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A result on robust boundedness

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Abstract: We give conditions under which, if, for *any fixed* p, all the solutions of (1) $\dot{x}=f(p, x)$ enter a compact set K_p (depending on p) after a finite time, then all the solutions of (2) $\dot{x}=f(p(t), x)+e(t)$ 'tend to' the moving compact set $K_{p(t)}$.

The differential equation (2) may be obtained when applying a time-varying control law to a system. Non-linear output tracking is concerned.

This may also apply to indirect adaptive control when you design such and adaptation law that you have *a priori* informations about $e(t)$, the equation error, and $p(t)$, the parameter estimate.

Keywords: Disturbed ordinary differential equation, Boundedness, Robustness, Slowly varying systems, Non-linear adaptive control, Non-linear systems stability.

1. The general result

Some motivations for studying equations (1) and (2) shall be given in Section 2. We state our results first.

1.1. Introduction; similar results

Our result consists in deducing from a property *for any fixed p*, of the (autonomous) ordinary differential equation

$$\dot{x}=f(p, x) \tag{1}$$

a slightly weaker property of the (non-autonomous) equation

$$\dot{x}=f(p(t), x)+e(t). \tag{2}$$

This is a robustness result with respect to two kinds of perturbations: $e(t)$, an additive perturbation, and $\dot{p}(t)$, which makes the system time-varying.

Many robustness results exist about *local* stability properties of an attractive point or set. In [6, §3.8 to 3.10], the robustness of different kinds of stabilities with respect to C^0-small or L^k-small perturbations are studied. In [11, Th. 6, §5.6], slowly varying systems are dealt with (i.e. the case when $e=0$ and $p=t$): if $\|\partial f/\partial p\|<k\|x\|$, and the origin is an hyperbolic stable equilibrium of (1), then the same holds for (2). Total stability results are given in [7]. Let us mention at last Grobmann–Hartmann theorem (see for instance [10]), which is a very powerful tool for the study of hyperbolic equilibrium points.

In this paper, we are interested in the robustness of a *global* property, namely, the existence of a compact set to which all solutions enter after a finite time. For further precisions, see (H_0) in Section 1.3. We have not been able to find any result about the robustness of such a global property in the literature.

To get our result, we apply the Lyapunov second method, with the function $T(p, x)$ given by the time necessary to reach the compact set, following the flow of (1) for p frozen. This rather natural function is more convenient than those given by converse Lyapunov theorems (as Th. 3.15.5 in [6], for quasi-uniform ultimate boundedness). In particular, the dependence on p is easily obtained.

1.2. Notations

- Π is a bounded open subset of $\mathbf{R}^q$ (in which the parameter p evolves).
- $(p, x) \to f(p, x)$ is a C^1-map from $\overline{\Pi} \times \mathbf{R}^n$ to $\mathbf{R}^n$ ($\overline{\Pi}$ stands for the closure of the set Π).
- Φ_p is the flow of (1), i.e.

$$\frac{\partial}{\partial t}\left[\Phi_p^t(x)\right] = f\left(p, \Phi_p^t(x)\right), \quad \Phi_p^0(x) \equiv x. \tag{3}$$

- R is defined by

$$R(p, x, t) = \frac{\partial}{\partial x}\left[\Phi_p^t(s)\right]. \tag{4}$$

$R(p,\cdot,\cdot)$ is also the transition matrix of the system (1) linearized along the solutions, i.e.

$$\frac{\partial}{\partial t}[R(p, x, t)] = \frac{\partial f}{\partial x}\left(p, \Phi_p^t(x)\right)R(p, x, t), \quad R(p, x, 0) \equiv I_{R^n}. \tag{5}$$

1.3. Our result

Our main assumption about the frozen systems (1) ia assumption (H_0).

Primarily, it asks for any solution of (1) to enter a compact set K_p after a finite time. This is rather similar to *quasi-uniform ultimate boundedness* [8, Def 3.8; [6, §3.13.1 (B4)] which asks for a ball outside of which any solution spends only a finite amount of time; however, no care is taken there about the solutions getting inside and outside many times, whereas here, in (H_0), we demand the solutions to enter K_p only once: this is (7) (of course, constraining K_p to be a ball would then be too restrictive). Topological necessary conditions for this property (called there K-stability) to be met are given in [2]. Of course (this is the first part of (6)), K_p is supposed to remain in a bounded area when p varies.

In addition, the field $f(p,\cdot)$ is asked (in (8)) to point strictly inward K_p all along ∂K_p (its boundary). For the sake of simplicity, K_p is supposed to be the set where $G(p,\cdot)$ is negative, G being a smooth function. This makes precise "K_p smoothly depends on p".

