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A ROBUST ADAPTIVE MINIMUM VARIANCE CONTROLLER*

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Abstract. This paper addresses the twin questions of *performance* and *robustness* of an adaptive controller for single-input, single-output, linear, stochastic systems. The authors present an adaptive controller that has the following properties:

(1) Attaining *optimal* regulation and tracking in the *ideal*, minimum phase, known upper bound on system order, known sign and lower bound for the leading coefficient (b_0), positive real condition on noise case, and *self-tuning in a Cesaro sense* to a minimum variance regulator in the case of pure regulation.

(2) Providing *mean square stability* when the system is of minimum phase, with known upper bound on order *but not necessarily satisfying a positive real condition on the noise.*

(3) Providing *mean square stability* when the system is in a *graph topological neighborhood* (of computable size) of an ideal plant as in (1), and the statistical properties of the disturbance are violated.

(4) Continuing to *stabilize* the system when the *adaptation gain is prevented from vanishing.*

Key words. robustness, performance, adaptive control, optimal control, minimum variance control, graph topology, minimum variance regulator, self-tuning regulator

AMS(MOS) subject classification. 93C40

1. Introduction. Over the past 15 years, stochastic adaptive control theory has seen much development. The notable pioneering contributions of Aström and Wittenmark [2] and Ljung [14], [15] analyzed, respectively, the *possible* equilibrium values of the parameters to which an adaptive control law could converge, and the *stability* properties of these equilibrium points. This set the stage for the subsequent rigorous development of the foundations of the asymptotic theory of the so-called self-tuning controllers.

In 1981, Goodwin, Ramadge, and Caines [6] were able to successfully use some extensions of the martingale convergence theorem to show the convergence of a certain stochastic Lyapunov function. They were thus able to establish that for a variety of stochastic gradient algorithms the time average of the squared tracking error is almost surely optimal, a property we shall refer to as *self-optimality.* These results were then extended by similar arguments to some other algorithms; for example, an adaptive controller based on a modified least-squares estimate was analyzed by Sin and Goodwin [24]. In 1985, Becker, Kumar, and Wei [3] addressed the issue of convergence of the parameter estimates, and, in so doing, they also established the convergence of the adaptive regulator. By exploiting some geometric properties of the parameter estimate sequence, and some subsequent probabilistic analysis, they were able to show that while the parameter estimates converge almost surely (a.s.), they do *not* converge to their true values. Instead, the parameter estimate vector converges to a *random scalar multiple* of the true parameter vector. However, since the particular control law used for the regulation problem employs only *ratios* of estimates of individual parameters,

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the adaptive control law remains invariant under scaling of the parameter estimates. Hence convergence of the adaptive regulator to the true optimal regulator takes place almost surely. This result therefore proved the so-called *self-tuning* property of the adaptive regulator. Recently, this self-tuning property has been extended by Kumar and Praly [11] to the *tracking* problem, where the goal is not to regulate the system output to stay close to zero, but to *track* a given *reference trajectory* while *optimally rejecting* the noise entering into the system. The essential difference between the regulation and tracking problems is that in the former problem it is not necessary to estimate the coefficients of the colored noise polynomial (in the ARMAX representation of the system), while in the latter it is necessary to do so if we want to track arbitrary reference trajectories. Since an additional number of parameters have to be estimated in the tracking problem, it turns out that more "excitation" of the system is needed. This excitation is in turn guaranteed if the reference trajectory is sufficiently exciting of appropriately high order (see Kumar and Praly [11]). In many practically important situations, such as, for example, the *set-point problem*, however, the reference trajectory may only be a *constant* level, which is sufficiently exciting of *order one*, or some other trajectory that has a low order of excitation. In these situations, it turns out that not all the coefficients of the colored noise polynomial need be explicitly estimated, but rather knowledge of a smaller set of parameters derived from the coefficients is adequate. This allows the design of an adaptive tracker that uses a smaller dimension parameter estimator (which is still of larger dimension than is needed in the regulation problem). Such smaller dimensional adaptive trackers have also been proved to be *self-tuning* (see Kumar and Praly [11]).

The successful results quoted above essentially show that the adaptive regulators and trackers tune themselves to *optimal* regulators and trackers for the unknown system. Inevitably, such results are crucially dependent on making some "exact" assumptions about the unknown system being controlled. In particular, for such exact asymptotic optimality and strong convergence results to hold, it has been assumed that the stochastic system being controlled is linear, of minimum phase, of *known order*, and the disturbance entering into the system is a *stochastic process* satisfying some specified *statistical* properties.

Assumptions of the above type have been called "ideal" assumptions, and questions have been raised, especially in *deterministic* adaptive control (see Egardt [5], Rohrs, Valavani, Athans, and Stein [22], and Ioannou and Kokotovic [7]), about whether the adaptive controllers designed on the basis of these assumptions, and for which a successful "ideal" theory has been built, are *robust* with respect to these assumptions. Specifically, do "small' violations of these assumptions lead to drastically different behavior from that predicted by the ideal theory?

The *order* requirement arises since we must choose the dimension of the adaptive regulator before we can tune it. However, the true system need not necessarily be (and is frequently not) of the exact order that is assumed. It is well known that "small perturbations" of an nth-order linear system can lead to systems of arbitrarily high order.

Regarding the disturbance, the assumption made is that it is a *stochastic process* with a *rational spectral density*, and thus is representable as the output of a system driven by *white noise*. Moreover, the *order* of this "coloring" filter is assumed known. Finally, it is also assumed that the noise satisfies a certain "positive real" condition. This is essentially a requirement that the disturbance be close to a white noise and be not too colored. However, none of these assumptions need be strictly satisfied in practice. The positive realness condition also arises in recursive system identification using the pseudolinear regression method (see Solo [25], Ljung and Söderström [16], and Kumar and Varaiya [12]). It is basically a *pseudogradient condition* (see Ljung and Söderström [16] and Kumar [10]) guaranteeing that the direction in which the parameter estimates are recursively adjusted (in the types of recursive identification algorithms being employed) is appropriate.

Regarding the minimum phase restriction, it is well known (see Aström [1], Peterka [17], Shaked and Kumar [23], and Kumar and Varaiya [12]) that when a *stationary* control law that minimizes the output variance is used to control the system, then the control actions used become unbounded if the system is of *nonminimum phase.* However, for *adaptive control* where a *nonstationary*, nonlinear control law is used, it is not necessary that the minimum phase assumption be satisfied in order for the control inputs to be bounded. Hence the minimum phase assumption is a restrictive condition; it is easily violated by a very fast unstable zero that corresponds to a very small numerator perturbation of the transfer function.

Much attention has therefore been given in recent years to the issue of *robust adaptive control*, especially in deterministic adaptive control, to determine under what conditions signals in the system remain bounded under violations of assumptions (for example, see [8], [9], [20]). In the adaptive control of *stochastic* systems, however, noise is an essential feature of the system, and it is of interest not only to guarantee boundedness of signals, but it is also important to reject the noise optimally, or at least much of it. Thus, *performance* of the adaptive control algorithm in rejecting the corrupting noise, and thus tracking the desired reference trajectory with *small* tracking error, is also an important goal in stochastic adaptive control.

In this paper, therefore, we address the twin questions of *performance as well as robustness* of adaptive control laws for linear stochastic systems. In particular, we address the issue of adaptive controllers that are *performance-optimal* when the ideal assumptions are satisfied, *and* that are *robust with respect to perturbations* of the system from the ideal assumptions.

We will consider two types of *perturbations* of the system from optimality. First we consider perturbations of the coefficients of the colored noise polynomial that can be large and that allow gross violation of the positive real assumption. This problem has been treated by Egardt [5] for bounded noise and extended in Praly [18] for mean-square bounded noise.

Second, we consider *system perturbations.* Vidyasagar [26] has identified the appropriate topology on the set of linear systems, called the *graph topology*, which is the *weakest topology* such that there is a stabilizing linear controller for a nominal ideal system that remains stabilizing, and such that the closed-loop transfer function is continuous (uniformly over all frequencies) when perturbations with respect to this topology are allowed. Thus, for any given weaker topology which thus allows *more* perturbations, there is not necessarily any single linear control law that continues to maintain stability. Since self-tuning or adaptive control is really an online or real-time search over the space of linear controllers, we cannot expect to do better than allow for perturbations with respect to this graph topology. Thus while (nonadaptive) linear controllers are designed for perturbations with respect to the graph topology from a *given nominal system*, adaptive control laws should be designed to maintain stability with respect to the graph topology from all possible nominal systems. This indeed is the goal of this paper. We will achieve it by extending the approach of Praly [19] to the vanishing gain case.

Last, asymptotic optimality and convergence results for adaptive controllers rely on adaptive parameter adjustment schemes that use an asymptotically vanishing step-size, i.e., the gain converges to zero. However, to maintain the ability to adapt, the gain should be nonvanishing. Thus we also need to analyze the effect of nonvanishing gain on the ideal adaptive control algorithm.

In this paper we therefore exhibit an adaptive controller for linear stochastic systems that is *optimal for all ideal plants*, and remains stable with respect to violations of the positive real condition, and with respect to perturbations of the system, in the graph topology, from all ideal plants. Moreover, we show that stability is preserved when the gain is prevented from going to zero.

Specifically, we present an adaptive controller for which we prove the following performance and robustness properties:

(1) Attaining *optimal* regulation and tracking in the *ideal* case when the system is of minimum phase with a known upper bound on the system order, and when the coefficients of the colored noise polynomial satisfy a positive real condition (Theorem 5.1). In the case of the *regulation* problem, we also show that the adaptive controller *self-tunes in a Cesaro sense* to minimum variance regulator (Theorem 5.2).

(2) Providing *mean-square stability* when the system is of minimum phase with a known upper bound on the system order but *does not necessarily satisfy a positive real condition* (Theorem 4.6).

(3) Providing *mean-square stability* when the system is in a *graph topological neighborhood of computable size* of an ideal system as in (1) (Theorem 6.8).

(4) *Continuing to stabilize* the system when the *adaptive gain is prevented from vanishing to zero* (Theorem 7.7).

There are still many unresolved questions. Maybe the most important is to determine whether adaptive controllers without the modifications we have used are already robust, even though our modifications are well motivated. Moreover, we have not really been able to deal with the removal of the minimum phase assumption, even though, as we will show later, our adaptive controller is robust with respect to graph topological perturbations that do result in nonminimum phase systems.

2. The adaptive controller. In this section we present our adaptive controller. In the next five sections we analyze the effect of the adaptive controller when it is applied to a variety of systems satisfying varying assumptions. (Thus we are reversing the usual order of presentation, where the intended systems are first described before the adaptive controllers are defined!)

We will suppose that the system under control is a single-input, single-output system with input sequence $u(t)$ and output sequence $y(t)$. We will also suppose the following:

(A2.i) The reference trajectory $y^m(t)$ is bounded.

There are several fixed parameters that are chosen a priori. We choose the following:

(A2.ii) Three integers n_R, n_S, and n_C (which describe the dimensions of our adaptive controller, but not necessarily those of the system);

(A2.iii) Two positive numbers $0 < \lambda_0 < \lambda_1$ (which serve as bounds on certain eigenvalues);

(A2.iv) Three positive numbers $\rho > 0$, $\sigma_0 > 0$, and $K > 0$;

(A2.v) A parameter vector θ^c of dimension $(n_R + n_S + n_C + 2)$ whose first component is larger than or equal to σ_0;

(A2.vi) An integer $d \geqq 1$ (which models the delay but may not be equal to it).

We use the *regression vector* $\phi(t)$ defined as

$$\phi(t) := (u(t), \cdots, u(t-n_S), y(t), \cdots, y(t-n_R), y^m(t+d-1), \cdots, y^m(t+d-n_C))^T.$$

Given $\theta(n)$, $F(n)$, and $\rho(n) > 0$ for all $n \leqq t-1$, and having applied a new control input $u(t-1)$ and observed a new output $y(t)$, we *recursively* define the adaptive controller as follows:

(2.1) $\rho(t) := \rho(t-1) + \max(\rho, \|\phi(t-d)\|^2), \quad t \geqq 1 \quad$ (we choose $\rho(t) = 0$ for $t \leqq 0$),

$$\bar{\phi}(t-d) := \frac{\phi(t-d)}{\rho^{1/2}(t)}, \tag{2.2}$$

$$g(t) := \frac{1}{1+\bar{\phi}^T(t-d)F(t-d)\bar{\phi}(t-d)}, \tag{2.3}$$

$$e(t) := y(t) - \theta^T(t-d)\phi(t-d), \tag{2.4}$$

$$\bar{e}(t) := \frac{e(t)}{\rho^{1/2}(t)}, \tag{2.5}$$

$$F^1(t) := F(t-d) - g(t)F(t-d)\bar{\phi}(t-d)\bar{\phi}^T(t-d)F(t-d), \tag{2.6}$$

$$F(t) := \left(1 - \frac{\lambda_0}{\lambda_1}\right)F^1(t) + \lambda_0 I \quad (\text{we choose } \lambda_0 I \leqq F(0) \leqq \lambda_1 I), \tag{2.7}$$

$$\theta^1(t) := \theta(t-d) + g(t)F(t-d)\bar{\phi}(t-d)\bar{e}(t), \tag{2.8}$$

$$\theta^2(t) := \theta^1(t) + \max(0, \sigma_0 - s_0^1(t))\frac{F_1(t)}{F_{11}(t)} \tag{2.9}$$

where

$s_0^1(t) :=$ first element of the vector $\theta^2(t)$,

$F_1(t) :=$ first column of the matrix $F(t)$,

$F_{11}(t) := (1, 1)$th element of $F(t)$,

$$\theta(t) := \theta^c + (\theta^2(t) - \theta^c)\min\left(1, \frac{K\lambda_1}{\lambda_0\|\theta^2(t) - \theta^c\|}\right). \tag{2.10}$$

Finally, the control input is defined implicitly through

$$\theta^T(t)\phi(t) = y^m(t+d). \tag{2.11}$$

Explanation of adaptive control algorithm. There are essentially only three features of our adaptive control law that are different from the usual adaptive control laws.

Normalization. The sequence $\rho(t)$ is a *normalization* (or *scaling*) sequence. The vector $\bar{\phi}(t-d)$ obtained by normalizing (i.e., dividing) $\phi(t-d)$ by $\rho^{1/2}(t)$ is then the *normalized* regression vector, and similarly $\bar{e}(t)$ is the *normalized prediction error.* These normalized signals are then used to update the parameter estimates.

Condition number bounding. The matrix $F(t)$ is what is usually called the "covariance matrix." It is well known in recursive identification (see Lai and Wei [13] and Kumar and Varaiya [12]) that if the condition number of the so-called "covariance matrix" remains bounded as $t \to \infty$, then the parameter estimates converge to their true values. Equation (2.7) ensures that the eigenvalues of $F(t)$ remain within the interval $[\lambda_0, \lambda_1]$, thereby keeping the condition number uniformly bounded. (In fact, as the reader can verify, any $F(t) \geqq F^1(t)$ satisfying the property that its eigenvalues lie in the interval $[\lambda_0, \lambda_1]$ can be used.)

Parameter estimate projection. Finally there is a set of two modifications that ensure that the parameter estimates are kept bounded, while at the same time making sure that the first component of the vector $\theta(t)$ (which is an estimate of the so-called "high-frequency gain" of the system) is kept positive and bounded below. This is done in two stages. The first stage, (2.9), ensures that the first component is larger than σ_0. The second stage, (2.10), keeps the parameter estimates inside the sphere with center θ^c and radius $K\lambda_1/\lambda_0$ by projecting them radially onto the surface of the sphere whenever they wander outside.

