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About finite nonlinear zeros for decouplable systems

B. D'ANDREA, L. PRALY

Centre d'Automatique et d'Informatique, Ecole Nationale Supérieure des Mines de Paris, 35 rue Saint-Honoré, 77305 Fontainebleau, France

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Abstract: We establish that the eigenvalues of the gradient at an equilibrium point of the 'zero-dynamics' defined by Byrnes and Isidori [1], are nothing but the finite linear zeros of the linearized system at the equilibrium, if the nonlinear system can be input–output decoupled by feedback and its linearization is controllable. This property allows us to describe (and to give an algorithm to find) the output functions leading to stability while using a linear model following controller. We study on an example the problem of both stability and maximal linearization.

Keywords: Zeros, Nonlinear control, Feedback decoupling and partial linearization.

Introduction

It is now well known how to impose a decoupled linear dynamic behavior to functions of the state whose number may in general be equal to the number of inputs (see [3,4,7,11]). However, if this number is smaller than the dimension of the state, then part of the state is made unobservable. Consequently, stability cannot be easily guaranteed.

By an appropriate choice of local coordinates (see [8]) it can be seen that the closed loop nonlinear system is made of the desired linear system plus an extra nonlinear subsystem whose state is unobservable. With analogy to the linear case, Byrnes and Isidori [1] have called 'zero-dynamics', the restriction of the closed loop dynamic to this unobservable part. Isidori and Moog [9], have remarked that this definition is equivalent to their definition as the dynamics of 'reduced inverse systems' or as the 'zero-output-constrained dynamics'. However, as in the linear case (see [10]), for this equivalence to hold, some other condition is needed (for example: the decoupling matrix is non singular at the equilibrium). But, even in this case, the above definitions are difficult to use. In particular, they do not allow an easy characterization of those functions of the state leading to stable equilibria. In this direction, a further interesting question is the possibility to both stabilize the equilibrium and to obtain by feedback the largest linear system (see [12]).

In this paper, we restrict our attention to decoupable systems. In Section 1, we characterize the 'zero dynamics' around the equilibrium by introducing the notion of 'finite nonlinear zeros'. In Section 2, we prove that the finite nonlinear zeros are nothing but the finite zeros of the linear system obtained by linearizing the open loop nonlinear system around the equilibrium. In Section 3, we characterize the set of functions of the state leading to stability and we propose an algorithm to place the poles at the equilibrium. Finally, in Section 4, we study on an example how can be handled both stability and maximal linearization.

1. Finite nonlinear zeros

We consider the following system on $\mathbf{R}^n$:

$$\dot{x} = f_0(x) + f(x)u \tag{1}$$

where x is the state and u is the input m-vector. The vector field f_0 and the matrix field f are assumed to be sufficiently smooth.

Let 0 be a singular point of f_0, i.e. $f_0(0) = 0$.

Given $h = (h_j)$, m sufficiently smooth functions of the state, with

$$h(0) = 0, \tag{2}$$

our control objective is to obtain the following linear decoupled dynamical behavior:

$$\frac{\mathrm{d}^{\rho_i+1}h_i(x)}{\mathrm{d}t^{\rho_i+1}} + \sum_{j=0}^{\rho_i} \phi_j^i \frac{\mathrm{d}^j h_i(x)}{\mathrm{d}t^j} = 0, \quad i = 1, \ldots, m, \tag{3}$$

where the ϕ_j^i are arbitrary constants and, for each i in $\{1,\ldots,m\}$, ρ_i is defined, from the system (1), as the smallest integer k such that $L_f L_{f_0}^k h_i$ is not identically zero, $L_g h$ denoting the Lie derivative of h with respect to g, i.e.

$$L_g h(x) = \sum_{i=1}^{n} g^i \frac{\partial h}{\partial x_i}. \tag{4}$$

We call (NL) the system (1) with the functions h as output functions:

$$\begin{cases} \dot{x} = f_0(x) + f(x)u, \\ y = h(x). \end{cases} \tag{NL}$$

We introduce the so-called 'decoupling matrix' Δ, whose i-th row is

$$\left(L_f L_{f_0}^{\rho_i} h_i\right), \quad i = 1,\ldots,m. \tag{5}$$

Throughout the paper we have the following assumption:

$$\Delta(0) \text{ is non-singular.} \tag{H1}$$

This implies that $m \le n$.

