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Mathematics of Control, Signals, and Systems

On Periodic Solutions of Adaptive Systems in the Presence of Periodic Forcing Terms*

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Abstract. We consider a discrete-time system consisting of a linear plant and a periodically forced feedback controller whose parameters are slowly adapted. Using degree theory, we give sufficient conditions for the existence of periodic solutions. Using linearization methods, we give sufficient conditions for their (in)stability provided the adaptation is slow enough. We then study when the degree theoretic conditions for the existence are satisfied by d-steps-ahead adaptive controllers in the presence of unmodeled dynamics and a persistently exciting periodic reference output.

Key words. Adaptive control, Unmodeled dynamics, Averaging analysis, Periodic solutions, Degree theory.

Introduction

It is well known that adaptive linear control performs satisfactorily when no undermodeling is involved, i.e., when the order of the plant to be controlled, or an upper bound of it, is known, and the controller takes it into account, or, more generally, under the "exact matching assumption," i.e., when one value of the parameter somehow perfectly represents the plant to be controlled (see [GS] or part of [ABJ*] for a study of this "ideal" case). One of the present problems in adaptive control is characterizing asymptotic performance in the more general and practical case of imperfect modeling when no "matching assumption" is satisfied (e.g., when the order of the plant is allowed to be higher than that of the controller). Here performance refers to the behavior of asymptotic solutions and the stability of these solutions.

It is reasonable to study the performance under *stationary*, *quasi-period*, or *periodic extraneous inputs*, there being nothing meaningful to say when nonstationary inputs are involved. In addition, we only deal here, as does most of the literature, with *slow adaptation* [BMP], [PR], [RK], [RPK]. In Section 1.1 we show the link between small extraneous inputs and slow adaptation; this often legitimizes the slow adaptation assumption.

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If the closed-loop system were linear, the performance could be studied as follows: in the case when no eigenvalue is on the unit circle, there would be one stationary (or quasi-periodic, or periodic) solution, and its (global exponential) stability would be characterized by the location of the eigenvalues; this would describe the behavior of *all the solutions*. The closed-loop system, in the adaptive case, is nonlinear. Trying nevertheless to do something similar means (1) find a stationary (or quasi-periodic, or periodic) solution, and (2) get information about the behavior of the other solutions. This can be done, at least in a neighborhood of the particular solution found above, by means of studying its (exponential) stability.

Averaging analysis has been applied for this problem; see [BFS], [ABJ*], or [BMP]. Assuming the existence of an isolated (and even hyperbolic) equilibrium point for a certain averaged differential equation, an answer to the first point above is obtained, i.e., the existence of a quasi-periodic solution whose exponential stability/instability is studied via the properties of the equilibrium point. The trouble is that the assumption about the existence of an equilibrium point for the "averaged differential equation" is very difficult to check even in the case of periodic inputs.

In [RPK], or in [ABJ*], the same results are obtained under a less restrictive assumption; what is assumed is just the existence of a so-called "tuned" solution, which is something weaker tha an equilibrium point for the averaged differential equation. Under some other assumptions, it turns out to be an approximation of a quasi-periodic (or periodic if the inputs are assumed to be periodic) solution, and the sequel follows. Again the problem is that the existence of this "tuned solution" is not automatic.

In [RK] a value of this tuned solution is proposed: defining it as minimizing a cost function works quite well if the minimum is small enough.

In [PR] it is shown that the fixed point of the averaged differential equation exists, if the inputs are chosen in a proper class of inputs, the so-called "test-inputs."

In this paper we take a first step towards checking, on known adaptive controllers with unmodeled dynamics, the existence and stability of this stationary solution in the case of periodic small extraneous inputs: we give (in Section 1) sufficient conditions for the existence and stability/instability of a periodic solution of the closed-loop system, and (in Section 2) we check the sufficient conditions for the existence on a wide class of known direct adaptive controllers. This second point is new compared with previous literature: the point in our paper is that we *establish* the existence of the stationary solution itself, i.e., the existence of an equilibrium point for the averaged differential equation according to the averaging technique terminology.

The method used here consists of considering the initial condition of a periodic solution with period K as a fixed point of the K-advance map (see (29), Definition 3) and studying this map in the case of slow adaptation as a perturbation of the case where the parameters are not adapted; the implicit function theorem may be used, but it needs assumptions which are not checkable in the example presented in Section 2. This motivates the use of topological continuation methods (degree theory).

1. A General Adaptive Control System

1.1. *Problem Statement*

In this section we deal with the following discrete-time closed-loop system:

$$(S_\varepsilon)\quad \begin{cases} x(t+1) = A(\theta(t))x(t) + B(\theta(t))v(t), \\ \theta(t+1) = \theta(t) + \varepsilon C(x(t+1), \theta(t), \varepsilon, t), \end{cases} \qquad v(t) \in \mathbb{R}^q, \quad x(t) \in \mathbb{R}^n, \quad \theta(t) \in \mathbb{R}^p, \tag{1}$$

where Ω is an open subset of $\mathbb{R}^p$, $A(\cdot)$ is a C^2-map from Ω to $M_{n,n}(\mathbb{R})$, $B(\cdot)$ is a C^2-map from Ω to $M_{n,q}(\mathbb{R})$, $C(\cdot, \cdot, \cdot, t)$ being a C^2-map uniformly in t, from $\mathbb{R}^n \times \Omega \times \mathbb{R}$ to $\mathbb{R}^p$, and $C(x, \theta, \varepsilon, \cdot)$ is K-periodic; $(v(t))$ is a given K-periodic sequence in $\mathbb{R}^q$, and n, p, q, and K are fixed integers.

System (1) represents an adaptive controller in feedback with a linear time-invariant plant, where v is the reference output (to be tracked) and θ is the generalized parameter estimate. For example, if a least-squares algorithm with forgetting factor is used, the form (1) is obtained by incorporating the columns of the covariance matrix in the θ-vector. If an indirect pole placement were used, the function $A(\theta)$ would incorporate the operation of solving the linear system given by the Bezout identity (see, e.g., Section 1.3 of [BMP]). In (1), x contains the state of the plant and of the controller.

As explained in the introduction, we assume here that v is periodic with period K and our aim is to find a K-periodic solution (x, θ) of (S_ε) and study its stability. For this, we also require C to be K-periodic.

Note that the set Ω is not usually assumed to be all $\mathbb{R}^p$ because some values of the parameters are singular for the control problem; for example, they may model uncontrollable systems, e.g., the Bezout equation may no longer be solved (indirect pole placement).

Let us explain the use of the rather simple form (1) and the slow adaptation assumption. It turns out that most adaptive controllers in feedback with a linear time-invariant system with arbitrary finite order and extraneous additive disturbance may be represented by the more general equation

$$\begin{aligned} Y(t+1) &= A(\theta(t))Y(t) + B(\theta(t))r(t), \\ \theta(t+1) &= \theta(t) + C(Y(t), \theta(t), w(t), \lambda(t)), \end{aligned} \tag{2}$$

where the adaptation is not necessarily slow and the adaptation law varies in a class indexed by λ. Now, slow adaptation may be forced by forcing C to be small, or it may come from the smallness of the extraneous input w as shown by the following computation. In most cases, the C function satisfies (at least locally)

$$C(\sqrt{\varepsilon}x, \theta, \sqrt{\varepsilon}v, \lambda) = \varepsilon C(x, \theta, v, \varepsilon\lambda) \qquad \text{for all} \quad \varepsilon > 0. \tag{3}$$

Consequently, if the forcing term w satisfies $\|w(t)\| \le \sqrt{\varepsilon}$, the transformation $\sqrt{\varepsilon}x = Y$, $\sqrt{\varepsilon}v = w$ leads to

$$\begin{aligned} x(t+1) &= A(\theta(t))x(t) + B(\theta(t))v(t), \\ \theta(t+1) &= \theta(t) + \varepsilon C(x(t), \theta(t), v(t), \varepsilon\lambda(t)). \end{aligned}$$

The smaller ε (i.e., the forcing term) is, the slower θ is adapted. In fact, property (3) means that a small forcing term results in (locally) slow adaptation. In addition, we denote the dependence of C on v and λ by an explicit time dependence and just write $C(x, \theta, \varepsilon, t)$. This completely accounts for the form of (1).

1.2. *Existence of Periodic Solutions*

Theorems 1 and 2 give two different sets of sufficient hypotheses for (S_ε) to have K-periodic solutions for small values of ε. In both theorems the important hypothesis is about the zeros of the "determining field" E.

Definition 1. The map E, called the *Determining Field*, is defined, for any θ in Ω such that no Kth root of unity is an eigenvalue of $A(\theta)$, by

$$E(\theta) = \sum_{t=0}^{K-1} C(x(t), \theta, 0, t), \tag{4}$$

where $(x(t), \theta)$ is the only K-periodic solution of (S_0).

An easy computation yields

$$E(\theta) = \sum_{t=0}^{K-1} C\left([I - A(\theta)^K]^{-1} \sum_{j=0}^{t-1} A(\theta)^{t-j-1} B(\theta) v(j), \theta, 0, t\right). \tag{5}$$

Here E is defined on the open subset of Ω where $A(\theta)^K - I$ is nonsingular. It is a vector field on this open set and acts as the averaged field ($\dot{\theta} = E(\theta)$ is the averaged ordinary differential equation) in averaging theory.

We now state a much more technical definition. There may be initial conditions such that θ gets out of Ω before K steps, which would cause problems since A, B, and C are only defined for θ in Ω; this motivates:

Definition 2. We call U_ε the open subset of $\mathbb{R}^n \times \mathbb{R}^p$ consisting of all the (x, θ) such that if $(x(0), \theta(0)) = (x, \theta)$, then the corresponding solution $(x(t), \theta(t))$ of (S_ε) is such that $\theta(t)$ remains in Ω for all t in $[0, K]$.

Note that if (x, θ) is the initial condition of K-periodic solution, it must be in U_ε.

With $C(x, \theta, \varepsilon, \cdot)$ and $v(\cdot)$ being K-periodic, a solution $(x(t), \theta(t))$ of (S_ε) is K-periodic if and only if $(x(K), \theta(K))) = (x(0), \theta(0))$. Looking for K-periodic solutions of (S_0) is very simple because in (S_0), θ is frozen, and the remaining system is linear: for any θ such that $I - A(\theta)^K$ is nonsingular, there is one and only one K-periodic solution $(x(t), \theta)$ of (S_0); its explicit expression is used in (5). Of course, when $\varepsilon \neq 0$, θ may vary in (S_ε), so that, when writing $(x(K), \theta(K)) = (x(0), \theta(0))$, the θ-equation is no longer degenerate. Therefore, the system (S_ε) often has fewer periodic solutions than (S_0). In fact, only a few of the K-periodic solutions of (S_0) continue into K-periodic solutions of (S_ε); the following theorems identify these "good" periodic solutions of (S_0).