Remarks on modifications. The reasonableness of the modifications of normalization and eigenvalue bounding can be seen from the following calculation. Normal *unmodified* adaptive control laws using least-squares parameter estimates would use the (d interlaced) recursions

$$\theta(t)=\theta(t-d)+\frac{R^{-1}(t-d)\phi(t-d)}{1+\phi^T(t-d)R^{-1}(t-d)\phi(t-d)}(y(t)-\theta^T(t-d)\phi(t-d)),$$

$$R(t)=R(t-d)+\phi(t-d)\phi^T(t-d).$$

These recursions are clearly equivalent to

$$\theta(t)=\theta(t-d)+\frac{\left(\frac{R(t-d)}{\rho(t)}\right)^{-1}\frac{\phi(t-d)}{\rho^{1/2}(t)}}{1+\frac{\phi^T(t-d)}{\rho^{1/2}(t)}\left(\frac{R(t-d)}{\rho(t)}\right)^{-1}\frac{\phi(t-d)}{\rho^{1/2}(t)}}\frac{(y(t)-\theta^T(t-d)\phi(t-d)}{\rho^{1/2}(t)},$$

i.e.,

$$\theta(t)=\theta(t-d)+\frac{\left(\frac{R(t-d)}{\rho(t)}\right)^{-1}\bar{\phi}(t-d)}{1+\bar{\phi}^T(t-d)\left(\frac{R(t-d)}{\rho(t)}\right)^{-1}\bar{\phi}(t-d)}\bar{e}(t).$$

Thus we see that modified adaptive control uses $F(t-d)$ instead of $(R(t-d)/\rho(t))^{-1}$. This is reasonable since $R(t-d)/\rho(t)\leqq I$, and $F(t-d)$ also has a lower-bounded minimum eigenvalue. Hence both $R^{-1}(t-d)/\rho(t)$ and $F(t-d)$ are of the *same order* and grow at the same rate. Last, the bounding of the *maximum eigenvalue* of $F(t-d)$ is a reasonable effort at keeping the condition number bounded.

An intuitive rationale for the introduction of normalization is the following. Let us consider the case where the system is not of the order assumed. Then, generally we can assume that the system can be represented in the following form (which also allows infinite-dimensional systems):

$$y(t)=ay(t-1)+bu(t-1)+\sum_{i=2}^{t}(\alpha_i y(t-i)+\beta_i u(t-i))$$

where the summation represents the portion of the system dynamics that has not been modeled. Then, under the assumption that $n_S=n_R=0$, we have $\phi(t-1)=(u(t-1),y(t-1))$, and so for any $\theta=(\theta_1,\theta_2)^T$,

$$y(t)-\phi^T(t-1)\theta=(a-\theta_2)y(t-1)+(b-\theta_1)u(t-1)+\sum_{i=2}^{t}(\alpha_i y(t-i)+\beta_i u(t-i)).$$

This *modeling error* may be unbounded irrespective of the choice of θ. However, the neglected component can be bounded by

$$\left|\sum_{i=2}^{t}(\alpha_i y(t-i)+\beta_i u(t-i))\right| \leqq \sqrt{2}\left(\sum_{i=2}^{t}\alpha_i^2+\beta_i^2\right)^{1/2}\left(\sum_{i=2}^{t}y^2(t-i)+u^2(t-i)\right)^{1/2}$$

by the Cauchy-Schwarz inequality. Noting that $\sum_{i=2}^{t}(y^2(t-i)+u^2(t-i)) \leqq \rho(t)$, where $\rho^{1/2}(t)$ is the *normalization factor*, we see that $|\bar{y}(t)-\bar{\phi}^T(t-1)\theta| \leqq M_0$, when $\{\alpha_i\}$ and $\{\beta_i\}$ are in l_2. Hence the error due to mismodeling is bounded when we use the normalized quantities instead of the original variables. This is the heuristic reason for our use of normalization.

The purposeful bounding of the parameter estimates (by keeping them in a certain sphere), which is our last modification, does not cause any problems, at least when the "true parameter vector" is known to satisfy a similar bound, thus allowing convergence of the parameter estimates to their "true values" if that is necessary. As we show later, there need not even be a "true parameter vector" for this modification to be reasonable. In fact, Egardt [5] has shown that some sort of parameter boundedness is necessary for good behavior. Similarly, keeping the first component of the parameter estimates bounded below is tolerable at least when the true parameter vector also has the same lower bound on its first component.

It should be noted that our bounding of the eigenvalues of $F(t)$ is somewhat similar to the case of the *stochastic gradient algorithm* (see Becker, Kumar, and Wei [3]). In fact, the stochastic gradient algorithm is a special case of our modified adaptive controller that is obtained when we choose $\lambda_0=\lambda_1$ in (2.7). In general, however, we expect that the initial transient performance of the adaptive controller will be closer to the least-squares algorithm, but that the asymptotic convergence rate will be governed by that of the gradient algorithm, although we have not been able to establish either of these results analytically.

The modifications present in our adaptive controller, which were first proposed in Praly [21], therefore, all stem from reasonable motivations. In what follows we actually show the power of these modifications in a variety of situations.

3. Some properties of the adaptive controller. Interestingly enough (and very useful to us), the adaptive controller that we defined earlier satisfies some useful conditions *irrespective* of the system to which it is applied.

Let us define Θ as the intersection of the closed sphere with center θ^c and radius K, with the closed half-space $s_0 \geqq \sigma_0$ (where $s_0=$ first component of vector $\theta \in \Theta$). Note that by construction (see (A2.v)) θ^c belongs to Θ. For any $\theta \in \Theta$, we define the *prediction error* by

$$w_\theta(t) := y(t)-\theta^T\phi(t-d) \tag{3.1}$$

and its *normalized* version by $\bar{w}_\theta(t) := w_\theta(t)/\rho^{1/2}(t)$.

We wish to emphasize that the results of this section are obtained without any assumptions on $\bar{w}_\theta(t)$. The following preliminary results are of much interest, and will be very useful to us. Since they are a direct consequence of our definitions, their proofs are omitted.

LEMMA 3.1.
(i) $1 \geqq g(t) \geqq 1/(1+\lambda_1)$;
(ii) *If* $\theta \in \Theta$, *then* $\|\theta(t)-\theta\| \leqq K_1$, *for some constant* K_1;
(iii) $\rho(T) \geqq \sum_{t=1}^{T}\|\phi(t-d)\|^2$.

Proof. The proof is trivial. □

It should be noted that $g(t)$ is the only eigenvalue of the matrix $[I-g(t)F(t-d)\bar{\phi}(t-d)\bar{\phi}^T(t-d)]$ that is not equal to 1. Since (2.8) can be rewritten as $\theta^1(t)=[I-g(t)F(t-d)\bar{\phi}(t-d)\bar{\phi}^T(t-d)]\theta(t-d)+g(t)F(t-d)\bar{\phi}(t-d)\bar{y}(t)$, it is then clear that $g(t)$ tells us how *contractive* the homogeneous part of this update equation is, and (i) provides a lower bound on the rate of convergence of the parameters. Statement (ii) above merely makes note of the fact that $\theta(t)$ is kept bounded.

LEMMA 3.2. *Define a Lyapunov function* $V_\theta(t):=(\theta(t)-\theta)^T F^{-1}(t)(\theta(t)-\theta)$, *for* $\theta\in\Theta$. *Then*

$$V_\theta(t)\leqq V_\theta(t-d)+\bar{w}_\theta^2(t)-g(t)\bar{e}^2(t).$$

Proof.

Step 1. $(F^1(t))^{-1}=F^{-1}(t-d)+\bar{\phi}(t-d)\bar{\phi}^T(t-d)$. After some algebra and (3.1), we have

$$(\theta^1(t)-\theta)^T(F^1(t))^{-1}(\theta^1(t)-\theta)=V_\theta(t-d)+\bar{w}_\theta^2(t)-g(t)\bar{e}^2(t). \tag{3.2}$$

Since $(F^1(t))^{-1}\geqq F^{-1}(t)$ and because of (3.2), we have

$$(\theta^1(t)-\theta)^T F^{-1}(t)(\theta^1(t)-\theta)\leqq V_\theta(t-d)+\bar{w}_\theta^2(t)-g(t)\bar{e}^2(t). \tag{3.3}$$

Step 2. Let $\Delta'(t)=(\theta^2(t)-\theta)^T F^{-1}(t)(\theta^2(t)-\theta)-(\theta^1(t)-\theta)^T F^{-1}(t)(\theta^1(t)-\theta)$. Then some algebra yields $\Delta'(t)=(\theta^2(t)+\theta^1(t)-2\theta)^T F^{-1}(t)(\theta^2(t)-\theta^1(t))$.

Now we consider two cases.

Case 1. If $\sigma_0\leqq s_0^1(t)$ then $\theta^1(t)=\theta^2(t)$ and so $\Delta'(t)=0$.

Case 2. If $\sigma_0>s_0^1(t)$ then

$$\begin{aligned}\Delta'(t)&=(\theta^2(t)+\theta^1(t)-2\theta)^T e_1\frac{\sigma_0-s_0^1(t)}{F_{11}(t)}\\&=\frac{\sigma_0-s_0^1(t)}{F_{11}(t)}[s_0^1(t)+(\sigma_0-s_0^1(t))+s_0^1(t)-2s_0]\leqq 0\quad(\text{since } s_0^1(t)<\sigma_0<s_0)\end{aligned}$$

where $e_1=(1,0,\cdots,0)^T$.

Hence, in any case we have

$$(\theta^2(t)-\theta)^T F^{-1}(t)(\theta^2(t)-\theta)\leqq V_\theta(t-d)+\bar{w}_\theta^2(t)-g(t)\bar{e}^2(t). \tag{3.4}$$

Step 3. For convenience, let M_1 and d_1 denote

$$M_1:=(\theta^2(t)-\theta)^T F^{-1}(t)(\theta^2(t)-\theta)\quad\text{and}\quad d_1:=K\frac{\lambda_1}{\lambda_0}\frac{1}{\|\theta^2(t)-\theta^c\|}.$$

Now consider two cases again.

Case 1. If $d_1\geqq 1$, then $\theta(t)=\theta^2(t)$ and so $M_1=V_\theta(t)$.

Case 2. If $d_1<1$, using (2.10) and the Cauchy-Schwarz inequality, then

$$\begin{aligned}M_1-V_\theta(t)&=(\theta^2(t)+\theta(t)-2\theta)^T F^{-1}(t)(\theta^2(t)-\theta(t))\\&\geqq(1-d_1^2)\|\theta^2(t)-\theta^c\|^2\frac{1}{\lambda_1}-2\frac{K}{\lambda_0}(1-d_1)\|\theta^2(t)-\theta^c\|\\&=\frac{\|\theta^2(t)-\theta^c\|^2}{\lambda_1}[1-d_1^2-2(1-d_1)d_1]\geqq 0.\end{aligned}$$

Hence, in any case, $M_1\geqq V_\theta(t)$ and the result follows. $\square$

The above recursive bound on the "Lyapunov function" will be useful subsequently.

LEMMA 3.3.

(i) $s_0(t) \geqq \sigma_0$;

(ii) $\|\theta(t)\| \leqq \|\theta^c\| + K(\lambda_1/\lambda_0) =: R$;

(iii) $e(t) = y(t) - y^m(t)$;

(iv) $\|\theta(t) - \theta(t-d)\| \leqq \sqrt{\lambda_1}(1 + \sqrt{\lambda_1}/\sqrt{\lambda_0})|\bar{e}(t)|$;

(v) *For any* $\theta \in \Theta$, $e^2(t) \leqq \rho(t)(1+\lambda_1)(V_\theta(t-d) - V_\theta(t)) + (1+\lambda_1)w_\theta^2(t)$ *and* $0 \leqq V_\theta(t) \leqq V_4 := 1/\lambda_0(K + K(\lambda_1/\lambda_0))^2$.

Proof. Formulas (i)-(iii) follow almost by definition.

(iv) Because $\|g(t)F(t-d)\bar{\phi}(t-d)\| \leqq \sqrt{\lambda_1}/2$, it follows that

$$\|\theta^1(t) - \theta(t-d)\| \leqq \frac{\sqrt{\lambda_1}}{2}|\bar{e}(t)|. \tag{3.5}$$

From the algorithm, we can easily see that

$$(\theta^2(t) - \theta^1(t))^T F^{-1}(t)(\theta^2(t) - \theta^1(t)) \leqq \frac{(\sigma_0 - s_0^1(t))^2}{\lambda_0}, \tag{3.6}$$

$$\|\theta^2(t) - \theta^1(t)\|^2 \leqq \frac{\lambda_1}{\lambda_0}\|\theta(t-d) - \theta^1(t)\|^2, \tag{3.7}$$

$$\|\theta(t) - \theta^2(t)\| \leqq \|\theta^2(t) - \theta(t-d)\|. \tag{3.8}$$

Using (3.7) and (3.5), we have

$$\|\theta^2(t) - \theta^1(t)\|^2 \leqq \frac{\lambda_1^2}{4\lambda_0}|\bar{e}(t)|^2. \tag{3.9}$$

Combining (3.8), (3.9), and (3.5), we have

$$\begin{aligned}\|\theta(t) - \theta(t-d)\| &\leqq \|\theta(t) - \theta^2(t)\| + \|\theta^2(t) - \theta(t-d)\| \leqq 2\|\theta^2(t) - \theta(t-d)\| \\ &\leqq \sqrt{\lambda_1}\left(1 + \frac{\sqrt{\lambda_1}}{\sqrt{\lambda_0}}\right)|\bar{e}(t)|.\end{aligned}$$

(v) From Lemmas 3.2 and 3.1(i),

$$\frac{\bar{e}^2(t)}{1+\lambda_1} \leqq g(t)\bar{e}^2(t) \leqq V_\theta(t-d) - V_\theta(t) + \bar{w}_\theta^2(t),$$

and the bound on $e^2(t)$ follows readily. On the other hand,

$$V_\theta(t) \leqq \frac{1}{\lambda_0}\|\theta(t) - \theta\|^2 \leqq \frac{1}{\lambda_0}\left(\|\theta^c - \theta\| + K\frac{\lambda_1}{\lambda_0}\right)^2$$

and the claimed bound follows, since $\|\theta^c - \theta\| \leqq K$ due to the requirement that $\Theta \in \Theta$. □

The first result above merely states that the subsequent projection onto the surface of the sphere continues to preserve the property (i). The fourth result above gives a bound on the *increments* of the parameter estimates in terms of the normalized errors, while the last result gives a *bound* on the normalized errors themselves.

This last result is fundamental. It shows that insofar as the norms of the sequences are concerned, the adaptation law may be regarded as a *static gain* operator with inputs $w_\theta(t)$, $\sqrt{\rho(t)}$ and output $e(t)$. The gain from $w_\theta^2(t)$ to $e^2(t)$ is simply $(1+\lambda_1)$, which increases as the *speed* of adaptation measured by the largest eigenvalue λ_1 is concerned. It tells us that the error given by the parameter estimates will be smaller than $\sqrt{1+\lambda_1}$ times the error given by *any* vector $\theta \in \Theta$. The gain from $\rho(t)$ to $e^2(t)$ is $(1+\lambda_1)(V_\theta(t-d)-V_\theta(t))$. Suppose now that, due to the boundedness of $V_\theta(\cdot)$, the "mean" value of $V_\theta(t-d)-V_\theta(t)$ is close to zero. Then boundedness of $e^2(t)$ will follow from the small gain theorem [4] if the operator $e(t)\to\sqrt{\rho(t)}$ has bounded gain and the operator $e(t)\to w_\theta(t)$ is an operator whose gain multiplied by $\sqrt{1+\lambda_1}$ is smaller than 1. Moreover, since this result holds for *all* $\theta\in\Theta$, we have

$$\frac{1}{T}\sum_{i=t}^{t+T} e^2(i) \leqq \frac{1}{T}\sum_{i=t}^{t+T}\rho(i)(1-\lambda_1)(V_\theta(i-d)-V_\theta(i))+(1+\lambda_1)\min_{\theta\in\Theta}\frac{1}{T}\sum_{i=t}^{t+T} w_\theta^2(i)$$

for all $t\geqq d$ and T. This tells us why optimality can reasonably be expected to hold.