Assumption (H_0). (a) $\Phi_p^t(x)$ is well defined for any $p \in \Pi$, $x \in \mathbf{R}^n$, $t > 0$.

(b) There exists a C^1-function $G: \Pi \times \mathbf{R}^n \to \mathbf{R}$, such that:

(b1) The sets

$$L = \{(x, p) \mid G(p, x) \leqslant 0\} \quad \text{and} \quad K_p = \{x \mid G(p, x) \leqslant 0\} \quad (\text{for any } p) \tag{6}$$

are compact.

(b2) There exists a function $T: \Pi \times \mathbf{R}^n \to \mathbf{R}$ (this is the time needed to reach K_p), such that

$$\forall (p, x, t) \quad t < T(p, x) \Rightarrow \Phi_p^t(x) \notin K_p, \tag{7a}$$

$$t > T(p, x) \Rightarrow \Phi_p^t(x) \in K_p^\circ \quad (\text{the interior set of } K_p). \tag{7b}$$

(c) There exists a positive constant C such that, for ξ in ∂K_p,

$$\frac{\partial G}{\partial x}(p, \xi) f(p, \xi) < -C. \tag{8}$$

Let us now define the other hypotheses needed for the theorems:

(i) *No finite escape reverse time*: For any p in Π, any x in $\mathbf{R}^n$ and any negative t, $\Phi_p^t(x)$ is well defined.

(ii) *Bounded p-sensitivity*: $\|(\partial f/\partial p)(p, x)\|$ is bounded for p in Π and x in $\mathbf{R}^n - K_p$,

(iii) *Uniform quasi-instability in reverse time*: $\|R(p, x, T(p, x))\|$ is bounded for p in Π and x in $\mathbf{R}^n - K_p$ or, equivalently, with $\xi = \Phi_p^{T(p,x)}(x)$ and $t = T(p, x)$, $\|[R(p, \xi, -t)]^{-1}\|$ is bounded for p in Π, ξ in ∂K_p and t positive.

(iii′) *Uniform exponential instability in reverse time*: There exist positive constants α, ω such that $\|R(p,x, T(p, x))\| < \alpha e^{-\omega T(p,x)}$ for any p in Π and x in $\mathbf{R}^n - K_p$ or, equivalently, $\|[R(p, \xi, -t)]^{-1}\| < \alpha e^{-\omega t}$ for any p in Π, ξ in ∂K_p and t positive.

(iv) *L^k-bounded perturbations*: $\dot{p}$ belongs to L^1 and e belongs to L^k for some k ($1 \leqslant k < +\infty$), i.e. $\int_0^\infty |e(t)|^k \, dt < +\infty$ and $\int_0^\infty |\dot{p}(t)| \, dt < +\infty$.

(iv′) *Bounded perturbations*: There exists a positive constant a, smaller than 1, such that

$$\frac{\alpha g_x}{C} |e(t)| + \frac{1}{C}\left(g_p + \frac{\alpha}{\omega} g_x f_p\right) |\dot{p}(t)| < 1 - a,$$

$$g_x = \max_{(p,x)\in L} \left\| \frac{\partial G}{\partial x}(p, x) \right\|, \quad g_p = \max_{(p,x)\in L} \left\| \frac{\partial G}{\partial p}(p, x) \right\|, \quad f_p = \max_{(p,x)\in L} \left\| \frac{\partial f}{\partial p}(p, x) \right\|,$$

where α and ω are defined in (iii′) and C in (8).

Comments. All the assumptions are about the frozen systems or about f, e and $\dot{p}$ explicitly. (H_0) is the main assumption about the frozen unperturbed system (1). (i) to iv) are needed to prove that when it holds for (1), something like it holds for (2).

The first point in (H_0) only prevents the solutions from going to infinity before entering K_p. In fact, if the flow is well defined up to $T(p, x)$, (7b) implies that it is always defined.

(i) means, in picturesque words, that "no bounded set can be reached from infinity in finite time". It makes T infinite at infinity.

(iv) or (iv′) make precise how e and p are weak perturbations. In particular, the bounds in (iv′) tell us how $|e|$ and $|\dot{p}|$ should be small.

(ii) and (iii) (or (iii′)) are the two really restrictive assumptions on the structure of the 'frozen systems' (1).

(ii) is rather strong. In particular, it excludes the case $f(p, x) = A(p)x$ where $A(p)$ would be a matrix depending on p: our theorems unfortunately do not include slowly varying linear systems studied for example in Vidyasagar [11].