In Theorem 1 we show that if θ is a nondegenerate (hyperbolic) zero of E, then the periodic solution $(x(t), \theta)$ of (S_ε) does continue into a periodic solution. We have not been able to check this assumption for the example of Section 2.5, based on a d-steps-ahead adaptive scheme; this is the motivation for Theorem 2, which employs weaker hypotheses using degree theory, satisfied in the (wide) class of adaptive controllers described in Section 2. Let us state the theorems precisely:

Theorem 1. *If a θ_0 in Ω can be found such that*

(i) $E(\theta_0) = 0$,
(ii) $E'(\theta_0)$ *is nonsingular* (E' *denotes the derivative of* E), *and*
(iii) $I - A(\theta_0)^K$ *is nonsingular,*

then there is an ε_0 and a continuous map

$$\begin{aligned}[-\varepsilon_0, \varepsilon_0] &\to \mathbb{R}^n \times \mathbb{R}^p,\\ \varepsilon &\mapsto (x_\varepsilon, \theta_\varepsilon)\end{aligned} \tag{6}$$

such that $(x_\varepsilon, \theta_\varepsilon)$ is the initial condition of a K-periodic solution of (S_ε) (of course, $(x_\varepsilon, \theta_\varepsilon)$ is in U_ε); for $\varepsilon \neq 0$, it is locally the only initial condition of a K-periodic solution of (S_ε).

Theorem 2. *If an open bounded set V can be found such that $\bar{V} \subset \Omega$ and*

(i) *E never vanishes on ∂V (the boundary of V),*
(ii) $\deg(E, V, 0) \neq 0$, *and*
(iii) $\det(A(\cdot)^K - I)$ *never vanishes on V,*

then there is an $\varepsilon_0 > 0$ and an $R > 0$ such that, for $|\varepsilon| < \varepsilon_0$, $\overline{\mathscr{B}_{\mathbb{R}^n}(0, R) \times V} \subset U_\varepsilon$, and (S_ε) has (at least) one periodic solution with initial condition (x, θ) in $\mathscr{B}_{\mathbb{R}^n}(0, R) \times V$.

The proofs are given in Appendix A.

Remarks. (1) The notation $\deg(f, U, \alpha)$ stands for the degree of the map f on the open set U relatively to the value α. For a full definition, see [L2] or [M].

(2) The hypotheses of Theorem 2 are weaker than those of theorem 1: from (i) and (ii) of Theorem 1, θ_0 is a nondegenerate zero of E, so that defining V as a small ball around θ_0, the degree is ± 1, which gives (ii) of Theorem 2; (i) and (iii) of Theorem 2 are immediate consequences of (ii) and (iii) of Theorem 1, respectively.

1.3. *Stability of the Periodic Solutions*

The map E also allows us to characterize the (in)stability of the periodic solutions given by Theorem 1:

Theorem 3. *If a θ_0 in Ω can be found such that*

(i) $E(\theta_0) = 0$,
(ii′) $E'(\theta_0)$ *has no purely imaginary eigenvalue, and*
(iii′) $A(\theta_0)$ *has no eigenvalue on the unit circle,*

then there is an ε_0 *and a continuous function of* ε *in* $[0, \varepsilon_0]$, $(x_\varepsilon, \theta_\varepsilon)$, *which is the initial condition of a K-periodic solution of* (S_ε). *For* $\varepsilon \neq 0$, *it has the same number of (exponentially) stable and unstable directions as* $E'(\theta_0)$ *(considered as a vector field) plus* $A(\theta_0)$ *(considered as a diffeomorphism).*

In particular, if θ_0 *is an exponentially stable zero of the vector field E and* $A(\theta_0)$ *is an exponentially stable matrix, then, for* ε *positive small enough,* (S_ε) *has an exponentially stable periodic solution; otherwise (assuming* (i), (ii′), *and* (iii′)) *it is unstable (i.e., it has at least one unstable direction).*

The proof is given in Appendix A.

Remarks. (1) Of course, it is understood that $\theta(t)$ remains in Ω, i.e., that $\theta(0)$ is in U_ε; this is true for ε small enough because, for $\varepsilon = 0$, θ is frozen in Ω.

(2) (ii′) and (iii′) imply (ii) and (iii) of Theorem 1, so this theorem only deals with the solutions given by Theorem 1.

2. Application to a *d*-step-ahead Adaptive Control System with Unmodeled Dynamics

2.1. *Statement of the Adaptive Control System*

We consider the ARMA representation of a plant:

$$D(q^{-1})y = [q^{-1}N(q^{-1})]u, \qquad t \geq 0, \tag{7}$$

where $u = (u(t))_{t \in \mathbb{N}}$ and $y = (y(t))_{t \in \mathbb{N}}$ are the input and output sequences, q^{-1} is the unit delay operator, and D and N are real polynomials whose degrees are respectively d^* and n^*.

The control law is given (implicitly) by the following equation, which can be solved for $u(t)$ if the first component of $\theta(t)$ is nonzero (this turns out to have some importance, see (13)):

$$\theta(t)^T\Phi(t) = y^M(t + d), \tag{8}$$

where Φ is given by

$$\Phi^T(t) = (u(t), u(t-1), \ldots, u(t-n^{**}), y(t), y(t-1), \ldots, y(t-d^{**})) \tag{9}$$

$(y^M(t))_{t \in \mathbb{N}}$ is a reference output known d steps in advance (the control objective is to have y track y^M).

For later notational convenience V is the polynomial vector given by

$$V^T(X) = (D(X), XD(X), \ldots, X^{n^{**}}D(X), XN(X), \ldots, X^{d^{**}+1}N(X)) \tag{10}$$

In (8), θ is the $(n^{**} + d^{**} + 2)$-dimensional parameter vector, adapted as follows:

$$\theta(t) = \theta(t-1) + \varepsilon\frac{y(t) - \theta(t-1)^T\Phi(t-d)}{1 + \varepsilon\Phi(t-d)^T\Phi(t-d)}\Phi(t-d). \tag{11}$$

2.2. *Computation of the Determining Field E*

The closed-loop system consists of (7), (11), and (8). To use the previous section, we must mold the system description into an (S_ε)-like system. In fact, defining $x(t)$ as a vector containing enough (and at least as much as $\Phi(t)$) present and past values of u and y, (7) and (8) may be rewritten into the following (possibly nonminimal) state–space representation:

$$x(t+1) = A(\theta(t))x(t) + B(\theta(t))y^M(t+d) \tag{12}$$

which contains only two nontrivial equations: those giving $y(t+1)$ and $u(t+1)$. The latter comes from (8) and is not defined when the first component of θ, say $\theta^{(1)}$, is zero; $A(\theta)$ and $B(\theta)$ are therefore defined if and only if $\theta^{(1)}$ is nonzero. This illustrates the usefulness of assuming, in Section 1, that A and B are only defined on Ω; here,

$$\Omega = \{\theta \in \mathbb{R}^p, \theta^{(1)} \neq 0\}. \tag{13}$$

Of course, C is given by

$$C(x, \theta, \varepsilon) = \frac{y(t) - \theta(t-1)^T\Phi(t-d)}{1 + \varepsilon\Phi(t-d)^T\Phi(t-d)}\Phi(t-d). \tag{14}$$

We now apply Section 1, (S_ε) being given by (12) and (11). Theorems 1–3 of Section 1 do not really need the exact form of A and B: all the conditions only involve the determining field E and the nondegeneracy of $A(\theta)^k - I$. We now need the expression for the field E:

Lemma 4. *System (S_ε) being given by* (12) *and* (11), *the determining field E has the following expression*:

$$E(\theta) = \sum_{j=0}^{K-1} |\hat{y}_j^M|^2 \frac{z^{(1-d)j}N(z^j) - \theta^T V(z^j)}{|\theta^T V(z^j)|^2} V(z^{-j}), \tag{15}$$

where V is given by (10), $z = e^{2i\pi/K}$ $(i^2 = -1)$, *and the $\hat{y}_j^M$'s are the Fourier coefficients of $y^M(t+d)$*:

$$y^M(t+d) = \frac{1}{\sqrt{K}} \sum_{j=0}^{K-1} \hat{y}_j^M z^{-jt}. \tag{16}$$

All the proofs of this section are given in Appendix B.

2.3. *The "Exact Modeling" Case*

This case has been thoroughly treated, e.g., [GS]; we take

$$n^{**} \geq n^*, \qquad d^{**} \geq d^* - 1, \qquad d \geq 1, \tag{17}$$

$$X^{d-1} \text{ divides } N, \tag{18}$$

$$XN(X) \text{ and } D(X) \text{ are coprime.} \tag{19}$$

The two first conditions are, strictly speaking, exact modeling, and the third one is the controllability of the plant. Without any assumptions on the reference input y^M, it is proven in [GS] that if

$$\frac{N(X)}{X^{d-1}} \text{ has all its zeros outside the unit disk,} \tag{20}$$

then all the signals are bounded and the output $y(t)$ converges to $y^M(t)$. If, in addition, the reference output is sufficiently rich of order $n^{**} + d^{**} + 2$, which in the periodic case is equivalent to

$$(y^M) \text{ has at least } n^{**} + d^{**} + 2 \text{ nonzero Fourier coefficients} \tag{21}$$

(see [BS]), then the output (y) *exponentially* converges to (y^M), *and* the parameter estimate exponentially converges to the true value (Θ defined in (22)); this is proved in [BS].

Our techniques easily prove local exponential convergence in this case. Under assumptions (19), (17), and (18), there is one value of θ, say Θ, such that (7) may be written as

$$\Theta^T \Phi(t) = y(t + d), \tag{22}$$

this is the "d-step-ahead predictor form" (see, e.g., [GS]). For such a Θ, we have the following *polynomial* identity:

$$X^d \Theta^T V(X) \equiv X N(X),$$

so that, obviously, $E(\Theta) = 0$. Furthermore, $E'(\Theta)$ is easy to compute; differentiating (15), we get

$$E'(\Theta) = -\sum_{j=0}^{K-1} \frac{|\hat{y}_j^M|^2}{|\theta^T V(z^j)|^2} V(z^j) V(z^{-j})^T$$

which is negative definite when (21) is satisfied (by an argument very similar to Lemma 10). This and Theorems 1 and 3 imply the existence of a periodic solution for the closed-loop system. Moreover, since $N(X^{-1})/X^{1-d}$ is the characteristic polynomial of $A(\Theta)$ if (20) is satisfied, this solution is exponentially stable. It is not difficult to see that even for $\varepsilon > 0$ this solution gives $\theta(t) = \Theta$ and $y(t) = y^M(t)$.