In this case, a direct computation (see [3,4,7,11]) shows that the following state feedback allows us to meet the objective, in a neighborhood of 0:

$$u(x) = \Delta^{-1}(\Phi\xi - \Delta_0), \tag{6}$$

where Φ is the following block-diagonal matrix $m \times n_\lambda$ of the ϕ_j^i's:

$$\Phi = \operatorname{diag}(\Phi_1,\ldots,\Phi_m), \quad \Phi_i = \left(\phi_0^i,\ldots,\phi_{\rho_i}^i\right),$$

and

$$n_\lambda = \sum_{i=1}^{m} (\rho_i + 1). \tag{7}$$

Δ_0 is the m-vector with i-th component $\Delta_0^i = L_{f_0}^{\rho_i+1} h_i$ and finally, ξ is a vector in $\mathbf{R}^{n_\lambda}$ defined by

$$\xi = \left(h_1,\ldots,L_{f_0}^{\rho_1} h_1,\ldots,h_m,\ldots,L_{f_0}^{\rho_m} h_m\right)'. \tag{8}$$

Remark. An important result which can be found in [8] is that, when $\Delta(0)$ is invertible, $\partial\xi/\partial x$ has rank n_λ on a neighborhood of 0. Moreover, the coordinates transformation $x \to (\xi, \zeta)$ allows us to rewrite the closed loop system (NL)–(6) in

$$\begin{cases} (\Sigma I) \begin{cases} \dot{\xi} = A\xi \\ y = K\xi, \end{cases} \\ \dot{\zeta} = g_0(\xi, \zeta), \end{cases} \tag{9}$$

where ζ in $\mathbf{R}^q$ with $q = n - n_\lambda$, is introduced to complete (when necessary, i.e. when $n_\lambda < n$) ξ into a coordinate chart. A and K are block-diagonal matrices:

$$A = \operatorname{diag}(A_1,\ldots,A_m),$$
$$K = \operatorname{diag}(K_1,\ldots,K_m),$$

$$A_i = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & & 1 \\ \phi_0^i & \phi_1^i & \phi_2^i & \ldots & \phi_{\rho_i}^i \end{pmatrix},$$

$$K_i = (1, 0,\ldots,0), \quad K_i \in (\mathbf{R}^{\rho_i+1})^*.$$

The subsystem (ΣI) with dimension n_λ is as desired linear and split into m independent subsystems. Each subsystem can be stabilized by a suitable choice of the ϕ_j^i's.

On the other hand, the remaining part of the state (ζ in these new coordinates), has a nonlinear dynamic and is unobservable from y.

This remark has motivated the next definition:

Definition 1 [1,2]. We call asymptotic unobservable submanifold, the q-dimensional submanifold

$$N = \{x \in X \mid \xi(x) = 0\}.$$

From (9) we see that whenever x belongs to N, $\dot{\xi} = A\xi$ is zero. Therefore N is an invariant submanifold of (NL)–(6).

Proposition 1 [1,2]. *The restriction to N of the closed loop vector field $\bar{f}$,*

$$\bar{f} = f_0 + f\Delta^{-1}(\Phi\xi - \Delta_0),$$

is independent of Φ and tangent to N.

As a consequence, the following definition makes sense:

Definition 2. We call $\hat{f}$ the vector field of N, induced by the restriction of $\bar{f}$ to N. From (9) we have

$$\hat{f}(\zeta) = g_0(0, \zeta).$$

The triangular form of (9) allows us to state the following proposition:

Proposition 2. *The eigenvalues of the gradient of* $\bar{f}$ *at the equilibrium point,* $\nabla\bar{f}(0)$, *are the* n_λ *poles of the desired linear subsystem* (ΣI) *and the* $n - n_\lambda$ *eigenvalues of the gradient of* $\hat{f}$ *at* 0.

With analogy to the linear case:

Definition 3. We call *finite nonlinear zeros at an equilibrium point* the eigenvalues of $\nabla\hat{f}$ at this equilibrium point.

Remark. Byrnes and Isidori have called $\hat{f}$ the 'zero-dynamics' (see [1]). The 'finite nonlinear zeros at an equilibrium point' are the eigenvalues of the 'zero-dynamics' at this equilibrium point.

2. Main result

We denote by

$$\begin{cases} \dot{X} = FX + GU, \\ Y = HX, \end{cases} \tag{L}$$

the linear system obtained by linearizing (NL) around zero, i.e.:

$$F = \frac{\partial f_0}{\partial x}(0), \quad G = f(0), \quad H = \frac{\partial h}{\partial x}(0). \tag{10}$$

Recall that $f_0(0) = 0$, $h(0) = 0$, and (H1): $\Delta(0)$ is invertible, which implies that G has full rank m.