4. Stability in ideal, not necessarily positive real case. In this section we analyze the performance of the adaptive controller when it is applied to minimum phase ARMAX systems of *known* order. We do *not* make the usual positive-real assumption on the coefficients of the colored noise polynomial; in fact, we do not even assume any *stochastic* properties of the disturbance except for mean-square boundedness. Nevertheless we show that the adaptive controller proposed in the previous section mean-square *stabilizes* the system. (In the next section we show that stability result can be strengthened to one of *optimality* when a positive real condition is satisfied.)

We consider therefore the following ideal system:

$$A(q^{-1})y(t)=q^{-d}B(q^{-1})u(t)+C(q^{-1})w(t), \qquad t\geqq 1 \tag{4.1}$$

where

$$A(q^{-1})=1+\sum_{i=1}^{n_A} a_i q^{-i}, \quad B(q^{-1})=\sum_{i=0}^{n_B} b_i q^{-i}, \quad b_0\neq 0, \quad C(q^{-1})=1+\sum_{i=1}^{n_C} c_i q^{-i}.$$

Note that we assume the following:

(A4.i) Positive numbers λ_0, λ_1, delay d and reference output $y^m(t)$ are the same as that used in the adaptive controller (see § 2).

We only assume that the noise or disturbance $\{w(t)\}$ is mean-square bounded, i.e.,

(A4.ii) $$\limsup_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T} w^2(t)\leqq K<\infty \quad \text{a.s.}$$

Regarding the polynomials A, B, and C, we make the following assumptions.

(A4.iii) $B(z)$ has all zeros outside the closed unit disk.

Given $A(q^{-1})$, $B(q^{-1})$, and $C(q^{-1})$, there exist polynomials $S^*(q^{-1})$, $R^*(q^{-1})$, and $Q^*(q^{-1})$ so that

$$S^*(q^{-1})A(q^{-1})+q^{-d}R^*(q^{-1})B(q^{-1})=B(q^{-1}), \tag{4.2}$$

$$S^*(q^{-1})=Q^*(q^{-1})B(q^{-1}). \tag{4.3}$$

We will assume that, with n_S and n_R corresponding to the choices in the adaptive controller, we have

(A4.iv) $S^*(q^{-1})$ is of degree n_S,

(A4.v) $R^*(q^{-1})$ is of degree n_R, i.e.,

$$S^*(q^{-1})=\sum_{i=0}^{n_S} s_i^* q^{-i} \quad \text{and} \quad R^*(q^{-1})=\sum_{i=0}^{n_R} r_i^* q^{-i}.$$

We define

$$\theta^*=(s_0^*,\cdots,s_{n_S}^*,r_0^*,\cdots,r_{n_R}^*,0,\cdots,0)^T, \tag{4.4}$$

and make the following assumptions:

(A4.vi) $(\theta^c-\theta^*)^T(\theta^c-\theta^*)\leqq K$;

(A4.vii) $s_0^*\geqq\sigma_0$, where s_0^* is the first component of θ^*.

With these assumptions in hand, we can proceed to our proof of stability. In what follows we denote the components of $\theta(t)$, the parameter estimate, by

$$\theta(t)=(s_0(t),\cdots,s_{n_S}(t),r_0(t),\cdots,r_{n_R}(t),-c_1(t),\cdots,-c_{n_C}(t))^T. \tag{4.5}$$

As we have observed at the end of § 3, we must first understand how the normalizing sequence $\rho(\cdot)$ is related to $e^2(t)$, the sum of the squares of $y(\cdot)$ and $u(\cdot)$.

LEMMA 4.1.

$$T\rho\leqq\rho(T)\leqq K_2\left(\sum_{t=1}^{T}y^{m^2}(t)+\sum_{t=1}^{T-1}(e^2(t)+w^2(t))\right)+T\rho \quad \textit{for some constant } K_2.$$

Proof. Let

$$\phi^r(t):=(u(t-1),\cdots,u(t-n_S),y(t),\cdots,y(t-n_R),y^m(t+d-1),\cdots,y^m\cdot(t+d-n_C))^T,$$

$$\theta^r(t):=(s_1(t),\cdots,s_{n_S}(t),r_0(t),\cdots,r_{n_R}(t),-c_1(t),\cdots,-c_{n_C}(t))^T.$$

Note that these "reduced" vectors are obtained by removing the first component from the vectors $\phi(t)$ and $\theta(t)$. From Lemma 3.3(iii) and assumption (A4.iii), we obtain that

$$\sum_{t=1}^{T}\|\phi^r(t-d)\|^2\leqq C_1\sum_{t=1}^{T-1}(y^{m^2}(t)+e^2(t)+w^2(t)), \tag{4.6}$$

for some constant C_1. From (2.1) we have

$$\rho(T)\leqq T\rho+\sum_{t=1}^{T}(u^2(t-d)+\|\phi^r(t-d)\|^2). \tag{4.7}$$

Using (2.11), (4.6), and Lemma 3.3(ii), we get

$$\begin{aligned}\sum_{t=1}^{T}u^2(t-d)&=\sum_{t=1}^{T-d}\left(\frac{y^m(t+d)-\theta^{rT}(t)\phi^r(t)}{s_0(t)}\right)^2\\ &\leqq\frac{2}{\sigma_0^2}\sum_{t=1}^{T}y^{m^2}(t)+\frac{2R^2}{\sigma_0^2}\sum_{t=d+1}^{T}\|\phi^r(t-d)\|^2\\ &\leqq C_2\sum_{t=1}^{T}(y^{m^2}(t)+e^2(t)+w^2(t)) \quad \text{for some constant } C_2.\end{aligned} \tag{4.8}$$

When we combine (4.6)-(4.8) the result follows. □

The following is a technical result that we use below.

LEMMA 4.2. *Let $v(t)\geqq0$ be a sequence of positive real numbers for all $t\geqq1$. If $1/T\sum_{t=1}^{T}v(t)\leqq V$, for all $T\geqq1$, then*

(i) $$\sum_{t=q+1}^{q+k}\frac{v(t)}{t}\leqq V\left(1+\log\frac{q+k}{q}\right) \quad \textit{where } q\geqq1;$$

(ii) $$\sum_{t=q+1}^{q+k}\frac{v(t)}{t^\alpha}\leqq\frac{V}{1-\alpha}(q+k)^{1-\alpha} \quad \textit{where } q\geqq1,\quad 0\leqq\alpha<1.$$

Proof. Let $X(T)=(1/T)\sum_{t=1}^{T} v(t)$; then $v(t)=tX(t)-(t-1)X(t-1)$.

(i)
$$\sum_{t=q+1}^{q+k} \frac{v(t)}{t}=\sum_{t=q+1}^{q+k} X(t)-X(t-1)+\frac{1}{t}X(t-1)$$
$$=X(q+k)-X(q)+\sum_{t=q+1}^{q+k}\frac{X(t-1)}{t}$$
$$\leqq V\left(1+\sum_{t=q+1}^{q+k}\frac{1}{t}\right)\leqq V\left(1+\log\frac{q+k}{q}\right);$$

(ii)
$$\sum_{t=q+1}^{q+k}\frac{v(t)}{t^\alpha}=\sum_{t=q+1}^{q+k}\frac{t}{t^\alpha}X(t)-\frac{(t-1)}{t^\alpha}X(t-1)$$
$$\leqq (q+k)^{1-\alpha}X(q+k)+\sum_{t=q+1}^{q+k}(t-1)\left(\frac{1}{(t-1)^\alpha}-\frac{1}{t^\alpha}\right)X(t-1).$$

If $0\leqq\alpha<1$, then $t^\alpha-(t-1)^\alpha\leqq\alpha(t-1)^{\alpha-1}$. Therefore we have
$$\sum_{t=q+1}^{q+k}(t-1)\left(\frac{1}{(t-1)^\alpha}-\frac{1}{t^\alpha}\right)\leqq\alpha\sum_{t=q+1}^{q+k}t^{-\alpha}\leqq\frac{\alpha}{1-\alpha}(q+k)^{1-\alpha}.$$

Hence the result follows. □

LEMMA 4.3. *For any α, $0\leqq\alpha<1$, and with $V_\theta(\cdot)$ the sequence shown to be bounded in Lemma 3.3, there exists a constant C such that*
$$\sum_{t=q+1}^{q+k}t^\alpha(V_\theta(t-d)-V_\theta(t))\leqq C(q+k)^\alpha,\qquad q\geqq d\geqq 1.$$

Proof. The proof is by induction. Consider the case where $d=1$. Then,
$$\sum_{t=q+1}^{q+k}t^\alpha(V_\theta(t-1)-V_\theta(t))\quad(\text{where } 0\leqq V_\theta(t)\leqq V_4 \text{ from Lemma 3.3})$$
$$\leqq q^\alpha V_4+\alpha V_4\sum_{t=q+1}^{q+k}(t-1)^{\alpha-1}\leqq 2V_4(q+k)^\alpha.$$

The induction is now on d. Suppose that for $i=1,\cdots,d-1$ there exist C_i such that
$$\sum_{t=q+1}^{q+k}t^\alpha(V_\theta(t-i)-V_\theta(t))\leqq C_i(q+k)^\alpha,\qquad q\geqq d.$$

Then, let us consider
$$\sum_{t=q+1}^{q+k}t^\alpha(V_\theta(t-d)-V_\theta(t))$$
$$=\sum_{t=q+1}^{q+k}t^\alpha(V_\theta(t-d)-V_\theta(t-d+1))+\sum_{t=q+1}^{q+k}t^\alpha(V_\theta(t-d+1)-V_\theta(t))$$
$$\leqq\sum_{t=q+1}^{q+k}t^\alpha(V_\theta(t-d)-V_\theta(t-d+1))+C_{d-1}(q+k)^\alpha$$
$$\leqq 2(q+k)^\alpha(d-1)V_4+(2V_4+C_{d-1})(q+k)^\alpha=:C_d(q+k)^\alpha$$
and the induction is complete. □

We now reinterpret Lemma 3.3(v) to show that $\bar{e}(t)$ is small in the mean-square sense. This will then show that the operator $e(t)\to\sqrt{\rho(t)}$ can be considered small in the mean static gain operator, at least as far as norms are considered. Unfortunately, this property holds true only for time intervals where $\rho(t)$ is much larger than t.

LEMMA 4.4. *There exist almost surely finite random variables* $\tilde{L}$ *and* $\tilde{w}$ *such that for some given* $\varepsilon > 0$, $q+1 \leqq t \leqq q+k$, $q \geqq 1$, *if* $t/\rho(t) \leqq \varepsilon$ *then* $\sum_{t=q+1}^{q+k} \bar{e}^2(t) \leqq \tilde{L} + \varepsilon \tilde{w}(1+\lambda_1) \log(q+k)/q$.

Proof. From Lemma 3.3(v) and Lemma 3.2, we have

$$\sum_{t=q+1}^{q+k} \bar{e}^2(t) \leqq C_0 + (1+\lambda_1) \min_{\theta \in \Theta} \left(\sum_{t=q+1}^{q+k} \bar{w}_\theta^2(t) \right) \tag{4.9}$$

for some constant C_0. By the definition of θ^*, we have

$$w_{\theta^*}(t) = Q^*(q^{-1}) C(q^{-1}) w(t). \tag{4.10}$$

Because $w(t)$ is almost surely mean-square bounded, i.e., $\limsup_T (1/T) \sum_{t=1}^T w^2(t) < \infty$ a.s., $Q^*(q^{-1})$ and $C(q^{-1})$ are polynomials, from (4.10) we see that there exists an almost surely finite random variable $\tilde{w}$ such that

$$\sup \frac{1}{T} \sum_{t=1}^{T} w_{\theta^*}^2(t) \leqq \tilde{w} \quad \text{a.s.} \tag{4.11}$$

If $t/\rho(t) \leqq \varepsilon$, then $\bar{w}_{\theta^*}^2(t) \leqq \varepsilon(w_{\theta^*}^2(t)/t)$ for $t \in [q+1, q+k]$, $q \geqq 1$. Combining this inequality, Lemma 4.2(i), and (4.11), we have

$$\sum_{t=q+1}^{q+k} \bar{e}^2(t) \leqq C_0 + (1+\lambda_1)\varepsilon\tilde{w}\left(1 + \log\frac{q+k}{q}\right) = \tilde{L} + \varepsilon\tilde{w}(1+\lambda_1)\log\frac{q+k}{q}. \qquad \square$$

With Lemmas 4.1 and 4.4 now established, we are in a position to "close the loop." To do so we need an appropriate version of the small gain theorem given in the next result.

LEMMA 4.5 (Bellman-Gronwall Lemma). *If* $\rho(T) \leqq \tilde{M}_4 T + M_2 \rho(T_0) + \gamma \sum_{t=T_0+1}^{T-1} \bar{e}^2(t) \rho(t)$, *then*

$$\rho(T) \leqq \tilde{M}_4 T + M_2 \rho(T_0) \prod_{t=T_0+1}^{T-1} (1+\gamma \bar{e}^2(t)) + \gamma \tilde{M}_4 \sum_{t=T_0+1}^{T-1} t\bar{e}^2(t) \prod_{i=t+1}^{T-1} (1+\gamma\bar{e}^2(i))$$

for some positive constant M_2 *and some positive random variable* $\tilde{M}_4$.

Proof. The proof uses mathematical induction and we provide a sketch. For $T = T_0 + 1$ statement is obviously true. Suppose that the statement is true for $T_0 + 1 \leqq T \leqq T_1$, then $\rho(T_1+1) \leqq \tilde{M}_4(T_1+1) + M_2\rho(T_0)X_1 + \gamma\tilde{M}_4 X_2$, where

$$X_1 := 1 + \gamma \sum_{t=T_0+1}^{T_1} \bar{e}^2(t) \prod_{j=T_0+1}^{t-1} (1+\gamma\bar{e}^2(j)) = \prod_{j=T_0+1}^{T_1} (1+\gamma\bar{e}^2(j))$$

and

$$\begin{aligned} X_2 &:= \sum_{t=T_0+1}^{T_1} t\bar{e}^2(t) + \gamma \sum_{t=T_0+1}^{T_1} \bar{e}^2(t) \sum_{j=T_0+1}^{t-1} j\bar{e}^2(j) \prod_{i=j+1}^{t-1} (1+\gamma\bar{e}^2(i)) \\ &= \sum_{t=T_0+1}^{T_1} t\bar{e}^2(t) \prod_{i=t+1}^{T_1} (1+\gamma\bar{e}^2(i)). \qquad \square \end{aligned}$$

We now show that the adaptive controller mean square stabilizes the system under the assumptions stated at the beginning of this section. Note that we are *not* assuming a positive real condition on the noise.

THEOREM 4.6. *For system* (4.1), *subject to the assumptions* (A4.i)-(A4.vii), *our algorithm ensures that*:

(i) $\limsup_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} y^2(t) < \infty$ *a.s.*,

(ii) $\limsup_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} u^2(t) < \infty$ *a.s.*

Proof. For $\tilde{w}$ (given by (4.11)), K_2 (given by Lemma 4.1), and λ_1, there exist random variables $\tilde{\varepsilon}$ and $\tilde{\alpha}$ such that

$$\tilde{\alpha} = 1 - K_2 \tilde{\varepsilon} \tilde{w}(1+\lambda_1) \quad \text{and} \quad 0 < \tilde{\alpha} < 1 \quad \text{a.s.} \tag{4.12}$$

Suppose now that there exists a time interval $(T_0, T_1]$ such that

$$\frac{T}{\rho(T)} \leqq \tilde{\varepsilon} \quad \text{for all } T \in (T_0, T_1] \quad \text{and} \quad T_0/\rho(T_0) > \tilde{\varepsilon}$$

where T_1 may be infinite. (Note that if such an interval does not exist, then we are done.)