(iii) (or (iii′)) mean: in reverse time, let x be the point reached at time t, starting from ξ on the boundary of K_p; then, starting close to ξ and following the flow during the same (reverse) time, you do not end up too close to x, uniformly with respect to t. In Section 2.4, an example is given in which (iii) appears not to be superfluous.

We may now state our two theorems:

Theorem 1. *If* (H_0), (i), (ii), (iii), (iv) *hold, then for each solution $x(t)$ of* (2), *one can find a time t_0 such that*:

(a) $x(t_0) \in K_{p(t_0)}$.

(b) After t_0, $x(t)$ is sometimes outside $K_{p(t)}$, but both the length of the time intervals in which this occurs and the maximum distance between $x(t)$ and $K_{p(t)}$ during these intervals tend to zero, and, as a consequence, $\operatorname{dist}(x(t), K_{p(t)})$ *tends to zero.*

Theorem 2. *If* (H_0), (i), (ii), (iii′), (iv′) *hold, then for each solution $x(t)$ of* (2), *one can find a time t_0 after which $x(t)$ is always inside $K_{p(t)}$.*

1.4. *A counter-example in which assumption* (iii) *fails*

ψ being the following C^1-map from $\mathbf{R}$ to $\mathbf{R}$:

$$\psi(s) = \begin{cases} 0 & \text{on} \,]-\infty, 0], \\ s^2(3-2s) & \text{on} \, [0, 1], \\ 1 & \text{on} \, [1, +\infty[, \end{cases}$$

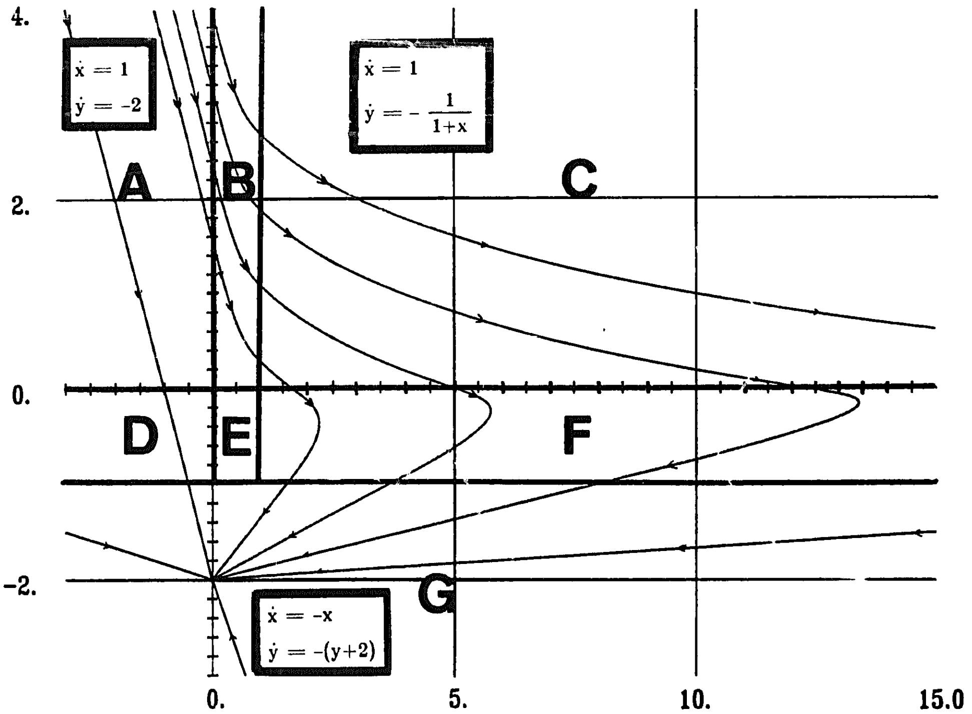

Fig. 1. Phase portait of $\dot{x} = f(x)$.

we define f and e by the following formulae (Figure 1 represents the phase portrait of $\dot{x} = f(x)$):

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \psi(y+1)\psi(x)\begin{bmatrix} 1 \\ -1/(1+x) \end{bmatrix} + \psi(y+1)[1-\psi(x)]\begin{bmatrix} 1 \\ -2 \end{bmatrix} + [1-\psi(y+1)]\begin{bmatrix} -x \\ -(y+2) \end{bmatrix},$$

$$e(t) = \begin{bmatrix} 0 \\ 1/(1+t) \end{bmatrix}.$$

The compact set K is defined by $G(x, y) = x^2 + (y+2)^2 - \frac{1}{4}$; K is the ball with center $(0, -2)$ and radius $\frac{1}{2}$.

f does not depend on p so that (1) is written $\dot{x} = f(x)$ and (2) is written $\dot{x} = f(x) + e(t)$. What exactly happens is:
- the point $(0, -2)$ is a global attractor of (1), so that (H_0) is satisfied.
- some solutions of (2) go to infinity, so that the conclusion of our theorems fail.
- (i), (ii) and (iv) are satisfied; (iii) fails (if not, Theorem 1 would be false!). This points out that hypothesis (iii) is necessary.

