A COUNTER-EXAMPLE TO THE ROBUSTNESS OF THE PROPERTY "ANY SOLUTION IS BOUNDED"

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ABSTRACT

We give a vector field f on $\mathbf{R}^2$ with a globally attractive fixed point. It is nowhere small except near this fixed point (its norm is greater that one outside a bowl). Nevertheless, an arbitrarily C^0-small perturbation makes some solution go to infinity.

The study is presented as an annex of [1], namely as an example pointing out the necessaryness of an hypothesis in a robustness theorem exposed in this paper. We first recall this theorem; then we show that all its hypothesis except one are satisfied, and that its conclusion (boundedness of all the solutions of the perturbed equation) fails. The missing hypothesis has to do with the boundedness of the transitions matrix of the linearized unperturbed system along reverse time trajectories. We see how and where it fails.

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REFERENCE :

[1] J.-B. POMET / L.PRALY *A Result on Robust Boundedness.* To appear, 1987

1. RECALL OF THE THEOREM 1 IN [1]

This result consists in deducing from a property *for any fixed* p, of the (autonomous) ordinary differential equation

$$\dot{x} = f(p,x) \tag{1}$$

a slightly weaker property of the (non-autonomous) o.d.e.

$$\dot{x} = f(p(t),x) + e(t) \tag{2}$$

1.1. Notations

- Π is a bounded open subset of $\mathbf{R}^q$ (in which the parameter p evolves)
- $(p,x) \rightarrow f(p,x)$ is a C^1- map from $\overline{\Pi}\times\mathbf{R}^n$ to $\mathbf{R}^n$ ($\overline{\Pi}$ stands for the closure of the set Π).
- Φ_p is the flow of (1), i.e. $\quad \dfrac{\partial}{\partial t}\left[\Phi_p^t(x)\right] = f(p,\Phi_p^t(x)) \quad ; \quad \Phi_p^0(x) \equiv x \qquad (11)$
- R is defined by: $$R(p,x,t) = \frac{\partial}{\partial x}\left[\Phi_p^t(x)\right] \tag{12}$$

$R(p,.,.)$ is also the transition matrix of the system (1) linearized along the solutions, i.e.:

$$\frac{\partial}{\partial t}\left[R(p,x,t)\right] = \frac{\partial f}{\partial x}(p,\Phi_p^t(x)) \,.\, R(p,x,t) \quad ; \quad R(p,x,0) \equiv I_{\mathbf{R}^n} \tag{13}$$

1.2. The theorem

Assumption (H_o)

- $\Phi_p^t(x)$ is well defined for any $p\in\Pi$, $x\in\mathbf{R}^n$ $\quad t>0$
- There exists a C^1-function $G : \Pi\times\mathbf{R}^n \rightarrow \mathbf{R}$, such that

  1- $L = \{(x,p) \mid G(p,x) \leq 0\}$ and $K_p = \{x \mid G(p,x) \leq 0\}$ (for any p) are compact sets. $\qquad (15)$

  2- There exists a function T $\Pi\times\mathbf{R}^n \rightarrow \mathbf{R}$ (this is the time needed to reach K_p), such that

$$\forall (p,x,t) \quad \begin{array}{l} t < T(p,x) \Rightarrow \Phi_p^t(x) \notin K_p \\ t > T(p,x) \Rightarrow \Phi_p^t(x) \in \overset{\circ}{K}_p \ \text{(the interior set of } K_p) \end{array} \tag{16}$$

- There exists a positive constant C such that, for ξ in ∂K_p, $\quad \dfrac{\partial G}{\partial x}(p,\xi).f(p,\xi) < -C \qquad (18)$

Hypothesis (*i*) *to* (*iv*)

(*i*) *"No finite escape reverse time"*: For any p in Π , any x in $\mathbf{R}^n$ and any negative t, $\Phi_p^t(x)$ is well defined.

(*ii*) *"Bounded p-sensitivity"*: $\| \frac{\partial f}{\partial p}(p,x) \|$ is bounded for p in Π and x in $\mathbf{R}^n - K_p$

(*iii*) *"Uniform quasi instability in reverse time"*: $\| R(p,x,T(p,x)) \|$ is bounded for p in Π and x in $\mathbf{R}^n - K_p$ or, equivalently, with $\xi = \Phi_p^{T(p,x)}(x)$ and $t=T(p,x)$, $\| [R(p,\xi,-t)]^{-1} \|$ is bounded for p in Π, ξ in ∂K_p and t positive.