Because $y^m(t)$ is uniformly bounded and $w(t)$ is almost surely mean-square bounded, by means of Lemma 4.1, there exists $\tilde{M}_1$ such that $\rho(T) \leqq \tilde{M}_1 T + K_2 \sum_{t=1}^{T-1} e^2(t)$ almost surely.

Since $e(t) = (\theta^* - \theta(t-d))^T \phi(t-d) + w_{\theta^*}(t)$, from Lemma 3.1 and (4.11) there exist M_2 and $\tilde{M}_3$ such that $\sum_{t=1}^{T_0} e^2(t) \leqq M_2 \rho(T_0) + \tilde{M}_3 T_0$.

Using this inequality and the Bellman-Gronwall Lemma, we have

$$\rho(T) \leqq \tilde{M}_4 T + M_5 \rho(T_0) \prod_{t=T_0+1}^{T-1} (1 + K_2 \bar{e}^2(t)) + K_2 \tilde{M}_4 \sum_{t=T_0+1}^{T-1} t\bar{e}^2(t) \prod_{i=t+1}^{T-1} (1 + K_2 \bar{e}^2(i)).$$

From Lemma 4.4 and (4.12), we have $\prod_{t=q+1}^{q+k} (1 + K_2 \bar{e}^2(t)) \leqq e^{K_2 \tilde{L}} ((q+k)/q)^{1-\tilde{\alpha}}$. Therefore there exists an almost surely finite random variable $\tilde{M}_6$ such that

$$\rho(T) \leqq \tilde{M}_6 \left[T + \rho(T_0) \left(\frac{T-1}{T_0} \right)^{1-\tilde{\alpha}} + (T-1)^{1-\tilde{\alpha}} \sum_{t=T_0+1}^{T-1} t^{\tilde{\alpha}} \bar{e}^2(t) \right]. \tag{4.13}$$

Choosing $\theta = \theta^*$ in Lemma 3.2, we have $(1/(1+\lambda_1))\bar{e}^2(t) \leqq (V_{\theta^*}(t-d) - V_{\theta^*}(t)) + \bar{w}_{\theta^*}^2(t)$. Hence we get

$$\sum_{t=T_0+1}^{T-1} t^{\tilde{\alpha}} \bar{e}^2(t) \leqq (1+\lambda_1) \sum_{t=T_0+1}^{T-1} t^{\tilde{\alpha}} (V_{\theta^*}(t-d) - V_{\theta^*}(t) + \bar{w}_{\theta^*}^2(t)).$$

From Lemmas 4.3 and 4.2(ii), we have

$$\sum_{t=T_0+1}^{T-1} t^{\tilde{\alpha}} \bar{e}^2(t) \leqq C_1 (T-1)^{\tilde{\alpha}} + (1+\lambda_1) \tilde{\varepsilon} \sum_{t=T_0+1}^{T-1} \frac{w_{\theta^*}^2(t)}{t^{1-\tilde{\alpha}}} \leqq \tilde{M}_7 (T-1)^{\tilde{\alpha}},$$

for some $\tilde{M}_7$. We can rewrite (4.13) as $\rho(T)/T \leqq \tilde{M}_6 (1 + 1/\tilde{\varepsilon} + \tilde{M}_7)$. Hence there exists a random variable $\tilde{\varepsilon}_1$ such that

$$\frac{T}{\rho(T)} \geqq \tilde{\varepsilon}_1 > 0. \tag{4.14}$$

From Lemma 3.1(iii), we know that $1/\tilde{\varepsilon} > (1/T) \sum_{t=1}^{T-d} y^2(t)$. This implies that $\limsup_{T\to\infty} 1/T \sum_{t=1}^{T} y^2(t) < \infty$ a.s. Similarly, $\limsup_{T\to\infty} (1/T) \sum_{t=1}^{T} u^2(t) < \infty$ a.s. □

5. Optimality in the ideal, positive real case. Now we turn attention to the so-called *ideal case*, where the noise *satisfies a positive real condition*, and show that the preceding stability results can be improved to prove that the sample mean-square variance of the output error is actually *optimal*. Also, in the case of *regulation*, we prove that the adaptive controller *self-tunes in a Cesaro sense* to a minimum variance regulator.

Given the system in the previous section, let us suppose that the polynomials

$$S(q^{-1}) := \sum_{i=0}^{n_S} s_i q^{-i} \quad \text{and} \quad R(q^{-1}) := \sum_{i=0}^{n_R} r_i q^{-i}$$

satisfy the equations

$$S(q^{-1})A(q^{-1}) + q^{-d}R(q^{-1})B(q^{-1}) = C(q^{-1})B(q^{-1}), \tag{5.1}$$

$$S(q^{-1}) = Q(q^{-1})B(q^{-1}). \tag{5.2}$$

Then we can define $\theta^0(t) := (s_0, \cdots, s_{n_S}, r_0, \cdots, r_{n_R}, -c_1, \cdots, -c_{n_C})^T$.

Regarding the noise $\{w(t)\}$ we assume the following:

(A5.i) It is a martingale difference sequence on a probability space (Ω, F, P).

Specifically, denoting by F_t the sub-σ-algebra generated by the observation up to and including time t. We assume that:

(A5.ii) $E\{w(t)|F_{t-1}\} = 0$ a.s.;

(A5.iii) $E\{w^2(t)|F_{t-1}\} = \sigma^2$ a.s.;

(A5.iv) $\sup_t E\{|w(t)|^{2+\delta}|F_{t-1}\} < \infty$ a.s. for some $\delta > 0$.

Next, let $v(t) := Q(q^{-1})w(t)$; then

$$E\{v^2(t+d)|F_t\} = \sigma^2 \sum_{i=0}^{d-1} q_i^2 =: v^2 \quad \text{a.s.} \quad \text{where } Q(q^{-1}) := \sum_{i=0}^{d-1} q_i q^{-i}. \tag{5.3}$$

Clearly the minimum tracking variance is v^2 (see Kumar and Varaiya [12]). We now show that our adaptive controller achieves this optimal tracking performance.

THEOREM 5.1. *Suppose that the system* (4.1) *satisfies assumptions* (A4.i), (A4.iii)-(A4.vii), *and* (A5.i)-(A5.iv). *Furthermore, assume the positive realness condition* $\sup_\omega |C(e^{i\omega}) - 1| < 1/\sqrt{1+\lambda_1}$ *and also that* $\theta^0 \in \Theta$. *Then* $\lim_{T\to\infty} 1/T \sum_{t=d}^{T+d} (y(t) - y^m(t))^2 = v^2$ *almost surely.*

Proof. From (5.1) and (4.1), it is easy to see that $(e(t) - v(t))$ is F_{t-d}-measurable. Now let

$$z(t-d) := e(t) - v(t), \tag{5.4}$$

$$b(t) := (\theta^0 - \theta(t))^T \phi(t), \tag{5.5}$$

$$h(t) := b(t) - z(t); \tag{5.6}$$

then it is easy to see that $C(q^{-1})z(t) = b(t)$. Hence $h(t) = (C(q^{-1}) - 1)z(t)$.

Because $C(e^{i\omega})$ is strictly inside the circle with center 1, and radius $1/\sqrt{1+\lambda_1}$, there exists a positive ε such that $\sum_{j=1}^t (z^2(j)/(1+\lambda_1) - h^2(j)) \geqq \varepsilon \sum_{j=1}^t z^2(j)$ for all t.

Let us define a function

$$S(t) := \sum_{j=d+1}^{t} \left(\frac{z^2(j-d)}{1+\lambda_1} - h^2(j-d) - \varepsilon z^2(j-d) \right), \qquad t \geqq d+1 \tag{5.7}$$

with $S(d) := 0$. Obviously, $S(t) \geqq 0$ for $t \geqq d$ and

$$S(t) - S(t-1) = \frac{z^2(t-d)}{1+\lambda_1} - h^2(t-d) - \varepsilon z^2(t-d) \quad \text{for } t \geqq d+1.$$

Since $w_{\theta^0}(t)=e(t)-b(t-d)$, from Lemma 3.2 and (5.4) we get

$$\begin{aligned}V_{\theta^0}(t) &\leqq V_{\theta^0}(t-d)+(\bar{e}(t)-\bar{b}(t-d))^2-g(t)\bar{e}^2(t)\\ &= V_{\theta^0}(t-d)+(\bar{z}(t-d)+\bar{v}(t))^2-2(\bar{z}(t-d)+\bar{v}(t))\bar{b}(t-d)+\bar{b}^2(t-d)\\ &\quad -g(t)(\bar{z}(t-d)+\bar{v}(t))^2\end{aligned}$$

where $\bar{b}(t-d):=b(t-d)/\rho^{1/2}(t)$, $\bar{v}(t):=v(t)/\rho^{1/2}(t)$, and $\bar{z}(t-d):=z(t-d)/\rho^{1/2}(t)$.

Taking the conditional expectation and using Lemma 3.1(i),

$$\begin{aligned}E\{V_{\theta^0}(t)\,|\,F_{t-d}\} &\leqq V_{\theta^0}(t-d)+(1-g(t))\frac{v^2}{\rho(t)}+(\bar{z}(t-d)-\bar{b}(t-d))^2-\frac{1}{1+\lambda_1}\bar{z}^2(t-d)\\ &\leqq V_{\theta^0}(t-d)+(1-g(t))\frac{v^2}{\rho(t)}+\frac{S(t-1)-S(t)}{\rho(t)}-\varepsilon\bar{z}^2(t-d).\end{aligned}$$

However,

$$\frac{S(t-1)}{\rho(t)}\leqq\frac{S(t-1)}{\rho(t-1)},\qquad \frac{S(d)}{\rho(d)}\leqq\frac{M_1}{d\rho},$$

and so,

$$\sum_{t=d+1}^{T}E\{V_{\theta^0}(t)\,|\,F_{t-d}\}\leqq\sum_{t=d+1}^{T}V_{\theta^0}(t-d)+v^2\sum_{t=d+1}^{T}\frac{1-g(t)}{\rho(t)}+\frac{M_1}{d\rho}-\varepsilon\sum_{t=d+1}^{T}\bar{z}^2(t-d).$$

Because

$$\begin{aligned}\frac{1-g(t)}{\rho(t)} &= \frac{\bar{\phi}^T(t-d)F(t-d)\bar{\phi}(t-d)}{\rho(t)(1+\bar{\phi}^T(t-d)F(t-d)\bar{\phi}(t-d))}\\ &\leqq\frac{\lambda_1\phi^T(t-d)\phi(t-d)}{\rho(t)(\rho(t)+\lambda_1\phi^T(t-d)\phi(t-d))}\\ &\leqq\lambda_1\frac{\rho(t)-\rho(t-1)}{\rho^2(t)}\leqq\lambda_1\left(\frac{1}{\rho(t-1)}-\frac{1}{\rho(t)}\right).\end{aligned}$$

This implies that $\sum_{t=d+1}^{T}(1-g(t))/\rho(t)\leqq\lambda_1/d\rho$.

Taking unconditional expectation, and noting that $V_{\theta^0}(t)$ is bounded (surely), there exists M_2 such that $\varepsilon E\{\sum_{t=d+1}^{T}\bar{z}^2(t-d)\}\leqq M_2$. Hence

$$\sum_{t=d+1}^{\infty}\frac{(e(t)-v(t))^2}{\rho(t)}<\infty\quad\text{a.s.}\tag{5.8}$$

From Kronecker's lemma and (4.14), we get

$$\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}(e(t)-v(t))^2=0\quad\text{a.s.}\tag{5.9}$$

Hence $E\{(y(t)-y^m(t))^2\,|\,F_{t-d}\}=E\{(e(t)-v(t)+v(t))^2\,|\,F_{t-d}\}=(e(t)-v(t))^2+v^2$.

Now, continuing as in Lemma 7 of Becker, Kumar, and Wei [3], we get the desired result. □

It should be noted that as $(1+\lambda_1)$ increases, the *speed* of adaptation is increased. However, the condition $\sup_\omega|C(e^{i\omega})-1|<1/\sqrt{1+\lambda_1}$ then becomes more stringent, requiring that the noise be even closer to pure white noise. Hence we see that λ_1 allows a tradeoff between the rate of parameter convergence and the tolerance of the algorithm to colored noise.

In fact, we can even prove that the adaptive regulator self-tunes in a Cesaro sense to the set of optimum minimum variance regulators. To exhibit this result, we concentrate temporarily on the regulation problem. In this case,

$$y^m(t)=0 \quad \text{for every } t,$$

$$\theta^T(t)=(\theta_1(t),\cdots,\theta_{n_S+1}(t),\theta_{n_S+2}(t),\cdots,\theta_{n_S+n_R+2}(t)),$$

$$\phi^T(t)=(u(t),\cdots,u(t-n_S),y(t),\cdots,y(t-n_R))$$

and (2.11) can be rewritten as

$$\theta^T(t)\phi(t)=0. \tag{5.10}$$

Let us define $R'(q^{-1},\theta(t)):=\sum_{i=0}^{n_R}\theta_{n_S+2+i}(t)q^{-i}$ and $S'(q^{-1},\theta(t)):=\sum_{i=0}^{n_S}\theta_{i+1}(t)q^{-i}$. Then from (5.10), we have $u(t)=-(R'(q^{-1},\theta(t))/S'(q^{-1},\theta(t)))y(t)$.

Note that $D:=\{\theta\,|\,R'(q^{-1},\theta)S(q^{-1})=S'(q^{-1},\theta)R(q^{-1})\}$ is the set of parameters that yield a minimum variance regulator. We now have the following result on self-tuning in a Cesaro sense.

THEOREM 5.2. *For every open set* $O\supset D$, $\lim_{T\to\infty}1/T\sum_{t=1}^{T}1(\theta(t)\in O)=1$ *almost surely, where* $1(\cdot)$ *is the indicator function.*

Proof. Because $z(t)=y(t+d)-Q(q^{-1})w(t+d)=E\{y(t+d)\,|\,F_t\}$, from (5.9) we know that (14.i) in Becker, Kumar, and Wei [3] is true. From Lemma 3.3(iv), we have

$$\|\theta(t)-\theta(t-d)\|^2\leqq 2\lambda_i\left(1+\frac{\sqrt{\lambda_1}}{\sqrt{\lambda_0}}\right)^2\left[\frac{(e(t)-v(t))^2}{\rho(t)}+\frac{v^2(t)}{\rho(t)}\right].$$

It is easy to see that $\{X(t):=\sum_{i=1}^{t}(v^2(i)-v)/i;\ F_t\}$ is a martingale. Due to (A5.iv) $X(t)$ converges and so on $(v^2(t)-v)/t\to 0$ almost surely and this implies $v^2(t)/t\to 0$ almost surely. Because $1/\rho$ is the upper bound of $t/\rho(t)$ for every $t>0$, so $v^2(t)/\rho(t)=(v^2(t)/t)t/\rho(t)\to 0$ almost surely. Combining with (5.8), we get $\|\theta(t)-\theta(t-d)\|^2\to 0$ almost surely. Therefore (14.ii) in [3] is true.

From Lemma 3.3(ii), (5.10), Theorem 4.6(ii), and Lemma 3.3(i), we can see that (14.iii)-(14.vi) in [3] are true. Hence our result follows from Theorem 19(ii) in [3]. □

6. Robustness of optimal adaptive controller. Having proved in the previous section that our adaptive controller yields *optimal performance* for ideal systems, we now turn in this section to proving that the preceding adaptive controller is *robust*. This means that if mean-square stability holds for an ideal system Π_0 (and it does, as we have shown), it will continue to hold for all systems in an *open neighborhood* of Π_0. For this to make sense, we need to define a topology on the set of linear systems. We will consider the *graph topology* (see Vidyasagar [26]) and show that the adaptive controller applied to systems in a graph topological neighborhood retains mean-square stability. Furthermore, we will also give lower bounds on the size of these graph topological neighborhoods.