The complete study of this example is done in [9]. Let us sum it up:

(H_0), (i), (ii) *and* (iv) *are satisfied*:

(i) is true because f is underlinear.

(ii), (iv) are true because $\partial f/\partial p = 0$; $\int_0^\infty e^2(t)\,dt = 1$; $\dot{p} = 0$.

Let us check that (H_0) is satisfied: Point (a) is true because f is underlinear. Point (c) is true because the inner product is exactly -1. Finally, we must show that, starting from any point outside K, the solutions enter K after a finite time. In Figure 1, seven areas are represented, referred to as region A to region G and corresponding to the piecewise definition of f. One may convince oneself (or see [9]) that any solution spends finitely much time in finitely many areas before entering region G, where f is linear and pointing inward K.

The conclusion of the theorem fails:

This can be seen for in region C, the disturbed system is:

$$\dot{x}=1, \qquad \dot{y}=-\frac{1}{1+x}+\frac{1}{1+t},$$

so that any solution starting from an (x_0, y_0) in region C goes to infinity:

$$x(t)=x_0+t, \qquad y(t)=y_0-\operatorname{Log}\left(1+\frac{t}{1+x_0}\right)+\operatorname{Log}(1+t)>y_0.$$

Why?

Intuitively, this comes from the fact that the angle between f and the x-axis goes to zero as x goes to $+\infty$. Consequently, any vertical perturbation shall reverse, far enough on the x-axis, the sign of the vertical component of f, turning a situation in which the solution went from region C to region F, then to region G and finally to the compact set K, into a situation in which the solutions never leave region C.

In fact, this may be translated into (iii)'s failing: a rather long computation (see [9]) more or less derived from the former intuitive remark leads to

$$\|\Phi^{T((x,0))}((x,\varepsilon))-\Phi^{T((x,0))}((x,0))\|\geqslant\tfrac{1}{2}(1+x)\varepsilon+o(\varepsilon).$$

Hence

$$\left\|R((x,0),T((x,0)))\begin{bmatrix}0\\1\end{bmatrix}\right\|=\left\|\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\left[\Phi^{T((x,0))}((x,\varepsilon))-\Phi^{T((x,0))}((x,0))\right]\right\|\geqslant\tfrac{1}{2}(1+x)$$

which makes (iii) fail, because this tends to infinity when x does.

1.5. *The proofs*

For the proof of Theorems 1 and 2, we shall use the three following lemmas:

Lemma 1. *Under the Assumption* (H_0), *and as long as* $\Phi_p^t(x)$ *is well defined, we have*

$$T\big(p,\Phi_p^t(x)\big)=T(p,x)-t, \qquad \frac{\partial T}{\partial x}(p,x)f(p,x)=-1. \tag{9}$$

Proof. This is quite obvious. When you have been walking toward K_p for a time t, the time you need to reach K_p has decreased of exactly t. This can be written properly from (7). □

Lemma 2. *Under assumption* (H_0), *T is a C^1-function, and we have*

$$\frac{\partial T}{\partial x}(p,x)=-\frac{1}{\frac{\partial G}{\partial x}\big(p,\Phi_p^{T(p,x)}(x)\big)f\big(p,\Phi_p^{T(p,x)}(x)\big)}\frac{\partial G}{\partial x}\big(p,\Phi_p^{T(p,x)}(x)\big)R(p,x,T(p,x)), \tag{10}$$

$$\frac{\partial T}{\partial p}(p,x)=-\frac{1}{\frac{\partial G}{\partial x}\big(p,\Phi_p^{T(p,x)}(x)\big)f\big(p,\Phi_p^{T(p,x)}(x)\big)}\left[\frac{\partial G}{\partial p}\big(p,\Phi_p^{T(p,x)}(x)\big)\right.$$
$$\left.+\frac{\partial G}{\partial x}\big(p,\Phi_p^{T(p,x)}(x)\big)\int_0^{T(p,x)}R\big(p,\Phi_p^{\tau}(x),T(p,x)-\tau\big)\frac{\partial f}{\partial p}\big(p,\Phi_p^{\tau}(x)\big)\,\mathrm{d}\tau\right] \tag{11}$$