2.4. The Case of Unmodeled Dynamics

The results stated in this section are proved in Appendix B.

Let us consider first the case when the integer d (the delay of the controller) satisfies

$$K - d^{**} \leq d \leq K \quad (\text{modulo } K).$$

This implies the existence of an integer a such that

$$1 \leq a \leq d^{**} + 1, \qquad z^{1-d} = z^a.$$

Since $n^{**} + a + 1$ is less than $n^{**} + d^{**} + 2$, the least possible dimension of θ, we may define

$$\Theta = (\underbrace{0, \ldots, 0, 1}_{n^{**}+a+1\text{th row}}, 0, \ldots, 0)^T. \tag{23}$$

Then, from (10), the following polynomial identity holds:

$$X^a N(X) \equiv \Theta^T V(X). \tag{24}$$

It is clear that if the reference output is sufficiently rich, of order $n^{**} + d^{**} + 2$, Θ is the only zero for the expression of E given by (15) (compute $(\Theta - \theta)^T E(\theta)$; it is stricly positive when $\theta \neq \Theta$). Unfortunately, this Θ is not in Ω (see Definition 1), and, even worse, $A(\Theta)$ is not defined (because $\Theta^{(1)} = 0$, see Section 2.2). Hence, in this situation no periodic solution may be proved to exist *within* the framework of our theory. We therefore assume now that $1 \leq d \leq K - d^{**} - 1$ (modulo K), or, without loss of generality since we are considering K-periodic solutions,

$$1 \leq d \leq K - d^{**} - 1.$$

This is assumption (c) below. Notice that exact modeling plus sufficient richness imply (c): (18) implies that $d \leq n^* + 1$; (17) implies that $n^* + 1 \leq n^{**} + 1$; sufficient richness implies that $K \geq n^{**} + d^{**} + 2$, or equivalently $n^{**} + 1 \leq K - d^{**} - 1$; finally, sufficient richness plus exact modeling imply $1 \leq d \leq K - d^{**} - 1$. Hence there is no contradiction between Section 2.2 and (23) and (24).

Our main contribution consists in studying the situation of assumptions (a)–(c) which is beyond the scope of [GS] ((b) implies the existence of unmodeled dynamics).

Let us state these assumptions, recalling that $z = e^{2i\pi/K}$:

(a) $K \geq n^{**} + d^{**} + 2$.
(b) $n^{**} \leq n^* - 1$; $d^{**} \leq d^* - 1$.
(c) $1 \leq d \leq K - d^{**} - 1$.
(d) There is no zero term in the Fourier decomposition of the priodic entry y^M.
(e) No z^j (the Kth roots of 1) is a zero of N.
(f) The polynomials D and XN are coprime.
(g) The sum $\sum_{j=0}^{K-1} |\hat{y}_j^M| (z^{(1-d)j} N(z^j)/|\theta^T V(z^j)|^2) V(z^{-j})$ is zero for no θ in $\mathbb{R}^{n^{**}+d^{**}+2}$.
(h) $E(\theta) = \sum_{j=0}^{K-1} |\hat{y}_j^M|^2 ((z^{(1-d)j} N(z^j) - \theta^T V(z^j))/|\theta^T V(z^j)|^2) V(z^{-j})$ is zero for no θ in $\mathbb{R}^{n^{**}+d^{**}+2}$ whose first component is zero.

We have the following proposition:

Proposition 5. *Under assumptions* (a)–(g), *we can find G, an open bounded subset of* $\mathbb{R}^{n^{**}+d^{**}+2}$, *such that, E being give by* (15),

- *E does not vanish on ∂G (G's boundary),*
- *$A(\theta)^K - I$ defined by* (12) *is invertible for any θ in G, and*
- $\deg(E, G, 0) \neq 0$.

The main result is:

Theorem 6. *Under assumptions* (a)–(h), *the closed-loop system* (7), (11), (8) *has at least one K-periodic solution.*

Comments on the Assumptions

Assumption (a) is clearly satisfied when (c) is satisfied and the delay d is smaller than the (maximum) number of past values of u in $\Phi(t)$, because then $d \le n^{**} + 1$ and, from (c), $n^{**} + 1 \le K - d^{**} - 1$.

Assumption (b) means *unmodeled dynamics*. Precisely, there are not enough past values both of u and y in Φ to allow the existence of a Θ such that (22) matches (7).

Assumptions (c) and (d) together imply *sufficient richness* (SR) of order $n^{**} + d^{**} + 2$ of the input y^M; see [BS].

Assumptions (e) is implied by the plant having a stable inverse, as assumed in [BS].

Assumption (f) requires the plant (7) to be controllable. This clearly a necessary condition for proper control.

Assumptions (g) and (h) are difficult to interpret. The following result shows that (g) is "almost always" satisfied. Since we found nothing similar for (h), we study in an example (Section 2.5) below what it means.

Proposition 7. *Let K, d, n^*, d^*, n^{**}, and d^{**} be fixed such that* (a) *is satisfied. The set of (N, D, y^M) such that* (b)–(e) *are satisfied is a natural subset of an $(n^* + d^* + K + 2)$-dimensional real vector space. It is* almost all this space *both in the sense of measure theory and of Baire theory.*

2.5. *An Example*

We study the case

$$n^{**} = d^{**} = 0, \qquad d = 1,$$

i.e., the control law is

$$\theta_1(t)u(t) + \theta_2(t)y(t) = y^M(t+1) \tag{25}$$

and C (see (14)) is given by

$$C(x(t), \theta(t), \varepsilon(t)) = \frac{y(t) - \theta_1(t-1)u(t-1) + \theta_2(t-1)y(t-1)}{1 + \varepsilon[u^2(t-1) + y^2(t-1)]}\begin{pmatrix} u(t-1) \\ y(t-1) \end{pmatrix}. \tag{26}$$

Then a computation (or application of Lemma 4) gives the determining field

$$E\left(\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}\right) = \sum_0^{K-1} |\hat{y}_j^M|^2 \frac{N_j - \theta_1 D_j - \theta_2 z_j N_j}{|\theta_1 D_j + \theta_2 z_j N_j|^2}\begin{pmatrix} \bar{D}_j \\ \bar{z}_j \bar{N}_j \end{pmatrix}, \tag{27}$$

where the bar denotes complex conjugate and

$$z_j = z^j = e^{2ij\pi/K}, \qquad y^M(t+1) = \frac{1}{\sqrt{K}} \sum_{j=0}^{K-1} \hat{y}_j^M z_j, \qquad N_j = D(z_j), \qquad D_j = N(z_j).$$

The linear plant is still given by (7). Let us examine the meaning of assumptions (a)–(c) in this example:

1. Assumptions (a), (c), and (d) are satisfied if the period is at least 2.
2. Exact modeling occurs when N is constant and D has degree 1; if the degrees are larger, assumption (b) is satisfied.
3. From Proposition 7, we know that the set of polynomials N and D and input y^M such that (g) holds is a dense open subset with measure one in the set of all the possible N, D, and y^M; there is nothing further to say regarding this example.
4. Assumption (h) means that there is no θ_2 solution of ($E(\theta) = 0$ with $\theta_1 = 0$)

$$\frac{1}{|\theta_2|^2} \sum_0^{K-1} |\hat{y}_j^M|^2 \frac{D_j}{z_j N_j}(z_j - \theta_2) = 0,$$

$$\frac{1}{|\theta_2|^2} \sum_0^{K-1} |\hat{y}_j^M|^2 (z_j - \theta_2) = 0,$$

i.e., if the determinant of this linear system is nonzero:

$$\sum_{k=0}^{K-1} \sum_{l=0}^{K-1} |\hat{y}_k^M|^2 |\hat{y}_l^M|^2 \frac{D_k}{z_k N_k}(z_k - z_l). \tag{28}$$

This is a nonzero polynomial in the coefficients of N and D and the Fourier coefficients of y^M; see the explanation of this at the end of Appendix B. Hence for this example, (h) is "almost always" true as well as (g).

3. Conclusions

In Section 1 we have given general conditions for the existence and stability of periodic solutions of an adaptive system with periodic excitation. Being able to find a periodic solution (which is the natural ideal regime in this case) and its stability/instability provides thorough local knowledge of the closed-loop performance.

In Section 2 we have presented a very concrete and common class of adaptive controllers for poorly modeled plants. We were able to prove the existence of periodic solution(s), at least "almost always," but we have no tool to study its (their) stability, except in the "exact modeling" case, which is already well known. We conclude that this study is only a first step toward a complete description of the behavior of an adaptive controller in feedback with a high-order linear plant.

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Appendix A. Proof of the Results Stated in Section 1

We first prove Theorem 2 and then Theorems 1 and 3.

As noticed in Section 1.2, looking for K-periodic solutions of (S_ε) is the same as looking for *fixed points* of the *K-advance map* of (S_ε), which maps $(x(0), \theta(0))$ into $(x(K), \theta(K))$:

Definition 3. The *K-advance map* of (S_ε), $\mathscr{F}_\varepsilon$, is defined from U_ε (see Definition 2) to $\mathbb{R}^n \times \mathbb{R}^p$ as the map taking (x, θ) to $(x(K), \theta(K))$, where $(x(t), \theta(t))$ is the solution of (S_ε) with $(x(0), \theta(0)) = (x, \theta)$. In short,

$$\mathscr{F}_\varepsilon(x(0), \theta(0)) = (x(K), \theta(K)). \tag{29}$$

To understand the behavior of $\mathscr{F}_\varepsilon$ as ε tends to zero, we introduce another map $\mathscr{G}_\varepsilon$, such that

$$\mathscr{F}_\varepsilon = I + \begin{pmatrix} I & 0 \\ 0 & \varepsilon I \end{pmatrix} \circ \mathscr{G}_\varepsilon. \tag{30}$$

Definition 4. Let $\mathscr{G}_\varepsilon$ be the map from U_ε to $\mathbb{R}^n \times \mathbb{R}^p$ defined by

$$\mathscr{G}_\varepsilon(x(0), \theta(0)) = \left(x(K) - x(0), \sum_{t=0}^{K-1} C(x(t), \theta(t), \varepsilon, t) \right), \tag{31}$$

where $(x(t), \theta(t))$ is the solution of (S_ε) with $(x(0), \theta(0)) = (x, \theta)$.

A last definition to simplify the notations is

Definition 5. For any θ such that $A(\theta)^K - I$ is nonsingular, we let

$$\chi(\theta) = [I - A(\theta)^K]^{-1} \sum_{j=0}^{K-1} A(\theta)^{K-j-1} B(\theta) v(j). \tag{32}$$

$\chi(\theta)$ is then the only χ such that (χ, θ) is the initial condition of a K-periodic solution of (S_0).