(*iv*) "L^k*-bounded perturbations*": $\dot{p}$ belongs to L^1 and e belongs to L^k for some k ($1 \leq k < +\infty$), i.e. $\int_0^\infty | e(t) |^k \, dt < +\infty$ and $\int_0^\infty | \dot{p}(t) | \, dt < +\infty$

Theorem: If (H_o), (*i*), (*ii*), (*iii*), (*iv*) hold, **then** for each solution x(t) of (2), one can find a time t_o such that:

- $x(t_o) \in K_{p(t_o)}$.
- After t_o x(t) is sometimes outside $K_{p(t)}$, but both the length of the time intervals in which this occurs and the maximum distance between x(t) and $K_{p(t)}$ during these intervals tend to zero, and, as a consequence, *dist* (x(t) , $K_{p(t)}$) tends to zero.

For the proof, see [11] *where this is theorem 1.*

2. STUDY OF THE EXAMPLE

We give here a field f on $\mathbf{R}^2$ which doesn't depend on any parameter, but satisfies the hypothesis of the theorem (with a zero dependence on p) except (*iii*); we just add e(t) belonging to L^2, and the conclusion of the theorem fails : some solutions of

$$\dot{x} = f(x) + e(t)$$

go to infinity.Let us define this field f

2.1. The field and the perturbation

Let ψ be the following C^1-map from $\mathbf{R}$ to $\mathbf{R}$

$$\psi(s) = \begin{cases} s^2(3-2s) & \text{on } [0,1] \\ 0 & \text{on }]-\infty,0] \\ 1 & \text{on } [1,+\infty[\end{cases}$$

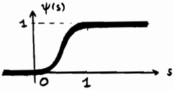

Fig. 1

we define the field f on $\mathbf{R}^2$ by the following formula

$$f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \psi(y+1)\psi(x)\begin{pmatrix} 1 \\ 1 \\ \frac{1}{1+x} \end{pmatrix} + \psi(y+1)[1-\psi(x)]\begin{pmatrix} 1 \\ -2 \end{pmatrix} + [1-\psi(y+1)]\begin{pmatrix} -x \\ -(y+2) \end{pmatrix}$$

and e by

$$e(t) = \begin{pmatrix} 0 \\ \frac{1}{1+t} \end{pmatrix}$$

The compact K is defined by $G(x,y) = x^2 + (y+2)^2 - \frac{1}{4}$ so that

K is the bowl with center (0,-2) and radius $\frac{1}{2}$

Fig.2 gives an idea of the solutions of $\dot{x}=f(x)$.

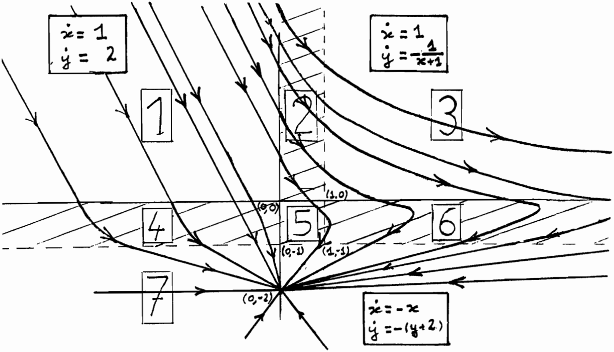

Fig 2

2.2. Assumption (H_o) and hypothesis (*i*),(*ii*),(*iv*) are satisfied

(*i*) is true because f is underlinear.

(*ii*),(*iv*) are true because $\frac{\partial f}{\partial p} = 0$; $\int_0^\infty e^2(t)dt = 1$; $\dot{p} = 0$

Let us check that (H_o) is satisfied:

-The first point is true because f is underlinear.

-The last point (18) is true because the inner product is exactly -1.

-Finally, we must show that, starting from any point outside K, the solutions enter K in a finite time. In fig. 2, seven areas are designed, referred to as $\boxed{1}$ to $\boxed{7}$ and corresponding to the piecewise definition of f.

K is in $\boxed{7}$ where f is a simple linear field, and it is clear that any point (x,y) in $\boxed{7}$ and outside K gets into K in a time $\frac{1}{2}\mathrm{Log}[4(x^2+(y+2)^2)]$, so that we just need to show that starting in any other area, we get into $\boxed{7}$ in a finite time.