or equivalently, with the notation

$$\xi(p, x) = \Phi_p^{T(p, x)}(x), \tag{12}$$

we have

$$\frac{\partial T}{\partial x}(p, x) = -\frac{1}{\frac{\partial G}{\partial x}(p, \xi(p, x))f(p, \xi(p, x))}\frac{\partial G}{\partial x}(p, \xi(p, x))[R(p, \xi(p, x), -T(p, x))]^{-1}, \tag{13}$$

$$\frac{\partial T}{\partial p}(p, x) = -\frac{1}{\frac{\partial G}{\partial x}(p, \xi(p, x))f(p, \xi(p, x))}\left[\frac{\partial G}{\partial p}(p, \xi(p, x)) + \frac{\partial G}{\partial x}(p, \xi(p, x))\int_0^{T(p, x)}[R(p, \xi(p, x), -\tau)]^{-1}\frac{\partial f}{\partial p}(p, \Phi_p^{-\tau}(\xi(p, x)) \,\mathrm{d}\tau\right]. \tag{14}$$

Proof. This is just the implicit function theorem applied to

$$G\left(p, \Phi_p^{T(p, x)}(x)\right) = 0.$$

Differentiating this equality, we get

$$\frac{\partial G}{\partial p}\,\mathrm{d}p + \frac{\partial G}{\partial x}\left[\frac{\partial}{\partial T}\left(\Phi_p^T(x)\right)\mathrm{d}T + \frac{\partial}{\partial x}\left(\Phi_p^T(x)\right)\mathrm{d}x + \frac{\partial}{\partial p}\left(\Phi_p^T(x)\right)\mathrm{d}p\right] = 0. \tag{15}$$

But $(\partial/\partial t)[\Phi_p^T(x)] = f(p, \Phi_p^T(x))$ (from the definition of the flow (see (3)); $(\partial/\partial x)[\Phi_p^T(x)] = R(p, x, T)$ (see (4)); and $(\partial/\partial p)[\Phi_p^T(x)] = S(p, x, T)$, where S is computed as follows: Just writing down that

$$\frac{\partial S}{\partial t} = \frac{\partial^2}{\partial p \partial t}\left[\Phi_p^t(x)\right] = \frac{\partial}{\partial p}\left[f\left(p, \Phi_p^t(x)\right)\right],$$

one gets S as the solution of

$$\frac{\partial}{\partial t}S(p, x, t) = \frac{\partial f}{\partial x}\left(p, \Phi_p^t(x)\right)S(p, x, t) + \frac{\partial f}{\partial p}\left(p, \Phi_p^t(x)\right), \quad S(p, x, 0) \equiv 0.$$

Using the variation of constants (just set $S(p, x, t) = R(p, x, t)Q(p, x, t)$ and compute Q), (5) yields

$$S(p, x, t) = \int_0^t R\left(p, \Phi_p^\tau(x), (t-\tau)\frac{\partial f}{\partial p}\left(p, \Phi_p^\tau(x)\,\mathrm{d}\tau.\right.\right. \tag{16}$$

Or ($s = t - \tau$, ξ being defined by (12)),

$$S(p, x, t) = \int_0^t [R(p, \xi(p, x), -s)]^{-1}\frac{\partial f}{\partial p}\left(p, \Phi_p^{-s}(\xi(p, x))\right)\mathrm{d}s. \tag{17}$$

Formulas (11) to (14) follow from (15) and the following. □

Now, we use an idea of Gronwall inequality to establish the following result:

Lemma 3. *If y is a C^1-function from* $[0, +\infty[$ *to R, such that*

$$\dot{y}(t) \leqslant -1 + \alpha(t) + (y(t) + a)\beta(t) \quad \text{whenever } y(t) \geqslant 0, \tag{18}$$

where

- *a is a positive real number, α and β are real continuous maps.*
- *$\alpha(\cdot)$ belongs to L^k for a certain k, $1 \leqslant k < +\infty$ (i.e. $[\int_0^\infty |\alpha|^k]^{1/k} = \alpha_k < +\infty$).*
- *$\beta(\cdot)$ belongs to L^1 (i.e. $\int_0^\infty |\beta| = \beta_1 < +\infty$).*

Then (j) *There is a time $t_0 > 0$ such that $y(t_0) \leqslant 0$.*

(jj) *After t_0, y may sometimes be positive again, but both the length of the time intervals in which this occurs and the maximum of y on these intervals tend to zero.*

(jjj) *As a consequence,* $\lim_{t\to\infty} \max\{y, 0\} = 0$.