To prove the existence of fixed points of $\mathscr{F}_\varepsilon$, we might think of using a continuation method starting from those of $\mathscr{F}_0$. Unfortunately, the fixed points of $\mathscr{F}_0$ are highly degenerate since the θ-part of $\mathscr{F}_0$'s derivative is the identity. However, from (30), it is clear that any zero of $\mathscr{G}_\varepsilon$ is a fixed point of $\mathscr{F}_\varepsilon$ (the converse being true only if $\varepsilon \neq 0$), and they turn out to be nondegenerate. These remarks lead us to the following method for proving Theorem 2:

1. Study [in Lemma 8] the case $\varepsilon = 0$, that is find zeros of $\mathscr{G}_0$ from zeros of E, more precisely show that the topological degree of $\mathscr{G}_0$ on a bounded open subset of $\mathbb{R}^n \times \Omega$ is nonzero if the degree of E is nonzero on a bounded open subset of Ω.
2. Show [in Lemma 9] that, for ε small enough, the degree of $\mathscr{G}_\varepsilon$ on the same subset is nonzero too, which guarantees the existence of some zeros for $\mathscr{G}_\varepsilon$.
3. Conclude that these zeros are fixed points of $\mathscr{F}_\varepsilon$, and therefore initial conditions of periodic solutions of (S_ε).

Lemma 8. *If V is an open, bounded set such that $\bar{V} \subset \Omega$,*

(i) $\det(A(\cdot)^K - I)$ *has a constant sign on V, and*
(ii) *E never vanishes on ∂V,*

then, for $R > 0$ large enough,

$$\deg(\mathscr{G}_0, \mathscr{B}_{\mathbb{R}^n}(0, R) \times V, 0) = \operatorname{sign}\{\det(A(\cdot)^K - I)\} \deg(E, V, 0). \tag{33}$$

Proof of Lemma 8. It is clear from (31), (32), and (5) that

$$\mathscr{G}_0(x, \theta) = 0 \qquad \text{if and only if} \quad x = \chi(\theta) \quad \text{and} \quad E(\theta) = 0. \tag{34}$$

This enables us to fix R: E being continuous, the set Z of all the zeros of E in V is closed and bounded, so it is compact; χ being continuous, $\chi(Z)$ is compact and therefore bounded, and we may choose R large enough to have

$$\chi(\{\theta \in V/E(\theta) = 0\}) \subset \mathscr{B}_{\mathbb{R}^n}(0, R/2). \tag{35}$$

It is clear from (34) that $\mathscr{B}_{\mathbb{R}^n}(0, R) \times V$ contains all the zeros (x, θ) of $\mathscr{G}_0$ with θ in V. Furthermore, we have no zero of $\mathscr{G}_0$ in $\partial(\mathscr{B}_{\mathbb{R}^n}(0, R) \times V)$ because

$$\partial(\mathscr{B}_{\mathbb{R}^n}(0, R) \times V) = \partial\mathscr{B}_{\mathbb{R}^n}(0, R) \times V \subset \mathscr{B}_{\mathbb{R}^n}(0, R) \times \partial V.$$

1. If $x \in \partial(\mathscr{B}_{\mathbb{R}^n}(0, R))$, from (35), there is no θ such that $x = \chi(\theta)$.
2. If $\theta \in \partial V$, $E(\theta) \neq 0$ (from (ii) in the hypothesis of Lemma 8).

Therefore, in these two cases, from (34), $\mathscr{G}_0(x, \theta) \neq 0$.

Let us now compute $\det[\mathscr{G}_0'(x, \theta)]$ for (x, θ) such that $\mathscr{G}_0(x, \theta) = 0$. ($\mathscr{G}_0'$ denotes the derivative of $\mathscr{G}_0$.) This is necessary for studying the degree (see (39)). For this purpose we let

$$x = \chi(\theta) + \xi; \tag{36}$$

note that

$$\mathscr{G}_0(x, \theta) = 0 \qquad \text{if and only if} \quad \xi = 0 \quad \text{and} \quad E(\theta) = 0.$$

We decompose $\mathscr{G}_0$ as follows:

$$(x, \theta) \to (\xi, \theta) \to \mathscr{G}_0(x, \theta).$$

The first map clearly has the derivative

$$\begin{pmatrix} I_n & -\chi'(\theta) \\ 0_{p,n} & I_p \end{pmatrix}.$$

The second map has the derivative, evaluated for $\xi = 0$,

$$\begin{pmatrix} A(\theta)^K - I_n & 0_{n,p} \\ R(\theta) & E'(\theta) \end{pmatrix};$$

we give some of the details of this computation: The top-left block is

$(\partial/\partial\xi)(x(K) - x(0))$ and a computation of (1) from 0 to K gives

$$x(K) - x(0) = (A(\theta)^K - I)(\chi(\theta) + \xi) + \sum_{j=0}^{K-1} A(\theta)^{K-j-1}B(\theta)v(j).$$

The bottom-left block, R, is a $p \times n$ matrix whose form is of no significance. The top-right block is zero for $x(T) - x(0) = 0$ when $\xi = 0$, no matter the value of θ. The bottom-right block is $(\partial/\partial\theta)[\sum_{t=0}^{K-1} C(x(t), \theta, 0, t)]$, where $(x(t), \theta)$ is the periodic solution of (S_0); this is exactly $E'(\theta)$.

So we have

$$\mathscr{G}_0'(x, \theta) = \begin{pmatrix} A(\theta)^K - I_n & 0_{n,p} \\ R(\theta) & E'(\theta) \end{pmatrix} \begin{pmatrix} I_n & -\chi'(\theta) \\ 0_{p,n} & I_p \end{pmatrix} \quad \text{when} \quad x = \chi(\theta). \tag{37}$$

And, therefore,

$$\det(\mathscr{G}_0'(x, \theta)) = \det(A(\theta)^K - I) \det(E'(\theta)) \qquad \text{when} \quad \mathscr{G}_0(x, \theta) = 0. \tag{38}$$

Now we must distinguish two cases:

Case 1. Zero is a regular value of E (i.e., $E'(\theta)$ is nonsingular whenever $E(\theta) = 0$). Then, since $\det(A(\cdot)^K - I)$ never vanishes on V, $(0, 0)$ is also a regular value in $\mathbb{R}^n \times \mathbb{R}^p$ for $\mathscr{G}_0$ (see (38)), hence (from Definition 1.1.4 of [L2])

$$\begin{aligned} \deg(\mathscr{G}_0, \mathscr{B}_{\mathbb{R}^n}(0, R) \times V, (0, 0)) &= \sum_{(x,\theta) \in \mathscr{B}_{\mathbb{R}^n}(0,R) \times V; \mathscr{G}_0(x,\theta)=0} \operatorname{sign} \det \mathscr{G}_0'(x, \theta) \\ &= \sum_{\theta \in V; E(\theta)=0} \operatorname{sign} \det(A(\theta)^K - I) \operatorname{sign} \det E'(\theta). \end{aligned} \tag{39}$$

This gives the lemma since

$$\deg(E, V, 0) = \sum_{\theta \in V; E(\theta)=0} \operatorname{sign} \det(E'(\theta)). \tag{40}$$

Case 2. Zero is not a regular value of E (i.e., for some θ in Ω, $E(\theta) = 0$ and $E'(\theta)$ is singular). Here, (39) and (40) cannot be used to compute the degrees of $\mathscr{G}_0$ and E because some determinants are zero.

However, Sard's theorem (Theorem 1.2.1 in [L2]) enables us to find a regular value of E, σ ($\sigma \in R^p$), as close as we want to the nonregular value 0. On the other hand, E being continuous and ∂V being compact, there is an $r > 0$ such that $\mathscr{B}_{\mathbb{R}^p}(0, r)$ contains no point of $E(\partial V)$, and, consequently, $\{0\} \times \mathscr{B}_{\mathbb{R}^p}(0, r)$ contains no point of $\mathscr{G}_0(\partial(\mathscr{B}_{\mathbb{R}^n}(0, R) \times V))$; hence, choosing $\|\sigma\| < r$, from Theorem 2.1.2 of [L2],

$$\deg(E, V, 0) = \deg(E, V, \sigma) \tag{41}$$

and

$$\deg(\mathscr{G}_0, \mathscr{B}_{\mathbb{R}^n}(0, R) \times V, 0) = \deg(\mathscr{G}_0, \mathscr{B}_{\mathbb{R}^n}(0, R) \times V, (0, \sigma)). \tag{42}$$

Besides, the same computation as in Case 1 with σ instead of $0_{\mathbb{R}^p}$ and $(0, \sigma)$ instead of $0_{\mathbb{R}^n \times \mathbb{R}^p}$ gives

$$\deg(\mathscr{G}_0, \mathscr{B}_{\mathbb{R}^n}(0, R) \times V, (0, \sigma)) = \operatorname{sign}\{\det(A(\cdot)^K - I)\} \deg(E, V, \sigma). \tag{43}$$

From (41), (42), and (43), we easily deduce the lemma in this case. ∎

We now extend the result of this lemma to some small positive ε:

Lemma 9. *If there exists a bounded open set V such that $\bar{V} \subset \Omega$,*

(i) $\det(A(\cdot)^K - I)$ *has a constant sign (and never vanishes) on V, and*
(ii) *E never vanishes on ∂V,*

then an $R > 0$ and an $\varepsilon_0 > 0$ can be found such that, for $|\varepsilon| < \varepsilon_0$,

(α) $\overline{\mathscr{B}_{\mathbb{R}^n}(0, R) \times V} \subset U_\varepsilon$,
(β) *$\mathscr{G}_\varepsilon$ never vanishes on $\partial(\mathscr{B}_{\mathbb{R}^n}(0, R) \times V)$, and*
(γ) $\deg(\mathscr{G}_\varepsilon, \mathscr{B}_{\mathbb{R}^n}(0, R) \times V, 0) = \deg(\mathscr{G}_0, \mathscr{B}_{\mathbb{R}^n}(0, R) \times V, 0)$.

Proof of Lemma 9. Let

$$c_0 = \sup_{(x,\theta,\varepsilon) \in \mathscr{B}_{\mathbb{R}^n}(0,R) \times V \times [0,1]} \|C(x, \theta, \varepsilon, t)\|.$$

As long as $\theta(t)$ can be computed, i.e., $\theta(t-1) \in U_\varepsilon$, we have

$$\|\theta(t) - \theta(t-1)\| \le \varepsilon c_0 \quad \text{and} \quad \|\theta(t) - \theta(0)\| \le k c_0 |\varepsilon| \le K c_0 |\varepsilon|.$$

If we choose, for instance, $\varepsilon < (1/2Kc_0)\,\mathrm{dist}(\partial\Omega, \bar{V})$, then ($\alpha$) is satisfied.

