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Almost exact modelling assumption in adaptive linear control

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Modern adaptive controllers are known to give bounded solutions when the system 'normalized' unmodelled effects are bounded by a small constant. This paper studies this unusual characterization of uncertainties. We show it encompasses more classical approaches. We discuss how feedback and by-passing may allow this assumption to be satisfied. We conclude by proposing the notion of almost exactly linearly modelled systems.

1. Introduction

Modern adaptive controllers, in particular those incorporating so-called normalization, guarantee boundedness (but possibly not stability) of all the solutions when placed in feedback with a physical process for which the main sufficient property is as follows.

Assumption A

Let n_A, n_B, μ, γ be two integers and two positive real numbers, respectively, obtained from the adaptive controller, with γ strictly positive and μ strictly smaller than one. The physical process input–output signals satisfy

$$\left.\begin{aligned} |A(q^{-1})y(t) - B(q^{-1})u(t-1)| &\leqslant \gamma s(t) + \beta(t) \\ s(t)^2 &= \mu^2 s(t-1)^2 + u(t-1)^2 + y(t-1)^2 \end{aligned}\right\} \tag{1}$$

where β is a bounded sequence and A, B are polynomials in the unit delay operator q^{-1} whose coefficients, in $\mathbb{R}^{n_A}$, $\mathbb{R}^{n_B}$ respectively, are arbitrary but lie in a compact set given by the controller.

Assumption A is unusual compared with those used in linear system theory, such as singular perturbations or norm-restricted multiplicative or additive uncertainties, for example. The topic of this report is to discuss the meaning of this assumption, to study what class of physical processes can be represented this way and to see how it is related to more standard approaches.

The main aspect of Assumption A is the fact that it involves two components available to the designer:

— the collection of the input–output signals (u, y) which is the only way the physical process is known,

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— the model (A, B), element of a parametrized family of mathematical dynamical systems.

This parametrized family of dynamical systems is chosen for providing an adequate description of the physical process without being excessively complicated. However, this limited complexity and the idealization of representing a physical process by a mathematical dynamic system lead to inability to explain exactly the output signals from the input signals. Consequently, approximation is involved which motivates Assumption A. This assumption is meant to qualify, if not to quantify, the allowable misfit between this limited complexity idealization and the physical process, as observed from its input–output signals.

A fundamental factor affecting this misfit is the way the input–output signals we are trying to relate are measured. For instance, if the mean values are the only measurements, a (possibly non-linear) gain will be sufficient to explain the input–output relation. Let u_p, y_p be the physical process input and output sequences, respectively. From these 'primitive' signals, we define measurements z as the signals given by

$$U_z(q^{-1})z(t) = V_z(q^{-1})y_p(t) - W_z(q^{-1})u_p(t) \tag{2}$$

where U_z, V_z, W_z are respectively prime polynomials, U_z is monic (i.e. $\lim_{z\to\infty} U_z(z^{-1}) = 1$) and 'exponentially stable'. In other words, z is the output of an exponentially stable completely reachable finite dimensional time-invariant linear system with the physical process input–output signals as inputs. In particular, we define a measured input sequence u, by

$$U_u(q^{-1})u(t) = V_u(q^{-1})y_p(t) - W_u(q^{-1})u_p(t) \tag{3}$$

where, in this case, W_u is monic to allow the computation of u_p by

$$W_u(q^{-1})u_p(t) = V_u(q^{-1})y_p(t) - U_u(q^{-1})u(t) \tag{4}$$

Similarly, we define a measured output sequence y by

$$U_y(q^{-1})y(t) = V_y(q^{-1})y_p(t) - W_y(q^{-1})u_p(t-1) \tag{5}$$

Note the delay in u_p. The roles of (U_u, V_u, W_u) and (U_y, V_y, W_y) will be defined by studying the misfit between the process limited complexity idealization and the physical process itself.

In § 2, we define the model, our process limited complexity idealization. In § 3, we give a mathematical description of a physical process candidate for satisfying Assumption A. In § 4, we propose a measurement system aimed at making the measured process satisfy Assumption A effectively. Then § 5 summarizes our results by introducing the notion of almost exactly modelled processes and by giving their properties. Finally, we give in § 6 some notes and references related to our topic.

2. Model

The model is an idealization of the physical input–output relation process. It is used for the design of the adaptive control law and for the evaluation of the 'ideal' closed-loop system behaviour. We have mentioned our choice of the model as an element of a parametrized family of mathematical dynamical systems. However, both the parameter fitting problem and the control law design impose a limited complexity system. The model is consciously only an approximate process description. Typically for realizing the compromise between admissible complexity and better approximation, we prefer a model of a pragmatic mathematical nature motivated by the input–output relation representation more than a deduction from known basic physical laws motivated by the description of the mechanisms involved in this relation. Even more, in the adaptive control context the input–output relation needs only to be represented as far as it is sufficient for meeting the control objective at each point in time.

Typically, the model is a discrete-time linear time-invariant finite-dimensional system, leading to an adaptive linear controller. Noticing that unobservable modes do not modify the input–output signals, complete observability can be assumed. There are many equivalent ways to represent such a system. With the complexity minimality requirement, we prefer canonical forms among which the more convenient for our specific application happens to be the following.

2.1. *Backward shift operator observable representation*

With u, y the measured input and output signals, respectively, we describe the model by

$$A(q^{-1})y(t) = B(q^{-1})u(t-1) + C(q^{-1})v(t) \tag{6}$$

where A, B, C are polynomials in q^{-1}, A, C being monic and C exponentially stable. We use $u(t-1)$ instead of $u(t)$ to express the necessary delay present somewhere in the closed-loop system. The model family of dynamical systems among which we are looking for our model is completely determined by choosing n_A, n_B, n_C the degrees of A, B, C respectively. Within this family, a model is obtained by (possibly implicitly) choosing the polynomial coefficients. v is an extra sequence needed fully to explain the measured output y from the measured input u. Namely $C(q^{-1})v(t)$ is the part of $y(t)$ which cannot be explained only from knowledge of $\{y(\tau), u(\tau), \tau < t\}$. In fact, the correct definition of v is as follows.

Given the model polynomial A, B, C on the one hand and the measured input–output signals on the other, v is defined by

$$C(q^{-1})v(t) = A(q^{-1})y(t) - B(q^{-1})u(t-1) \tag{7}$$

Consequently v depends on the model and is called the modelling error. We can think of our model as being a good model if all the meaningful information of the input–output relation has been extracted. This means that knowledge of $\{y(\tau), u(\tau), v(\tau), \tau < t\}$ should give no information on the actual value $v(t)$. We say (in a very loose sense) that v is unpredictable. We could appeal to the stochastic framework to define this notion (see Goodwin and Sin 1984): v is said to be unpredictable if v is a sequence of integrable random variables on a probability space such that if $\mathsf{F}(t)$ is the increasing sequence of sub-σ-fields generated by $\{v(\tau), u(\tau), y(\tau), \tau \leqslant t\}$, we have

$$E(v(t)/\mathsf{F}(t-1)) = 0 \quad \text{a.s.} \tag{8}$$

$$E(v(t)^2/\mathsf{F}(t-1)) < +\infty \quad \text{a.s.} \tag{9}$$

Unfortunately, in practice, the approximation inherent with any modelling implies the inability of reaching this absolute property of unpredictability. To be more pragmatic, it is sufficient to define a property related to an objective and to express the idea that this particular objective can still be achieved. Assumption A in the

Introduction has been proved to be sufficient for replacing this unpredictability property as far as the boundedness problem is concerned. However, it is known to be insufficient for more specific performance problems.

3. Process

To allow a mathematical description to encompass more physical phenomena, weaker structures must be used. Typically, inequalities replace equalities. Looking for a wider class of physical process candidate for satisfying Assumption A, we consider linearly dominated systems characterized as follows.

Definition 1

Let μ be a given positive constant strictly smaller than one; a process with input u_p, output y_p is said to be a *linearly dominated system* if y_p can be scaled, namely if there exist a bounded sequence β, depending on the initial conditions, and a positive constant γ such that for any input sequence u_p, we have at each time t

$$|y_p(t)| \leqslant \gamma s_p(t) + \beta(t) \tag{10}$$

where s_p is defined by

$$\begin{aligned} s_p(t)^2 &= \sum_{i=0}^{t-1} \mu^{2(t-1-i)}(|y_p(i)|^2 + |u_p(i)|^2) \\ &= \mu^2 s_p(t-1)^2 + (|y_p(t-1)|^2 + |u_p(t-1)|^2) \end{aligned} \tag{11}$$

Inequality (10) expresses that the output at time t can be bounded in terms of the past inputs and outputs weighted by a forgetting factor. Formally, the square of the process output is dominated by the output of

$$\frac{\gamma^2 q^{-1}}{1 + (\mu^2 - \gamma^2)q^{-1}}$$

with the square of the process input as input. This process representation is unusual, but it has been shown to be well adapted to our topic. To get a better grip of this definition, let us state several properties.