Considering the direction of f on the frontiers, it is clear that the only possible changes of area are those designed on fig.3

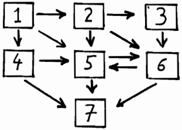

Fig. 3

so that it is impossible to pass infinitely many times in any area: this is obvious on fig. 3 except for $\boxed{5}$ and $\boxed{6}$ but $\dot{y}<0$ in these areas and you can go 5-6 in the upper part of the frontier and 6-5 in the lower part (see the equations) so that the only possibility to get many times through an area is to go 5-6-5. So, we only need to check that we always remain a finite time in a given area

$\boxed{1}$ and $\boxed{2}$: this is obvious because $\dot{x}=1$ so that starting from (x_o,y_o) in $\boxed{1}$ or $\boxed{2}$, you get into $\boxed{3}$ or $\boxed{6}$ at time $1-x_o$ (>0) , unless you have already gone into $\boxed{4}$ or $\boxed{5}$

$\boxed{3}$: here, the solutions are

$$\begin{cases} x(t) = x_o+t \\ y(t) = y_o-\mathrm{Log}(1+\dfrac{t}{1+x_o}) \end{cases}$$

so that you must get into $\boxed{6}$ in a time $e^{y_o}-1$

$\boxed{4}$ and $\boxed{5}$: there, $\dot{y} < -\frac{1}{2}$ so that you get into $\boxed{7}$ in a time smaller than 2, unless you got into $\boxed{6}$ meanwhile.

$\boxed{6}$:here, $\dot{y}>0$, so that if we didn't go into $\boxed{7}$ y would have a limit y_o $(-1<y_o<0)$, and x should be sometimes as large as we want, but for $y<\frac{y_o}{2}$ (for example) and x large, it results $\dot{x}<0$ so that we would have $x \to +\infty$. $y \to y_o$ $\dot{y} \to 0$ but this is impossible for $\dot{y} = -\frac{\psi(y+1)}{1+x} - [1-\psi(y+1)](y+2)$ would tend to $-[1-\psi(y_o+1)](y_o+2)$ which is not zero. We have proved that f satisfies H_o.

2.3. The conclusion of the theorem fails

We have checked all the hypothesis, except (*iii*), but the conclusion of the theorem fails: in $\boxed{3}$, the field is

$$\begin{cases} \dot{x} = 1 \\ \dot{y} = -\frac{1}{1+x} + \frac{1}{1+t} \end{cases}$$

and the solutions

$$\begin{cases} x(t) = x_o + t \\ y(t) = y_o - \mathrm{Log}(1+\frac{t}{1+x_o}) + \mathrm{Log}(1+t) \ > \ y_o \end{cases}$$

so that y doesn't get down, and x goes to infinity.

2.4. How (*iii*) fails

In fact, this non robustness comes from the fact that the angle between f and the x-axis goes to zero as x goes to $+\infty$ so that any vertical perturbation may reverse, far enough, the sign of the vertical component of f, turning a situation in which the flow got from $\boxed{3}$ to $\boxed{6}$ and then to $\boxed{7}$ and the point (0,-2) into a situation in which the flow never gets out of $\boxed{3}$ for x large enough.

Assumption (*iii*) tells something like this : let x enter K_p at a the point ξ (i.e. $\Phi_p^{T(p,x)}(x) = \xi$).Then, if you move slightly x to a neighbour y, ξ moves to $\Phi_p^{T(x,p)}(y)$ which is not too far from ξ (the ratio is bounded). Such a condition seems, intuitively important for our global property: if it fails, when you start from very far, a slight motion (brought by e(t) or p(t)) may have too important consequences on the future.

We shall now try to see how and where (*iii*) fails. It shall turn out that

$$\| R(\ (x_o,0)\quad T(\ (x_o,0)\ )\ )\ \| \to +\infty \quad \text{when} \quad x_o \to +\infty \tag{20}$$

which, of course, is enough to contradict (*iii*) To prove (20), we shall show that

$$\| R(\ (x_o,0)\quad T(\ (x_o,0)\ )\ )\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \| \to +\infty \quad \text{when } x_o \to +\infty \tag{21}$$

First of all, let M_o be the point $(x_o,0)$ t_1 be the time t at which $\Phi^t(x_o)$ enters $\boxed{7}$ and $\Phi^{t_1}(x_o) = N_0 = (x_1,-1)$ (see fig.4). Then

$$R(\, M_o,T(M_o)\,) \;=\; R(\, N_o \quad T(N_o)-t_1\,)\;.\;R(\, M_o,t_1\,) \tag{22}$$

but, since the solution is, from N_o, in $\boxed{7}$ where the field is a simple linear one, we may compute

$$R(\, N_o\;,\; T(M_o)-t_1\,) \;=\; R(N_o,T(N_o)) \;=\; \frac{1}{2\sqrt{1+x_1^2}}\; Id_{R^2} \tag{23}$$

We shall now get a lower bound of

$$\| R(M_o,t_1)\,\binom{0}{1} \| \;=\; \lim_{\epsilon\to 0}\; \frac{1}{\epsilon}\; \|\Phi^{t_1}(M_\epsilon) \quad \Phi^{t_1}(M_o)\,\|$$

where $M_\epsilon = (x_o,\epsilon)$; we set , in addition, $N_\epsilon = \Phi^{t_1}(M_\epsilon)$