Proof. *Step* 1. Let $t_2 > t_1 > 0$ be such that $y(t) \geqslant 0$ for any t in $[t_1, t_2]$ (if no such t_1, t_2 exist, the lemma is obviously true). In the following, t stands for any real number in $[t_1, t_2]$. Let us perform a sort of a variation of constants: we *define the function h* by

$$y(t) = h(t) + y(t_1)\,\mathrm{e}^{\int_{t_1}^t \beta} - \int_{t_1}^t \mathrm{e}^{\int_s^t \beta}\,\mathrm{d}s + \int_{t_1}^t [\alpha(s) + a\beta(s)]\,\mathrm{e}^{\int_s^t \beta}\,\mathrm{d}s. \tag{19}$$

Then, from (19), $h(t_1) = 0$ and, from (18) and (19), $\dot{h} \leqslant 0$ in $[t_1, t_2]$; hence $h \leqslant 0$ in $[t_1, t_2]$, and

$$\begin{aligned} y(t) &\leqslant -\int_{t_1}^t \mathrm{e}^{\int_s^t \beta}\,\mathrm{d}s + \mathrm{e}^{\int_{t_1}^t \beta} y(t_1) + \int_{t_1}^t |\alpha(s)|\,\mathrm{e}^{\int_s^t \beta}\,\mathrm{d}s + a\int_{t_1}^t |\beta(s)|\,\mathrm{e}^{\int_s^t \beta}\,\mathrm{d}s \\ &\leqslant -\mathrm{e}^{-\beta_1}(t - t_1) + \mathrm{e}^{\beta_1}\left[\int_{t_1}^\infty |\alpha|^k\right]^{1/k} (t - t_1)^{1-1/k} + \mathrm{e}^{\beta_1}\left[y(t_1) + a\int_{t_1}^\infty |\beta|\right]. \end{aligned} \tag{20}$$

We may derive two inequalities from (20), which is true for any t in $[t_1, t_2]$:

1. Noticing that

$$\max_{\tau \in R}\left[-p\tau + q\tau^{(1-1/k)} + r\right] = \frac{1}{k}\frac{q^k}{p^{k-1}}\left(1 - \frac{1}{k}\right)^{k-1} + r,$$

we get

$$\max_{[t_1, t_2]} y(t) \leqslant \frac{1}{k}\left(1 - \frac{1}{k}\right)^{k-1} \mathrm{e}^{-\beta_1}\left[\int_{t_1}^\infty |\alpha|^k\right] + \mathrm{e}^{\beta_1}\left[y(t_1) + a\int_{t_1}^\infty \beta\right]. \tag{21}$$

2. Noticing that

$$-p\tau + q\tau^{(1-1/k)} + r = \left[-\frac{p\tau}{2} + q\tau^{(1-1/k)}\right] + \left[-\frac{p\tau}{2} + r\right]$$

and that this is negative when τ is bigger than $2r/p$ and $(2q/p)^k$, we get, since $y(t_2)$ must be positive,

$$t_2 - t_1 \leqslant \max\left\{2\,\mathrm{e}^{2\beta_1}\left[y(t_1) + \int_{t_1}^\infty |\beta|\right], (2\,\mathrm{e}^{2\beta_1})^k \int_{t_1}^\infty |\alpha|^k\right\}. \tag{22}$$

Step 2. Now, the point (j) of the lemma is given by (22), taking $t_1 = 0$: no t_2 bigger than $2\mathrm{e}^{2\beta_1}[y(0) + \beta_1]$ and $(2\mathrm{e}^{2\beta_1})^k \alpha_k$ may be such that y remains positive all over $[0, t_2]$; hence the existence of a time t_0 such that $y(t_0)$ is negative.

Point (jj) (and consequently (jjj)) is obtained as follows: considering that α belongs to L^k and β to L_1, the upperbounds given in (21) and (22) both tend to zero as t_1 tends to infinity. This means that both the maximum value of y on $]t_1, +\infty[$ and the maximum possible length of an interval after t_1 on which y remains positive tend to zero as t_1 tends to infinity. □

We shall now state the proofs of Theorem 1 and 2:

Proof of Theorem 1. $x(t)$ being a solution of (2), we consider the function (of time) ρ defined by

$$\rho(t) = T(p(t), x(t)). \tag{23}$$

With (9), we get

$$\dot\rho(t) = -1 + \frac{\partial T}{\partial x}(p(t), x(t))e(t) + \frac{\partial T}{\partial p}(p(t), x(t))\dot p(t). \tag{24}$$

Considering (13), (14) and bounds taken from (8), (ii), (iii), and the fact that $(p, \xi(p, x))$ lies in the compact set L (see (6)) so that any function of $(p, \xi(p, x))$ is bounded, we get constants C_1, C_2, C_3, such that

$$\left|\frac{\partial T}{\partial x}(p(t), x(t))\right| \leqslant C_1, \qquad \left|\frac{\partial T}{\partial p}(p(t), x(t))\right| \leqslant C_2\rho(t) + C_3,$$

and

$$\dot\rho(t) \leqslant -1 + C_1 |e(t)| + (C_2\rho(t) + C_3) |\dot p(t)|. \tag{25}$$

We may now apply Lemma 3 with $y = \rho$.