Property 1

Any discrete-time linear time-invariant finite-dimensional system is a linear dominated system for any μ, $0 < \mu < 1$, where the corresponding β sequence can be chosen to be μ-exponentially decaying.

Proof

Choose any μ, $0 < \mu < 1$. From the canonical structure theorem, this system can be represented in the following state-space form:

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ x_3(t+1) \end{bmatrix} = \begin{bmatrix} A_{11} & 0 & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} u_p(t) \tag{12}$$

$$y_p(t) = (0 \quad 0 \quad C_3)\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \tag{13}$$

where the pair (A_{33}, C_3) is completely observable and the eigenvalues of A_{22} are in the open disc of radius μ. From the complete observability property, we can find K such that the eigenvalues of $(A_{33} - KC_3)$ are in the open disc of radius μ. Then let us consider the following non-minimal representation of (12), (13) with $\hat{x}_3(0) = 0$:

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ x_3(t+1) - \hat{x}_3(t+1) \\ \hat{x}_3(t+1) \end{bmatrix} = \begin{bmatrix} A_{11} & 0 & A_{13} & A_{13} \\ 0 & A_{22} & A_{23} & A_{23} \\ 0 & 0 & A_{33} - KC_3 & 0 \\ 0 & 0 & 0 & A_{33} - KC_3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) - \hat{x}_3(t) \\ \hat{x}_3(t) \end{bmatrix} + \begin{bmatrix} B_1 & 0 \\ B_2 & 0 \\ 0 & 0 \\ B_3 & K \end{bmatrix} \begin{bmatrix} u_p(t) \\ y_p(t) \end{bmatrix} \tag{14}$$

With the properties of A_{22}, $A_{33} - KC_3$, there exist positive constants α, λ with $\lambda < \mu$, such that, with $\|\cdot\|$ denoting the euclidan norm,

$$\begin{bmatrix} A_{22} & A_{23} & A_{23} \\ 0 & A_{33} - KC_3 & 0 \\ 0 & 0 & A_{33} - KC_3 \end{bmatrix}^n \leqslant \alpha\lambda^n, \quad \forall n \geqslant 0 \tag{15}$$

Hence applying the variation of constants formula and taking the euclidean norm, we obtain, for some positive constants γ_1, β_1,

$$\left\| \begin{matrix} x_2(t) \\ x_3(t) \end{matrix} \right\| \leqslant \beta_1 \lambda^t \left\| \begin{matrix} x_2(0) \\ x_3(0) \end{matrix} \right\| + \gamma_1 \sum_{n=0}^{t-1} \lambda^{t-1-n} (|u_p(n)|^2 + |y_p(n)|^2)^{1/2} \tag{16}$$

With the Cauchy–Schwarz inequality, we obtain:

$$\sum_{n=0}^{t-1} \left(\frac{\lambda}{\mu}\right)^{t-1-n} \mu^{t-1-n} (|u_p(n)|^2 + |y_p(n)|^2)^{1/2}$$

$$\leqslant \left[\sum_{n=0}^{t-1} \left(\frac{\lambda}{\mu}\right)^{2(t-1-n)}\right]^{1/2} \left[\sum_{n=0}^{t-1} \mu^{2(t-1-n)} (|u_p(n)|^2 + y_p(n)|^2)\right]^{1/2} \tag{17}$$

$$\leqslant \frac{\mu}{\sqrt{\mu^2 - \lambda^2}} s_p(t) \tag{18}$$

The result follows from (13). □

Remark 1

In the course of proving this property, we have established that the state components, which are either in the observable subspace or in the unobservable subspace but with an 'unobservable' pole strictly smaller than μ, can be scaled by s_p:

$$\left\| \begin{matrix} x_2(t) \\ x_3(t) \end{matrix} \right\| \leqslant \gamma s_p(t) + \beta(t) \tag{19}$$

For this reason, we have the following definition.

Definition 2

We call $s_p(t)$ a *scaling signal* for the system.

Its main property is its availability from the process input–output signals. As soon as μ is given, we know how to compute $s_p(t)$ and therefore have the possibility of scaling all the signals in the process. In order to prove that this scaling property extends to measurements given by time-varying systems, let us prove the following lemma.

Lemma 1

Let w be a sequence scaled by s_p; let $C(t)$ be a monic and exponentially μ-stable time-varying polynomial. The sequence v defined by

$$C(t, q^{-1})v(t) = w(t) \tag{20}$$

is scaled by s_p with a sequence $\beta\mu$-exponentially decaying if the same holds for w.

Proof

Since $C(t)$ is monic and exponentially μ-stable, there exist positive constants β_1 (depending on the initial conditions), γ_1, λ_1 and λ, $\lambda_1 < \lambda < \mu$, such that, for any sequence w:

$$|v(t)| \leqslant \beta_1 \lambda_1^t + \gamma_1 \sum_{i=0}^{t} \lambda_1^{t-i} |w(i)| \tag{21}$$

However, since w can be scaled by s_p, there exist γ and a bounded sequence β such that:

$$|v(t)| \leqslant \beta_1 \lambda_1^t + \gamma_1 \sum_{i=0}^{t} \lambda_1^{t-i} (\gamma s_p(i) + \beta(i)) \tag{22}$$

The conclusion follows since, $\mu^{-2t} s_p(t)^2$ being an increasing sequence, the Cauchy–Schwarz inequality yields:

$$\sum_{i=0}^{t} \lambda_1^{t-i} s_p(i) \leqslant \left[\sum_{i=0}^{t} \left(\frac{\lambda_1}{\lambda} \right)^{2(t-i)} \right]^{1/2} \mu^t \left[\sum_{i=0}^{t} \frac{\lambda^{2(t-i)}}{\mu^{2(t-i)}} \mu^{-2i} s_p(i)^2 \right]^{1/2} \tag{23}$$

$$\leqslant \frac{\lambda\mu}{(\lambda^2 - \lambda_1^2)^{1/2} (\mu^2 - \lambda^2)^{1/2}} s_p(t) \tag{24}$$

and

$$\mu^{-t} \sum_{i=0}^{t} \lambda_1^{t-i} \beta(i) \leqslant \sup_i \left\{ \frac{\beta(i)}{\mu^i} \right\} \sum_{i=0}^{t} \left[\frac{\lambda_1}{\mu} \right]^{t-i} \leqslant \frac{\mu}{\mu - \lambda_1} \operatorname*{Sup}_i \frac{\beta(i)}{\mu^i} \Big\} \tag{25}$$

□

Property 2

Let u_p, y_p be respectively the input and output of a linearly dominated system; let $A_p(t)$, $B_p(t)$, $C_p(t)$ be any time varying polynomials with bounded coefficients, $C_p(t)$ monic and exponentially μ-stable. The measurement v defined by

$$C_p(t, q^{-1})v(t) = A_p(t, q^{-1})y_p(t) - B_p(t, q^{-1})u_p(t-1) \tag{26}$$

is scaled by s_p, namely, there exist some constant γ and bounded sequence β, independent of u_p, y_p, such that

$$|v(t)| \leqslant \gamma s_p(t) + \beta(t) \tag{27}$$

Moreover, β is a μ-exponentially decaying sequence if the same holds for the system.

Proof

Since u_p and y_p are scaled by s_p and the coefficients of $A_p(t)$, $B_p(t)$ are bounded, w defined by

$$w(t) = A_p(t, q^{-1})y_p(t) - B_p(t, q^{-1})u_p(t-1) \tag{28}$$

is scaled by s_p. The conclusion follows from Lemma 1. □

Property 3

For a process with u_p, y_p as input and output respectively, if there exist time-varying polynomials $A_p(t)$, $B_p(t)$, $C_p(t)$ with bounded coefficients and $A_p(t)$ and $C_p(t)$ monic, such that the measurement v given by

$$C_p(t, q^{-1})v(t) = A_p(t, q^{-1})y_p(t) - B_p(t, q^{-1})u_p(t-1) \tag{29}$$

satisfies, for some constant γ some bounded sequence β and some μ $0 \leqslant \mu < 1$,

$$|v(t)| \leqslant \gamma s_p(t) + \beta(t) \tag{30}$$

then the process is a linearly dominated system.