The map

$$(x, \theta, \varepsilon) \mapsto \mathscr{G}_\varepsilon(x, \theta)$$

is continuous from $\overline{\mathscr{B}_{\mathbb{R}^n}(0, R) \times V} \times [0, +\infty)$ to $\mathbb{R}^n \times \mathbb{R}^p$ so it is uniformly continuous, for instance, on $\overline{\mathscr{B}_{\mathbb{R}^n}(0, R) \times V} \times [0, 1]$; hence,

$$\mathscr{G}_\varepsilon \text{ converges uniformly on } \mathscr{B}_{\mathbb{R}^n}(0, R) \times V \text{ to } \mathscr{G}_0 \quad \text{when} \quad \varepsilon \to 0 \tag{44}$$

From (ii), (34), and (35) we deduce that $\mathscr{G}_0$ never vanishes on $\partial(\mathscr{B}_{\mathbb{R}^n}(0, R) \times V)$. Furthermore, (44) and the compactness of $\partial(\mathscr{B}_{\mathbb{R}^n}(0, R) \times V)$ clearly imply the existence of an ε_0 such that (β) is satisfied for any $\varepsilon < \varepsilon_0$.

Condition (β) being proved, $(t \mapsto \mathscr{G}_t)$ is a homotopy on $\mathscr{B}_{\mathbb{R}^p}(0, R) \times V$ joining $\mathscr{G}_0$ to $\mathscr{G}_\varepsilon$, which, thanks to (β) (see Theorem 1.3.5(3) of [L2]), gives (γ). ∎

Proof of Theorem 2. The degree on V is the sum of the degrees on all its connected components (see Theorem 2.2.1(1) of [L2]), so that (ii) implies that E's degree relative to 0 is nonzero on at least one of V's connected components, say V'. From (iii), $\det(A(\cdot)^K - I)$ has a constant sign on V', so that, with V' instead of V, asumption (i) in Lemmas 8 and 9 is satisfied and, using these lemmas, we get, for ε small enough,

$$\deg(\mathscr{G}_\varepsilon, \mathscr{B}_{\mathbb{R}^n}(0, R) \times V', 0) \neq 0,$$

which means that $\mathscr{G}_\varepsilon$ has att least one zero in $\mathscr{B}_{\mathbb{R}^n}(0, R) \times V'$.

Since a zero of $\mathscr{G}_\varepsilon$ is a fixed point of $\mathscr{F}_\varepsilon$ and a fixed point of $\mathscr{F}_\varepsilon$ gives a periodic solution of (S_ε), the theorem is proved. ∎

Now, the proof of Theorem 1 is quite simple, and that of Theorem 3 needs only a few computations:

Proof of Theorem 1. Considering (38) and the assumptions, the implicit function theorem clearly gives the result. ∎

Proof of Theorem 3. From continuity, the K-periodic solution has the same (in)stability property as the corresponding fixed point of the K-advance map. Then we apply the well-known first approximation stability theorem (see [L1]): a sufficient condition for (in)stability of a fixed point is given by the position with respect to the unit circle of the eigenvalues of the derivative of this map, evaluated at the fixed point; this is precisely the matrix $\mathscr{F}_\varepsilon'(x_\varepsilon, \theta_\varepsilon)$. Let us compute it.

Differentiating (30) we get

$$\mathscr{F}_\varepsilon'(x_\varepsilon, \theta_\varepsilon) = I + \begin{pmatrix} I & 0 \\ 0 & \varepsilon I \end{pmatrix} \mathscr{G}_\varepsilon'(x_\varepsilon, \theta_\varepsilon), \tag{45}$$

where $(x_\varepsilon, \theta_\varepsilon)$ is the initial condition of the periodic solution for (S_ε), defined in (6).

On the other hand, we notice that from Theorem 1 and the continuous differentiability of A, B, C, x_ε, and θ_ε, the Hadamard Lemma (see [AE]) yields the existence, in a neighborhood of zero, of a bounded function $\Delta(\varepsilon)$, satisfying

$$\mathscr{G}_\varepsilon'(x_\varepsilon, \theta_\varepsilon) = \mathscr{G}_0'(x_0, \theta_0) + \varepsilon\Delta(\varepsilon). \tag{46}$$

Now, from (37), if we define S by

$$S = \begin{pmatrix} I & -\chi'(\theta) \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \varepsilon I \end{pmatrix} \mathscr{G}_\varepsilon'(x_\varepsilon, \theta_\varepsilon) \begin{pmatrix} I & \chi'(\theta) \\ 0 & I \end{pmatrix}, \tag{47}$$

we obtain

$$S = \begin{pmatrix} A(\theta_0)^K - I + \varepsilon\Delta_1(\varepsilon) & \varepsilon\Delta_2(\varepsilon) \\ \varepsilon\Delta_3(\varepsilon) & \varepsilon E'(\theta_0) + \varepsilon^2\Delta_4(\varepsilon) \end{pmatrix}, \tag{48}$$

where $\Delta_i(\varepsilon)$, $i = 1, 4$, are bounded on a neighborhood of zero. Now we apply Lemma 1 of [K]: since $A(\theta_0)^K - I$ is nonsingular, there exists a function $L(\varepsilon)$ bounded on a neighborhood of zero such that S may be put in triangular form by the change of basis

$$T = \begin{pmatrix} I & L(\varepsilon) \\ 0 & I \end{pmatrix}$$

in such a way that

$$TST^{-1} = \begin{pmatrix} A(\theta_0)^K - I + \varepsilon(\Delta_1 + \varepsilon L\Delta_3) & 0 \\ \varepsilon\Delta_3 & \varepsilon E'(\theta_0) + \varepsilon^2(\Delta_4 - \Delta_3 L) \end{pmatrix}. \tag{49}$$

Relations (45)–(49) lead to

$$M\mathscr{F}_\varepsilon'(x_\varepsilon, \theta_\varepsilon)M^{-1} = \begin{pmatrix} A(\theta_0)^K + \varepsilon(\Delta_1 + \varepsilon L\Delta_3) & 0 \\ \varepsilon\Delta_3 & I + \varepsilon E'(\theta_0) + \varepsilon^2(\Delta_4 - \Delta_3 L) \end{pmatrix}$$

with

$$M = \begin{pmatrix} I & L(\varepsilon) \\ 0 & I \end{pmatrix} \begin{pmatrix} I & -\chi'(\theta_0) \\ 0 & I \end{pmatrix}.$$

It follows that the eigenvalues of $\mathscr{F}_\varepsilon'(x_\varepsilon, \theta_\varepsilon)$ are $\lambda\{A(\theta_0)^K\} + o(1)$ and $1 + \varepsilon \operatorname{Re} \lambda\{E'(\theta_0)\} + o(\varepsilon)$, where $o(1)$ and $o(\varepsilon)/\varepsilon$ are continuous functions of ε which tend to zero as ε tends to zero. This finishes the proof. ■

Appendix B. Proof of the Results Stated in Section 2

Proof of Lemma 4. We have to compute the vector field E, the particular system (S_ε) considered being given by Section 2. From (4) of Definition 1 we have

$$E(\theta) = \sum_{t=0}^{K-1} (y(t) - \theta^T \Phi(t-d))\Phi(t-d), \tag{50}$$

where $y(t)$ and $\Phi(t)$ are the only K-periodic solutions of the system (7), (8) such that $\theta(t) = \theta$. As these $y(t)$ and $u(t)$ are periodic, we may write their Fourier decomposition

$$y(t) = \frac{1}{\sqrt{K}} \sum_{j=0}^{K-1} \hat{y}_j z^{-jt}, \qquad \Phi(t) = \frac{1}{\sqrt{K}} \sum_{j=0}^{K-1} \hat{\Phi}_j z^{-jt}, \tag{51}$$

and, applying the Parseval identity to (50), we get

$$E(\theta) = \sum_{j=0}^{K-1} (\hat{y}_j - z^{dj}\theta^T \hat{\Phi}_j)\overline{z^{dj}\hat{\Phi}_j}. \tag{52}$$

Let us now compute the $\hat{y}_j$ and $\hat{\Phi}_j$'s by using the $\hat{y}_j^M$'s. In order to make the notations clearer, let us define the X-polynomial vectors Z_u and Z_y by

$$Z_u^{\mathrm{T}} = (1, X, \ldots, X^{n^{**}}, 0, \ldots, 0) \quad \text{and} \quad Z_y^{\mathrm{T}} = (0, \ldots, 0, 1, X, \ldots, X^{d^{**}}). \tag{53}$$

Then, from (9) and (10), we have the following expressions for the polynomial vector V and the time-dependent vector Φ:

$$V(X) = D(X)Z_u(X) + XN(X)Z_y(X), \tag{54}$$

$$\Phi(t) = [Z_u(q^{-1})u + Z_y(q^{-1})y](t). \tag{55}$$

Then, from (7), (8), and (54),

$$\theta^T V(q^{-1})u(t) = D(q^{-1})y^M(t+d), \tag{56}$$

$$\theta^T V(q^{-1})y(t) = q^{-1}N(q^{-1})y^M(t+d), \tag{57}$$

and, from (56), (57), and (55),

$$\theta^T V(q^{-1})\Phi(t) = V(q^{-1})y^M(t+d). \tag{58}$$

Now, from (51), (57), (8), and (58), we get

$$\theta^T \hat{\Phi}_j = \hat{y}_j^M, \qquad \hat{y}_j = \frac{z^j N(z^j)\hat{y}_j^M}{\theta^T V(z^j)}, \qquad \hat{\Phi}_j = \frac{\hat{y}_j^M}{\theta^T V(z^j)} V(z^j). \tag{59}$$

Equations (59) and (52) give (15) and the lemma. ■

Before proceeding with the proof of Proposition 5 and Theorem 6, let us state the following technical lemma:

Lemma 10. *Under assumptions* (b), (c), *and* (f), *any* $n^{**} + d^{**} + 2$ *vectors taken from among all the* $V(z^j)$, *for* $0 \le j \le K - 1$, *are linearly independent in* $\mathbb{C}^{n^{**}+d^{**}+2}$.