We make a second assumption:

The linear system (L) is controllable. (H2)

In this case, an equivalent representation is given by the controller polynomial form (see [10]), with s the derivation operator:

$$\begin{cases} P(s)\eta = U, \\ Y = R(s)\eta, \end{cases} \tag{11}$$

where η is a partial state. In particular, this implies

$$R(s)P(s)^{-1} = H(sI - F)^{-1}G. \tag{12}$$

Definition 4 [10]. *We call finite linear zeros, the finite zeros of* (L), *namely the complex values z for which* $\det(R(z))$ *is zero.*

Let us now give our main result.

Theorem. *Under the assumptions* (H1) *and* (H2), *the finite nonlinear zeros at the equilibrium point are the finite linear zeros.*

For the proof of this theorem we introduce the following two sets:

$$P_1 = \{\text{eigenvalues of (NL)–(6) linearized around zero}\},$$

$$P_2 = \{\text{poles of } (\Sigma I)\} \cup \{\text{finite linear zeros}\}.$$

Let us first prove the following proposition:

Proposition 3. *Under the previous conditions* (H1) *and* (H2), *we have the following equality*:

$$P_1 = P_2.$$

Proof. We linearize the equation of the closed loop system (NL)–(6), around the equilibrium point to obtain

$$\dot{X} = \left[\frac{\partial f_0}{\partial x}(0) + f(0)\frac{\partial u}{\partial x}(0)\right]X + \frac{\partial f}{\partial x}(0) \otimes Xu(0)$$

where $\otimes$ is the contracting tensor product.

Since, by assumption $f_0(0)$ and $h(0)$ are zero, with (6) the same holds for $u(0)$. Hence,

$$\dot{X} = \left[\frac{\partial f_0}{\partial x}(0) + f(0)\frac{\partial u}{\partial x}(0)\right]X. \tag{13}$$

This means that the linearized closed loop system (13) is nothing but (L) in closed loop with

$$U = \frac{\partial u}{\partial x}(0)X. \tag{14}$$

Let us make $(\partial u/\partial x)(0)$ explicit. From (6), u satisfies

$$\Delta(x)u(x) = Z$$

with

$$Z^i = -L_{f_0}^{\rho_i+1}h_i - \sum_{j=0}^{\rho_i} \phi_j^i L_{f_0}^j h_i$$

and

$$\Delta_j^i(x) = L_{f_j} L_{f_0}^{\rho_i} h_i(x).$$

differentiating each member of the equality $\Delta(x)u(x) = Z$ we obtain, with $u(0) = 0$,

$$\Delta(0)\frac{\partial u}{\partial x}(0) = \frac{\partial Z}{\partial x}(0),$$

which gives

$$\frac{\partial u}{\partial x}(0) = \Delta^{-1}(0)\frac{\partial Z}{\partial x}(0).$$

We will obtain the expressions of $\Delta(0)$ and $(\partial Z/\partial x)(0)$ by showing by induction on every integer k that

$$\frac{\partial}{\partial x}\left(L_{f_0}^k h_i(x)\right)\Big|_{x=0} = \frac{\partial h_i}{\partial x}(0)\left(\frac{\partial f_0}{\partial x}(0)\right)^k. \tag{15}$$

(a) (15) is true for $k = 0$.

(b) Let us assume (15) holds for k. We have

$$\frac{\partial}{\partial x}\left(L_{f_0}^{k+1} h_i(x)\right)\Big|_{x=0} = \frac{\partial}{\partial x}\left(L_{f_0} L_{f_0}^k h_i(x)\right)\Big|_{x=0} = \frac{\partial}{\partial x}\langle \mathrm{d}L_{f_0}^k h_i;\ f_0\rangle\Big|_{x=0}.$$

Since $f_0(0)$ is zero, this equals

$$\langle \mathrm{d}L_{f_0}^k h_i;\ \nabla f_0\rangle\Big|_{x=0}.$$

By the induction assumption, this is equal to

$$\frac{\partial h_i}{\partial x}(0)\left(\frac{\partial f_0}{\partial x}(0)\right)^k \frac{\partial f_0}{\partial x}(0) = \frac{\partial h_i}{\partial x}(0)\left(\frac{\partial f_0}{\partial x}(0)\right)^{k+1}.$$