Let Γ be the set of proper rational functions $F(q)$ whose poles are all in the open unit disk. Γ is equipped with the norm $\gamma(\cdot)$, defined as $\gamma(F):=\sup_{|q|=1}|F(q)|$, for all $F\in\Gamma$. For a sequence $x(t)$, we define its l_2-norm as $\|x\|_T^2:=\sum_{t=1}^{T}x^2(t)$.

We will prove that our adaptive controller stabilizes nonideal systems Π if they satisfy the following assumptions:

(A6.i) Let the system Π be described by the equation

$$A(q)y(t)=B(q)u(t-1)+C(q)w(t),\qquad t\geqq 1 \tag{6.1}$$

where A, B, $C \in \Gamma$, A, B are coprime, B/A is a proper rational function, $A(\infty) = 1 = C(\infty)$, and the noise is a stochastic process that satisfies merely $\sup_T (1/T \sum_{t=1}^{T} w^2(t))^{1/2} \leqq V$, where V is a *deterministic* finite number. (This clearly holds if, for example, the noise is bounded.) We also assume that $|y^m(t)| \leqq M$ for all $t > 0$ and $y^m(t) = u(t) = y(t) = w(t) = 0$ for all $t \leqq 0$.

Because $B(q)$ is an analytic function outside the unit disk, we can write a Laurent series $B(q) = \sum_{i=0}^{\infty} h_i q^{-i}$ and, for $d \geqq 2$, set $P(q^{-1})$ equal to $\sum_{i=0}^{d-2} h_i q^{-i}$; otherwise it equals zero, and $D(q) := \sum_{i=0}^{\infty} h_{i+d-1} q^{-i}$.

It is easy to see that

$$B(q) = P(q^{-1}) + q^{1-d} D(q). \tag{6.2}$$

Note that for the ideal system Π_0 (as in § 5), $P_0(q^{-1}) = 0$ and $D_0(q) = B_0(q^{-1}) \in \Gamma$. Because Π_0 is minimum phase, $D_0(q)$ is strictly stably invertible. Motivated by this, we assume the following:

(A6.ii) $D(q)$ is an invertible element of Γ (i.e., $D(q)$ and $D^{-1}(q)$ belong to Γ, or we can say $D(q)$ is a *unit* of Γ).

For a system Π, we define

$$T(q) = (A(q), \cdots, q^{-n_S} A(q), q^{-1} B(q), \cdots, q^{-(n_R+1)} B(q), 0, \cdots, 0)^T.$$

With $D(q)$ a unit of Γ, for every θ we can define a new element of Γ by

$$H_\theta(q) := 1 - D^{-1}(q) T^T(q) \theta. \tag{6.3}$$

Clearly $\gamma(H_\theta)$ is a continuous function of θ. Hence we can choose $\tilde{\theta} \in \Theta$ so that $\gamma(H_{\tilde{\theta}}) \leqq \gamma(H_\theta)$ for all $\theta \in \Theta$. Next we assume the following:

(A6.iii) $\gamma(H_{\tilde{\theta}}) < \gamma_h$, where $\gamma_h = 1/\gamma_4 = 1/\sqrt{1+\lambda_1}$;

(A6.iv) $\gamma(P) < (\gamma_h - \gamma(H_{\tilde{\theta}})) / (\gamma(D^{-1})(k_1 \gamma(D^{-1}) \gamma(A) + k_2 + k_3(\gamma_h - \gamma(H_{\tilde{\theta}}))) \gamma_3^{d-1}))$, where k_1, k_2, and k_3 are strictly positive constants given in the Table 1 in the Appendix (as is γ_3 also).

We illustrate these assumptions by the following two examples.

Example 1. Consider the adaptive controller with $n_S = 0$, $n_R = 0$, $n_C = 0$, i.e., $\phi(t) = (u(t), y(t))$. We now examine the above assumptions by allowing only one parameter to vary. Consider $\theta = (1, r)$, which denotes that the adaptive controller is associated with the idealized plant

$$(1 + r q^{-1}) y(t) = u(t-1) + w(t).$$

However, suppose that the true plant is given by

$$A(q) = 1 + a_1 q^{-1} + a_2 q^{-2}, \quad B(q) = 1, \quad C(q) = 1.$$

Then straightforward computations give

$$T(q) = (1 + a_1 q^{-1} + a_2 q^{-2}, q^{-1})^T, \qquad D(q) = 1,$$
$$H_\theta(q) = -q^{-1}(a_1 + r + a_2 q^{-1})$$

and it follows that

$$\gamma(H_\theta) = \sup_{0 \leqq \alpha \leqq 2\pi} \sqrt{(a_1+r)^2 + 2a_2(a_1+r)\cos\alpha + a_2^2} = |a_1 + r| + |a_2|.$$

Therefore we get $\tilde{\theta} = (1, -a_1)^T$, $H_{\tilde{\theta}} = -a_2 q^{-2}$, and $\gamma(H_{\tilde{\theta}}) = |a_2|$.

Hence, for this problem, with the expression of γ_h, assumption (A6.iii) is just equivalent to $|a_2| < 1/\sqrt{1+\lambda_1}$.

Note that all the other assumptions are satisfied. Thus by this inequality we see that λ_1 also allows a tradeoff between the rate of parameter convergence and the size of the allowed value of a_2 (see also the last comment of § 5).

Example 2. To illustrate that our assumptions do not require the system to be of minimum phase, we now consider $n_S=1$, $n_R=0$, $n_C=0$, $d=2$, i.e., $\phi(t)=(u(t), u(t-1), y(t))$. We now study the assumptions by the variation of two parameters, so suppose that $\theta=(1, s, r)^T$; this clearly corresponds to an adaptive controller for an idealized plant:

$$(1+aq^{-1})y(t)=q^{-1}u(t-1)+w(t).$$

However, suppose that the true plant is $(1+aq^{-1})y(t)=(b+q^{-1})u(t-1)+w(t)$. Then we have

$$T(q)=(1+aq^{-1}, q^{-1}(1+aq^{-1}), (b+q^{-1})q^{-1}), \qquad D(q)=1,$$

$$P(q^{-1})=b, \qquad H_\theta(q)=-q^{-1}(a+s+rb+(as+r)q^{-1}).$$

Therefore we get

$$\gamma(H_\theta)=|a+s+rb|+|as+r|, \quad \gamma(H_{\tilde\theta})=0, \quad \tilde\theta=\left(1, -\frac{a}{1-ab}, \frac{a^2}{1-ab}\right).$$

Hence all the assumptions are satisfied if (A6.iv) holds, i.e.,

$$|b|<\frac{\gamma_h}{k_1(1+|a|)+k_2+k_3\gamma_h\gamma_3},$$

which reduces to

$$|b|\left\{\sqrt{2(1+\lambda_1)}R(1+|a|)+R\sqrt{1+\lambda_1}+2\sqrt{2}\left\{\left(1+\frac{2R^2}{\sigma_0^2}\right)\left[\frac{\sup|w(t)|}{\sqrt{\rho}}+2(|a|+|b|)+2\right]^2 + 3+\frac{\sup y^{m^2}}{\rho}\left[1+\frac{2}{\sigma_0^2}(1+R^2)\right]+\frac{2R^2}{\sigma_0^2}\right\}^{1/2}\right\}<1.$$

Note that the actual plant has a zero at $-1/b$. Thus we see that if the plant is nonminimum phase, then we can model the unstable zeros by delays, provided these zeros are *large* enough. We notice that by reducing the size of the parameter domain (i.e., by decreasing R and increasing σ_0), we allow smaller unstable zeros. This is a manifestation of the well-known fact that high gains may cause problems in the presence of unmodeled dynamics.

We see also that the threshold for the unmodeled unstable zero depends on the l_∞-norm of the forcing signals w and y^m of the closed-loop system. This is a manifestation of its nonlinear nature. However, since these norms are divided by $\sqrt{\rho}$, we can overcome this difficulty by choosing the threshold ρ in (2.1) proportional to the square of these norms.

We consider a graph topology constructed from the set Γ. All the properties of [26] can be rederived here. Specifically, this topology is the weakest one such that feedback stability is robust and closed-loop transfer functions are continuous (with respect to the "sup" norm). Since this topology on the collection of systems Π follows from the topology on Γ^3, our robustness result follows from the following theorem.

THEOREM 6.1. *The set of* (A, B, C) *satisfying assuptions* (A6.ii)–(A6.iv) *is open.*

Proof. The set

$$\left\{(\bar{A}, \bar{B}, \bar{C}, F, H, G) \mid \bar{A}, \bar{B}, \bar{C}, F, H, G \in \Gamma, F \in U, \gamma(H) < \gamma_h, \right.$$

$$\left. \gamma(G) < \frac{\gamma_h - \gamma(H)}{\gamma(F)(k_1\gamma(F)\gamma(\bar{A}) + k_2 + k_3(\gamma_h - \gamma(H)))\gamma_3^{d-1}(\bar{A}, \bar{B}, \bar{C})}\right\}$$

is an open set of Γ^6 where U denotes the set of units of Γ. This holds since U is an open subset of Γ, and the mapping $F^{-1} \to F$ is continuous on U (see [26]).

Let us prove that the mapping $(A, B, C) \to P$ is continuous. Using the Cauchy-Schwarz inequality, and Parseval's theorem (see [4]) we have $\gamma(P) \leqq \sum_{i=0}^{d-2} |h_i| \leqq \sqrt{d-1}\gamma(B)$. Since the mapping $(A, B, C) \to P$ is linear, and as we have just shown, also bounded, this proves that it is continuous. This implies that the mapping $(A, B, C) \to D$ is continuous, and $(A, B, C) \to H_\theta$ is continuous, for any fixed θ. Hence, for any fixed θ, the mapping $(A, B, C) \to (D, H_\theta, P)$ is continuous.

Therefore the set

$$\psi_\theta := \left\{(A, B, C) \mid D \in U, \gamma(H_\theta) < \gamma_h, \right.$$

$$\left. \gamma(P) < \frac{\gamma_h - \gamma(H_\theta)}{\gamma(D^{-1})(k_1\gamma(D^{-1})\gamma(A) + k_2 + k_3(\gamma_h - \gamma(H_\theta)))\gamma_3^{d-1}(A, B, C)}\right\}$$

is open, and therefore $\bigcup_{\theta \in \Theta} \psi_\theta$ is also open. The result follows. □

Before showing the proof of Theorem 6.8, the main robustness theorem, we need some results. As Lemma 3.3 shows, it is sufficient to prove that the operator $e(t) \to \sqrt{\rho(t)}$ has a finite gain and the operator $e(t) \to w_{\tilde{\theta}}(t)$ has a gain bounded by $1/\sqrt{1+\lambda_1}$. In what follows, we use a number of positive constants α_i, β_i, γ, δ_i, V_i, and k_i, given in Table 1 in the Appendix, that depend on $\gamma(A)$, $\gamma(B)$, $\gamma(C)$, M, V, K, R, n_C, n_S, n_R, μ, d, ρ, λ_0, λ_1, and σ_0.

LEMMA 6.2.

(i) $\rho(t) \geqq \rho(t-1)$;

(ii) $\|\phi\|_T^2 \leqq \rho(T+d) \leqq \|\phi\|_T^2 + T\rho + V_1$;

(iii) $\frac{1}{2}(\|u\|_t + \|y\|_t) \leqq \|\phi\|_t \leqq \gamma_1\|u\|_t + \gamma_2\|y\|_t + \sqrt{t}\alpha_1$;

(iv) $\|w\|_T \leqq \sqrt{T}V \leqq (V/\sqrt{\rho})\rho^{1/2}(T)$.

Proof. Formulae (i) and (iv) are immediate.

(ii) Since $\rho(t+d) - \rho(t+d-1) \leqq \rho + \|\phi(t)\|^2$, we have $\sum_{t=1}^T (\rho(t+d) - \rho(t+d-1)) \leqq T\rho + \|\phi\|_T^2$. Choosing $V_1 := \rho(d)$, we find that $\rho(T+d) \leqq T\rho + \|\phi\|_T^2 + V_1$.

(iii) The left-hand inequality is obvious; for the right-hand side,

$$\|\phi\|_t^2 \leqq (1+n_S)\|u\|_t^2 + (1+n_R)\|y\|_t^2 + n_C t M^2.$$

Now choosing $\gamma_1 := \sqrt{1+n_S}$, $\gamma_2 := \sqrt{1+n_R}$, and $\alpha_1 := M\sqrt{n_C}$, we get the result. □

LEMMA 6.3. $\rho(t+1) \leqq \gamma_3^2 \rho(t)$.

Proof.

$$|y(t+1)| \leqq w(t+1)| + \gamma(B)\|u\|_t + \gamma(A-1)\|y\|_t + \gamma(C-1)\|w\|_t. \tag{6.4}$$

Define $\phi^r(t) := (u(t-1), \cdots, u(t-n_S), y(t), \cdots, y(t-n_R), y^m(t+d-1), \cdots, y^m(t+d-n_C))^T$ and $\theta^r(t) := (s_1(t), \cdots, s_{n_S}(t), r_0(t), \cdots, r_{n_R}(t), -c_1(t), \cdots, -c_{n_C}(t))^T$. Then the control law becomes

$$u(t) = \frac{y^m(t+d) - \theta^{rT}(t)\phi^r(t)}{s_0(t)}. \tag{6.5}$$

By Lemma 6.2(ii) we have

$$\|\phi^r(t)\|^2 \leqq \rho(t+d-1)+y^2(t)+M^2. \tag{6.6}$$

Putting (6.5) and (6.6) together, we have

$$u^2(t) \leqq \frac{2}{\sigma_0^2}(M^2+R^2(\rho(t+d-1)+y^2(t)+M^2)). \tag{6.7}$$

Next, with Lemma 6.2(ii), we have

$$\|\phi(t-d+1)\|^2 \leqq \rho(t)+u^2(t-d+1)+y^2(t-d+1)+M^2. \tag{6.8}$$

From (6.4) and Lemma 6.2, we have

$$\begin{aligned} y^2(t-d+1) &\leqq [|w(t-d+1)|+(\gamma(B)+\gamma(A-1))(\|u\|_{t-d}+\|y\|_{t-d}) \\ &\qquad +\gamma(C-1)\|w\|_{t-d}]^2 \\ &\leqq \left(\frac{V}{\sqrt{\rho}}+2(\gamma(B)+\gamma(A-1))+\frac{\gamma(C-1)}{\sqrt{\rho}}V\right)^2\rho(t). \end{aligned} \tag{6.9}$$

Combining (6.7)-(6.9), we have

$$\begin{aligned} \rho(t+1) &\leqq \rho(t)+\rho+\|\phi(t-d+1)\|^2 \\ &\leqq \left(2+\frac{2}{\sigma_0^2}R^2\right)\rho(t)+\left(1+\frac{2}{\sigma_0^2}R^2\right)y^2(t-d+1)+M^2+\rho+\frac{2}{\sigma_0^2}M^2(1+R^2) \\ &\leqq \left(1+\frac{2}{\sigma_0^2}R^2\right)\left[\frac{V}{\sqrt{\rho}}+2(\gamma(B)+\gamma(A-1))+\gamma\frac{(C-1)}{\sqrt{\rho}}V\right]^2\rho(t) \\ &\quad +\left[2+\frac{2}{\sigma_0^2}R^2+\frac{M^2}{\rho}+1+\frac{2}{\rho\sigma_0^2}M^2(1+R^2)\right]\rho(t) \\ &=: \gamma_3^2\rho(t). \end{aligned}$$

□

LEMMA 6.4.

$$\|u\|_{t-d} \leqq \gamma(D^{-1})[\gamma(A)\|y\|_t+\gamma(C)\|w\|_t+\gamma(P)\|u\|_{t-1}].$$

Proof. Using (6.2), we have $D(q)u(t-d)=A(q)y(t)-C(q)w(t)-P(q^{-1})u(t-1)$, that is, $u(t-d)=D^{-1}(q)\{A(q)y(t)-C(q)w(t)-P(q^{-1})u(t-1)\}$. The result follows. □

The next lemma is immediate.