Proof

Since $A_p(t)$ is monic, (29) can be rewritten as

$$y_p(t) = [q(1 - A_p(t, q^{-1}))]y_p(t-1) + B_p(t, q^{-1})u_p(t-1) + C_p(t, q^{-1})v(t) \tag{31}$$

which means that $y_p(t)$ is a 'finite' linear combination, with bounded coefficients, of terms which can be bounded in terms of $s_p(t)$. The conclusion readily follows. □

Remark 2

(i) With Properties 2 and 3, we see that if a particular measurement v_0, given by some triple $(A_{p0}(t), B_{p0}(t), C_{p0}(t))$ is scaled by s_p, then any measurement v given by $(A_p(t), B_p(t), C_p(t))$ is scaled by s_p if the coefficients of these time-varying polynomials are bounded and $C_p(t)$ is exponentially μ-stable.

(ii) One could propose an alternative definition of linearly dominated systems: 'A process is a linearly dominated system if one can find some time-varying polynomials $A_p(t)$, $B_p(t)$, $C_p(t)$ with bounded coefficients, $C_p(t)$ exponentially μ-stable and $A_p(t)$, $C_p(t)$ monic, such that for the measurement v given by

$$C_p(t, q^{-1})v(t) = A_p(t, q^{-1})y(t) - B_p(t, q^{-1})u_p(t-1) \tag{32}$$

there exist a constant γ and a bounded sequence β satisfying

$$|v(t)| \leqslant \gamma s_p(t) + \beta(t) \tag{33}$$

This definition is more attractive for the similarity of (32) and the model equation (6). Unfortunately it is in fact very ambiguous for the arbitraryness of the triple of polynomials $(A_p(t), B_p(t), C_p(t))$. Using one triple instead of another would simply change β and γ.

It is also worth noticing that though (32) and (6) can be similar, replacing the unpredictability property of v for the model, by inequality (33) for the process allows us to encompass many more effects. We illustrate this aspect by means of examples.

Example 1: *Bounded disturbance*

Let the process be described by

$$y_p(t) = -ay_p(t-1) + bu_p(t-2) + n(t), \quad |(n(t))| \leqslant \beta \tag{34}$$

Clearly (32), (33) are satisfied by choosing

$$\left.\begin{aligned} &A_p(q^{-1}) = 1 + aq^{-1}, \quad B_p(q^{-1}) = bq^{-1}, \quad C_p(q^{-1}) = 1 \\ &\mu > 0, \quad \gamma = 0, \quad v(t) = n(t) \end{aligned}\right\} \tag{35}$$

Also, we can check that we have a linearly dominated system. Indeed

$$s_p(t)^2 \geqslant |y_p(t-1)|^2 + \mu^2 |u_p(t-2)|^2 \tag{36}$$

and (10) is satisfied since

$$|y_p(t)|^2 \leqslant 3(a^2|y_p(t-1)|^2 + b^2|u_p(t-2)|^2 + |n(t)|^2) \tag{37}$$

$$|y_p(t)| \leqslant \sqrt{3}\,\text{Max}\,\{|a|, |b|\} s_p(t) + \beta\sqrt{3} \tag{38}$$

Note that β is a constant in this case.

Example 2: *Infinite dimensional system*

Let the process be described by

$$y_p(t) = -ay_p(t-1) + bu_p(t-2) + \sum_{i=3}^{t} (a_i y_p(t-i) + b_i u_p(t-i)) \tag{39}$$

where the infinite impulse responses a_i, b_i satisfy, for all i and with $\lambda < 1$

$$|a_i \lambda^{-i}| \leqslant \varepsilon, \quad |b_i \lambda^{-i}| \leqslant \varepsilon \tag{40}$$

Let us take

$$A_p(q^{-1}) = 1 + aq^{-1}, \quad B_p(q^{-1}) = bq^{-1}, \quad C_p(q^{-1}) = 1, \quad \lambda < \mu < 1 \tag{41}$$

we get

$$v(t) = \sum_{i=3}^{t} (a_i y_p(t-i) + b_i u_p(t-i)) \tag{42}$$

Hence:

$$|v(t)| \leqslant \varepsilon \sum_{i=3}^{t} \lambda^i (|y_p(t-i)| + |u_p(t-i)|) \tag{43}$$

$$\leqslant \varepsilon \left(\sum_{i=3}^{t} \left(\frac{\lambda}{\mu}\right)^{2i} \right)^{1/2} \left(2 \sum_{i=3}^{t} \mu^{2i} (|y_p(t-i)^2 + (u_p(t-i))|^2 \right)^{1/2} \tag{44}$$

where we have used the Cauchy–Schwarz inequality in the last step. It follows that (33) holds with

$$\gamma = \varepsilon\sqrt{2}\frac{1}{\sqrt{\mu^2 - \lambda^2}}, \quad \beta = 0 \tag{45}$$

Example 3: *Non-linearities*

Let the process be described by

$$y_p(t) = -ay_p(t-1) + \frac{bu_p(t-2)^3}{1 + u_p(t-2)^2} \tag{46}$$

We notice that the non-linear function

$$f(u) = \frac{u^3}{1+u^2} \tag{47}$$

is linearly dominated since

$$|f(u) - u| \leqslant \tfrac{1}{2} \tag{48}$$

This linear domination property of f implies that it belongs to the process. Taking:

$$A_p(q^{-1}) = 1 + aq^{-1}, \quad B_p(q^{-1}) = bq^{-1}, \quad C_p(q^{-1}) = 1 \tag{49}$$

yields:

$$v(t) = \frac{-bu_p(t-2)}{1 + u_p(t-2)^2} \tag{50}$$

Also, (33) holds with

$$\gamma = 0, \quad \beta = \frac{|b|}{2} \tag{51}$$

Another very important property of linearly dominated systems is that, when placed in feedback with a linearly dominated controller, the signals cannot grow faster than exponentially. Namely, we have the following property.

Property 4

Let u_p be the input of a linearly dominated system with output y_p and such that, with β_c a bounded sequence:

$$|u_p(t)| \leqslant \gamma_c(s_p(t) + |y_p(t)|) + \beta_c(t) \tag{52}$$

There exists a constant M and a bounded sequence α such that:

$$s_p(t) \leqslant Ms_p(t-1) + \alpha(t) \tag{53}$$

Remark 3

With arguments similar to those used in the proof of Property 3, we can see that (52) holds if the input u_p is given by

$$R(t, q^{-1})u_p(t) = -S(t, q^{-1})y_p(t) + T(t, q^{-1})u_m(t) \tag{54}$$

where $R(t)$, $S(t)$, $T(t)$ are time-varying polynomials with bounded coefficients, $R(t)$ is monic and u_m is a bounded set point sequence.

Proof

With (10), (11) and (52), we easily obtain

$$s_p(t)^2 \leqslant s_p(t-1)^2(\mu^2 + 2\gamma^2(1 + 3\gamma_c^2) + 3\gamma_c^2) + 2\beta(t)^2(1 + 3\gamma_c^2) + 3\beta_c(t)^2 \tag{55}$$

The conclusion follows taking the square root. □

Up to now, we have established that a linearly dominated system can incorporate a wide class of phenomena and is characterized by the existence of an (available) scaling signal. Let us now study linearly dominated systems in terms of graph topology, i.e. the weakest topology in which feedback exponential μ-stability is defined on open neighbourhoods of time-invariant linear systems and closed-loop transfer functions are continuous (see Vidyasagar 1985). Though the general scope of this report prevents us from pursuing this topic too far, this will allow us to relate the results, obtained for adaptive linear controllers, to those obtained for time-invariant linear controllers.