Proof of Lemma 10. We have $V^T = (1, X, X^2, \dots, X^\delta)P^T$ (for one positive δ) where P is the following $\delta \times n^{**} + d^{**} + 2$ matrix:

$$P = \begin{array}{c} \begin{array}{ccc} \underbrace{\hphantom{d_0 \quad \ddots \quad (0) \quad d_0}}_{\substack{n^{**}+1 \\ \text{columns}}} & & \underbrace{\hphantom{0 \quad \ddots \quad 0}}_{\substack{d^{**}+1 \\ \text{columns}}} \end{array} \\ \begin{pmatrix} d_0 & & & 0 & (0) & \\ d_1 & \ddots & (0) & n_0 & \ddots & \\ \vdots & & \ddots & n_1 & \ddots & 0 \\ \vdots & & d_0 & \vdots & \ddots & n_0 \\ d_{d^*} & & d_1 & n_{n^*} & & n_1 \\ & \ddots & \vdots & & \ddots & \vdots \\ & (0) \quad \ddots & \vdots & & (0) & n_{n^*} \\ & & d_{d^*} & & & \end{pmatrix} \end{array}.$$

Matrix P has full rank when (f) is true since its columns are extracted (because $n^{**} < \deg XN = n^* + 1$ and $d^{**} < \deg D = d^*$) from the Sylvester matrix of the coprime polynomials D and XN (see [W]). This proves that the components of V are $n^{**} + d^{**} + 2$ linearly independent polynomials, and the result then follows from the properties of the Vandermonde matrix given by the $n^{**} + d^{**} + 2$ different z^j. ■

Proof of Proposition 5. All the real difficulties in this proof are contained in Lemma 11 given in Appendix C. To use it, we must first point out that our $E(\theta)$, given by (15), looks like the $E(x)$ of Lemma 11, defined in (70). In order to see this, consider (15), and just select together the jth term and the $(K - j)$th, which is its complex conjugate (for each component). This gives

$$E(\theta) = \sum_{j=0}^{\mathrm{Int}(K/2)} \lambda_j \frac{u_j - Q_j\theta}{\theta^T Q_j \theta}, \tag{60}$$

where the λ_j's are given by

$$\lambda_j = |\hat{y}_j^M|^2 \tag{61}$$

and the u_j's and Q_j's are given by

$$u_j = 2\,\mathrm{Re}(z^{(1-d)j} N(z^j) V(z^{-j})), \tag{62}$$

$$Q_j = 2\,\mathrm{Re}(V(z^{-j}) V(z^j)^T) \tag{63}$$

for $0 < j < K/2$,

$$u_0 = N(1)V(1), \tag{64}$$

$$Q_0 = V(1)V(1)^T, \tag{65}$$

and, if K is even,

$$u_{K/2} = N(-1)V(-1), \tag{66}$$

$$Q_{K/2} = V(-1)V(-1)^T. \tag{67}$$

Considering (60), we apply Lemma 11 with $p = n^{**} + d^{**} + 2$, the λ_i, Q_i, and u_i's being given by (61)–(67). Let us check that the assumptions of this lemma are satisfied:

- The Q_j are obviously positive symmetric (see (65), (67), and (63)).
- Assumption (H_1) is satisfied since the image of Q_j is spanned by the real and imaginary parts of the complex vector $V(z^j)$, and u_j is a real linear combination of them.
- Assumption (H_2) is satisfied because Lemma 10 tells us that the Im Q_j span $\mathbb{R}^{n^{**}+d^{**}+2}$; in addition, every Q_j is positive.
- Assumption (H_3) is satisfied from (61) and (d).
- Assumption (H_4) is exactly (g).
- Assumption (H_5) derives from (e) and (f): if one $y \neq 0$ is in the kernel of some Q_j, then, since the Q_j are symetric, it is orthogonal to the corresponding Im Q_j; this implies, from expressions (65), (67), and (63) and Lemma 10, that the sum of these Im Q_j is not the whole space and is a direct one, which finally, from (H_1), implies that the corresponding u_j are independent.

Applying Lemma 11 then gives a bounded open subset of $\mathbb{R}^{n^{**}+d^{**}+2}$, $G(r, R, \eta)$, described in (71), such that E does not vanish on the boundary of this set and E's degree on this set is nonzero. In addition, $A(\theta)^K - I$ is invertible for any θ in $G(r, R, \eta)$; if not, there would be θ in $G(r, R, \eta)$, x_0 of suitable dimension (in (12)), and j $(0 \leq j \leq K - 1)$ such that $A(\theta)x_0 = z^j x_0$, so that $x(t) = x_0 z^{tj}$ would be a solution of (12) for $y^M = 0$. Then $\Phi(t) = \Phi_0 z^{tj}$ would be a solution of (58) for $y^M = 0$ and some Φ_0 extracted from x_0. This would imply $\theta^T V(z^j) = 0$ and, from (63), $Q_j\theta = 0$, which, from (53), is impossible if θ is in $G(r, R, \eta)$. ∎

Proof of Theorem 6. Since G, given by Proposition 5, is bounded and E is continuous and does not vanish on the hyperplane $\{\theta^{(1)} = 0\}$, it does not vanish on $H' = \{\theta \in G; |\theta^{(1)}| < \alpha\}$ for α small enough either. Then let

$$V = G(r, R, \eta) - H'. \tag{68}$$

The hypotheses of Theorem 2 are satisfied:

- $\bar{V} \subset \Omega$ because $\bar{V} \cap \{\theta^{(1)} = 0\} = \varnothing$ (from (68)).
- E does not vanish on ∂V because it vanishes neither on ∂G (from Proposition 5) nor on H' (because α has been chosen small enough).
- $\deg(E, V, 0)$ is equal to $\deg(E, G, 0) - \deg(E, H', 0)$ from (68), and $\deg(E, H', 0)$ is zero because, from Proposition 5, E does not vanish in H'.
- $\det(A(\cdot)^K - I)$ never vanishes in V, because, from Proposition 6, it does not vanish in G. ∎

Proof of Proposition 7. We consider (N, D, y^M) as a vector in $\mathbb{R}^{n^*+d^*+K+2}$ made of the coefficients of the polynomials N and D and the real and imaginary parts of the Fourier coefficients of the K-periodic sequence y^M, and we show that to satisfy (e) and (d), we must eliminate a finite union of sets whose measure is zero and whose closure has an empty interior:

Assumption (e): $N(z^j) = 0$ is a hyperplane in $\mathbb{R}^{n^*+d^*+K+2}$, we call it H_b.

Assumption (f) requires the "Sylvester Resultant" of XN and D to be nonzero (see [W]). This is a nonidentically zero polynomial in the coefficients of N and D. Let us denote the nontrivial algebraic hypersurface where it vanishes by H_c.

Assumption (d) is true outside the union of K hyperplanes; let us denote this union by H_e.

Assumption (g): The sum can be written

$$\sum_{j=0}^{k} \frac{|\hat{y}_j^M|^2}{|\theta^T V(z^j)|^2} \operatorname{Re}[z^{(1-d)j} N(z^j) V(z^{-j})] \qquad \text{with} \quad K = 2k \quad \text{or} \quad 2k+1. \tag{69}$$

(1) If $k + 1 \le n^{**} + d^{**} + 2$, (g) is obviously satisfied if the vectors $\operatorname{Re}[z^{(1-d)j} N(z^j) V(z^{-j})]$ are linearly independent. This is true if some determinant, which is a polynomial in the coefficients of N and D, is not zero. This polynomial is not identically zero since $\operatorname{Re}[z^{(1-d)j} N(z^j) V(z^{-j})]$ *are* linearly independent for $N = 1$ and $D = X^{d^{**}}$. Let H_d be the nontrivial algebraic hypersurface where it vanishes.

(2) If $k + 1 \ge n^{**} + d^{**} + 3$, let H_d be the nontrivial algebraic hypersurface where the $n^{**} + d^{**} + 2$ first $\operatorname{Re}[z^{(1-d)j} N(z^j) V(z^{-j})]$ are linearly independent. Since (69) (or the sum in (g)) is homogeneous with respect to θ, we do not change the meaning of (g) by compelling θ to be on the unit sphere of $\mathbb{R}^{n^{**}+d^{**}+2}$ (denoted $S^{n^{**}+d^{**}+1}$). Hence, the set of the (N, D, y^M) for which (g) fails is exactly $\pi(J)$ where π is the canonical projection from $\mathbb{R}^{n^*+d^*+K+2} \times S^{n^{**}+d^{**}+1}$ onto $\mathbb{R}^{n^*+d^*+K+2}$ and

$$J = \{(N, D, y^M, \theta) \in \mathbb{R}^{n^*+d^*+K+2} \times S^{n^{**}+d^{**}+1} / (69) = 0\}.$$

Now, with

$$\begin{aligned} J_1 = \{(N, D, y^M, \theta) \in \mathbb{R}^{n^*+d^*+K+2} \times S^{n^{**}+d^{**}+1} / (N, D, y^M) \notin H_d \text{ and} \\ (N, D, y^M) \notin H_e \text{ and } (69) = 0\}, \end{aligned}$$

we have, clearly,

$$\pi(J) \subset \pi(J_1) \cup H_d.$$

We have proved that the points where (e), (f), (g), or (d) fail are all in

$$H_b \cup H_c \cup H_d \cup H_e \cup \pi(J_1),$$

where $\pi(J_1)$ is empty in the case when $k + 1 \le n^{**} + d^{**} + 2$. The H_b, H_c, H_d, and H_e are nontrivial algebraic hypersurfaces, i.e., they are the set of the zeros of a nonidentically zero polynomial; it is clear that their measure is zero and that they are closed with an empty interior. We end by proving, if $k + 1 \le n^{**} + d^{**} + 3$, that $\pi(J_1)$ has measure zero and has a closure with an empty interior:

(1) $\pi(J_1)$ has measure zero because it is the projection of J_1 onto $\mathbb{R}^{n^*+d^*+K+2}$ and

J_1 is an $(n^* + d^* + K + 1)$-dimensional manifold. Proof of this fact: J_1 being entirely outside H_d, any (N, D, y^M) in J_1 is such that the $n^{**} + d^{**} + 2$ first $\mathrm{Re}[z^{(1-d)j}N(z^j)V(z^{-j})]$ are independent; therefore (69) $= 0$ gives the $n^{**} + d^{**} + 2$ first $|\hat{y}_j^M|$ as a function of the coefficients of N, D, and the other $|\hat{y}_j^M|$; these $n^{**} + d^{**} + 2$ independent nondegenerate (because we are outside H_e) equations give J_1 as a submanifold of $\mathbb{R}^{n^*+d^*+K+2} \times S^{n^{**}+d^{**}+1}$ with codimension $n^{**} + d^{**} + 2$, i.e., with dimension $n^* + d^* + K + 1$.