LEMMA 6.5.

(i) $\|y\|_t \leqq \|y^m\|_t+\|e\|_t$;

(ii) $\|\phi(t)\| \leqq \alpha_1$ *for* $t \leqq 0$, *where* $\alpha_1 = M\sqrt{n_C}$.

We therefore see from Lemmas 6.2, 6.4, and 6.5 that the operator $e(t)\to\sqrt{\rho(t)}$ has a bounded l_2-gain, neglecting $\gamma(P)$.

LEMMA 6.6.

$$\|w_{\tilde\theta}\|_t \leqq \gamma(H_{\tilde\theta})\|y\|_t+\gamma(D^{-1})\gamma(P)\beta_1(\|\phi\|_{t-1}+\alpha_1)+\gamma(D^{-1})\alpha_3\|w\|_t+\gamma(D^{-1})\alpha_2M\sqrt{t}.$$

Proof.

$$\tilde\theta^TT(q)y(t)=\tilde\theta^T[B(q)(\phi(t-1)-W(q^{-1})y^m(t+d-1))+C(q)U(q^{-1})w(t)]^T,$$

where

$$U(q^{-1}) := (1,\cdots,q^{-n_s},0,\cdots,0,0,\cdots,0)^T,$$

$$W(q^{-1}) := (0,\cdots,0,0,\cdots,0,q^{-1},\cdots,q^{-n_C})^T.$$

From (3.1), (6.6), and the equality above, we have

$$D(q)w_{\tilde\theta}(t)=[D(q)-T^T(q)\tilde\theta]y(t)+P(q^{-1})\tilde\theta^T\phi(t-1)$$
$$-\tilde\theta^TB(q)W(q^{-1})y^m(t+d-1)+\tilde\theta^TC(q)U(q^{-1})w(t).$$

Using (6.3), we get

$$w_{\tilde\theta}(t)=H_{\tilde\theta}(q)y(t)+D^{-1}(q)P(q^{-1})\tilde\theta^T\phi(t-1)+D^{-1}(q)\tilde\theta^TC(q)U(q^{-1})w(t)$$
$$-D^{-1}(q)\tilde\theta^TB(q)W(q^{-1})y^m(t+d-1).$$

Hence

$$\|w_{\tilde\theta}\|_t\leqq\gamma(H_{\tilde\theta})\|y\|_t+\gamma(D^{-1})\gamma(P)R(\|\phi\|_{t-1}+\alpha_1)$$
$$+\gamma(D^{-1})R\gamma(B)\sqrt{n_C}M\sqrt{t}+\gamma(D^{-1})R\gamma(C)\sqrt{1+n_S}\,\|w\|_t.$$

If we choose $\alpha_2:=R\gamma(B)\sqrt{n_C}$, $\beta_1:=R$, and $\alpha_3:=R\gamma(C)\sqrt{1+n_S}$, the result follows. □

If we neglect the last three terms, this lemma and Lemma 6.5 tell us that the gain of operator $e(t)\to w_{\tilde\theta}(t)$ is $\gamma(H_{\tilde\theta})$.

LEMMA 6.7.

$$\|y^m\|_t\leqq M\sqrt{t}\leqq\alpha_4\rho^{1/2}(t)\quad \text{where } \alpha_4:=M/\sqrt{\rho}.$$

Proof. From the fact that $|y^m(t)|\leqq M$ and $T\rho\leqq\rho(T)$, the result follows. □

Our result on the robustness of the adaptive controller with respect to the graph topology is given by the following theorem.

THEOREM 6.8. *Under assumptions* (A6.i)–(A6.iv), *the adaptive controller in feedback with the system* Π, *yields mean-square bounded inputs and outputs.*

Proof. As observed at the end of § 3, and due to the lemmas above, we can use a small gain argument.

Because $\rho(T)/T\geqq 1/T\sum_{t=1}^{T-d}y^2(t)$, if we can prove that there exists N so that

$$N>\frac{\rho(T)}{T}, \tag{6.10}$$

then $\limsup_{T\to\infty}1/T\sum_{t=1}^{T}y^2(t)<\infty$. Similarly, we will have $\limsup_{T\to\infty}\cdot 1/T\sum_{t=1}^{T}u^1(t)<\infty$.

We now prove (6.10).

From Lemma 3.3(v), we get $\|e\|_T^2\leqq(1+\lambda_1)\sum_{t=1}^{T}(V_{\tilde\theta}(t-d)-V_{\tilde\theta}(t))\rho(t)+(1+\lambda_1)\|w_{\tilde\theta}\|_T^2$. Let

$$\Delta^2(T):=\max\left(0,\sum_{t=1}^{T}(V_{\tilde\theta}(t-d)-V_{\tilde\theta}(t))\frac{\rho(t)}{\rho(T)}\right)\quad\text{and}\quad\gamma_4:=(1+\lambda_1)^{1/2}; \tag{6.11}$$

then

$$\|e\|_T\leqq\gamma_4\Delta(T)\rho^{1/2}(T)+\gamma_4\|w_{\tilde\theta}\|_T. \tag{6.12}$$

From Lemma 6.6 and Lemma 6.2(ii), (iii), we have

$$\begin{aligned}\|w_{\tilde\theta}\|_t\leqq{}&\gamma(H_{\tilde\theta})\|y\|_t+\gamma(D^{-1})\gamma(P)\beta_1(\rho^{1/2}(t-1+d)+\alpha_1)\\&+\gamma(D^{-1})\alpha_3V\sqrt{t}+\gamma(D^{-1})\alpha_2M\sqrt{t}.\end{aligned} \tag{6.13}$$

Substituting (6.13) into (6.12), with Lemmas 6.5 and 6.7 we obtain

$$\|e\|_T\leqq[\gamma_4\Delta(T)+\gamma_4\gamma(D^{-1})\gamma(P)\beta_1\gamma_3^{d-1}]\rho^{1/2}(T)+\gamma_4\gamma(D^{-1})\gamma(P)\beta_1\alpha_1$$
$$+\gamma_4\gamma(H_{\tilde\theta})\|e\|_T+[\gamma_4\gamma(H_{\tilde\theta})M+\gamma_4\gamma(D^{-1})(\alpha_3V+\alpha_2M)]\sqrt{T}.$$

Choose $\beta_2 := \beta_1\gamma_4$, $V_2 := \gamma_4 M$, $\delta_1 := \gamma_4\alpha_3$, $\beta_3 := \gamma_4\beta_1\alpha_1$, $\alpha_5 := \gamma_4\alpha_2 M$; then

(6.14)
$$(1-\gamma_4\gamma(H_{\tilde\theta}))\frac{\|e\|_T}{\sqrt{T}} \leqq (\gamma_4\Delta(T)+\beta_2\gamma(D^{-1})\gamma(P)\gamma_3^{d-1})\frac{\rho^{1/2}(T)}{\sqrt{T}} + V_2\gamma(H_{\tilde\theta}) + \delta_1\gamma(D^{-1})V + \alpha_5\gamma(D^{-1}) + \frac{\gamma(D^{-1})\gamma(P)\beta_3}{\sqrt{T}}.$$

From Lemmas 6.4 and 6.2 we have
$$\|u\|_{T-d} \leqq \gamma(D^{-1})[\gamma(A)\|y\|_T + \gamma(C)V\sqrt{T} + 2\gamma(P)\rho^{1/2}(T+d-1)].$$
Using this inequality and Lemma 6.2, we have
$$\rho^{1/2}(T) \leqq \gamma_1\gamma(D^{-1})\gamma(A)\|y\|_T + \gamma_1\gamma(D^{-1})\gamma(C)\sqrt{T}V + 2\gamma_1\gamma(D^{-1})\gamma(P)\rho^{1/2}(T)\gamma_3^{d-1} + \gamma_2\|y\|_T + (\alpha_1+\sqrt{\rho})\sqrt{T-d} + \sqrt{V_1}.$$
Choose $\beta_4 = 2\gamma_1$, $\delta_2 = \gamma_1\gamma(C)$, $V_3 = \sqrt{\rho}$, and $\gamma_5 = \gamma_1\gamma(A)$; then

(6.15)
$$(1-\gamma(D^{-1})\gamma(P)\gamma_3^{d-1}\beta_4)\frac{\rho^{1/2}(T)}{\sqrt{T}} \leqq (\gamma(D^{-1})\gamma_5+\gamma_2)\frac{\|e\|_T}{\sqrt{T}} + (\gamma(D^{-1})\delta_2 V + \alpha_1 + V_3 + \gamma(D^{-1})\gamma_6 + \gamma_7) + \frac{\sqrt{V_1}}{\sqrt{T}}$$

where we let $\gamma_6 := \gamma_5 M$, $\gamma_7 := \gamma_2 M$.

From assumption (A6.iii), there exists $\varepsilon > 0$, so that $1-\gamma(H_{\tilde\theta})\gamma_4 = \varepsilon$. Substituting (6.14) into (6.15), we get

(6.16)
$$\left(1-\gamma(D^{-1})\gamma(P)\beta_4\gamma_3^{d-1} - \frac{(\gamma(D^{-1})\gamma_5+\gamma_2)}{\varepsilon}(\gamma_4\Delta(T)+\beta_2\gamma_3^{d-1}\gamma(D^{-1})\gamma(P))\right)\frac{\rho^{1/2}(T)}{\sqrt{T}} \leqq M_1\frac{1}{\sqrt{T}} + M_0$$

where
$$M_0 := \gamma(D^{-1})\delta_2 V + \alpha_1 + V_3 + \gamma(D^{-1})\gamma_6 + \gamma_7 + \frac{(\gamma(D^{-1})\gamma_5+\gamma_2)}{\varepsilon}[V_2\gamma(H_{\tilde\theta}) + \alpha_5\gamma(D^{-1}) + \gamma(D^{-1})\delta_1 V],$$
$$M_1 := \sqrt{V_1} + \frac{(\gamma(D^{-1})\gamma_5+\gamma_2)}{\varepsilon}\gamma(D^{-1})\gamma(P)\beta_3.$$

From assumption (A6.iv), we know that $\gamma(P)\gamma(D^{-1})(\beta_4+\beta_2((\gamma(D^{-1})\gamma_5+\gamma_2)/\varepsilon))\gamma_3^{d-1} < 1$. For convenience, define $\eta := 1-\gamma(D^{-1})\gamma(P)(\beta_4+\beta_2((\gamma(D^{-1})\gamma_5+\gamma_2)/\varepsilon))\gamma_3^{d-1} > 0$. Choose some fixed δ such that

(6.17)
$$0 < \delta < \frac{\eta\varepsilon}{\gamma_4(\gamma_2+\gamma_5\gamma(D^{-1}))}.$$

Case 1. For each time T such that $\Delta(T) \leqq \delta$, (6.16) can be rewritten as
$$\left(\eta - \frac{\gamma_4(\gamma_2+\gamma_5\gamma(D^{-1}))}{\varepsilon}\Delta(T)\right)\frac{\rho^{1/2}(T)}{\sqrt{T}} \leqq M_0 + \frac{M_1}{\sqrt{T}}.$$
Let
$$M_2 := \eta - \frac{\gamma_4(\gamma_2+\gamma_5\gamma(D^{-1}))}{\varepsilon}\delta > 0 \quad (\text{from (6.17)});$$

then

$$\frac{\rho^{1/2}(T)}{\sqrt{T}} \leqq \frac{1}{M_2}\left(M_1 \frac{1}{\sqrt{T}} + M_0 \right).$$

Hence $\rho(T)/T$ is bounded almost surely.

Case 2. Consider some time interval, say (T_0, T_1), such that

$$\frac{\rho^{1/2}(T_0)}{\sqrt{T_0}} \leqq \left(\frac{M_1}{\sqrt{T_0}} + M_0 \right) \frac{1}{M_2},$$

$$\frac{\rho^{1/2}(T)}{\sqrt{T}} > \left(\frac{M_1}{\sqrt{T}} + M_0 \right) \frac{1}{M_2}$$

(6.18) for every $T \in (T_0, T_1)$ (where $T_1 - T_0$ may be infinite),

$$\frac{\rho^{1/2}(T_1)}{\sqrt{T_1}} \leqq \left(\frac{M_1}{\sqrt{T_1}} + M_0 \right) \frac{1}{M_2}.$$

On such intervals, we necessarily have $\Delta(T) > \delta$ for every $T \in (T_0, T_1)$. From (6.11), $\Delta(T) > \delta$ yields

$$\sum_{t=1}^{T} \left(V_{\tilde{\theta}}(t-d) - V_{\tilde{\theta}}(t) \right) \frac{\rho(t)}{\rho(T)} > \delta^2. \tag{6.19}$$

Define $W_{\tilde{\theta}}(T) = \sum_{t=T-d+1}^{T} V_{\tilde{\theta}}(t)$, $T \geqq 0$. Note that $W_{\tilde{\theta}}(T) - W_{\tilde{\theta}}(T-1) = V_{\tilde{\theta}}(T) - V_{\tilde{\theta}}(T-d)$ for $T \geqq 1$. Because $0 \leqq V_{\tilde{\theta}}(t) \leqq V_4$, We have

$$0 \leqq W_{\tilde{\theta}}(T) \leqq dV_4. \tag{6.20}$$

From (6.19), we know that

$$\sum_{t=1}^{T} \left(W_{\tilde{\theta}}(t-1) - W_{\tilde{\theta}}(t) \right) \frac{\rho(t)}{\rho(T)} > \delta^2. \tag{6.21}$$

We define

$$W_{\tilde{\theta}}^{av}(T) = \left(\sum_{t=1}^{T} W_{\tilde{\theta}T} \frac{\rho(t+1) - \rho(t)}{\rho(T+1)} \right) + \frac{W_{\tilde{\theta}}(0)\rho(1)}{\rho(T+1)}.$$

Note the following:

(i) From (6.20), $0 \leqq W_{\tilde{\theta}}^{av}(T) \leqq dV_4$;

(ii) From (6.21), $\sum_{t=1}^{T} (W_{\tilde{\theta}}(t-1) - W_{\tilde{\theta}}(t))(\rho(t)/\rho(T)) > \delta^2$;

(iii) $$W_{\tilde{\theta}}^{av}(T) = W_{\tilde{\theta}}^{av}(T-1) \frac{\rho(T)}{\rho(T+1)} + \left(1 - \frac{\rho(T)}{\rho(T+1)} \right) W_{\tilde{\theta}}(T);$$

(iv) From (ii) and (iii),

$$W_{\tilde{\theta}}^{av}(T-1) - W_{\tilde{\theta}}^{av}(T) > \delta^2 \frac{\rho(T+1) - \rho(T)}{\rho(T+1)} \geqq \left(\frac{\delta}{\gamma_3} \right)^2 \frac{\rho(T+1) - \rho(T)}{\rho(T)}, \qquad T \in (T_0, T_1).$$

hence for any $T \in (T_0, T_1)$, we have $\sum_{t=T_0+1}^{T} (\rho(t+1) - \rho(t))/\rho(t) \leqq (\gamma_3/\delta)^2 V_4 d$. Since $\rho(t)$ is increasing,

$$\log \frac{\rho(T)}{\rho(T_0+1)} \leqq \left(\frac{\gamma_3}{\delta} \right)^2 V_4 d \quad \text{or} \quad \frac{\rho^{1/2}(T)}{\rho^{1/2}(T_0+1)} \leqq \exp\left(\frac{1}{2} \left(\frac{\gamma_3}{\delta} \right)^2 V_4 d \right).$$

From Lemma 6.3 and (6.18), we get

$$\frac{\rho^{1/2}(T)}{\sqrt{T}} \leqq \frac{\gamma_3}{M_2}\left(M_1 \frac{1}{\sqrt{T_0}} + M_0\right) \exp\left(\frac{1}{2}\left(\frac{\gamma_3}{\delta}\right)^2 V_4 d\right).$$

Cases 1 and 2 tell us $\rho^{1/2}(T)/\sqrt{T}$ is bounded. □

7. Stability with nonvanishing adaptive gains. In the previous sections the gain of the parameter estimates, or equivalently the stepsize of the adaptation algorithm, has been allowed to converge asymptotically to zero. Indeed this is necessary if we asymptotically want to achieve optimal tracking. However, this vanishing gain also causes the adaptive controller to have asymptotically *diminishing* ability to adjust to *system changes*. Hence, in practice, adaptation gains are frequently prevented from going to zero. Therefore in this section we address the nonvanishing gain case of our adaptive controller.