First we notice that, possibly up to a change of β in (33), one can add to $v(t)$ in (32) (with (A_p, B_p, C_p) time invariant), any μ-exponentially decaying sequence $\delta(t)$ satisfying

$$C_p(q^{-1})\delta(t) = 0 \tag{56}$$

This justifies rewriting (32) as

$$v(t) = P(q)y_p(t) - Q(q)u_p(t-1) \tag{57}$$

where P, Q are exponentially μ-stable proper fractions, P being monic. This type of representation in terms of stable fractions is called the factorization approach and gives the context in which one can define the graph topology. For this topology, a basic neighbourhood of a system represented by (P, Q) is simply defined by the set

$$\mathbf{V}((P, Q), \varepsilon) = \left\{(P_1, Q_1) \Big/ \operatorname*{Sup}_{|z| \geqslant \mu} \{|P_1(z) - P(z)|^2 + \mu^{-2}|Q_1(z) - Q(z)|\} < \varepsilon^2 \right\} \tag{58}$$

Property 5

Given μ, γ and A_p, B_p, C_p polynomials in q^{-1}, A_p, C_p monic, C_p exponentially μ-stable, the set of linear time-invariant finite-dimensional systems whose input–output signals satisfy for some μ-exponentially decaying sequence β

$$C_p(q^{-1})v(t) = A_p(q^{-1})y_p(t) - B_p(q^{-1})u_p(t-1) \tag{59}$$

$$|v(t)| < \gamma s_p(t) + \beta(t) \tag{60}$$

contains the open neighbourhood

$$\mathbf{V}\left(\left(\frac{A_p}{C_p}, \frac{B_p}{C_p}\right), \frac{\gamma}{\mu}\right)$$

of the graph topology of exponential μ-stability.

Proof

The main point of this proof is to notice that for any exponentially μ-stable proper fraction P, we have as a consequence of Plancherel's theorem (with zero initial conditions)

$$\sum_{t=0}^{T} \mu^{-2t}|P(q)u(t)|^2 \leqslant \sup_{|z|\geqslant\mu} \{|P(z)|^2\} \sum_{t=0}^{T} \mu^{-2t}|u(t)|^2 \tag{61}$$

Then let P, Q be defined by

$$P(q) = \frac{A_p(q^{-1})}{C_p(q^{-1})}, \quad Q(q) = \frac{B_p(q^{-1})}{C_p(q^{-1})} \tag{62}$$

and consider $\mathbf{V}((P, Q), (\gamma/\mu))$. Our proof will be established if any process in this neighbourhood satisfies (59), (60). Hence let (P_1, Q_1) represent such a process, i.e. its output is obtained by

$$P_1(q)y_p(t) = Q_1(q)u_p(t-1) \tag{63}$$

With (59), v is defined by

$$v(t) = P(q)y_p(t) - Q(q)u_p(t-1) \tag{64}$$

$$= (P(q) - P_1(q) \quad Q_1(q) - Q(q))\begin{bmatrix} y_p(t) \\ u_p(t-1) \end{bmatrix} \tag{65}$$

In the following, we assume zero initial conditions. As already mentioned, the only consequence of this assumption would be a modification of β (thanks to the exponential μ-stability with $\mu < 1$). We notice that, P and P_1 being monic, $q(P - P_1)$ is proper. Then, applying a two-dimensional version of inequality (61), we obtain

$$\sum_{t=0}^{T} \mu^{-2t}|v(t)|^2 \leqslant \sup_{|z|\geqslant\mu} \{|P(z) - P_1(z)|^2 + \mu^{-2}|Q(z) - Q_1(z)|^2\}$$
$$\times \sum_{t=0}^{T-1} \mu^{-2t}(|y_p(t)|^2 + |u_p(t)|^2) \tag{66}$$

Since (P_1, Q_1) belongs to $\mathbf{V}((P, Q), (\gamma/\mu))$, with (58), we obtain

$$\mu^{-2T}|v(T)|^2 \leqslant \sum_{t=0}^{T} \mu^{-2T}|v(t)|^2 < \left(\frac{\gamma}{\mu}\right)^2 \sum_{t=0}^{T-1} \mu^{-2t}(|y_p(t)|^2 + |u_p(t)|^2) \tag{67}$$

The conclusion follows by multiplying by μ^{2T}. □

With this fact, we have established that if a property holds for a set of systems satisfying (59) and (60), it is preserved in the presence of (i.e. we say is robust to) any (sufficiently small) linear time-invariant perturbation for which linear-feedback exponential μ-stability is preserved and the closed-loop transfer function remains continuous.

It is important to notice the role of μ in the above analysis. In Property 1, we have remarked that state components in an unobservable subspace associated with an eigenvalue larger or equal in modulus to μ may not be scaled by s_p. However, as mentioned earlier, unobservability does not affect input–output signals. It is not the case of unreachability. This property corresponds to the existence of a common factor of A_p and B_p in (60) or of both the numerator and denominator of P and Q in (57). Since P and Q must be exponentially μ-stable in our framework, we cannot consider as a small perturbation the fact of introducing a nearly unreachable mode corresponding to an eigenvalue larger than or equal to μ. This point will be illustrated in the following section.

4. Misfit between model and process and the measurement system

The process is known to be a linearly dominated system as defined by (10). However based on some complexity consideration we fix the degrees n_A, n_B, n_C and therefore define the model family of § 2. Is it possible to find a model of the process within this class? More precisely, is it possible to determine the coefficients of A, B, C such that the measurement v, given by

$$C(q^{-1})v(t) = A(q^{-1})y(t) - B(q^{-1})u(t-1) \tag{68}$$

where u, y are measured input and output respectively, is an 'unpredictable' sequence in the sense of (8)–(9) for example?

For instance, let us consider the process described by

$$\begin{aligned}(1-(\lambda+\varepsilon g_1)q^{-1})(1+aq^{-1}-\varepsilon a_1 q^{-2})y_p(t)\\ = (1-(\lambda+\varepsilon q_2)q^{-1})(-\varepsilon b_1+bq^{-1}-\varepsilon b_2 q^{-2})u_p(t-1)+n(t)\end{aligned} \tag{69}$$

where $n(t)$ is a sequence bounded by β_1. If ε were very small and $|\lambda|<1$ so that stable cancellation could occur, one would like to take a simpler model defined by

$$A(q^{-1}) = 1+aq^{-1}, \quad B(q^{-1}) = bq^{-1}, \quad C(q^{-1}) = 1 \tag{70}$$

taking the measured input–output equal to the process input–output. In fact, in doing so we neglect

(*a*) the nearly unreachable mode corresponding to the nearly cancellable $\lambda+\varepsilon g_1$, $\lambda+\varepsilon g_2$ pole-zero pair,

(*b*) the fast stable pole $\varepsilon a_1/a$,

(*c*) the fast stable zero $\varepsilon b_2/b$,

and we represent the fast unstable zero $b/\varepsilon b_1$ by a pure delay.

For this model, v is given by

$$v(t) = (1+aq^{-1})y_p(t) - bq^{-1}u_p(t-1) \tag{71}$$

Hence with (11), we obtain

$$\begin{aligned}(1-\lambda q^{-1})v(t) = \varepsilon[(g_1(1+aq^{-1}-\varepsilon a_1 q^{-2})+a_1(1-\lambda q^{-1})q^{-1})y_p(t-1)\\ +(g_2(\varepsilon b_1 - bq^{-1}+\varepsilon b_2 q^{-2})-(b_1+b_2 q^{-2})\\ \times(1-\lambda q^{-1}))u_p(t-1)]+n(t)\end{aligned} \tag{72}$$

Choosing μ, $1 > \mu > |\lambda|$, we can apply Property 2 and obtain the existence of a constant γ and a bounded sequence β such that

$$|v(t)| \leqslant \gamma s_{\mathrm{p}}(t) + \beta(t) \tag{73}$$

The constraint $\mu > |\lambda|$ illustrates the last remark of the previous Section: for our analysis to apply, the neglected nearly unreachable modes must be associated with eigenvalues strictly less than μ.

As predicted at the end of § 2, the 'unpredictability' property is not satisfied. Instead, with (15), we have a 'scaling' property. However, as mentioned in the Introduction, as far as boundedness is concerned this scaling property is sufficient provided the associated γ is sufficiently small. Hence the question: 'the model family being chosen in (12), how can we reduce γ (without increasing s_{p})?'

An answer is obtained from the general principle: 'To process data by a system with limited possibilities, they should be formated according to these possibilities'.

In our case, formating is obtained by the measurement. The main idea we wish to develop now is to consider the possibility of transforming the process by feedback, by-pass and filtering in order to allow a better fit between this transformed process, called the measured process, and an *a priori* fixed-complexity model.