The result of Lemma 3 is enough to get point (a) of the theorem because when $\rho(t)$ is negative, $x(t)$ is inside $K_{p(t)}$. To deduce (b) and (c) of the theorem from (jj) and (jjj) of Lemma 3, we shall now derive a bound of the distance between x and K_p in terms of $T(p, x)$.

From Lemma 3, $\rho(t)$ tends to zero as t tends to infinity; hence, there exists a time t_1 such that when (for instance) $t > t_1$, $\rho(t)$ is less than 1, which means that either $x(t)$ is in $K_{p(t)}$ or $0 \leqslant T(p(t), x(t)) \leqslant 1$. In the former case, dist$(x(t), K_{p(t)})$ is zero. For the latter, we need to bound dist$(x(t), K_{p(t)})$: let

$$M = \max_{(p, x)\in\psi(L\times[0, 1])} \|f(p, x)\|$$

where ψ is the map defined on $\mathbb{R}^q \times \mathbb{R}^n \times \mathbb{R}$ (from (H_0) and (i), the flow always exists) by $\psi(p, x, t) = \Phi_p^t(x)$. Then M exists because, L being a compact set and Φ continuous, $\psi(L \times [0, 1])$ is a compact set. Hence, for t larger than t_1, $(p(t), x(t))$ is in $\psi(L \times [0, 1])$ (from the definitions), so that $\|f(p(t), x(t))\|$ remains smaller than M, and

$$\mathrm{dist}(x(t), K_{p(t)}) \leqslant \int_0^{T(p, x)} \|f(\Phi_{p_\tau}(x), p)\| \,\mathrm{d}\tau \leqslant MT(p, x) \leqslant M\rho(t). \tag{26}$$

This completes the proof. □

Proof of Theorem 2. We use the same comparison function ρ as in the proof of Theorem 1; (24) is true again. Now, comparing (13), (14), (iii′) and (iv′), we get

$$\dot\rho(t) < -a < 0 \quad \text{when } \rho(t) \geqslant 0.$$

Hence the theorem (t_0 is at most $\rho(0)/a$). □

2. Two different motivations for our results

2.1. *Following a trajectory*

Consider the plant

$$\dot x = A(x) + B(x)u, \tag{27}$$

and let $(K_p)_{p\in\Pi}$ be a family of 'targets' (subsets of the state-space) smoothly depending on p Suppose that one can, for any (fixed) possible value of p, design a control law $u=\alpha(p, x)$ such that any solution of

$$\dot{x}=f(p, x) \triangleq A(x)+B(x)\alpha(p, x) \tag{28}$$

enters K_p after a finite time and stays inside afterwards.

Now, one wants to 'follow' a moving target $K_{p(t)}$, and one simply uses $u=\alpha(p(t), x)$ at any time. This gives

$$\dot{x}=f(p(t), x)=A(x)+B(x)\alpha(p(t), x), \tag{29}$$

which is (2) with $e=0$ (or e could be there as a noise). Theorem 1 or 2 may now apply to this time-varying system under conditions on A, B and α. We shall only give very simple sufficient conditions for the hypothesis of Theorem 2 to be met:

For example, consider, as d'Andréa and Lévine in [3], the problem of following a given trajectory $x_{p(t)}$ for a robot arm ($p=x_p$; K_p is the ball with radius r around x_p). In this case, $B(x)$ is bounded, and $\alpha(p, x)=C(x)(x-x_p)+D(x)$ where $C(x)$ is bounded. Then:
- (H_0), (i) and (iii′) are satisfied if, for any p, $\alpha(p, x)$ makes x_p a hyperbolic stable point of (28).
- (ii) requires $B(x)(\partial\alpha/\partial p)(p, x)$ to be bounded; this is $B(x)C(x)$.
- (iv′) requires the maximum speed along the reference trajectory to be small enough, the required smallness depending on the maximum of $B(x)C(x)$, the sharpness of the exponential convergence in (28), and the size of the balls K_p.

Under these conditions, from Theorem 2, the control law $u(p(t), x)$ shall bring $x(t)$ close to $x_{p(t)}$ (precisely, $\|x-x_p\|<r$) after a finite time.