Using (15) we get the expression of $\Delta(0)$:

$$\begin{aligned} L_{f_j} L_{f_0}^{\rho_i} h_i(0) &= \langle \mathrm{d}L_{f_0}^{\rho_i} h_i;\ f_j\rangle\,|_{x=0} \\ &= \frac{\partial L_{f_0}^{\rho_i} h_i}{\partial x}(0) f_j(0) \\ &= \frac{\partial h_i}{\partial x}(0)\left(\frac{\partial f_0}{\partial x}(0)\right)^{\rho_i} f_j(0) \end{aligned}$$

and the expression of $(\partial Z^i/\partial x)(0)$:

$$\begin{aligned} \frac{\partial Z^i}{\partial x}(0) = &-\frac{\partial h_i}{\partial x}(0)\left(\frac{\partial f_0}{\partial x}(0)\right)^{\rho_i+1} \\ &-\sum_{j=0}^{\rho_i} \phi_j^i \frac{\partial h_i}{\partial x}(0)\left(\frac{\partial f_0}{\partial x}(0)\right)^j. \end{aligned}$$

Now, following (14), let us apply to the system (L) the following law:

$$\begin{aligned} U &= \frac{\partial u}{\partial x}(0)X + \Delta^{-1}(0)W \\ &= \Delta^{-1}(0)\left(\frac{\partial Z}{\partial x}(0)X + W\right). \end{aligned} \tag{16}$$

The outputs Y satisfy

$$\frac{\mathrm{d}^{(\rho_i+1)}}{\mathrm{d}t^{(\rho_i+1)}} Y_i + \sum_{j=0}^{\rho_i} \phi_j^i \frac{\mathrm{d}^j}{\mathrm{d}t^j} Y_i = W^i, \quad i = 1, \ldots, m, \tag{17}$$

and therefore denoting by $T(s)$ the closed loop transfer $W \to Y$, $\det(T^{-1}(s))$ is the polynomial given by

$$\det\left(T^{-1}(s)\right) = \prod_{i=1}^{m}\left(s^{\rho_i+1} + \sum_{j=0}^{\rho_i} \phi_j^i s^j\right), \tag{18}$$

whose roots are exactly the poles of (ΣI).

On the other hand, from Wolovich (see [13], Section 7.2), if Q denotes the equivalence transformation which gives the controllable companion form of (L), the closed loop transfer $T(s)$ can also be written, with $R(s)$ and $P(s)$ given by (11),

$$T(s) = R(s)P_c^{-1}(s)\Delta^{-1}(0) \tag{19}$$

where

$$P_c(s) = P(s) - \hat{C}S(s)$$

with

$$S(s) = \begin{pmatrix} 1 & 0 & \dots & 0 \\ \vdots & & & \vdots \\ s^{d_1-1} & 0 & & \\ 0 & 1 & & \\ \vdots & \vdots & \ddots & \vdots \\ & s^{d_2-1} & & 1 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & s^{d_m-1} \end{pmatrix}$$

the d_i being the controllability indices of (L) ($d_i \geq 1$ since G has full rank m) and $\hat{C} = CQ$ where $C = -(\partial u/\partial x)(0)$.

Using this expression of $T(s)$ we obtain

$$\det\left(T^{-1}(s)\right)\det\left(R(s)\right) = \det\left(\Delta(0)\right)\det\left(P_c(s)\right).$$

The conclusion follows with this equality, Definition 3, the invertibility of $\Delta(0)$ and the fact that the poles of the closed loop system are exactly the roots of $\det(P_c(z)) = 0$. □

Proof of the Theorem. We know by Proposition 2 that

$$P_1 = \{\text{poles of } (\Sigma I)\} \cup \{\text{eigenvalues of } \nabla \hat{f}(x_c)\}.$$

On the other hand,

$$P_2 = \{\text{poles of } (\Sigma I)\} \cup \{\text{finite linear zeros}\}.$$

From Proposition 3, $P_1 = P_2$ which concludes the proof. □

Remark. With (16) we have established the following commutative diagram:

$$\begin{array}{ccc} (\mathrm{NL}) & \xrightarrow{\text{linearization}} & (\mathrm{L}) \\ \downarrow & & \downarrow \\ (\mathrm{NL})\text{–}(6) & \xrightarrow[\text{linearization}]{} & (\mathrm{L})\text{–}(16) \end{array}$$

where each vertical arrow is obtained by applying the feedback law allowing us to meet the linear decoupled behaviour objective (3).