For specificity, in the above example assume that λ is known and $n(t)$ is constant:

$$(1 - q^{-1})n(t) = 0 \tag{74}$$

We choose the following model (still two parameters):

$$A(q^{-1}) = (1 + aq^{-1})(1 - q^{-1}), \quad B(q^{-1}) = bq^{-1}, \quad C(q^{-1}) = 1 \tag{75}$$

and the following measurements:

$$u(t) = \frac{(1 - \lambda q^{-1})(1 - q^{-1})}{1 - \lambda} u_{\mathrm{p}}(t), \quad y(t) = \frac{(1 - \lambda q^{-1})}{1 - \lambda} y_{\mathrm{p}}(t) \tag{76}$$

Now, v is given by:

$$v(t) = (1 + aq^{-1})(1 - q^{-1})y(t) - bq^{-1}u(t-1) \tag{77}$$

and, by (69), it satisfies:

$$\begin{aligned} v(t) = \varepsilon(1 - q^{-1})[&(g_1(1 + aq^{-1} - \varepsilon a_1 q^{-2}) + a_1(1 - \lambda q^{-1})q^{-1})y_{\mathrm{p}}(t-1) \\ &+ (g_2(\varepsilon b_1 - bq^{-1} + \varepsilon b_2 q^{-2}) \\ &- (b_1 + b_2 q^{-2})(1 - \lambda q^{-1}))u_{\mathrm{p}}(t-1)] \end{aligned} \tag{78}$$

Hence $v(t)$ no more depends on $n(t)$ or $\{u_{\mathrm{p}}(\tau), y_{\mathrm{p}}(\tau), n(t), \tau \leqslant t-1\}$ but only on $\{y_{\mathrm{p}}(t-1), \ldots, y_{\mathrm{p}}(t-4), u_{\mathrm{p}}(t-1), \ldots, u_{\mathrm{p}}(t-4)\}$. In particular, this means that if the input–output signals were large in the past, say at time $t-5$, then, in the former case, $v(t)$ is influenced by those large terms (though weighted by λ^5). In the latter case, this influence is removed.

Notice that our measurement procedure is a disguised way of reintroducing complexity in the model. However, the model incorporates free parameters to be adapted on line whereas the measurement system does not.

In the general case, the measured input–output signals u and y are defined by

$$\begin{bmatrix} U_y(q^{-1}) & y(t) \\ U_u(q^{-1}) & u(t) \end{bmatrix} = \begin{bmatrix} V_y(q^{-1}) & -q^{-1}W_y(q^{-1}) \\ V_u(q^{-1}) & -W_u(q^{-1}) \end{bmatrix} \begin{bmatrix} y_{\mathrm{p}}(t) \\ u_{\mathrm{p}}(t) \end{bmatrix} \tag{79}$$

and, given (A, B, C) as an element of the model family, we obtain its associated modelling error v by

$$C(q^{-1})v(t) = A(q^{-1})y(t) - B(q^{-1})u(t-1) \tag{80}$$

The problem of choosing this element being taken care of by the adaptation law, here we are interested in choosing the measurement system for any possible model. A first objective is clearly as follows.

Objective 1

The modelling error v should be made as 'unpredictable' as possible. Practically, the process effects that cannot be represented, or that we choose not to represent, by the model should be made as unobservable as possible by the measurement system. Or, equivalently, the measured signals should be as insensitive as possible to the unmodelled effects.

To understand how this can be achieved, let us assume that the process is exactly a finite dimensional linear time-invariant observable system, i.e.

$$A_p(q^{-1})y_p(t) = B_p(q^{-1})u_p(t-1) \tag{81}$$

for some polynomials A_p, B_p, with A_p monic. In this case, the measured input–output signals are related by

$$\begin{aligned}(A_p(q^{-1}) \quad -q^{-1}B_p(q^{-1}))&\begin{bmatrix} -W_u(q^{-1}) & q^{-1}W_y(q^{-1}) \\ -V_u(q^{-1}) & V_y(q^{-1}) \end{bmatrix} \\ &\times \begin{bmatrix} U_y(q^{-1}) & 0 \\ 0 & U_u(q^{-1}) \end{bmatrix} \begin{bmatrix} y(t) \\ u(t) \end{bmatrix} = 0 \end{aligned} \tag{82}$$

That is

$$\bar{A}(q^{-1})y(t) = \bar{B}(q^{-1})u(t-1) \tag{83}$$

where

$$\bar{A} = (-A_p W_u + q^{-1} B_p V_u)U_y \tag{84}$$

$$\bar{B} = (-A_p W_y + q^{-1} B_p V_y)U_u \tag{85}$$

Hence

(*a*) the measurement y, obtained by by-passing the process, may be used to move and/or add zeros;

(*b*) the measurement u, obtained by feedback around the process, may be used to move and/or add poles.

Clearly, reduced-complexity models may be obtained by conjunction of both measurements leading to stable pole-zero cancellation. However, by the same token, this shows the drawback of this measurement system, namely the possibility of creating an unstable pole-zero cancellation. In such a (very unlikely) case, the measured process is not stabilizable even if the process is.

Another objective, assigned to the measurement system, is as follows.

Objective 2

The control law will be designed for the model to impose some properties on its input–output signals, i.e. on the measured signals. The measurement system should be such that these properties are transferred to the actual process input–output signals. The least requirement is: 'boundedness of u, y' implies 'boundedness of u_p, y_p'.

Let us see how these objectives can be met.

About Objective 1, typically the unmodelled effects are divided into two components: unmodelled dynamics and exogenous signals.

The unmodelled dynamics prevent the restricted complexity model from fitting the process frequency response at all frequencies. On the other hand, in Objective 2 we are usually interested in the properties of the input–output signals only in a restricted frequency range. Practically, we may define this frequency range as a finite set of values given by the zeros of a monic polynomial D with all its zeros on the unit circle. Then Objective 2 may be: '$y(t)$ and $y_p(t)$ should have the same amplitude and phase at each frequency given by a zero of D, whatever the corresponding amplitude and phase of u_p may be'. Invoking linearity of the measurement system and Fourier decomposition, we write this objective as

$$\forall u_p \text{ s.t. } D(q^{-1})u_p(t)=0, \quad D(q^{-1})y(t)=D(q^{-1})y_p(t)=0 \Rightarrow y(t)=y_p(t) \tag{86}$$

For example, if we choose

$$D(q^{-1})=1-q^{-1} \tag{87}$$

(86) implies equality of the d.c. components of y and y_p.

We now have the following property.

Property 6

Let U_y and D be relatively prime; then (86) is satisfied if there exist two polynomials $\bar{V}_y$, $\bar{W}_y$ such that

$$U_y=V_y+\bar{V}_yD, \quad W_y=\bar{W}_yD \tag{88}$$

Proof

We have

$$U_y(q^{-1})(y(t)-y_p(t))=-\bar{V}_y(q^{-1})D(q^{-1})y_p(t)-q^{-1}\bar{W}_y(q^{-1})D(q^{1})u_p(t) \tag{89}$$

Hence

$$\begin{aligned} D(q^{-1})y(t)=D(q^{-1})y_p(t)=D(q^{-1})u_p(t)=0 &\Rightarrow U_y(q^{-1})(y(t)-y_p(t)) \\ &= D(q^{-1})(y(t)-y_p(t))=0 \end{aligned} \tag{90}$$

But, D and U_y being relatively prime, there exist two polynomials α, β such that

$$\alpha D+\beta U_y=1 \tag{91}$$

Applying this operator identity to $y(t)-y_p(t)$ gives the result (90). □

With this property, choosing U_y exponentially stable, we can rewrite the measurement y as (up to the addition of an exponentially decaying sequence):

$$y(t)=\left(1-\frac{\bar{V}_y(q^{-1})D(q^{-1})}{U_y(q^{-1})}\right)y_p(t)-\frac{\bar{W}_y(q^{-1})}{U_y(q^{-1})}D(q^{-1})u_p(t-1) \tag{92}$$

This expression can be understood as follows.

Interpret

$$\left(1-\frac{\bar{V}_y D}{U_y}\right) \quad \text{and} \quad \frac{\bar{V}_y D}{U_y}$$

as pass band filters in the frequency range of interest and its complements, respectively. Then we have $y=y_p$ in the bandwidth of

$$\left(1-\frac{\bar{V}_y D}{U_y}\right)$$

and $\bar{V}_y y=-q^{-1}\bar{W}_y u_p$ in its complement. Writing a model for y, this allows us to fit model and process in the bandwidth. However, outside the fitting may also be obtained trivially by choosing $\bar{W}_y$, $\bar{V}_y$ so that $-z^{-1}\bar{W}_y/\bar{V}_y$ is simply the model transfer function.