We choose μ such that $0<\mu<1$. Let Γ' be the set of proper rational functions $F(q)$ whose poles are all in the open disk of radius μ. Γ' is equipped with the norm

$$\gamma(F) = \sup_{|q|=\mu} |F(q)|, \qquad F \in \Gamma'.$$

For a sequence $x(t)$, we define its $l_2(\mu)$-norm as $\|x\|_T^2 := \sum_{t=1}^T \mu^{-2t} x^2(t)$. Note that if $F(q) \in \Gamma'$ and $z(t) = F(q)x(t)$, then $\|z\|_T \leqq \gamma(F)\|x\|_T$.

Let us consider system II, which can be described as follows:

(A7.i) We suppose that the true system satisfies

$$A(q)y(t) = B(q)u(t-1) + C(q)w(t), \qquad t \geqq 1 \tag{7.1}$$

where A, B, $C \in \Gamma'$, A, B are coprime, B/A is a proper rational function, and $A(\infty) = 1 = C(\infty)$. Regarding the noise $w(t)$ we will merely assume that it is bounded, $|w(t+1)| \leqq V$, where V is a *deterministic* finite positive number.

As before we will also assume that $|y^m(t)| \leqq M$ for all $t>0$ and

$$y^m(t) = u(t) = y(t) = w(t) = 0 \quad \text{for all } t \leqq 0.$$

Because $B(q)$ is an analytic function outside the disk of radius μ, we can write a Laurent series $B(q) = \sum_{i=0}^\infty h_i q^{-i}$ and state that if $d \geqq 2$, then $P(d^{-1}) = \sum_{i=0}^{d-2} h_i q^{-i}$; otherwise it equals zero and $D(q) = \sum_{i=0}^\infty h_{i+d-1} q^{-i}$. It is easy to see that $B(q) = P(q^{-1}) + q^{1-d} D(q)$. Then we assume the following:

(A7.ii) $D(q)$ is an invertible element of Γ'.

As before, we define

$$T(q) := (A(q), \cdots, q^{-n_S} A(q), q^{-1} B(q), \cdots, q^{-(n_R+1)} B(q), 0, \cdots, 0)^T.$$

For any θ, we define $H_\theta(q) := 1 - D^{-1}(q) T^T(q) \theta$ and choose $\tilde{\theta} \in \Theta$ so that $\gamma(H_{\tilde{\theta}}) \leqq \gamma(H_\theta)$, for all $\theta \in \Theta$. We also make the following two assumptions:

(A7.iii) $\gamma(H_{\tilde{\theta}}) < \gamma_h$ where $\gamma_h := 1/\gamma_4$;

(A7.iv) $\gamma(P) < (\gamma_h - \gamma(H_{\tilde{\theta}}))/(\gamma(D^{-1})(k_4 \gamma(D^{-1})\gamma(A) + k_5 + k_6(\gamma_h - \gamma(H_{\tilde{\theta}})))\gamma_{10}^{d-1})$ where k_4, k_5, and k_6 are strictly positive constants given in Table 1 in the Appendix (as is γ_{10} also).

Because the mapping $F(q)\to F(\mu q)$ is an isomorphism on the field of rational functions, all the properties of [26] can be used here. In particular we obtain a topology that is the weakest one, such that feedback μ-exponential stability is robust.

THEOREM 7.1. *The set of* (A, B, C) *satisfying assumptions* (A7.ii)-(A7.iv) *is open.*

Proof. The proof is the same as in Theorem 6.1 except that we need to prove that the mapping $(A, B, C)\to P$ is continuous. Note first that $\gamma(P)\leqq\sum_{i=0}^{d-2}|h_i|\mu^{-i}$ and $\sum_{i=0}^{d-2}|h_i|\mu^{-i}\leqq\sqrt{d-1}(\sum_{i=0}^{\infty}h_i^2\mu^{-2i})^{1/2}$. Now, using Parseval's theorem, we have $\sum_{i=0}^{\infty}h_i^2\mu^{-2i}\leqq\sup_{|q|\geqq\mu}|B(q)|^2$. Hence $\gamma(P)\leqq\sqrt{d-1}\gamma(B)$ proves that the mapping $(A, B, C)\to P$ is continuous. □

Now we define a *new* normalization sequence:

$$\rho(t)=\mu^2\rho(t-1)+\max(\rho, \|\phi(t-d)\|^2), \qquad t\geqq 1 \tag{7.2}$$

where $\rho(t)=0$ if $t\leqq 0$ and $0<\rho<\infty$.

It is important to note that in going from (2.1), where we had simply $\mu^2=1$, to (7.2), we have made our assumptions more restrictive. This can be seen by comparing Theorems 6.1 and 7.1, In the latter we need μ-exponential stability, whereas in the former mere exponential stability is sufficient. In particular, this means that in the latter case we cannot neglect a pole-zero pair that nearly cancels and that corresponds to an eigenvalue larger than μ in modulus. We can also note that for the first example of § 6, we now obtain the restriction $|a_2|<\mu/\sqrt{1+\lambda_1}$.

The following lemmas are essentially similar to those in § 6, and so we abbreviate the proofs.

LEMMA 7.2.

(i) $\mu^{-T}|w(T)|\leqq\|w\|_T\leqq\delta_3\mu^{-T}V$;

(ii) $\mu^{-2t}\rho(t)\geqq\mu^{-2(t-1)}\rho(t-1)$;

(iii) $\frac{1}{2}(\|u\|_T+\|y\|_T)\leqq\|\phi\|_T\leqq\gamma_9\|u\|_T+\gamma_{11}\|y\|_T+\alpha_6\mu^{-T}$;

(iv) $\|\phi\|_T\leqq\gamma_8\mu^{-(T+d)}\rho^{1/2}(T+d)\leqq\|\phi\|_T+V_6\mu^{-T}+V_5$.

Proof. (i) The proof follows from the definition of the norm $\|w\|_T$ as $\|w\|_T=\sum_{t=1}^{T}\mu^{-2t}w^2(t)$ and from (A7.i), which assumes $|w(t+1)|\leqq V$.

Inequality (ii) follows from (7.2).

(iii) $\|\phi\|_T^2\leqq(1+n_S)\mu^{-2n_S}\|u\|_T^2+(1+n_R)\mu^{-2n_R}\|y\|_T^2+n_CM^2(\mu^{-2T}/(1-\mu^2))$. Now choose $\gamma_9=\sqrt{1+n_S}\mu^{-n_S}$, $\gamma_{11}=\sqrt{1+n_R}\mu^{-n_R}$, and $\alpha_6=M(\sqrt{n_C}/\sqrt{1-\mu^2})$ and the result follows.

(iv) $\mu^{-2(T+d)}\rho(T+d)\geqq\sum_{t=1}^{T+d}\mu^{-2t}\|\phi(t-d)\|^2\geqq\mu^{-2d}\|\phi\|_T^2$.

When we choose $\gamma_8=\mu^d$, the left inequality in (iv) follows. On the other hand, when we use

$$\sum_{t=1}^{T+d}\mu^{-2t}(\|\phi(t-d)\|^2+\rho)\geqq\mu^{-2(T+d)}\rho(T+d)-\rho(0),$$

$$\mu^{-2d}\left(\frac{1-\mu^{2d}}{1-\mu^2}\alpha_1^2+\|\phi\|_T^2\right)\geqq\sum_{t=1}^{T+d}\mu^{-2t}\|\phi(t-d)\|^2 \quad (\text{where } \alpha_1:=M\sqrt{n_C}),$$

$$\sum_{t=1}^{T+d}\mu^{-2t}\rho=\frac{\mu^{-2}(1-\mu^{-2(T+d)})}{1-\mu^2}p,$$

the right-hand side inequality in (iv) holds if we choose

$$V_5=\alpha_1\frac{\sqrt{1-\mu^{2d}}}{\sqrt{1-\mu^2}} \quad\text{and}\quad V_6=\frac{\sqrt{\rho}}{\sqrt{1-\mu^2}}. \qquad □$$

LEMMA 7.3. $\rho(t-1)\leqq\gamma_{10}^2\rho(t)$.

Proof. If we use $|y(t+1)| \leqq |w(t+1)| + \mu t\gamma(B)\|u\|_t + \mu^{t+1}\gamma(A-1)\|y\|_t + \mu^{t+1}\gamma(C-1)\|w\|_t$ instead of (6.4), and

$$y^2(t-d+1) \leqq [|w(t-d+1)| + (\gamma(B)+\mu\gamma(A-1))\mu^{t-d} \cdot (\|u\|_{t-d} + \|y\|_{t-d}) + \mu^{t-d+1}\gamma(C-1)\|w\|_{t-d}]^2$$
$$\leqq \left[2(\gamma(B)+\mu\gamma(A-1))\mu^{-d}\gamma_8 + (1+\gamma(C-1)\mu)\delta_3\frac{V}{\sqrt{\rho}}\right]^2 \rho(t)$$

instead of (6.9), then we get the desired result. □

LEMMA 7.4.

$$\|u\|_{T-d} \leqq \mu^d\gamma(D^{-1})(\gamma(A)\|y\|_T + \gamma(C)\|w\|_T + \mu^{-1}\gamma(P)\|u\|_{T-1}).$$

Proof. Because $D(q)u(t-d) = A(q)y(t) - C(q)w(t) - P(q^{-1})u(t-1)$, we have $\|u\|_{T-d} \leqq \mu^d\gamma(D^{-1})(\|Ay\|_T + \|Cw\|_T + \mu^{-1}\|Pu\|_{T-1})$. □

The next result is immediate.

LEMMA 7.5.

(i) $\|y\|_T \leqq \|y^m\|_T + \|e\|_T$;

(ii) $\|y^m\|_T \leqq M(\mu^{-2T}(1-\mu^{2T})/(1-\mu^2))^{1/2} \leqq \alpha_8\mu^{-T}$.

LEMMA 7.6.

$$\|w_{\tilde{\theta}}\|_T \leqq \gamma(H_{\tilde{\theta}})\|y\|_T + \gamma(D^{-1})\gamma(P)\beta_6(\|\phi\|_{T-1} + \alpha_1) + \gamma(D^{-1})\alpha_9\mu^{-T} + \gamma(D^{-1})\alpha_{10}\|w\|_T.$$

Proof. The proof is similar to that of Lemma 6.6. □

THEOREM 7.7. *Under assumptions* (A7.i)–(A7.iv), *the adaptive controller in feedback with the system* II, *yields bounded inputs and outputs almost surely.*

Proof. From Lemma 7.2(iv) and (iii), we have $\frac{1}{2}\mu^{-T}y(T) \leqq \mu^{-T}\rho^{1/2}(T+d)$. If we can now prove that there exists $N < \infty$ almost surely so that

$$\rho^{1/2}(T) < N \quad \text{for } T > 0, \tag{7.3}$$

then clearly $y(T) \leqq 2\rho^{1/2}(T+d) < 2N$, so we will have shown that $y(T)$ is bounded almost surely. A similar situation holds regarding $u(T)$ also.

So we only need to prove (7.3). From Lemma 3.3(v), we have

$$\|e\|_T^2 \leqq (1+\lambda_1)\mu^{-2T}\rho(T)\Delta^2(T) + (1+\lambda_1)\|w_{\tilde{\theta}}\|_T^2$$

where

$$\Delta^2(T) := \max\left(0, \sum_{t=1}^{T} (V_{\tilde{\theta}}(t-d) - V_{\tilde{\theta}}(t))\mu^{2(T-t)}\frac{\rho(t)}{\rho(T)}\right). \tag{7.4}$$

Thus

$$\|e\|_T \leqq \gamma_4\mu^{-T}\rho^{1/2}(T)\Delta(T) + \gamma_4\|w_{\tilde{\theta}}\|_T \quad \text{where } \gamma_4 = \sqrt{1+\lambda_1}. \tag{7.5}$$

From Lemmas 7.6 and 7.2, we have

$$\begin{aligned}\|w_{\tilde{\theta}}\|_T \leqq{}& \gamma(H_{\tilde{\theta}})\|y\|_T + \gamma(D^{-1})\gamma(P)\beta_7\mu^{-T}\rho^{1/2}(T)\gamma_{10}^{d+1} \\ &+ \gamma(D^{-1})\gamma(P)\delta_4 + \gamma(D^{-1})(\alpha_9 + \alpha_{10}\delta_3 V)\mu^{-T}\end{aligned} \tag{7.6}$$

where $\beta_7 := \beta_6\gamma_8\mu^{1-d}$ and $\delta_4 := \beta_6\alpha_1$. Substituting (7.6) into (7.5), we obtain

$$\begin{aligned}\|e\|_T \leqq \gamma_4\mu^{-T}\rho^{1/2}(T)\Delta(T) + \gamma_4\gamma(H_{\tilde\theta})\|y\|_T + \gamma(D^{-1})\gamma(P)\beta_8\mu^{-T}\rho^{1/2}(T)\gamma_{10}^{d-1}\\ + \gamma(D^{-1})\gamma(P)\delta_5 + \gamma(D^{-1})(\alpha_{11} + \alpha_{12}V)\mu^{-T}\end{aligned} \tag{7.7}$$

where $\beta_8 := \gamma_4\beta_7$, $\delta_5 := \gamma_4\delta_4$, $\alpha_{11} := \alpha_9\gamma_4$, and $\alpha_{12} := \alpha_{10}\delta_3\gamma_4$.

From Lemma 7.5, we have $\|y\|_T \leqq \|y^m\|_T + \|e\|_T \leqq \alpha_8\mu^{-T} + \|e\|_T$. Using this, we rewrite (7.7) as

$$\begin{aligned}(1-\gamma_4\gamma(H_{\tilde\theta}))\|e\|_T \leqq [\gamma_4\Delta(T) + \gamma(D^{-1})\gamma(P)\beta_8\gamma_{10}^{d-1}]\mu^{-T}\rho^{1/2}(T) + \gamma(D^{-1})\gamma(P)\delta_5\\ + [\gamma_4\gamma(H_{\tilde\theta})\alpha_8 + \gamma(D^{-1})(\alpha_{11} + \alpha_{12}V)]\mu^{-T}.\end{aligned} \tag{7.8}$$

From Lemmas 7.4, 7.2, and 7.3, we have

$$\|u\|_{T-d} \leqq \gamma(D^{-1})[\gamma_{15}\|y\|_T + \gamma(P)\delta_7\mu^{-T}\rho^{1/2}(T)\gamma_{10}^{d-1} + \delta_8\mu^{-T}V]$$

where $\gamma_{15} := \mu^d\gamma(A)$, $\delta_7 := 2\gamma_8\mu$, and $\delta_8 := \gamma(C)\mu^d\delta_3$.