3. Choice of the *h* functions for stability

In some cases, the h functions are imposed by physical considerations. However, if the control objective is only to stabilize the equilibrium point, without any imposed h functions, the state feedback law (6) can still be used. The ability of choosing h to guarantee stability is proved in the following proposition:

Proposition 4. *If* (L) *is controllable, then one can always find* $H = (\partial h/\partial x)(0)$ *so, that the equilibrium point is an exponentially stable solution of the system* (NL)–(6).

Proof. Using the same notations as in the proof of the theorem, we have

$$\hat{H}S(s) = R(s), \qquad H = \hat{H}Q. \tag{20}$$

Let us consider a polynomial $Z(s)$ with degree less than or equal to $\Sigma_{i=1}^{m} d_i - m$ factorized in

$$Z(s) = P_1^m(s) P_2^{m-1}(s) \cdots P_m(s).$$

Let $U_1(s)$ and $U_2(s)$ be unimodular matrices. We define

$$R(s) = U_1(s) \begin{pmatrix} P_1(s) & & & \\ & P_1(s)P_2(s) & & \\ & & \ddots & \\ & & & P_1(s)\cdots P_m(s) \end{pmatrix} U_2(s).$$

If $U_1(s)$ and $U_2(s)$ are such that the i-th column degree of $R(s)$ is less than d_i, then H can be obtained from (20). $Z(s)$ being arbitrary, the proposition is proved. □

Remarks. (a) Noticing that the above expression of R is its Smith form, the arguments used in this proof give a complete description of the set of matrices H leading to stable zeros when the polynomial $Z(s)$ is constrained to have its roots in the left half complex plane. In the scalar case equation (20) can be directly used, taking for $\hat{H}$ the coefficients of the polynomial numerator of the transfer function. In the multivariable case, the above expressions are more difficult to handle.

(b) A priori $H = (\partial h/\partial x)(0)$ will have components on the whole state. Consequently, around the equilibrium, the ρ_i will generically be zero, and the linear system obtained by feedback be of minimal dimension m.

We propose an algorithm which gives H by choosing the poles of the set P_1, and imposing $\rho_i = 0$, $i = 1, \ldots, m$.

It is based on the following fact:

Proposition 5. *Under the assumption* $\rho_i = 0$, H_i, *the i-th row of the matrix H, is a left eigenvector of* $F - GC$, *associated with the eigenvalue* $\lambda_i = -\phi_0^i$, $i = 1, \ldots, m$.

Proof. From (16) it appears that the matrix C is given by

$$C = -\frac{\partial u}{\partial x}(0) = (HG)^{-1}(HF + \phi_0 H). \tag{21}$$

Consequently,

$$H(F - GC) = -\phi_0 H. \quad □ \tag{22}$$

Under the controllability assumption of the pair (F, G), the following algorithm can be used for choosing the h functions so as to insure a pole-placement for the linear system, obtained by linearizing (NL)–(6):

- Find a matrix C to place the poles of $F - GC$ in the left half complex plane to stabilize (L). This is always possible from the controllability assumption of (F, G).
- Choose m of these poles, and denote them λ_i, $i = 1, \ldots, m$, (these m poles will correspond to those assigned by the ϕ_0^i).

– Solve the equations $H_i(F - GC) = \lambda_i H_i$ for $i = 1, \dots, m$.
– The matrix H obtained by the superposition of the rows H_i is such that, if HG is non-singular, P_1 is equal to the set of the poles of $F - GC$.

4. Example

While equilibrium point stability is important, guaranteed behavior for a larger subsystem is also attractive. This leads us to look for nonlinear functions h, leading to both stability and maximal linearization.

We illustrate this aspect by an example coming from robotics. Consider a link of mass m_1, length $2l_1$ and inertia I_1, turning in a vertical plane around an horizontal axis with angle θ_1. The joint is controlled by a motor. This link is topped with a stick of mass m_2, length $2l_2$ and inertia I_2, turning freely at its extremity, with an angle θ_2. This system can be described by:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = \left(\Gamma_1(x_1) + JR^2\right)^{-1}\left(-x_2'\Gamma_2(x_1)x_2 \right. \\ \qquad \left. - QVR^2 x_2 + \Gamma_0(x_1) + Qu\right), \end{cases} \tag{23}$$

$$x_1 = (\theta_1, \theta_2)', \quad x_2 = (\dot{\theta}_1, \dot{\theta}_2)', \quad Q = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{24}$$

J is the motor inertia, V its viscous friction coefficient, R its reduction ratio, $\Gamma_1(x_1)$ is the inertia matrix, the vector $x_2'\Gamma_2 x_2$ represents the centrifugal and Coriolis torques and the vector Γ_0 the gravity torques.