Let us now treat the problem of corrupting exogenous signals. Among these signals the ones that will prevent v from being unpredictable are those strongly autocorrelated and in particular the sequences n that are solutions of (i.e. the purely deterministic component in the Wold stationary process decomposition—Ash and Gardner 1975):

$$E(q^{-1})n(t)=0 \tag{93}$$

where E is monic with all its zeros on the unit circle.

For specificity, let us assume that the process can be described by:

$$A_{\mathrm{p}}(q^{-1})y_{\mathrm{p}}(t)=B_{\mathrm{p}}(q^{-1})u_{\mathrm{p}}(t-1)+n(t) \tag{94}$$

with A_{p} monic and n satisfying (93). We wish to remove the dependence of (i.e. to decouple) y_{p} on n.

According to Objective 2, this should be made at least in the frequency range of interest. However, if y_{p} equals y in this frequency range, it is sufficient to decouple the measured signal y from n.

Since

$$\begin{bmatrix} U_y(q^{-1})y(t) \\ U_u(q^{-1})u(t) \end{bmatrix}=\begin{bmatrix} V_y(q^{-1}) & -q^{-1}W_y(q^{-1}) \\ V_u(q^{-1}) & -W_u(q^{-1}) \end{bmatrix}\begin{bmatrix} y_{\mathrm{p}}(t) \\ u_{\mathrm{p}}(t) \end{bmatrix} \tag{95}$$

Proceeding as in (82), (94) can be rewritten as

$$\bar{A}(q^{-1})y(t)=\bar{B}(q^{-1})u(t-1)+\Delta(q^{-1})n(t) \tag{96}$$

where Δ is the determinant

$$\Delta=q^{-1}V_u W_y-V_y W_u \tag{97}$$

Property 7

Assume that (94) holds. Given u, any sequence y that is a solution of (96) for some n satisfying (93), is also a solution for any other n satisfying (93) iff there exists a polynomial $\bar{\Delta}$ such that

$$\Delta=\bar{\Delta}E \tag{98}$$

Proof

If. With (96) and (98), we readily obtain

$$\bar{A}(q^{-1})y(t) = \bar{B}(q^{-1})u(t-1) + \bar{\Delta}(q^{-1})E(q^{-1})n(t) = \bar{B}(q^{-1})u(t-1) \tag{99}$$

Only if. Let n_1, n_2 be sequences satisfying (93). The same sequence y being obtained for n_1, n_2, we have

$$\begin{aligned}\bar{A}(q^{-1})y(t) &= \bar{B}(q^{-1})u(t-1) + \Delta(q^{-1})n_1(t) \\ &= \bar{B}(q^{-1})u(t-1) + \Delta(q^{-1})n_2(t)\end{aligned} \tag{100}$$

This implies

$$\Delta(q^{-1})(n_1(t) - n_2(t)) = 0 \tag{101}$$

In particular, choosing for n_2 the zero sequence, we have established

$$\forall n_1 \quad \text{s.t. } E(q^{-1})n_1(t) = 0, \quad \Delta(q^{-1})n_1(t) = 0 \tag{102}$$

The conclusion follows. □

With (96) and Property 1, we have also established the following property.

Property 8

If the process is any linear system which can be described by (94), with a corrupting exogenous signal n satisfying (93), then, choosing the measurement system so as to satisfy (98), the measured process is a linearly dominated system. More precisely, for any μ, $0 < \mu < 1$, there exist a constant γ and a μ-exponentially decaying sequence β such that

$$|y(t)| \leqslant \gamma s(t) + \beta(t) \tag{103}$$

with

$$s(t)^2 = \sum_{i=0}^{t-1} \mu^{2(t-1-i)}(|y(i)|^2 + |u(i)|^2) \tag{104}$$

In practice E is unknown; but with Objective 2, restricting our interest to the frequency range defined by D, we take

$$E = D \tag{105}$$

With Property 6, this yields

$$\bar{\Delta}D = \Delta = q^{-1}V_u\bar{W}_yD - U_yW_u + \bar{V}_yW_uD \tag{106}$$

Consequently, D should be divisible by U_yW_u. However, choosing U_y to be exponentially stable, we have to take

$$W_u = \bar{W}_uD \tag{107}$$

In these conditions, the measurement u may be rewritten as

$$\bar{W}_u(q^{-1})D(q^{-1})u_p(t) = U_u(q^{-1})u(t) - V_u(q^{-1})y_p(t) \tag{108}$$

This expression can be seen as an application of the internal model principle: 'The polynomial acting on the process input in the control law should have in factor the annihilating polynomial of both the set point and the exogenous disturbances'.

In Objective 2, we have also mentioned that the measurement system should imply the process signal boundedness from the measured signal boundedness.

We now know that

$$\begin{bmatrix} \bar{\Delta}(q^{-1})y_p(t) \\ \bar{\Delta}(q^{-1})D(q^{-1})u_p(t) \end{bmatrix} = \begin{bmatrix} \bar{W}_u(q^{-1}) & -q^{-1}\bar{W}_y(q^{-1}) \\ -V_u(q^{-1}) & U_y(q^{-1}) - \bar{V}_y(q^{-1})D(q^{-1}) \end{bmatrix} \begin{bmatrix} U_y(q^{-1})y(t) \\ U_u(q^{-1})u(t) \end{bmatrix} \tag{109}$$

Hence, if u, y are bounded, so are $\bar{\Delta}y_p$, $\bar{\Delta}Du_p$, and if $\bar{\Delta}$ is exponentially stable, y_p and Du_p are bounded. Therefore we have the following property.

Property 9

Assume that V_y, W_y, W_u satisfies (88), (107), if $q^{-1}V_u\bar{W}_y - V_y\bar{W}_u$ is exponentially stable then y_p is bounded whenever u, y are bounded. Moreover u_p is bounded if D is chosen such that u_p is bounded whenever y_p and Du_p are bounded.

Remark 4

To understand this last assumption, notice that if the process is described by

$$A_p(q^{-1})y_p(t) = B_p(q^{-1})u_p(t-1) + n(t) \tag{110}$$

with A_p monic and n bounded, then the condition holds if B_p and D are relatively prime. Indeed in this case there exist two polynomials α, β such that, with D monic,

$$\alpha q^{-1}B_p + \beta D = 1 \tag{111}$$

Hence

$$u_p(t) = \alpha(q^{-1})A_p(q^{-1})y_p(t) + \beta(q^{-1})D(q^{-1})u_p(t) - \alpha(q^{-1})n(t) \tag{112}$$

from which the condition follows.

In fact, by extension, we have the following property.

Property 10

(i) Assume that V_y, W_y, W_u satisfy (88), (107); if $q^{-1}V_u\bar{W}_y - V_y\bar{W}_u$ is exponentially μ-stable then there exist a μ-exponentially decaying sequence β_y (depending on the initial conditions) and a constant γ_y such that

$$\left(\sum_{i=0}^{t-1} \mu^{2(t-1-i)}|y_p(i)|^2\right)^{1/2} \leqslant \gamma_y s(t) + \beta_y(t) \tag{113}$$

with s defined in (104). Moreover if D is chosen such that

$$\left(\sum_{i=0}^{t-1} \mu^{2(t-1-i)}|u_p(i)|^2\right)^{1/2} \leqslant \gamma_u\left(\sum_{i=0}^{t-1} \mu^{2(t-1-i)}(|y_p(i)|^2 + |D(q^{-1})u_p(i)|^2)\right)^{1/2} + \beta_u(t) \tag{114}$$

for some positive constant γ_u and bounded sequence β_u (depending on the initial conditions), then there exist a bounded sequence β_p (depending on the initial conditions) and a constant γ_p such that

$$s_p(t) \leqslant \gamma_p s(t) + \beta_p(t) \tag{115}$$

(ii) If U_y, U_u are exponentially μ-stable, there exist a bounded sequence β and a constant γ such that

$$s(t) \leqslant \gamma s_p(t) + \beta(t) \tag{116}$$

Remark 5

(i) Equations (115) and (116) show that s and s_p can be exchanged. In particular, we can use s, computed in terms of measured signals, instead of s_p as a scaling signal (see Remark 1).

(ii) Again notice that (114) holds if (110) holds and B_p and D are relatively prime.

