\sigma \delta.$$

Therefore, with Lemma 7 and Lemma B.5 on p. 118 of [A4], our property holds if

$$1 - \varepsilon\zeta'' + \varepsilon^2 \frac{1+\varepsilon v'}{1-\varepsilon\zeta''}(v_1 + c_1^\sigma \delta) < 1.$$

In conclusion, our theorem holds if ε and δ satisfy, for v given by Theorem 5 of [F3],

$$0 < \zeta' < \zeta - \varepsilon\frac{(c_1^\sigma)^2}{2}, \qquad \zeta'' = \zeta' - [(1+\varepsilon v)(1-\varepsilon\zeta')w]^{1/2}, \qquad v' = v(1-\varepsilon\zeta') - \zeta'',$$

$$w < \frac{\zeta'^2}{(1+\varepsilon v)(1-\varepsilon\zeta')},$$

$$\frac{1+\varepsilon v'}{1-\varepsilon\zeta''}\frac{v_0 + \varepsilon c_2^\sigma \delta^2}{\zeta''} \le \delta \le \frac{\eta}{\varepsilon} - c\frac{p}{2},$$

$$1 - \varepsilon\zeta'' + \varepsilon^2 \frac{1+\varepsilon v'}{1-\varepsilon\zeta''}(v_1 + c_1^\sigma \delta) < 1. \qquad \blacksquare$$

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References

[ABJ] B. D. O. Anderson, R. R. Bitmead, C. R. Johnson, P. V. Kokotovic, R. L. Kosut, I. M. Y. Mareels, L. Praly, and B. D. Riedle, *Stability of Adaptive Systems: Passivity and Averaging Analysis*, MIT Press, Cambridge, MA, 1986.

[A1] K. J. Åström, Analysis of Rohrs' counter-example to adaptive control, *Proceedings of the 22nd IEEE Conference on Decision and Control*, San Antonio, TX, 1983, pp. 982–987.

[A2] K. J. Åström, Interactions between excitation and unmodeled dynamics in adaptive control, *Proceedings of the 23rd IEEE Conference on Decision and Control*, Las Vegas, NV, 1984, pp. 1276–1281.

[AE] J. P. Aubin and I. Ekeland, *Applied Nonlinear Analysis*, Wiley, New York, 1984.

[A3] B. Aulbach, A reduction principle for nonautonomous differential equations, *Arch. Math.*, **39** (1982), 217–232.

[A4] B. Aulbach, *Continuous and Discrete Dynamics Near Manifolds of Equilibria*, Lecture Notes in Mathematics, Vol. 1058, Springer-Verlag, Berlin, 1984.

[BMP] A. Benveniste, M. Metivier, and P. Priouret, *Algorithmes Adaptatifs et Approximations Stochastiques: Theórie et Applications*, Masson, Paris, 1987.

[BSA] M. Bodson, S. Sastry, B. D. O. Anderson, I. Mareels, and R. R. Bitmead, Nonlinear averaging theorems, and the determination of parameter convergence rates in adaptive control, *Systems Control Lett.*, 7 (1986), 145–157.

[DF] D. P. Derevitskii and A. L. Fradkov, Two models for analyzing the dynamics of adaptation algorithms, *Automat. Remote Control*, **35** (1974), 59–67.

[E] B. Egardt, *Stability of Adaptive Controllers*, Lecture Notes in Control and Information Sciences, Vol. 20, Springer-Verlag, Berlin, 1979.

[F1] R. P. Fedorenko, Derivation and justification of equations in slow time (the stroboscopic method), *USSR Comput. Math. and Math. Phys.*, **14** (1974), 81–118.

[F2] N. Fenichel, Persistence and smoothness of invariant manifolds for flows, *Indiana Math. J.*, **21** (1971), 193–226.

[F3] J. J. Fuchs, On the good use of the spectral radius of a matrix, *IEEE Trans. Automat. Control*, **27** (1982), 1134–1135.