(2) Let (N_p, D_p, y_p^M) be a sequence in $\pi(J_1)$ converging to $(N_\infty, D_\infty, y_\infty^M)$ in $\mathbb{R}^{n^*+d^*+K+2}$. There is a sequence (θ_p) such that $(N_p, D_p, y_p^M, \theta_p)$ is in J_1. Since θ_p remains in $S^{n^{**}+d^{**}+1}$, we may assume that θ_p converges to one θ_∞. If no $\theta_\infty^T V_\infty(z^j)$ is zero, then, by continuity, $(N_\infty, D_\infty, y_\infty^M, \theta_\infty)$ is a zero of (69) so that $(N_\infty, D_\infty, y_\infty^M)$ is in $\pi(J_1)$. If some $\theta_\infty^T V_\infty(z^j)$ are zero, $(N_\infty, D_\infty, y_\infty^M)$ must be in H_b, H_c, or H_e because if not, from (e) and (d), the terms in (69), in which the denominator tends to zero, do tend to infinity when $p \to \infty$ because the numerator does not tend to zero and, from (f) and Lemma 10, the $\mathrm{Re}[z^{(1-d)j}N_\infty(z^j)V_\infty(z^{-j})]$ such that $\theta_\infty^T V_\infty(z^j) = 0$ are linearly independent: the component on each of these $\mathrm{Re}[z^{(1-d)j}N_\infty(z^j)V_\infty(z^{-j})]$ would tend to infinity, which is impossible since (69) is zero for any p. We have proved that

$$\overline{\pi(J_1)} = J_1 \cup H_b \cup H_c \cup H_e.$$

Hence $\overline{\pi(J_1)}$ has measure zero and, consequently, its interior is empty. ∎

Explanation of the End of the Example in Section 2.5. We have only to prove that expression (28) is a nonzero polynomial. It is nonzero because when, for instance $D = X^2 + 1$, $N = X^2$, and $y^M(t) = z^t + z^{-t}$, then (28)'s value is $2\sin^2(4\pi/K)$.

Appendix C. The Key Technical Lemma

We use $(\cdot|\cdot)$ to denote the inner product on $\mathbb{R}^p$: $(x|y) = x^T y$. We consider:

$(u_i)_{i\in\{1\ldots N\}}$, a family of real p-vectors,
$(Q_i)_{i\in\{1\ldots N\}}$, a family of real positive symmetric $p \times p$ matrices, and
$(\lambda_i)_{i\in\{1\ldots N\}}$, a family of real numbers.

We define the map E from $\mathbb{R}^p - \bigcup_1^N \mathrm{Ker}\, Q_i$ to $\mathbb{R}^p$ and the set $G(r, R, \eta)$ (for $R > 0$, $r > 0$, and $\eta > 0$) by

$$E(x) = \sum_1^N \lambda_i \frac{u_i - Q_i x}{(x|Q_i x)}, \tag{70}$$

$$G(r, R, \eta) = \{x \in \mathbb{R}^p | (\text{for all } i \in \{1, \ldots, N\})\ r < \|x\| < R \text{ and } \|Q_i x\| > \eta\}. \tag{71}$$

All the results of Section 2 rely upon the following lemma.

Lemma 11. *Under assumptions* (H_1)–(H_5), *the degree of the map* E *on* $G(r, R, \eta)$ *is odd (hence nonzero) for* r *and* η *small enough and* R *large enough. In particular,* E *has (at least) one zero.*

Assumptions (H_1)–(H_5) are defined below:

(H_1) For all $i \in \{1 \dots N\}$, $u_i \in \operatorname{Im} Q_i$.
(H_2) $\sum_{i=1}^N Q_i$ is nonsingular.
(H_3) For all $i \in \{1, \dots, N\}$, $\lambda_i > 0$.
(H_4) For all $y \in (\mathbb{R}^p - \bigcup_1^N \operatorname{Ker} Q_i)$, $\sum_1^N \lambda_i(u_i/(y|Q_i y)) \neq 0$.
(H_5) For all $I \subset \{1, \dots, N\}$, $[\sum_{i \in I} Q_i$ is not invertible$]$ implies $[(u_i)_{i \in I}$ is a linearly independent family$]$.

To prove this lemma, we define, for $t > 0$, the maps

$$E_t(x) = \|x\|^4 \sum_1^N \lambda_i \frac{u_i - (Q_i x + tx)}{(x|Q_i x) + t\|x\|^2}.$$

Note that, for $t \neq 0$, E_t is a C^1-map all over $\mathbb{R}^p$. In order to prove Lemma 11, we first establish the following result concerning the E_t maps:

Lemma 12. *There exist strictly positive real numbers r_0, R_0, η_0, and t_0 such that, for any $r < r_0$, $R > R_0$, $\eta < \eta_0$, $t < t_0$, and any t in $]0, t_0]$ and x in $\mathbb{R}^p$, we have*

$$E_t(x) = 0 \quad \textit{implies} \quad [x = 0 \text{ or } x \in G(r, R, \eta)].$$

Proof of Lemma 12. If the result was false, there would exist four sequences (r_n), (R_n), (t_n), and (η_n) in $]0, +\infty[$ and (x_n) in $\mathbb{R}^p - \{0\}$ satisfying the following six properties:

$$\left.\begin{array}{rl}
\text{(i)} & r_n \to 0,\\
\text{(ii)} & R_n \to +\infty,\\
\text{(iii)} & \eta_n \to 0,\\
\text{(iv)} & t_n \to 0,\\
\text{(v)} & (\text{for all } n \in \mathbb{N})\ E_{t_n}(x_n) = 0 \text{ and } x_n \neq 0,\\
\text{(vi)} & (\text{for all } n \in \mathbb{N})\ \|x_n\| \geq R_n \text{ or } \|x_n\| \leq r_n \text{ or (there exists } i \in \{1, \dots, N\}\\
& \text{such that } \|Q_i x_n\| \leq \eta_n).
\end{array}\right\} \tag{72}$$

Since $x_n \neq 0$, let

$$\rho_n = \|x_n\|, \qquad y_n = \frac{1}{\rho_n} x_n. \tag{73}$$

Eventually extracting from (x_n) a subsequence also denoted (x_n), (y_n) converges to one y with

$$y_n \to y, \qquad \|y\| = 1. \tag{74}$$

To show the contradiction, we assume (72)(i)–(v) to be true, and show that (72(vi) is then necessarily false, i.e., under assumptions (72)(i)–(v), eventually extracting a subsequence from (x_n), we have, for any n and i, $\|x_n\| < R_n$ and $\|x_n\| > r_n$ and $\|Q_i x_n\| > \eta_n$. This is done in two steps:

Step 1. *Conditions* (72)(i)–(v) *imply that* (x_n) *is bounded.* If it were unbounded, a subsequence of it (denoted (x_n) too) would satisfy

$$\rho_n \to +\infty. \tag{75}$$

Then, from (70) and (73), we would get

$$E_{t_n}(x_n) = \rho_n^3 \sum_{i=1}^{N} \lambda_i \frac{\rho_n^{-1}u_i - Q_i y_n - t_n y_n}{(y_n|Q_i y_n) + t_n}.$$

Let us divide by ρ_n^3, do the scalar product with y, and separate the sum into the terms such that $Q_i y \neq 0$ and those such that $Q_i y = 0$. When $Q_i y = 0$, we have $(u_i|y) = (y|Q_i y_n) = 0$ because (from (H_1)) Q_i is symetric and $u_i \in \operatorname{Im} Q_i$. We would finally get

$$\frac{(E_{t_n}(x_n)|y)}{\rho_n^3} = \sum_{Q_i y \neq 0} \lambda_i \frac{\rho_n^{-1}(y|u_i) - (y|Q_i y_n) - t_n(y|y_n)}{(y_n|Q_i y_n) + t_n} - \sum_{Q_i y = 0} \lambda_i \frac{t_n(y|y_n)}{t_n + (y_n|Q_i y_n)}.$$

On the other hand, from (72)(v),

$$E_{t_n}(x_n) = 0,$$

so that we would have

$$\text{(for all } n \in \mathbb{N}) \qquad \sum_{Q_i y \neq 0} \lambda_i \frac{\rho_n^{-1}(y|u_i) - (y|Q_i y_n) - t_n(y|y_n)}{(y_n|Q_i y_n) + t_n} = \sum_{Q_i y = 0} \lambda_i \frac{t_n(y|y_n)}{t_n + (y_n|Q_i y_n)}. \tag{76}$$

Now, in (76), the left-hand term tends to $-\sum_{Q_i y \neq 0} \lambda_i$, which, considering (H_2) and (H_3), is negative and nonzero; besides, the right-hand term is positive when n is large because t_n and $(y_n|Q_i y_n)$ are always positive and $(y|y_n) \to 1$. Therefore (76) cannot be satisfied for n large. We have proved that (x_n) is bounded.

Step 2. With (x_n) *being bounded, conditions* (72)(i)–(v) *imply that* (x_n) *(or a subsequence of it) has a limit* x *such that* $Q_i x \neq 0$ *for all* i. With $\rho_n = \|x_n\|$ being bounded, we have, may be only for a subsequence of (x_n),

$$\rho_n \to \rho \geq 0, \qquad \text{hence,} \quad x_n \to x \quad \text{with} \quad x = \rho y.$$

From (70) and (73), we get

$$\begin{aligned} \rho_n^{-2} E_{t_n}(x_n) &= \sum_{1}^{N} \lambda_i \frac{u_i - \rho_n(Q_i y_n + t_n y_n)}{t_n + (y_n|Q_i y_n)} \\ &= \sum_{Q_i y \neq 0} \lambda_i \frac{u_i - \rho_n(Q_i y_n + t_n y_n)}{t_n + (y_n|Q_i y_n)} + \sum_{Q_i y = 0} \lambda_i \frac{u_i}{t_n + (y_n|Q_i y_n)} \\ &\quad - \rho_n \sum_{Q_i y = 0} \lambda_i \frac{Q_i y_n + t_n y_n}{t_n + (y_n|Q_i y_n)}. \end{aligned} \tag{77}$$

We notice about the first sum that

$$\sum_{Q_i y \neq 0} \lambda_i \frac{u_i - \rho_n(Q_i y_n + t_n y_n)}{t_n + (y_n|Q_i y_n)} \to \sum_{Q_i y \neq 0} \lambda_i \frac{u_i - \rho Q_i y}{(y|Q_i y)}, \tag{78}$$

thus it is bounded.