Proof

(i) From (108) we notice that we can write

$$y_p(t) = \left[\frac{\bar{W}_u(q^{-1})U_y(q^{-1})}{\bar{\Delta}(q^{-1})} \quad -q^{-1}\frac{\bar{W}_y(q^{-1})U_u(q^{-1})}{\bar{\Delta}(q^{-1})} \right] \begin{bmatrix} y(t) \\ u(t) \end{bmatrix} + \delta(t) \tag{117}$$

with the sequence δ satisfying:

$$\bar{\Delta}(q^{-1})\delta(t) = 0 \tag{118}$$

From the assumption on $\bar{\Delta}$, δ is λ-exponentially decreasing for some $\lambda < \mu$ and

$$\sup_{|z| \geqslant \mu} \left\{ \left\| \frac{\bar{W}_u(z^{-1})U_y(z^{-1})}{\bar{\Delta}(z^{-1})} \quad -z^{-1}\frac{\bar{W}_y(z^{-1})U_u(z^{-1})}{\bar{\Delta}(z^{-1})} \right\| \right\} \leqslant \infty \tag{119}$$

With (61), this proves the existence of a constant γ_y, such that

$$\sum_{i=0}^{t-1} \mu^{-2i}|y_p(i) - \delta(i)|^2 \leqslant \frac{\gamma_y^2}{2} \sum_{i=0}^{t-1} \mu^{-2i}(|y(i)|^2 + |u(i)|^2) = \mu^{-2(t-1)}\frac{\gamma_y^2}{2} s(t)^2 \tag{120}$$

Equation (113) readily follows since

$$y_p(i)^2 \leqslant 2(y_p(i) - \delta(i))^2 + 2\delta(i)^2 \tag{121}$$

and

$$\begin{aligned} \mu^{-t}\left(\sum_{i=0}^{t-1} \mu^{2(t-1-i)}|\delta(i)|^2 \right)^{1/2} &\leqslant \frac{1}{\mu} \operatorname*{Sup}_i \left\{ \frac{|\delta(i)|}{\lambda^i} \right\} \left(\sum_{i=0}^{t-1} \left(\frac{\lambda}{\mu} \right)^{2i} \right)^{1/2} \\ &\leqslant \frac{1}{\mu - \lambda} \operatorname*{Sup}_i \left\{ \frac{|\delta(i)|}{\lambda^i} \right\} \end{aligned} \tag{122}$$

By exactly the same procedure, an inequality equivalent to (113) can be obtained for Du_p, i.e.

$$\left(\sum_{i=0}^{t-1} \mu^{-2(t-1-i)}|D(q^{-1})u_p(i)|^2 \right)^{1/2} \leqslant \bar{\gamma}_u s(t) + \bar{\beta}_u(t) \tag{123}$$

for some constant $\bar{\gamma}_u$ and μ-exponentially decaying sequence $\bar{\beta}_u$. Then (115) follows from (114) and the definition of s_p in (11).

(ii) is established in the same way from (81) with U_y or U_u playing the same role as $\bar{\Delta}$. □

Finally, let us treat the typical case where Objective 2 is written as a tracking problem. Namely, we want to impose the following property to the process output:

$$y_p(t) = y_d(t) \tag{124}$$

where y_d is a desired known output sequence. Again, relaxing this objective within a given frequency range, we suppose that

$$D(q^{-1})y_d(t) = 0 \tag{125}$$

Keeping in mind the idea of transforming the tracking problem into a regulation problem and applying Properties 6 to 10, we modify the measurement system into

$$\begin{bmatrix} U_y(q^{-1})y(t) \\ U_u(q^{-1})u(t) \end{bmatrix} = \begin{bmatrix} U_y(q^{-1}) - \bar{V}_y(q^{-1})D(q^{-1}) & -q^{-1}\bar{W}_y(q^{-1})D(q^{-1}) \\ V_u(q^{-1}) & -\bar{W}_u(q^{-1})D(q^{-1}) \end{bmatrix} \times \begin{bmatrix} y_p(t) - y_d(t) \\ y_p(t) \end{bmatrix} \tag{126}$$

We now have the following property.

Property 11

(i) Let us assume that the process is any linear system which can be described by (94) with A_p monic and a corrupting exogenous signal n satisfying (93). If the measurement system is given by (126) with y_d satisfying (125), then the measurement system is a linearly dominated system. More precisely, for any μ, $0 < \mu < 1$, there exist a constant γ and a μ-exponentially decaying sequence β such that

$$|y(t)| \leqslant \gamma s(t) + \beta(t) \tag{127}$$

with s given by (104).

(ii) If $q^{-1}V_u\bar{W}_y - V_y\bar{W}_u$ is exponentially μ-stable, then there exist a μ-exponentially decaying sequence β_y (depending on the initial conditions) and a constant γ_y such that

$$\left(\sum_{i=0}^{t-1} \mu^{2(t-1-i)}|y_p(i) - y_d(i)|^2\right)^{1/2} \leqslant \gamma_y s(t) + \beta_y(t) \tag{128}$$

with s defined in (104). Moreover if y_d is bounded and D is chosen such that (114) holds, then there exist a bounded sequence β_p (depending on the initial conditions) and a constant γ_p such that

$$s_p(t) \leqslant \gamma_p s(t) + \beta_p(t) \tag{129}$$

Proof

(i) As for Property 8, this follows from Property 1 and the fact that (94), (126) imply an equation of the type

$$\bar{A}(q^{-1})y(t) = \bar{B}(q^{-1})u(t-1) + \Delta(q^{-1})(n(t) - A_p(q^{-1})y_d(t)) \tag{130}$$

with $\bar{A}$, $\bar{B}$ some polynomials and

$$\Delta = \bar{\Delta}D \tag{131}$$

(ii) follows exactly as in Property 10, noticing that

$$|y_p(i)|^2 \leqslant 2(|y_p(i) - y_d(i)|^2 + |y_d(i)|^2) \tag{132}$$

□

With Properties 6 to 11, we have proposed a solution to meet Objectives 1 and 2. Let us complete this introduction to the notion of measured process by the following remarks.

Remark 6

(i) With Properties 6, 7, if V_u is zero, $\bar{A}$, defined in (33), can be divided by D. This allows us to write the model family as the triple (AD, B, C) with D given by the control objective.

(ii) The notion of measured input is also helpful for dealing with actuator limitations such as amplitude and/or speed constraints. For this, we introduce a distinction between (linearly) computed inputs and (actual) inputs. To be precise, let $u_c(t)$ be the computed measured input as computed by the controller at time t, whereas $u(t)$ is the actual measured input as sent back to the controller at time t. Similarly, let u_{pc}, u_p be the computed and actual process input respectively. We decompose the polynomial W_u into:

$$W_u = W_{us} - q^{-1} W_u' \tag{133}$$

where W_{us} is monic and exponentially stable. Its zeros characterize the so called 'tracking mode'. We choose to relate computed and actual inputs as

$$\begin{aligned} W_{us}(q^{-1})u_{pc}(t) = V_u(q^{-1})y_p(t) + W_u'(q^{-1})u_p(t-1) - u_c(t) \\ - [q(U_u(q^{-1}) - 1)]u(t-1) \end{aligned} \tag{134}$$

$$u_p(t) = f(u_{pc}(t)) \tag{135}$$

$$u(t) = W_{us}(q^{-1})(u_{pc}(t) - u_p(t)) + u_c(t) \tag{136}$$

where f describes the actuator limitations. We can check that

$$f(u_{pc}) = u_{pc} \Rightarrow u = u_c, \quad u_p = u_{pc} \tag{137}$$

On the other hand, if f is not the identity, this decomposition guarantees that u, u_{pc} are bounded if u_c, u_p, y_p are bounded.

(iii) More generally, the concept of the measurement u can be extended so as to allow linearization of some non-linearities by feedback.

5. Almost exact linear modelling

The previous Sections motivate us to introduce the main definition of this report. Given the integers n_A, n_B, n_C and given a real μ, $0 \leqslant \mu < 1$, we define a model family as the triple (A, B, C) of polynomials with degree n_A, n_B, n_C respectively, with A, C monic and C exponentially μ-stable.