From Lemma 7.2(iv), (iii), and this last inequality, we have

$$\begin{aligned}\mu^{-T}\rho^{1/2}(T) &\leqq \frac{\gamma_9}{\gamma_8}\|u\|_{T-d} + \frac{\gamma_{11}}{\gamma_8}\|y\|_{T-d} + \frac{\alpha_6}{\gamma_8}\mu^{-(T-d)} + \frac{V_5}{\gamma_8} + \frac{V_6}{\gamma_8}\mu^{-(T-d)}\\ &\leqq \left(\frac{\gamma_9}{\gamma_8}\gamma(D^{-1})\gamma_{15} + \frac{\gamma_{11}}{\gamma_8}\right)\|y\|_T + \frac{\gamma_9}{\gamma_8}\gamma(D^{-1})\gamma(P)\delta_7\mu^{-T}\rho^{1/2}(T)\gamma_{10}^{d-1}\\ &\quad + \frac{\gamma_9}{\gamma_8}\gamma(D^{-1})\delta_8\mu^{-T}V + \frac{1}{\gamma_8}\alpha_6\mu^d\mu^{-T} + \frac{V_5}{\gamma_8} + \frac{V_6}{\gamma_8}\mu^{-T}\mu^d.\end{aligned}$$

If we choose

$$\delta_9 := \frac{\gamma_9}{\gamma_8}\delta_7, \quad \gamma_{14} := \frac{\gamma_9}{\gamma_8}\gamma_{15}, \quad \gamma_{13} := \frac{\gamma_{11}}{\gamma_8}, \quad \delta_6 := \frac{\gamma_9}{\gamma_8}\delta_8,$$

$$\alpha_7 := \frac{1}{\gamma_8}\alpha_6\mu^d, \quad V_7 := \frac{V_5}{\gamma_8}, \quad V_8 := \frac{V_6}{\gamma_8}\mu^d,$$

then the inequality above becomes

$$\begin{aligned}&(1-\gamma(D^{-1})\delta_9\gamma(P)\gamma_{10}^{d-1})\mu^{-T}\rho^{1/2}(T)\\ &\leqq [\gamma(D^{-1})\gamma_{14}\alpha_8 + \gamma_{13}\alpha_8 + \gamma(D^{-1})\delta_6 V + \alpha_7 + V_8]\mu^{-T}\\ &\quad + [\gamma(D^{-1})\gamma_{14} + \gamma_{13}]\|e\|_T + V_7.\end{aligned} \tag{7.9}$$

From assumption (A7.iii) there exists $\varepsilon > 0$ such that $1 - \gamma(H_{\tilde\theta})\gamma_4 = \varepsilon$. Substituting (7.8) into (7.9), we have

$$\begin{aligned}&\Bigg[1 - \gamma(D^{-1})\delta_9\gamma(P)\gamma_{10}^{d-1} - \frac{\gamma(D^{-1})\gamma_{14} + \gamma_{13}}{\varepsilon}\\ &\quad \cdot (\gamma_4\Delta(T) + \gamma(D^{-1})\gamma(P)\beta_8\gamma_{10}^{d-1})\Bigg]\rho^{1/2}(T) \leqq M_0 + \mu^T M_1\end{aligned} \tag{7.10}$$

where

$$\begin{aligned}M_0 &:= \frac{\gamma(D^{-1})\gamma_{14} + \gamma_{13}}{\varepsilon}[\gamma_4\gamma(H_{\tilde\theta})\alpha_8 + \gamma(D^{-1})(\alpha_{11} + \alpha_{12}V)] + \gamma(D^{-1})\gamma_{14}\alpha_8 + \gamma_{13}\alpha_8\\ &\quad + \gamma(D^{-1})\delta_6 V + \alpha_7 + V_8,\\ M_1 &:= \frac{\gamma(D^{-1})\gamma_{14} + \gamma_{13}}{\varepsilon}\gamma(D^{-1})\gamma(P)\delta_5 + V_7.\end{aligned}$$

Now, assumption (A7.iv) implies

$$\gamma(P)\gamma(D^{-1})\left(\delta_9+\beta_8\frac{\gamma(D^{-1})\gamma_{14}+\gamma_{13}}{\varepsilon}\right)\gamma_{10}^{d-1}<1.$$

Hence, there exists $\eta>0$ so that

$$\eta:=1-\gamma(D^{-1})\gamma(P)\left(\delta_9+\frac{\gamma(D^{-1})\gamma_{14}+\gamma_{13}}{\varepsilon}\beta_8\right)\gamma_{10}^{d-1}.$$

Now we choose and fix a $\delta>0$ that satisfies $0<\delta<(\eta\varepsilon/(\gamma_4[\gamma(D^{-1})\gamma_{14}+\gamma_{13}]))$ and examine two cases.

Case 1. If for each time T, $\Delta(T)\leqq\delta$, then from (7.10),

$$\left(\eta-\frac{\gamma(D^{-1})\gamma_{14}+\gamma_{13}}{\varepsilon}\gamma_4\Delta T)\right)\rho^{1/2}(T)\leqq M_0+\mu M_1.$$

Hence

$$\rho^{1/2}(T)\leqq\frac{\varepsilon(M_0+\mu M_1)}{\eta\varepsilon-[\gamma(D^{-1})\gamma_{14}+\gamma_{13}]\delta\gamma_4}=:N_1$$

and so $\rho(T)$ is bounded.

Case 2. Suppose, however, that there is a certain time interval (T_0, T_1) such that

$$\begin{aligned}&\rho^{1/2}(T_0)\leqq N_1,\\ &\rho^{1/2}(T)>N_1\quad\text{for } T\in(T_0,T_1)\quad(\text{where } T_1-T_0 \text{ may be infinite}),\\ &\rho^{1/2}(T_1)\leqq N_1.\end{aligned}\tag{7.11}$$

On such an interval, we must have $\Delta(T)>\delta$ for $T\in(T_0, T_1)$. From (7.4) we get

$$\sum_{t=1}^{T}(V_{\tilde{\theta}}(t-d)-V_{\tilde{\theta}}(t))\mu^{2(T-d)}\frac{\rho(t)}{\rho(T)}<\delta^2.$$

Let us define $W_{\tilde{\theta}}(T):=\sum_{t=T-d+1}^{T}V_{\tilde{\theta}}(t)$, for $T\geqq0$, then

$$\sum_{t=1}^{T}(V_{\tilde{\theta}}(t-d)-V_{\tilde{\theta}}(t))\frac{\mu^{-2t}\rho(t)}{\mu^{-2T}\rho(T)}=\sum_{t=1}^{T}(W_{\tilde{\theta}}(t-1)-W_{\tilde{\theta}}(t))\frac{\mu^{-2t}\rho(t)}{\mu^{-2T}\rho(T)}>\delta^2.$$

Note that $0\leqq W_{\tilde{\theta}}(T)\leqq dV_4$, for $T\geqq0$.

If we now define

$$W_{\tilde{\theta}}^{av}(T):=\sum_{t=1}^{T}W_{\tilde{\theta}}(t)\frac{\mu^{-2(t+1)}\rho(t+1)-\mu^{-2t}\rho(t)}{\mu^{-2(T+1)}\rho(T+1)}+\frac{W_{\tilde{\theta}}(0)\mu^{-2}\rho(1)}{\mu^{-2(T+1)}\rho(T+1)},$$

then

$$\sum_{t=1}^{T}(W_{\tilde{\theta}}(t-1)-W_{\tilde{\theta}}(t))\frac{\mu^{-2t}\rho(t)}{\mu^{-2T}\rho(T)}=W_{\tilde{\theta}}^{av}(T-1)-W_{\tilde{\theta}}(T)>\delta^2.\tag{7.12}$$

Note that because $W_{\tilde\theta}(T) \leqq dV_4$, it follows that $W_{\tilde\theta}^{av}(T) \leqq dV_4$. It is easy to see that

$$W_{\tilde\theta}^{av}(T) = \frac{\mu^{-2T}\rho(T)}{\mu^{-2(T+1)}\rho(T+1)} W_{\tilde\theta}^{av}(T-1) + \left(1 - \frac{\mu^{-2T}\rho(T)}{\mu^{-2(T+1)}\rho(T+1)}\right) W_{\tilde\theta}(T).$$

Using (7.12) we have

$$\begin{aligned} &W_{\tilde\theta}^{av}(T-1) - W_{\tilde\theta}^{av}(T) \\ &\quad > \left(1 - \frac{\mu^{-2T}\rho(T)}{\mu^{-2(T+1)}\rho(T+1)}\right)(\delta^2 + W_{\tilde\theta}(T)) - \left(1 - \frac{\mu^{-2T}\rho(T)}{\mu^{-2(T+1)}\rho(T+1)}\right) W_{\tilde\theta}(T) \\ &\quad \geqq \left(\frac{\delta\mu}{\gamma_{10}}\right)^2 \left(\frac{\mu^{-2(T+1)}\rho(T+1) - \mu^{-2T}\rho(T)}{\mu^{-2T}\rho(T)}\right). \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{t=T_0+1}^{T} \frac{\mu^{-2(t+1)}\rho(t+1) - \mu^{-2t}\rho(t)}{\mu^{-2t}\rho(t)} &\leqq \sum_{t=T_0+1}^{T} \left(\frac{\gamma_{10}}{\delta\mu}\right)^2 [W_{\tilde\theta}^{av}(t-1) - W_{\tilde\theta}^{av}(t)] \\ &\leqq \left(\frac{\gamma_{10}}{\delta\mu}\right)^2 dV_4 \quad \text{for } T \in (T_0, T_1). \end{aligned}$$

Because $\mu^{-2t}\rho(t)$ is increasing with respect to time t, from the last inequality we get

$$\frac{\mu^{-2T}\rho(T)}{\mu^{-2(T_0+1)}\rho(T_0+1)} < \exp\left(\left(\frac{\gamma_{10}}{\delta\mu}\right)^2 dV_4\right).$$

and so again $\rho(T)$ is bounded.

When we combine Cases 1 and 2, the theorem is proved. □

8. Conclusions. Here we have analyzed the twin issues of obtaining both *good performance* and *robustness* out of an adaptive controller for linear stochastic systems.

For minimum phase plants of known order, with a known compact set containing a stabilizing regulator, and for which we know the sign and a lower bound for the leading coefficient of the control polynomial, we have shown that our adaptive controller yields *mean-square bounded* inputs and outputs. If the noise additionally satisfies a positive real condition, then we have shown that the adaptive controller is asymptotically *optimal* in the sense of *minimizing* output error variance. We have also presented a *graph topological neighborhood* of an ideal plant, such that the system is mean-square stabilized even when that system is not ideal and when the statistical properties of the noise are violated. This holds true whether the adpative controller is used in a vanishing or a nonvanishing gain mode.

Several open problems remain. It is still not known whether the standard self-tuning regulator using a *least-squares* parameter estimate is mean-square stable, let alone optimal. Moreover, it is not known whether the unmodified adaptive controller possesses good robustness properties. The first question deals essentially with the loss of identifiability, and the consequent unboundedness of the condition number of the so-called "covariance matrix," when the parameter estimates converge. Unfortunately the second issue cannot really be resolved until the first issue is better understood.

Appendix.

TABLE 1

$\alpha_1 = \sqrt{n_C}M$	$\beta_1 = R$
$\alpha_2 = R\gamma(B)\sqrt{n_C}$	$\beta_2 = \beta_1\gamma_4$
$\alpha_3 = R\gamma(C)\sqrt{1+n_S}$	$\beta_3 = \gamma_4\beta_1\alpha_1$
$\alpha_4 = M/\sqrt{\rho}$	$\beta_4 = 2\gamma_1$
$\alpha_5 = \alpha_2 M\gamma_4$	$\beta_5 = \gamma_4\sqrt{dV_4}$
$\alpha_6 = (\sqrt{n_C}/\sqrt{1-\mu^2})M$	$\beta_6 = R/\mu$
$\alpha_7 = (\alpha_6/\gamma_8)\mu^d$	$\beta_7 = \beta_6\gamma_8\mu^{1-d}$
$\alpha_8 = M/\sqrt{1-\mu^2}$	$\beta_8 = \gamma_4\beta_7$
$\alpha_9 = R\gamma(B)\sqrt{n_C}\mu^{-n_C}\alpha_8$	$\beta_9 = R\gamma(B)\sqrt{n_C}\mu^{-n_C}$
$\alpha_{10} = R\gamma(C)\sqrt{1+n_S}\mu^{-n_S}$	$V_1 = \rho(d)$
$\alpha_{11} = \alpha_9\gamma_4$	$V_2 = \gamma_4 M$
$\alpha_{12} = \alpha_{10}\delta_3\gamma_4$	$V_3 = \sqrt{\rho}$
$\delta_1 = \gamma_4\alpha_3$	$V_4 = 1/\lambda_0(\underline{K} + \underline{K}(\lambda_1/\lambda_0))$
$\delta_2 = \gamma_1\gamma(C)$	$V_5 = \alpha_1(\sqrt{1-\mu^{2d}}/\sqrt{1-\mu^2})$
$\delta_3 = 1/\sqrt{1-\mu^2}$	$V_6 = \sqrt{\rho}/\sqrt{1-\mu^2}$
$\delta_4 = \beta_6\alpha_1$	$V_7 = V_5/\gamma_8$
$\delta_5 = \gamma_4\delta_4$	$V_8 = \mu^d(V_6/\gamma_8)$
$\delta_6 = (\gamma_9/\gamma_8)\delta_8$	$k_1 = \beta_2\gamma_1/\gamma_4$
$\delta_7 = 2\gamma_8\mu$	$k_2 = \beta_2\gamma_2/\gamma_4$
$\delta_8 = \mu^d\delta_3\gamma(C)$	$\gamma_5 = \gamma_1\gamma(A)$
$\delta_9 = (\gamma_9/\gamma_8)\delta_7$	$\gamma_6 = \gamma_5 M$
$k_3 = \beta_4$	$\gamma_7 = \gamma_2 M$
$k_4 = \beta_8\gamma_{14}/\gamma_4$	$\gamma_8 = \mu^d$
$k_5 = \beta_8\gamma_{13}/\gamma_4$	$\gamma_9 = \mu^{-n_S}\sqrt{1+n_S}$
$k_6 = \delta_9$	$\gamma_{11} = \mu^{-n_R}\sqrt{1+n_R}$
$k_7 = \gamma_4$	$\gamma_{12} = \mu\gamma_1(\gamma_9/\gamma_8)$
$k_8 = \gamma_4\beta_6\gamma_{10}^d$	$\gamma_{13} = \gamma_{11}/\gamma_8$
$k_9 = \gamma_4\beta_9(M/\sqrt{\rho})$	$\gamma_{14} = \gamma_{15}(\gamma_9/\gamma_8)$
$k_{10} = \gamma_{10}\gamma_4(\delta_3/\sqrt{\rho})$	$\gamma_{15} = \mu^d\gamma(A)$
$\gamma_1 = \sqrt{1+n_S}$	$\gamma_h = 1/\sqrt{1+\lambda_1}$
$\gamma_2 = \sqrt{1+n_R}$	
$\gamma_4 = \sqrt{1+\lambda_1}$	

$$
\begin{aligned}
\gamma_3^2 &= 2 + \frac{2}{\sigma_0^2}R^2 + \frac{M^2}{\rho} + 1 + \frac{2}{\rho\sigma_0^2}M^2(1+R^2) \\
&\quad + \left(1 + \frac{2}{\sigma_0^2}R^2\right)\left[\frac{V}{\sqrt{\rho}} + 2(\gamma(B) + \gamma(A-1)) + \frac{\gamma(C-1)}{\sqrt{\rho}}V\right]^2,
\end{aligned}
$$

$$
\begin{aligned}
\gamma_{10}^2 &= 1 + \mu^2 + \frac{2}{\sigma_0^2}R^2 + \frac{1}{\rho}\left(\rho + \frac{2}{\sigma_0^2}M^2(1+R^2) + M^2\right) \\
&\quad + \left(1 + \frac{2}{\sigma_0^2}R^2\right)\left[2(\gamma(B) + \gamma(A-1))\mu^{-d}\gamma_8 + (1 + \gamma(C-1)\mu)\delta_3\frac{V}{\sqrt{\rho}}\right]^2
\end{aligned}
$$

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