We have $m = 1$, $n = 4$. The largest feedback linearizable subsystem from (23) with the feedback (6) has dimension $n_\lambda = 2$ with $\rho = 1$ (see D'Andrea and Levine [5,6]). It can be obtained for example by taking h a function in the angular variables, but independent of the velocities.

Hence in a first step let us choose $h(x)$ as $h(x_1)$. In this case H is of the form

$$H = [a \quad b \quad 0 \quad 0]. \tag{25}$$

To get also stability this H should solve (20) with the zeros of $R(s)$ in the left half complex plane.

Let us consider an equilibrium point x_c such that the stick is stabilized in its upper vertical position:

$$\begin{cases} \theta_1^c = -\theta_2^c, \\ \dot{\theta}_1^c = \dot{\theta}_2^c = 0. \end{cases} \tag{26}$$

The linearized system (L) around x_c is given by

$$F = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ f_{31} & f_{32} & f_{33} & 0 \\ f_{41} & f_{42} & f_{43} & 0 \end{pmatrix}, \quad G = \begin{pmatrix} 0 \\ 0 \\ g_1 \\ g_2 \end{pmatrix}. \tag{27}$$

Let $n(s)$ be the numerator of $(sI - F)^{-1}G$, applying the Routh criterion to $Hn(s)$ gives the following conditions on the components a and b of H:

$$\begin{cases} ag_1 + bg_2 > 0, \\ b(g_1 f_{43} - g_2 f_{33}) > 0, \\ a(g_2 f_{32} - g_1 f_{42}) + b(g_1 f_{41} - g_2 f_{31}) > 0. \end{cases}$$

This system has no solution in a and b if

$$\begin{cases} \dfrac{g_1(g_1 f_{43} - g_2 f_{33})}{g_1(g_1 f_{41} - g_2 f_{31}) - g_2(g_2 f_{32} - g_1 f_{42})} \le 0, \\ \dfrac{(g_1 f_{42} - g_2 f_{32})(g_1 f_{43} - g_2 f_{33})}{g_1(g_1 f_{41} - g_2 f_{31}) - g_2(g_2 f_{32} - g_1 f_{42})} \le 0. \end{cases}$$

Hence, for a non-zero Lebesgue measure set of (g_i, f_{ij}), stability cannot be obtained.

This leads us to look for h as functions of the whole state.

To minimize the dimension of the unobservable part or equivalently, to maximize the dimension of the linear system obtained after feedback, h must satisfy

$$L_f h(x) \equiv 0. \tag{28}$$

Since the input vector field is of the form

$$f = g_1(\theta_2)\frac{\partial}{\partial \dot{\theta}_1} + g_2(\theta_2)\frac{\partial}{\partial \dot{\theta}_2}, \tag{29}$$

(28) is satisfied by

$$h(x) = a\theta_1 + b\theta_2 + cg_2(\theta_2)\dot{\theta}_1 - cg_1(\theta_2)\dot{\theta}_2, \tag{30}$$

where a, b, c are arbitrary constants.

For such a function, we obtain

$$H = [a \quad b \quad cg_2 \quad -cg_1]. \tag{31}$$

Then, the Routh criterion applied to $Hn(s)$ gives,

$$\begin{cases} ag_1 + bg_2 > 0, \\ cg_2(g_2 f_{32} - g_1 f_{42}) > 0, \\ (g_2 f_{32} - g_1 f_{42})(a - b + cg_1) + cg_1 g_2 > 0. \end{cases}$$

These inequalities have a solution in a, b and c if

$$g_2(g_2 f_{32} - g_1 f_{42})^2(g_1 + g_2) \neq 0.$$

Hence, except may be for a set of zero Lebesgue measure of (g_i, f_{ij}), one can solve the stabilization problem.

This example points out the fact that it is possible both to stabilize the equilibrium and to obtain by feedback the largest linear system. However, this requires to find solutions of the partial differential equations:

$$L_{f_j} h^i(x) \equiv 0, \quad i, j = 1, \ldots, m. \tag{32}$$

These solutions must have a sufficient degree of freedom to allow satisfaction of the stability constraints.

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