Definition 3

We say that a process can be almost exactly linearly modelled if one can find a model within this model family such that the modelling error given by the measured signals is scaled by s obtained from these signals. Namely, there exist a positive constant γ, a bounded sequence β, depending only on the initial conditions, and (A, B, C), an element of the model family, such that the process output y_p satisfies, for any process input u_p,

$$|v(t)| \leqslant \gamma s(t) + \beta(t) \tag{138}$$

where

$$\begin{aligned} s(t)^2 &= \sum_{i=0}^{t-1} \mu^{2(t-1-i)}(|u(i)|^2 + |y(i)|^2) \\ &= \mu^2 s(t-1)^2 + |u(t-1)|^2 + |y(t-1)|^2 \end{aligned} \tag{139}$$

$$\begin{bmatrix} U_y(q^{-1})y(t) \\ U_u(q^{-1})u(t) \end{bmatrix} = \begin{bmatrix} V_y(q^{-1}) & -q^{-1}W_y(q^{-1}) \\ V_u(q^{-1}) & -W_u(q^{-1}) \end{bmatrix} \begin{bmatrix} y_p(t) - y_d(t) \\ u_p(t) \end{bmatrix} \tag{140}$$

$$C(q^{-1})v(t) = A(q^{-1})y(t) - B(q^{-1})u(t-1) \tag{141}$$

or, respectively, if Properties 6, 7 are applied with $E = D$ and $V_u = 0$

$$C(q^{-1})v(t) = A(q^{-1})D(q^{-1})y(t) - B(q^{-1})u(t-1) \tag{141'}$$

where y_d is a bounded sequence given by the control objective.

6. Comments

6.1. Following Properties 6 to 11, U_u, V_u, W_u and U_y, V_y, W_y may be chosen as

$$V_y = U_y - \bar{V}_y D, \quad W_y = \bar{W}_y D, \quad W_u = \bar{W}_u D \tag{142}$$

where, to obtain process signal boundedness from measured signal boundedness, it is sufficient to choose U_u, U_y, $V_y\bar{W}_u - q^{-1}V_u\bar{W}_y$ exponentially μ-stable monic polynomials and D satisfying (114) of Property 10. However, as explained in Remark 6, if V_u is zero, the model family should be the triple (AD, B, C).

6.2. Using Remark 5 and Property 2, we know that inequality (138) holds for linearly dominated systems as defined in Definition 1 and for which Property 1 and Examples 1, 2 and 3 are illustrations.

6.3. With Remark 2, we see that if a process can be almost exactly linearly modelled, then we can use in (141) time varying polynomials $A(t)$, $B(t)$, $C(t)$, with bounded coefficients and, conversely, if (141) and (138) hold for time varying polynomials, they hold for time invariant polynomials. In particular, we can always impose $C(t) = 1$. Using one triple or another changes the constant γ and the sequence β. This possibility of modifying γ is crucial for meeting Assumption A in the Introduction. However, restrictions have to be imposed on the time variations. For example:

$$\begin{aligned} A(t, q^{-1}) &= 1 - \frac{y(t)(y(t-1) + a(t)u(t-1))}{y(t-1)^2 + u(t-1)^2} q^{-1}, \\ B(t, q^{-1}) &= \frac{y(t)u(t-1) - a(t)y(t-1))}{y(t-1)^2 + u(t-1)^2} \\ C(t, q^{-1}) &= 1 \end{aligned} \tag{143}$$

give a zero modelling error. However, not only is this model non-causal and may have unbounded coefficients, but also its time variations may be very large. Actual proofs of the boundedness of all the solutions call for constraints on these time variations.

6.4. With Properties 11 (i) and 2, we know that β in (138) is a μ-exponentially decaying sequence if the measurement system (140) is chosen according to (142), y_d satisfies

$$D(q^{-1})y_d(t) = 0 \tag{144}$$

and if the process input–output relation is given by

$$A_p(q^{-1})y_p(t) = B_p(q^{-1})u_p(t-1) + n(t) \tag{145}$$

with A_p, B_p polynomials, A_p being monic and n a bounded sequence satisfying

$$D(q^{-1})n(t) = 0 \tag{146}$$

6.5. With Property 11 (ii), we know that $y_p - y_d$ tends to zero if the same holds for s.

6.6. With Property 4, we know that if a linear time varying controller with bounded coefficients is placed in feedback with this almost exactly modelled process, then the signals cannot grow faster than exponentially.

6.7. With Property 5, we know that all the linear invariant systems belonging to the neighbourhood

$$\mathbf{V}\left(\left(\frac{A}{C}, \frac{B}{C}\right), \frac{\gamma}{\mu}\right)$$

of the graph topology of exponential μ-stability satisfy (138).

6.8. Since on the one hand we expect a better fit between model and measured process (140), and on the other hand the properties of the measured signals can be transferred to the physical process signals, using Properties 6 to 11, it is sufficient to develop the theory for the measured process. However, recall that an unfortunate choice of the measurement system may render the measured process not stabilizable.

6.9. From the Introduction, we know adaptive controllers which, in closed loop with the measured process, guarantee boundedness of all the solutions if the corresponding process is almost exactly linearly modelled by the model family with a constant γ imposed by the controller. As mentioned in § 2, this establishes that the scaling property is sufficient to replace this boundedness problem by the unpredictability property. Moreover, with Property 4, this also establishes that boundedness of all the solutions is a robust property with respect to the graph topology of exponential μ-stability for linear systems.

7. Notes

Proofs of boundedness of all the solutions under Assumption A can be found in many recent papers (see de Larminat and Raynaud 1988 and Praly *et al.* 1989, for example). As far as we are aware, the first proof was given by Praly (1982 a). First extensions to time varying models can be found in de Larminat (1984) (see also

Landau and Dugard 1986) and Tsakalis and Ioannou (1986) for indirect schemes and direct schemes respectively.

However, though bounded, the solutions may be unstable and/or correspond to very bad performances. This has been observed by Egardt (1979) and Anderson (1985), for example, and some elementary cases have been analysed by Mareels and Bitmead (1986) and Praly and España (1987). On the other hand, Goodwin and Sin (1984) prove that good, if not optimal, performances are obtained for 'ideal' systems, i.e. those leading to unpredictable modelling errors.

The problem of modelling from process signals only is not particular to adaptive control. It has motivated Willems (1986 a, b and c) to define dynamical systems as a family of time series.

We have introduced the class of systems which can be almost exactly linearly modelled. For such systems, closed loop solution boundedness is established. We have shown that this way of representing unmodelled dynamics encompasses the more classical singular perturbations (Kokotovic *et al.* 1986) or norm bounded additive or multiplicative uncertainties (Vidyasagar 1985). However, for linear systems, these latter two uncertainty representations have also the advantage of allowing us to study performances.

The measurement system introduced in this report is only a formalization and a synthesis of many methods used in almost all implementations of adaptive and even linear controllers (see Harris and Billings 1981 and Åström and Wittenmark 1984). They have also been motivated by theoretical work, for example, among many others:

— Clarke and Gawthrop (1979) introduced by-passing to circumvent the unstable zero problem (see also M'Saad *et al.* 1985).
— Bar-Kana (1986), Gawthrop (1987) and Riedle and Kokotovic (1985) proposed by-passing to counteract the effects of unmodelled dynamics.
— Elliott and Goodwin (1984) and Gawthrop (1987) proposed using the internal model idea to take care of deterministic disturbances. Goodwin *et al.* (1988) proved that, in the presence of unmodelled dynamics (but with no unmodelled extraneous disturbance), this achieves asymptotic optimal performances.
— Åström and Wittenmark (1984) described how a measured input allows us to handle actuator limitations.

An important method not represented in this measurement system is adaptation signal filtering when it is different from the control signal filtering (see Anderson *et al.* 1986 and Egardt 1979).

The almost exact linear modelling assumption has been relaxed in the two following ways.

(*a*) To guarantee closed-loop solution boundedness, it is sufficient that, instead of (138), the mean value of the scaled modelling error be smaller than γ when the mean is taken on a time interval on which s is always larger than S and of length larger than T. To be precise (see Praly 1982 b): there exist a positive real S and an integer T such that for any (t, τ) in $\mathbf{I}_{S,T}(s)$, we have

$$\sum_{i=t+1}^{t+\tau} \left|\frac{v(i)}{s(i)}\right| \leqslant \gamma\tau \tag{147}$$

where

$$\mathbf{I}_{S,T}(s) = \{(t, \tau) \mid \tau \geqslant T \quad \text{and} \quad \forall\, i \in [t, t+\tau],\, s(i) \geqslant S\} \tag{148}$$

(*b*) Since (138) has to be satisfied by a fixed element of the model family for all process inputs u_p, the constant γ needed to satisfy this global property may be very large. In fact, most of the above results would hold even if the model were allowed to depend on the process input, provided that the polynomial coefficients are bounded uniformly in this process input. Unfortunately, such an assumption is meaningless as long as the process input is not specified. On the other hand, in practice, it would be sufficient to satisfy (147) for the actual process input. The possibility of working with a model or a bound γ related to an input is offered when studying the system around some particular solutions. This is the objective of the 'local analysis' (see Anderson *et al.* 1986).

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