2.2. A non-linear adaptive control problem

We consider the following family of models parametrized by θ:

$$\dot{z}=\theta^{\mathrm{T}}[A(z)+B(z)u], \tag{30}$$

and we suppose that for any value of θ, a control law $u=\alpha(\theta, z)$ is known, such that the closed-loop system, which may be written like (1):

$$\dot{z}=f(\theta, z) \triangleq \theta^{\mathrm{T}}\phi(\theta, z), \quad \phi(\theta, z) \triangleq A(z)+B(z)\alpha(\theta, z), \tag{31}$$

has the following boundedness property: any solution $z(t)$ enters a compact set K_θ after a finite time, and never leaves it afterwards.

Now, we want to control a plant which is one of these models, corresponding to an unknown value of θ, let θ^*. We estimate it by a $\hat{\theta}$ given by the following estimation law:

$$\dot{\hat{\theta}}=h(\hat{\theta}, z) \tag{32}$$

and the complete control law for (30) is (32) together with $u(t)=\alpha(\theta(t), z(t))$. The complete closed-loop system is then given by (32)–(33):

$$\dot{z}=\theta^{*\mathrm{T}}\phi(\theta, z)=\hat{\theta}^{\mathrm{T}}\phi(\hat{\theta}, z)+\varepsilon(\hat{\theta}, z) \tag{33}$$

where

$$\varepsilon(\theta, z)=[\theta^*-\theta]^{\mathrm{T}}\phi(\theta, z).$$

Now, let $(\theta(t), z(t))$ be a solution of the system (32)–(33); it is also a solution of its associated system defined by the following non-autonomous differential equation:

$$\dot{x}=f(p(t), x)+e(t) \tag{2}$$

with

$$p(t)=\begin{cases}\theta(t) & \text{if } t\in[0, \tau[,\\ 0 & \text{if } t\in[\tau, +\infty[,\end{cases} \qquad e(t)=\begin{cases}\varepsilon(\theta(t), z(t)) & \text{if } t\in[0,\tau[\\ 0 & \text{if } t\in[\tau, +\infty[,\end{cases} \tag{34}$$

where $[0, \tau[$ is the maximum right-interval on which the solution of (31)–(32) exists (τ might be infinite).

Theorem 1 or 2 may now be used if you have *a priori* estimations on e and p, which depend on the trajectory you follow (this is quite different from the former example in which $p(t)$ was an explicit data of the problem).

We may get these estimations in an idealistic case using, instead of (32), the following least-squares estimator, *unfortunately unrealizable since $\dot{z}$ is needed*:

$$\begin{cases} \dot{\theta} = -Q\phi(\theta, z)\left[\dot{z} - \theta^{\mathrm{T}}\phi(\theta, z)\right], \\ \dot{Q} = -Q\phi(\theta, z)\phi(\theta, z)^{\mathrm{T}}Q, \quad Q(0) = I. \end{cases} \tag{35}$$

Then,

$$e(t) = \dot{z}(t) - \theta(t)^{\mathrm{T}}\phi(\theta(t), z(t)) = \left[\theta^* - \theta(t)\right]^{\mathrm{T}}\phi(\theta(t), z(t)).$$

In this case, (see [5] for instance), studying the time variations of $(\theta^* - \theta(t))^{\mathrm{T}}Q(t)^{-1}(\theta(t)^* - \theta)$ shows that, regardless of the control laws $\alpha(\theta, z)$ any solution $(\theta(t), z(t))$ of (35)–(33) is such that $\dot{\theta}(t)$ belongs to $L^1([0, \tau[)$ and $e(t)$ belongs to $L^2([0, \tau[)$. Clearly, from (34), $\dot{p} \in L^1(\mathrm{R})$; $e \in L^2(\mathrm{R})$.

This satisfies hypothesis (iv); the assumption we made about (31) means that assumption (H_0) is satisfied; if, in addition, f satisfies conditions of 'no finite escape reverse time' (i), 'bounded p-sensitivity' (ii) and 'uniform quasi-unstability in reverse time' (iii), then Theorem 1 says that any solution $x(t)$ of (2) has the properties described in the conclusion of Theorem 1. In particular, it does not escape.

Consequently, $z(t)$ being a particular solution of (2), this proves that the solution (θ, z) exists till infinity ($\tau = +\infty$); moreover $z(t)$ enters the moving $K_{\theta(t)}$ and then only leaves it for little escapades, both their time-length and the maximum distance between $x(t)$ and $K_{\theta(t)}$ tending to zero as t tends to ∞.

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