[H1] J. K. Hale, *Ordinary Differential Equations*, 2nd edn., Krieger, Florida, 1980.

[H2] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Mathematics, Vol. 840, Springer-Verlag, Berlin, 1981.

[HPS] M. W. Hirsch, C. C. Pugh, and M. Shub, *Invariant Manifolds*, Lecture Notes in Mathematics, Vol. 583, Springer-Verlag, Berlin, 1976.

[K] A. Kelley, Stability of the center manifold, *J. Math. Anal. Appl.*, **18** (1967), 336–344.

[KAM] R. L. Kosut, B. D. O. Anderson, and I. M. Y. Mareels, Stability theory for adaptive systems: method of averaging and persistency of excitation, *IEEE Trans. Automat. Control*, **32** (1987), 26–34.

[KC] H. J. Kushner and D. S. Clark, *Stochastic Approximation Methods for Constrained and Unconstrained Systems*, Springer-Verlag, New York, 1978.

[L1] L. Ljung, Analysis of recursive stochastic algorithms, *IEEE Trans. Automat. Control*, **22** (1977), 551–575.

[LS] L. Ljung and T. Söderström, *Theory and Practice of Recursive Identification*, MIT Press, Cambridge, MA, 1983.

[L2] N. G. Lloyd, *Degree Theory*, Cambridge University Press, Cambridge, 1978.

[M] N. Minorsky, *Nonlinear Oscillations*, Van Nostrand, Princeton, NJ, 1962.

[O] G. S. Osipenko, Perturbation of invariant manifolds, *Differential Equations*, **4** (1985), 406–412.

[P1] V. A. Pliss, Principle reduction in the theory of the stability of motion, *Izv. Akad. Nauk. SSSR Ser. Mat.*, **28** (1964), 1297–1324 (in Russian).

[P2] V. A. Pliss, On the theory of invariant surfaces, *Differential Equations*, **2** (1966), 589–594.

[PCP] J.-P. Pomet, J.-M. Coron, and L. Praly, About periodic solutions of adaptive systems in presence of periodic forcing terms, CAI Report A186, January 1988, revised July 1988; also to appear in *Math. Control Signals Systems.*

[P3] L. Praly, A geometric approach for the local analysis of a one step ahead adaptive controller, *Proceedings of the 4th Yale Workshop on Applications of Adaptive Systems Theory*, New Haven, CT, 1985, pp. 173–178.

[P4] L. Praly, Local stability/instability of slowly adapted systems in presence of periodic inputs. CAI Report A156, July 1985.

[PP] L. Praly and J.-B. Pomet, Periodic solutions in adaptive systems: the regular case, *Proceedings of the 10th IFAC World Congress on Automatic Control*, Munich, 1987, Vol. 10, pp. 40–45.

[PR] L. Praly and D. Rhode, A local analysis of a one step ahead adaptive controller, *Proceedings of the 24th IEEE Conference on Decision and Control*, Ft. Lauderdale, FL, 1985, pp. 1862–1867.

[R] B. D. Riedle, Integral manifolds of slow adaptation, Ph.D. thesis, Department of Electrical Engineering, University of Illinois at Urbana-Champaign, 1986.

[RK1] B. D. Riedle and P. V. Kokotovic, A stability–instability boundary for disturbance-free slow adaptation with unmodeled dynamics, *IEEE Trans. Automat. Control*, **30** (1985), 1027–1030.

[RK2] B. D. Riedle and P. V. Kokotovic, Integral manifolds of slow adaptation, *IEEE Trans. Automat. Control*, **31** (1986), 316–324.

[RPK] B. D. Riedle, L. Praly, and P. V. Kokotovic, Examination of the SPR condition in output error parameter estimation, *Automatica*, **22** (1986), 495–498.

[S] M. Shub, Stabilité globale des systèmes dynamiques, *Astérisque*, **56** (1978).