We now consider two cases:

Case 1. If one $Q_i y$ at least is zero, the last two sums are not empty, but, from (H_5), the u_i such that $Q_i y = 0$ form an independent set; hence there exists a positive number C such that

$$\left\| \sum_{Q_i y = 0} \lambda_i \frac{u_i}{t_n + (y_n | Q_i y_n)} \right\|^2 > C \sum_{Q_i y = 0} \lambda_i^2 \frac{1}{[t_n + (y_n | Q_i y_n)]^2} \to +\infty. \tag{79}$$

Furthermore, about the third sum:

$$\begin{aligned}
&\left\| \rho_n \sum_{Q_i y = 0} \lambda_i \frac{Q_i y_n + t_n y_n}{t_n + (y_n | Q_i y_n)} \right\|^2 \\
&\quad \le \rho_n^2 \sum_{Q_i y = 0} \lambda_i^2 \frac{\|Q_i y_n + t_n y_n\|^2}{[t_n + (y_n | Q_i y_n)]^2} \\
&\quad \le \left(\sup_n \rho_n^2\right) \sup_{Q_i y = 0} \|Q_i y_n + t_n y_n\|^2 \sum_{Q_i y = 0} \lambda_i^2 \frac{1}{[t_n + (y_n | Q_i y_n)]^2}
\end{aligned}$$

but all the $\|Q_i y_n + t_n y_n\|^2$ such that $Q_i y = 0$ tend to 0, so that

$$\left\| \rho_n \sum_{Q_i y = 0} \lambda_i \frac{Q_i y_n + t_n y_n}{t_n + (y_n | Q_i y_n)} \right\|^2 = \varepsilon_n \left(\sum_{Q_i y = 0} \lambda_i^2 \frac{1}{[t_n + (y_n | Q_i y_n)]^2} \right) \quad \text{with} \quad \varepsilon_n \to 0, \tag{80}$$

then, from (77), (78), (79), and (80), we get

$$\|\rho_n^{-2} E_{t_n}(x_n)\| \to +\infty$$

but this is impossible because (72)(v) implies $E_{t_n}(x_n) = 0$.

Case 2. If no $Q_i y$ is zero, then the last two sums in (77) are empty and x is nonzero, because if it was zero ($\rho = 0$), we would have, from (77) and (78),

$$\rho_n^{-2} E_{t_n}(x_n) \to \sum_1^N \lambda_i \frac{u_i}{(y | Q_i y)}$$

but, from (72)(v), $E_{t_n}(x_n) = 0$. From (H_4), this is impossible. We have proved that x_n tends to an x such that no $Q_i x$ is zero, this is step 2. ■

Proof of Lemma 11. Let r_0, R_0, η_0, and t_0 be given by Lemma 12.

Step 1. It is clear that, if $r < r_0$, $R > R_0$, and $\eta < \eta_0$, the map

$$[0, 1] \times G(r, R, \eta) \to \mathbb{R}^p,$$

$$(s, x) \mapsto \|x\|^{4s} E(x)$$

is a smooth homotopy joining E restricted to $G(r, R, \eta)$ to E_0 restricted to $G(r, R, \eta)$ never vanishing for $x \in \partial G(r, R, \eta)$, so that (from Theorem 1.3.5(3) of [L2])

$$r < r_0, \quad R > R_0, \quad \eta < \eta_0 \quad \text{implies} \quad \deg(E, G(r, R, \eta), 0) = \deg(E_0, G(r, R, \eta), 0). \tag{81}$$

Similarily, from Lemma 12, if $r < r_0$, $R > R_0$, $\eta < \eta_0$, and $\varepsilon < t_0$, the map

$$[0, \varepsilon] \times G(r, R, \eta) \to \mathbb{R}^p,$$
$$(t, x) \mapsto E_t(x)$$

is a smooth homotopy allowing us to state (again from Theorem 1.3.5(3) of [L2])

$$\deg(E_0, G(r, R, \eta), 0) = \deg(E_\varepsilon, G(r, R, \eta), 0)$$

and, considering (81),

$$r < r_0, R > R_0, \eta < \eta_0, \varepsilon < t_0 \text{ implies } \deg(E, G(r, R, \eta), 0) = \deg(E_\varepsilon, G(r, R, \eta), 0). \tag{82}$$

Step 2. We need to compute $\deg(E_\varepsilon, G(r, R, \eta), 0)$. As it is easier to study $E(x)$ when x is large or close to 0, we first establish (83) according to which this degree depends only on the values of E for x large or close to 0.

Defining $\mathscr{B}_{\mathbb{R}^p}(0, R)$ as the ball in $\mathbb{R}^p$ with center 0 and radius R, it is clear that $G(r, R, \eta)$ is included in it, and, more precisely, that $\mathscr{B}_{\mathbb{R}^p}(0, R)$ is in fact the union of three disjoint open sets and their boundary: $G(r, R, \eta)$, $\mathscr{B}_{\mathbb{R}^p}(0, r)$, and $\bigcup_1^N \{x | r < \|x\| < R \text{ and } \|Q_i x\| < \eta\}$, so that, from Theorem 2.2.1(1) of [L2],

$$\begin{aligned}\deg(E_\varepsilon, \mathscr{B}_{\mathbb{R}^p}(0, R), 0) &= \deg(E_\varepsilon, G(r, R, \eta), 0) + \deg(E_\varepsilon, \mathscr{B}_{\mathbb{R}^p}(0, r), 0) \\ &\quad + \deg\left(E_\varepsilon, \bigcup_1^N \{x | \|x\| > r \text{ and } \|Q_i x\| < \eta\}, 0\right).\end{aligned}$$

But, from Lemma 12, the zeros of E_ε in $\mathscr{B}_{\mathbb{R}^p}(0, R)$ are all in $G(r, R, \eta)$ except 0, so that E_ε never vanishes in $\bigcup_1^N \{x | r < \|x\| < R \text{ and } \|Q_i x\| < \eta\}$. Hence the degree of E_ε with respect to this set is zero and

$$\deg(E_\varepsilon, G(r, R, \eta), 0) = \deg(E_\varepsilon, \mathscr{B}_{\mathbb{R}^p}(0, R), 0) - \deg(E_\varepsilon, \mathscr{B}_{\mathbb{R}^p}(0, r), 0). \tag{83}$$

Step 3. We notice (from (70)) that

$$(x | E_\varepsilon(x)) = \|x\|^4 \left(-\sum_1^N \lambda_i + \sum_1^N \lambda_i \frac{(u_i | x)}{(x | Q_i x) + \varepsilon \|x\|^2} \right)$$

but

$$\left\| \sum_1^N \lambda_i \frac{(u_i | x)}{(x | Q_i x) + \varepsilon \|x\|^2} \right\| \leq \frac{1}{\varepsilon \|x\|} \sum_1^N \lambda_i \|u_i\|.$$

So that, if $\varepsilon R \sum_1^N \lambda_i > 2 \sum_1^N \lambda_i \|u_i\|$, we can be sure that

$$x \in \partial \mathscr{B}_{\mathbb{R}^p}(0, R) \quad \text{implies} \quad (x | E_\varepsilon(x)) < 0.$$

This means that E_ε is pointing inward all along the R-sphere, so

$$(t, x) \mapsto -tx + (1 - t) E_\varepsilon(x)$$

is a smooth homotopy on $\mathscr{B}_{\mathbb{R}^p}(0, R)$ joining E_ε to $-I$ never vanishing on the

R-sphere, and, therefore, the degree of $-I$ being $(-1)^p$,

$$\varepsilon R > \frac{2\sum_1^N \lambda_i \|u_i\|}{\sum_1^N \lambda_i} \quad \text{implies} \quad \deg(E_\varepsilon, \mathscr{B}_{\mathbb{R}^p}(0, R), 0) = (-1)^p. \tag{84}$$

Step 4. We consider the smooth homotopy:

$$[0, 1] \times \mathscr{B}_{\mathbb{R}^p}(0, r) \to \mathbb{R}^p,$$

$$(s, x) \mapsto E_{\varepsilon,s}(x) = \|x\|^4 \sum_1^N \lambda_i \frac{u_i - s(Q_i x + \varepsilon x)}{(x|Q_i x) + \varepsilon\|x\|^2}.$$

From (H_4) and (H_5), we may derive the existence of two positive numbers ε_1 and C_1 such that

$$|\varepsilon| < \varepsilon_1 \qquad \text{if and only if} \quad \left\| \sum_1^N \lambda_i \frac{u_i}{(x|Q_i x) + \varepsilon\|x\|^2} \right\| \geq \frac{C_1}{\|x\|^2}$$

because, if this were wrong, there would be two sequences (x_n) and (ε_n) such that $\varepsilon_n \to 0$, $\|x_n\| = 1$ (because of the formula's homogeneity), and the left-hand term tends to zero as $n \to \infty$. Point $\bar{x}$ being an accumulation point of (x_n), this would contradict (H_4) if $\bar{x}$ belonged to no Ker Q_i and (H_5) if $\bar{x}$ belonged to some.

On the other hand,

$$\left\| \sum_1^N \lambda_i \frac{Q_i x + \varepsilon x}{(x|Q_i x) + \varepsilon\|x\|^2} \right\| \leq \frac{C_2}{\|x\|} \qquad \text{for some} \quad C_2 > 0.$$

Hence, choosing $r < C_2/2C_1 := r_1$, we have

$$x \in \mathscr{B}_{\mathbb{R}^n}(0, r) \quad \text{implies} \quad \|E_{\varepsilon,x}(x)\| > \frac{C_1}{2}\|x\|^2.$$

This proves that $E_{\varepsilon,s}$ never vanishes on $\partial\mathscr{B}_{\mathbb{R}^p}(0, r)$ so that, from Theorem 1.3.5(3) of [L2],

$$\deg(E_{\varepsilon,0}, \mathscr{B}_{\mathbb{R}^p}(0, r), 0) = \deg(E_{\varepsilon,1}, \mathscr{B}_{\mathbb{R}^p}(0, r), 0) \qquad \text{if} \quad r < r_1 \quad \text{and} \quad \varepsilon < \varepsilon_1,$$

but $E_{\varepsilon,1} = E_\varepsilon$, and, since $E_{\varepsilon,0}$ is even, its degree is even. It follows:

$$\text{if} \quad r < r_1 \quad \text{and} \quad \varepsilon < \varepsilon_1, \qquad \deg(E_\varepsilon, \mathscr{B}_{\mathbb{R}^p}(0, r), 0) \equiv 0 \quad [\text{mod } 2]. \tag{85}$$

Step 5. From (82)–(85) we get

$$\left.\begin{array}{l} r < \min(r_0, r_1) \\ \eta < \eta_0 \\ \varepsilon < \min(\varepsilon_0, \varepsilon_1) \\ R > \max\left(R_0, \dfrac{2\sum_1^N \lambda_i \|u_i\|}{\varepsilon \sum_1^N \lambda_i}\right) \end{array}\right\} \quad \text{implies} \quad \deg(E, G(r, R, \eta), 0) \equiv 1 \quad [\text{mod } 2].$$

Consequently, this degree is nonzero, and E, given by (70) has at least one zero inside $G(r, R, \eta)$. ■

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