To do this,let us study $\Phi^t(M_\epsilon)$ for $t\in[0,t_1]$

M_ϵ is in $\boxed{3}$ because $\epsilon>0$ Solving the field in $\boxed{3}$ we see that $\Phi^t(M_\epsilon)$ gets from $\boxed{3}$ to $\boxed{6}$ (y changes sign) at time

$$t_\epsilon = (x_o+1)(e^\epsilon-1) \tag{24}$$

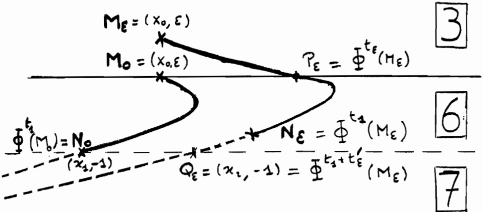

Fig 4

We set then $P_\epsilon = \Phi^{t_\epsilon}(M_\epsilon) = (x_o+t_\epsilon,0)$ (see fig.4 again). We have $t_\epsilon\to 0$ when $\epsilon\to 0$ so that for epsilon small enough, $t_\epsilon<t_1$ and we may get into $\boxed{6}$; the field then becomes

$$\begin{cases} \dot{x} = \psi(y+1) - [1-\psi(y+1)]x \\ \dot{y} = -\dfrac{\psi(y+1)}{1+x} - [1-\psi(y+1)](y+2) \end{cases}$$

$\Phi_t(M_\epsilon)$ can't go outside $\boxed{6}$ before the time t_1. To see this, let t_1+t_ϵ be the time t at which $\Phi^t(M_\epsilon)$ gets from $\boxed{6}$ to $\boxed{7}$ and set $Q_\epsilon = \Phi^{t_1+t_\epsilon'}(M_\epsilon) = (x_2,-1)$ (see fig. 4). Then y decreases from 0 to -1 within the time t_1 on the trajectory (M_o,M_ϵ), and within the time $t_1-t_\epsilon+t_\epsilon$ on the trajectory (P_ϵ,Q_ϵ); noticing that for the same value of y, $\dot{y}$ is weaker on (P_ϵ,Q_ϵ) than on (P_o,Q_o) [because (P_ϵ,Q_ϵ) is strictly on the right of (P_o,Q_o)], and that both curves may be paramatrized by y [because $\dot{y}>0$], we get $t_1 < t_1-t_\epsilon+t_\epsilon$ hence

$$0 < t_\epsilon < t_\epsilon$$

so that t_ϵ' is positive , and N_ϵ is "before" Q_ϵ, as designed on fig.4

Now, it is clear that, Φ being continuous,

$$t_\epsilon \to 0 \quad \text{and} \quad t_\epsilon' \to 0 \quad \text{as} \quad \epsilon \to 0$$

then, since

$$N_\epsilon = \Phi^{t_1}(M_\epsilon) = \Phi^{-t_\epsilon'}(Q_\epsilon)$$

we have

$$Q_\epsilon - N_\epsilon = -t_\epsilon' f(Q_\epsilon) + o(t_\epsilon') = \begin{pmatrix} t_\epsilon' x_2 + o(t_\epsilon') \\ t_\epsilon' + o(t_\epsilon') \end{pmatrix}$$

so that, noticing that $Q_\epsilon - N_o = \begin{pmatrix} x_2-x_1 \\ 0 \end{pmatrix}$,

$$\begin{aligned} \|\Phi^{t_1}(M_\epsilon)-\Phi^{t_1}(M_o)\| &= \|N_\epsilon - N_o\| = \|N_\epsilon - Q_\epsilon + Q_\epsilon - N_o\| \\ &= \sqrt{[x_2-x_1+t_\epsilon']^2 + x_2^2 t_\epsilon'^2} + o(t_\epsilon') \\ &\geq \left(\sqrt{1+x_1^2}\right) t_\epsilon + o(t_\epsilon) \qquad [\text{because } x_1<x_2 \text{ and } t_\epsilon<t_\epsilon'] \end{aligned} \tag{25}$$

Now, from (24) and (25), we get

$$\|\Phi^{t_1}(M_\epsilon)-\Phi^{t_1}(M_o)\| \geq \left[(1+x_o)\sqrt{1+x_1^2}\right]\epsilon + o(\epsilon)$$

hence

$$\| R(M_o,t_1)\begin{pmatrix}0\\1\end{pmatrix} \| \geq (1+x_o)\sqrt{1+x_1^2}$$

and , from (22) and (22),

$$\| R(\,(x_o,0) \quad T((x_o,0))\,)\begin{pmatrix}0\\1\end{pmatrix} \| \geq \frac{1+x_o}{2}$$

which does go to infinity as x goes to $+\infty$ This proves (21) and makes (*iii*) fail.